MODELING OF NATURAL TURBULENT CONVECTION IN AN ENCLOSURE WITH LOCALIZED HEATING

BY

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DECLARATION

I Josephat Kiplagat Kiprop declare that this is my work and has not been submitted to any university in part or whole for a degree award.

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I confirm that this work has been done by the student under my supervision

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DEDICATION

I dedicate this project to my parents James Kiprop and Esther Kiprop for their encouragement and support.

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First, I acknowledge God for His grace and care.

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ABSTRACT

This study model natural turbulent convection in a rectangular enclosure with localized heating. The equations used in modeling the flow are the continuity equation, the momentum equation, and the energy equation. These equations are decomposed using the Reynolds decomposition then the decomposed equations are non-dimensionalized and reduced using the Boussinesq assumptions. The model that is considered is a rectangular enclosure with the lower part of the face-wall being heated and the upper part of the face-wall being cooled. The other walls of the enclosure are adiabatic. The nonlinear differential questions obtained by using the k-ε model are solved using the finite difference technique and a computer program called Fluent 6.3.26 is used in the presentation of results in form of vector potentials and isotherms. The results of the study indicate that with the increase in the Rayleigh number there is an increase in the number of vortices and stream functions. With regard to the velocity magnitude, it is found that an increase in the Rayleigh number results in an increase in the turbulence hence implying that there is an increase in the velocity magnitude. In relation to the distribution of isotherms, it is found that the number of contours near the hot part of the enclosure are more and reduce towards the top part of the enclosure.

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NOMENCLATURE

Greek Symbols

ρ	Density
ε	turbulent dissipation
k	turbulent kinetic energy
ω	specific dissipation

Roman Symbols

Pr	Prandtl number
u_j	Velocity component
x_j	Function of position

Abbreviations

FOTRAN	formula translator
FVM	Finite Volume Method
LBM	Lattice Boltzmann Method
RANS	*Reynolds-averaged Navier Stokes*
RNG	Re-Normalisation Group
RSM	*Reynolds stress model*
SST	Shear stress transport
CPU	Central processing unit

CHAPTER ONE

INTRODUCTION

1.1 Background

A fluid is defined as a substance that undergoes deformation when an external force is subjected to it. The fluid may be a liquid or a gas. There are two categories of fluid mechanics namely fluid kinematics and fluid dynamics. Fluid kinematics involves the study of forces that are involved in the motion of fluids while fluid dynamics involves the study of states of fluid motion. The flow of fluids can be categorized depending on the properties of the fluid or depending on the properties of the flow. With regard to the properties of the fluid, there are ideal and real flows, incompressible and compressible flows. On the other hand, the types of flow depending on properties of flow could be laminar and turbulent flow, steady and unsteady flow, or uniform and non-uniform flow. An ideal fluid flow involves the flow of an ideal fluid whereby there is no resistance encountered in the flow while a real fluid flow occurs for a fluid that is viscous in nature and there is a certain amount of resistance to flow. In a uniform flow, the velocity and other parameters such as the pressure of the fluid do not change from one point to the other while in a non-uniform flow there are changes in the hydrodynamic parameters such as pressure and density from one point to the other as the fluid flows.

In an incompressible fluid flow, there is an assumption that the density of the fluid does not change over the flow path while in a compressible flow the density is a function of temperature and pressures in the flow field. A flow in which the fluid particles follow a smooth path and hence does not interfere with each other is called a laminar flow while a turbulent flow is a flow that is characterized by irregular flow and occurs in the instances that the velocity of the fluid is high. In a turbulent flow, there are whirlpools. A steady flow is one in which the fluid properties such as velocity, pressure, temperature, and density are independent of time while in an unsteady flow the fluid properties are functions of time. Specifically, this study will be based on turbulent flow.

The transfer of heat can occur in three ways namely conduction, convection, and radiation. In conduction, the transfer of heat occurs when two objects that have different temperatures come in contact with one another. The flow of heat moves from the object with the higher temperature to the one with a lower temperature. Convection is an efficient way of heat transfer in fluids and occurs as a result of differences in temperatures in varying areas of the fluid. Precisely, convection occurs when cooler fluids occupy the place for the warmer fluid resulting in a continuous circulation of fluids. In radiation, the flow of heat does not depend on the availability of any contact between the objects or fluid for heat to be transferred.

This study will mainly be based on the study of the transfer of heat through convection. In a natural convection, the occurrence of motion of a fluid result from the gravitational field that occurs due to the differences in density resulting from varying temperatures. Along these lines, there will be a buoyant force causing dense fluid parts to move to the lower region while the less dense fluid parts will move upwards. The investigation on the effects of the flow of fluids due to buoyancy is applicable in practical occurrences such as in the process of thermal insulation of buildings, cooling of electronic packages, as well as in passive heat removal systems of a liquid metal nuclear reactor among other engineering applications.

1.2 Problem Statement

There has been studies on the modeling of natural turbulent convection flow of fluids in an enclosure by difference researchers. However, the determination of fluid quantities such as velocity, temperature, pressure, and or density is challenging due to the presence of unknown turbulent correlations in the equations governing turbulent flows. This is attributed to the fact that the terms are nonlinear. Along these lines, there is need for the development of a model that could help in solving for the fluid quantities in a natural convection. The model that has been developed in this project allows for the determination of the velocity of the fluid in different parts of the enclosure and the temperature distribution in different parts of the enclosure.

1.3. Objectives

1.3.1. General Objectives

To model natural turbulent convection in an enclosure with Localized Heating

1.3.2. Specific Objectives

1. To determine the velocity of flow in different parts of the enclosure.
2. To determine the temperature variations in different parts of the enclosure.
3. To determine the effects of varying the Rayleigh number on the heat transfer.

1.4 Significance of the study

This study will be applicable in various engineering, farming or manufacturing practices. As such, the model will be useful for saving on the computing requirements that needed for solution of large problems of turbulent flows such as in thermal insulation of buildings and cooling of electronic packages. In addition, the results from this study will be useful in the determination and control of a micro-climate by ventilation, heating, and/ or cooling for thermal comfort in a poultry house. Furthermore, the data from this study will be applicable in the determination of optimum temperatures, pressure, and/ or velocity in a manufacturing plant for effective storage and preservation.

CHAPTER TWO

LITERATURE REVIEW

Different studies have been carried out on natural convection in enclosures. One of the studies that carried out an investigation on natural convection was the study by Alkshaish, and Esfahani (2017). In the study, the authors were focusing on the investigation of natural convection heat transfer in an enclosure that was of a rectangular shape. The side walls of the enclosure were insulated while the bottom wall and the top walls were kept at a constant temperature. In the investigation, Alkshaish, and Esfahani (2017) employed a single relaxation time Bhatnagar-Gross-Krook method for the Boltzmann method that was used. The results of the investigation showed that there were smooth isotherms for low Rayleigh numbers and there were strong vortices in the lower and the upper walls. The authors as well found that the increase in Rayleigh number results in an increase in the Nusselt number.

Altaç and Uğurlubilek (2016) investigated unsteady natural convection heat transfer in 2D and 3D rectangular enclosures using a numerical method. In the study, the rectangular enclosures were heated and cooled from opposing isothermal walls that were vertical to each other and the other side walls of the rectangular enclosure were assumed to be adiabatic and smooth. Further, the fluid that was used in the study was air and the flow considered was turbulent. Commercial software called FLUENT 6.3.26 was used in solving 2D and 3D continuity equation in the unsteady state, Reynolds-Average Navier-Stokes (RANS), as well as the averaged energy equation. The standard K-ε, Re-Normalization Group k-ε, Realizable k-ε, Reynolds Stress Model, standard k-ω, and the shear stress transport k- ω models were used. Precisely, the performance of turbulence models on heat transfer rates was investigated for 2D and square enclosures and for 3 dimension rectangular enclosures with the slenderness ratio of 1 and 10 respectively. The assessment of heat transfer rates was done by the surface averaged mean Nusselts numbers over the wall that was hot and empirical power-law correlations were deduced. It was noted that identical Nusselt number prediction up to $Ra=10^{10}$ were derived for 3D laminar and RANS models. There were no accurate predictions for the case of a 2D RANS models when there were large Rayleigh numbers. Further, it was concluded that accurate mean Nusselt numbers are yielded from 3D RANS models.

Further research on natural convection in an enclosure has been studied by Awuor (2012). Awuor (2012) investigated the performance of three numerical models namely K-ε, k-ω, and k-ω SST with the aim of ascertaining the better approximation to experimental data in the process of predicting heat transfer profiles in an air-filled cavity. In the study, the author used vorticity vector formulation in solving the momentum equation. It was noted that K-ε model was not an effective model in the case where the temperature gradient at the boundaries was high but was useful in the free stream flows. In addition, it was deduced that k-ω SST model gives accurate predictions under adverse pressure gradient and accounts for the transport shear stress. In terms of the convergence time, Awuor (2012) found that k-ω SST model was a more accurate model to be used in layer simulation at a high-temperature gradient in comparison to k-ε model and k-ω model. Awuor (2012) further obtained numerical data using k-ω SST model and noted that the room was stratified into the cold upper region, a warm lower region and a hot region between the window and heater. Edward *et al.* (2013) undertook a numerical study of turbulent natural convection flow in a 3-dimensional enclosure that had heaters on opposite sides two windows placed at each of the opposite adjacent walls. The governing equations of Newtonian fluid and the boundary conditions were discretized using three-point central difference approximation for the non-uniform mesh. The velocity profile and the temperature distribution results in the room were presented in graphs. The authors noted that the room was divided into numerous regions with the regions close to the heaters having higher temperatures as compared to the regions near the windows.

Goodarzi *et al*. (2014) have also investigated the effective and accurate numerical method for simulation of natural convection heat transfer. The study of a heated square enclosure was carried out and for that purpose a FORTRAN code on the basis of Lattice Boltzmann method was used. The discretization of the LBM equations was done using the finite difference method. FLUENT was used for comparison purpose in the simulation of the problem. The different discretization schemes such as First order upwind, second-order upwind, power law and QUICK while the finite volume solver used were SIMPLE and SIMPLEC algorithm then they were linked to the velocity pressure terms. The results derived from the model were compared with the experimental data. It was noted that less CPU usage time was required for the cases of using finite volume methods. In addition, accurate results were derived by the use of finite volume methods in comparison with LBM. Furthermore, the authors espoused that there was a faster convergence and higher accuracy when using a combination of the first order upwind and SIMPLEC. The convergence of the FVM discretization and pressure velocity linking methods converged after a closely similar number of iterations.

Khanal and Lei (2015) carried out a numerical investigation of the buoyancy induced turbulent air flow in an inclined passive wall solar chimney that was attached to a room. k- ε model was employed in modeling the air turbulence in the solar chimney system. The investigation was carried out over the Rayleigh number range of $1.36\times10^{13}\leq 1.36\times10^{16}$ and the angle of inclination was between 0^o and 6^o. The result in the study by Khanal and Lei (2015) indicated that there was a decrease in the amount of turbulent kinetic energy and turbulent intensity in the solar chimney with the increase in the angle of inclination.

Kimunguyi (2016) undertook a computational investigation of natural turbulent convective flow using primitive variables in the solution of the time average equations governing the flow instead of making use of the vorticity-vector potential formulation. In the study, Kimunguyi (2016) carried out a non-dimentionalization of the governing equations, the non-dimensional equations were then discretized using FVM and PISO and SIMPLEC algorithms were employed in the solution of these equations. The results of the study indicated that PISO method helped in the improvement of convergence time and speed. The method as well reduced the computational effort and there was a faster removal of the absolute error in the solution of flow. The author has concluded that the obtained velocity profiles and temperature profiles were vital for use in ventilation systems for instance in the process of modeling the flow of air in a room.

The study by Saeid (2018) focused on the investigation of the impacts of varying the fin shapes on the square enclosure that has a small heating strip the bottom wall. The other walls of the enclosure are adiabatic and a constant temperature of the walls was maintained. The researcher varied the area of the fin and found that there was variation in the heat transfer. The results in the study indicated that with the increase in Grashof number which was done by increasing the height of the fins, there was an enhancement of heat transfer. The authors found that with the use of triangular fin shapes there was a higher heat transfer as compared to the other shapes.

Safaei *et al*. (2016) did a study on turbulent natural convection and laminar mixed convection flow of air in a room. The results of the study were then compared with data of other scientists. In the study, the authors solved the flow as a turbulent mixed convection flow by employing RNG k-ε, Standard k-ε, and RSM. Finite volume method was used in the solution of the governing differential equations. Safaei *et al*. (2016) noted that the flow was stationary at the center of the room at high Richardson Numbers. In addition, it was deduced that an increase in Richardson

Number led to a reduction in the maximum of Nusselt and hence a smaller Richardson Number implies a more rate of transfer of heat.

Sajjadi and Kefayati (2015) undertook a Lattice Boltzmann simulation of turbulent natural convection with large eddy simulations in tall enclosures that were filled with air (Pr=0.71). High Rayleigh numbers ranging from 10^7 and 10^9 and an aspect ratio change between 0.5 and 2 were used in performing the calculations. The authors concluded that the average Nusselt number increased with the augmentation of Rayleigh numbers leading to a declination in the heat transfer in varying aspect ratios.

Wu and Lei (2015) studied turbulent natural convection in 2D and 3D with and without radiation transfer in heater cavities. Various Reynolds Average Navier-Stokes (RANS) turbulence model and the Discrete Ordinates radiation model were used in the numerical investigation. Further, Wu and Lei (2015) used quantitative and qualitative data for demonstration of the effects of three-dimensionality, thermal buoyancy condition and radiation transfer in surfaces that are horizontal. The derived simulation data from the numerical investigation are compared with the experimental data. The authors noted that when the radiation transfer was not accounted for the thermal boundary condition had an impact on the numerical solution on the horizontal surfaces. Moreover, it was deduced that variation of numerical results obtained when using three k-ε model were small. In the study, it was concluded that the k- ω SST model had the best overall performance while k-ω model had the worst performance.

Zimmermann and Groll (2014) undertook a numerical study on turbulent natural convection that had large eddy simulation. In the study, the acceleration in natural convection was driven by the differences in local densities and the pressure gradient. The increase in temperature gradients was used in determining the temperature distribution in the heated walls. In the numerical model, Zimmermann and Groll (2014) considered the change in density to occur due to change in temperature difference. The author made a comparison of the numerical results with the data from an experimental setup. It was noted that the temperature and the velocity showed an asymmetry as a result of the non-Boussinesq effects of the fluid. In the study, a recommendation for the study of an incompressible turbulent model simulation was made.

From the studies that have been carried out in the past as it can be seen in the literature work, there is little research on the impacts of change of Rayleigh number on the flow quantities with the use of k-ε model. As such, this study intended to provide a model for ascertaining the flow quantities in relation to the change in the Rayleigh number.

CHAPTER THREE

GOVERNING EQUATIONS

3.1 Introduction

The motion of fluids in a free convection is attributed to the buoyancy forces. In this section of the study, we will consider the equations governing a free convection. Precisely, the equations will govern a Newtonian fluid that experiences transfer of heat or mass. The governing equations are derived from the conservation principles namely the conservation of mass, the conservation of momentum, and the conservation of energy. Partial differential equations are used to represent the equations of a natural convection in turbulent flow. Taking this into consideration, a fluid that has the density ρ is a function of position x_j for j=1, 2, 3 and the velocity components of the fluid given by u_j where j=1, 2, 3, the derivation of the equations of conservation of mass, momentum, and energy have been presented by Currie (1974).

3.2 Continuity Equation

The continuity equation is based on the law of conservation of mass that states that the rate of increase of mass within a controlled volume is equal to the net rate of influx through the controlled surface. Along these lines, the continuity equation in Cartesian tensor notation as defined by Currie (1974) in general is:

$$\frac{\partial \rho}{\partial t} + \frac{\partial}{\partial x_j}\left(\rho u_j\right) = 0 \tag{3.1}$$

For steady state the rate of change of density of the Newtonian fluid is constant and hence the equation given in (3.1) reduces to;

$$\frac{\partial}{\partial x_j}\left(\rho u_j\right) = 0 \tag{3.2}$$

This equation is applicable for both compressible and incompressible flows and it implies that as the fluid flows, matter is neither created nor destroyed and hence the mass of the fluid does not change in the flow.

3.3 Momentum Equation

The momentum equation is derived from the Newton's second law of motion that states that the sum of the body forces and surface forces acting on a system is equal to the time rate of momentum of the system. As derived by Anderson *et al.* (1984) the momentum equation is given as;

$$\rho F_i + \frac{\partial}{\partial x_j}(\pi_{ij}) = \frac{\partial}{\partial t}(\rho u_i) + \frac{\partial}{\partial x_j}(\rho u_i u_j) \tag{3.3}$$

In this equation, the π_{ij} represents the stress tensor while ρF_i represents the body forces per unit volume. White (1974) indicates that for a Newtonian fluid, the stress tensor (π_{ij}) is represented as;

$$\pi_{ij} = -P\delta_{ij} + \tau_{ij} \tag{3.4}$$

In this case, the δ_{ij} is the Kronecker delta with values 1 for $i = j$ and 0 for $i \neq j$ while the term given by τ_{ij} is the viscous stress tensor given as;

$$\tau_{ij} = \mu\left(\frac{\partial u_i}{\partial x_j} + \frac{\partial u_j}{\partial x_i}\right) + \mu_s \delta_{ij}\frac{\partial u_k}{\partial x_k} \tag{3.5}$$

In which the terms μ is the first coefficient of viscosity while μ_s represents the second coefficients of viscosity. In this account, it can be deduced that;

$$\pi_{ij} = -P\delta_{ij} + \mu\left(\frac{\partial u_i}{\partial x_j} + \frac{\partial u_j}{\partial x_i}\right) + \mu_s \delta_{ij}\frac{\partial u_k}{\partial x_k} \tag{3.6}$$

Specifically, the electromagnetic forces are not taken into consideration in this work. As such, the only force being considered will be the gravitational force, let the gravitational force in the x_i direction be given by g_i and from equation (3.3) the force F_i and be written asg_i. The momentum equation for Newtonian fluid that is being considered in this work can be got by substituting equations (3.6) for the term given by π_{ij}in equation (3.3). This gives the momentum equation as;

$$\rho g_i - \frac{\partial P}{\partial x_i} + \frac{\partial}{\partial x_j}\left[\mu\left(\frac{\partial u_i}{\partial x_j} + \frac{\partial u_j}{\partial x_i}\right) + \mu_s \delta_{ij}\frac{\partial u_k}{\partial x_k}\right] = \frac{\partial}{\partial t}(\rho u_i) + \frac{\partial}{\partial x_j}(\rho u_i u_j) \tag{3.7}$$

The δ_{ij} is the Kronecker delta with values 1 for $i = j$ and 0 for $i \neq j$

3.4 Energy Equation

The energy equation is derived from the first law of thermodynamics that states that the rate of increase of energy in a system is equal to the heat added to the system and the work done on the system. According to Currie (1974), the energy equation is written as;

$$\rho\frac{Dh}{Dt} + \frac{\partial}{\partial x_j}(\rho u_j h) = \frac{\partial p}{\partial x_j} + \frac{\partial}{\partial x_j}(u_j p) - \frac{\partial q}{\partial x_j} + \Phi \tag{3.8}$$

Where the dissipation function (Φ) is given by;

$$\Phi = \tau_{ij}\frac{\partial u_i}{\partial x_j} \tag{3.9}$$

h =Specific enthalpy

q_i = Local rate of heat transfer per unit area

In the equation of the dissipation function given by Φ in (3.9) the heat produced by the external is neglected. The modeling of the heat has been done using the Fourier's law given by;

$$q_i = \lambda\frac{\partial T}{\partial x_j} \tag{3.10}$$

λ = Thermal conductivity

The simplification of equation (3.8) can be done by making use of the definition of$h$ as below;

$$h = e + \frac{p}{\rho} \tag{3.11}$$

e =Specific internal energy

In differential form, equation (3.11) is given as;

$$dh = de + \frac{1}{\rho}dp + pd\left(\frac{1}{\rho}\right) \quad (3.12)$$

By making use of the first and the second law of thermodynamics as presented by Hatsopolous and Keenan (1965), the change in specific energy de can be expressed as;

$$de = Tds - pd\left(\frac{1}{\rho}\right) \quad (3.13)$$

$s =$ Entropy

On substituting equation (3.13) in equation (3.12) we get;

$$dh = Tds - pd\left(\frac{1}{\rho}\right) + \frac{1}{\rho}dp + pd\left(\frac{1}{\rho}\right)$$

This implies that

$$dh = Tds + \frac{1}{\rho}dp \quad (3.14)$$

The entropy s depends on the temperature and pressure. This implies that it can be written as $s = s(T, P)$ and the total differentiation is given as;

$$ds = \left(\frac{\partial s}{\partial p}\right)_T dp + \left(\frac{\partial s}{\partial T}\right)_p dT \quad (3.15)$$

Using the generalized thermodynamics relation given by Hatsopolous and Keenan (1965) as;

$\left(\frac{\partial s}{\partial p}\right)_T = -\frac{\beta}{\rho}, \left(\frac{\partial s}{\partial T}\right)_p = \frac{c_p}{T}$ and $\left(\frac{\partial\left(\frac{1}{\rho}\right)}{\partial T}\right)_p = -\frac{\beta}{\rho}$ equation (3.15) is reduced to;

$$ds = -\frac{\beta}{\rho}dp + \frac{c_p}{T}dT \quad (3.16)$$

Whereby;

β= volumetric coefficient of expansion

c_p = Specific heat at constant pressure

On substituting equation (3.16) into equation (3.14) we get;

$$dh = T\left(-\frac{\beta}{\rho}dp + \frac{c_p}{T}dT\right) + \frac{1}{\rho}dp$$

Or

$$dh = c_p T + \frac{1}{\rho}(1 - \beta T)dp \tag{3.17}$$

On substituting equations (3.17) and (3.10) into equation (3.8) we can get the final form of energy equation given below as;

$$\frac{\partial}{\partial t}\left(\rho c_p T\right) + \frac{\partial}{\partial x_j}\left(\rho c_p u_j T\right) = \frac{\partial}{\partial x_j}\left(\lambda \frac{\partial T}{\partial x_j}\right) + \beta T\left(\frac{\partial p}{\partial t} + \frac{\partial u_j p}{\partial x_j}\right) + \Phi \tag{3.18}$$

The equations of continuity, momentum, and energy as given by equations (3.1), (3.7), and (3.18) respectively are in general form. One can use these equations and appropriate boundary conditions in the determination of the velocity component u_j, the densityρ, pressure p, and temperature T in a laminar or turbulent flows. However, the derivation of exact solutions for turbulent flows is not possible because the pressure and velocity are coupled in a complicated manner and there is lack of an equation that can be used explicitly in the discussion of pressure.

3.5 Reynolds Decomposition

The process of modeling a turbulent flow necessitates for the derivation of exact solutions of the governing equations. This can be difficult as attributed to the many scales that are involved in turbulent flows. The fluctuations that are seen are as a result of turbulent eddies and are not representatives of the mean flow. Taking this into account, there is a need for development of an approximation to reduce the complexity. An approximation called Reynolds-averaged Navier-Stokes equations (RANS equations) helps in the decomposition of the quantities of a flow to mean values or time-averaged values and the fluctuating values. Precisely, we need to take an average over a period of time that is long enough to smooth over the fluctuations but should be short enough to keep the trend. This is called the Reynolds decomposition. The process of Reynolds decomposition will allow us to substitute the sum of the steady and the fluctuating components and take the average value. The over-bar and prime are used to denote the mean value and the fluctuating value respectively. For instance, the property A could be a fluid property and we defined the time average variable $\bar{A}$ as follows;

$$\bar{A} = \frac{1}{T}\int_0^T A(x,t)\,\mathrm{dt} \tag{3.19}$$

Where;

T is a time that is large enough to have the quantity A being the same. The quantity may be written as follows;

$$A = \bar{A} + \acute{A} \tag{3.20}$$

Taking the time average of both sides of equation (3.20) yields

$$\bar{A} = \overline{\bar{A} + \acute{A}}$$

Or

$$\bar{A} = \bar{A} + \bar{\acute{A}}$$

This implies that

$$\bar{\acute{A}} = 0 \tag{3.21}$$

As such, the time average value of the fluctuating quantity is zero.

In this section we are going to undertake the derivation of equations of motion for the mean state in a turbulent flow. The first step in this process is putting down the equation of the instantaneous quantities before taking the time average for both sides. The basic idea here is that a quantity that is valid in an instance is as well valid on the average for a given time period. The final step is the simplification of the equations to ensure that the quantities appearing are only the mean quantities. Using the concept in equation (3.20) the fluid components can be decomposed into the sum of time average values and the fluctuating components. For example, $\rho = \bar{\rho} + \acute{\rho}$.

3.5.1 Continuity Equation

The equation of instantaneous quantities is;

$$\frac{\partial\rho}{\partial t}+\frac{\partial}{\partial x_j}\left(\rho u_j\right)=0$$

Now decomposing the variables of the equation of continuity equation using Reynolds decomposition we have;

$$\frac{\partial(\overline{\rho}+\acute{\rho})}{\partial t}+\frac{\partial(\overline{\rho}+\acute{\rho})(\overline{u_j}+\acute{u}_j)}{\partial x_j}=0 \tag{3.22}$$

On taking the time average of both sides of the equation we get the following;

$$\overline{\frac{\partial(\overline{\rho}+\acute{\rho})}{\partial t}+\frac{\partial(\overline{\rho}+\acute{\rho})(\overline{u_j}+\acute{u}_j)}{\partial x_J}}=0$$

Or

$$\frac{\overline{\partial(\overline{\rho}+\acute{\rho})}}{\partial t}+\frac{\overline{\partial(\overline{\rho}+\acute{\rho})(\overline{u_j}+\acute{u}_J)}}{\partial x_J}=0$$

Or

$$\frac{\partial\overline{\rho}}{\partial t}+\frac{\partial\,\overline{\acute{\rho}}}{\partial t}+\frac{\partial}{\partial x_j}(\overline{\overline{\rho}\overline{u}_J}+\overline{\overline{\rho}\acute{u}_J}+\overline{\overline{u}_J\acute{\rho}}+\overline{\acute{\rho}\acute{u}_J})=0$$

Or

$$\frac{\partial\overline{\rho}}{\partial t}+\frac{\partial\,\overline{\acute{\rho}}}{\partial t}+\frac{\partial}{\partial x_j}\left(\overline{\overline{\rho}\overline{u}_J}\right)+\frac{\partial}{\partial x_j}\left(\overline{\overline{\rho}\acute{u}_J}\right)+\frac{\partial}{\partial x_j}\left(\overline{\overline{u}_J\acute{\rho}}\right)+\frac{\partial}{\partial x_j}\left(\overline{\acute{\rho}\acute{u}_J}\right)=0$$

Now, making use of the idea from equation (3.21) that the time average value of the fluctuating quantity is zero yield;

$$\frac{\partial\overline{\rho}}{\partial t}+\frac{\partial}{\partial x_j}\left(\bar{\rho}\overline{u}_J\right)+\frac{\partial}{\partial x_j}\left(\overline{\acute{\rho}\acute{u}_J}\right)=0 \tag{3.23}$$

This is the Reynolds form of the continuity equation.

3.5.2 Momentum Equation

The momentum equation is given by equation (3.7) as;

$$\rho g_i-\frac{\partial P}{\partial x_i}+\frac{\partial}{\partial x_j}\left[\mu\left(\frac{\partial u_i}{\partial x_j}+\frac{\partial u_j}{\partial x_i}\right)+\mu_s\delta_{ij}\frac{\partial u_k}{\partial x_k}\right]=\frac{\partial}{\partial t}(\rho u_i)+\frac{\partial}{\partial x_j}(\rho u_i u_j)$$

Using Reynolds decomposition with the knowledge that the kronecker delta δ_{ij}=1 or 0, this equation can be written as follows;

$$-\frac{\partial(\bar{P}+\acute{P})}{\partial x_i}+(\bar{\rho}+\acute{\rho})g_i+\frac{\partial}{\partial x_j}\left[\mu\left(\frac{\partial(\overline{u_i}+\acute{u}_i)}{\partial x_j}+\frac{\partial(\overline{u_j}+\acute{u}_j)}{\partial x_i}\right)+u_s\frac{\partial(\overline{u_k}+\acute{u}_k)}{\partial x_k}\right]=\frac{\partial(\bar{\rho}+\acute{\rho})(\overline{u_i}+\acute{u}_i)}{\partial t}+\frac{\partial}{\partial x_j}(\bar{\rho}+\acute{\rho})(\overline{u_i}+\acute{u}_i)(\overline{u_j}+\acute{u}_j)$$

Taking the time average on both sides of this equation yields;

$$\overline{-\frac{\partial(\bar{P}+\acute{P})}{\partial x_i}+(\bar{\rho}+\acute{\rho})g_i+\frac{\partial}{\partial x_j}\left[\mu\left(\frac{\partial(\overline{u_i}+\acute{u}_i)}{\partial x_j}+\frac{\partial(\overline{u_j}+\acute{u}_j)}{\partial x_i}\right)+u_s\frac{\partial(\overline{u_k}+\acute{u}_k)}{\partial x_k}\right]}=$$

$$\overline{\frac{\partial(\bar{\rho}+\acute{\rho})(\overline{u_i}+\acute{u}_i)}{\partial t}+\frac{\partial}{\partial x_j}(\bar{\rho}+\acute{\rho})(\overline{u_i}+\acute{u}_i)(\overline{u_j}+\acute{u}_j)}$$

Or

$$-\overline{\frac{\partial(\bar{P}+\acute{P})}{\partial x_i}}+\overline{(\bar{\rho}+\acute{\rho})g_i}+\overline{\frac{\partial}{\partial x_j}\left[\mu\left(\frac{\partial(\overline{u_i}+\acute{u}_i)}{\partial x_j}+\frac{\partial(\overline{u_j}+\acute{u}_j)}{\partial x_i}\right)+u_s\frac{\partial(\overline{u_k}+\acute{u}_k)}{\partial x_k}\right]}=\overline{\frac{\partial(\bar{\rho}+\acute{\rho})(\overline{u_i}+\acute{u}_i)}{\partial t}}+$$

$$\overline{\frac{\partial}{\partial x_j}(\bar{\rho}+\acute{\rho})(\overline{u_i}+\acute{u}_i)(\overline{u_j}+\acute{u}_j)}$$

$$-\overline{\frac{\partial(\bar{P}+\acute{P})}{\partial x_i}}=-\frac{\partial}{\partial x_i}\left(\overline{\bar{P}+\acute{P}}\right)=-\frac{\partial}{\partial x_i}\left(\bar{P}+\bar{\acute{P}}\right)=-\frac{\partial\bar{P}}{\partial x_i}$$

$$\overline{(\bar{\rho}+\acute{\rho})g_i}=\left(\overline{\bar{\rho}+\acute{\rho}}\right)g_i=\left(\bar{\bar{\rho}}+\bar{\acute{\rho}}\right)g_i=\bar{\rho}g_i$$

$$\overline{\frac{\partial(\overline{u_i}+\acute{u}_i)}{\partial x_j}}=\frac{\partial}{\partial x_j}\left(\overline{\overline{u_i}+\acute{u}_i}\right)=\frac{\partial}{\partial x_j}\left(\overline{u_i}+\overline{\acute{u}_i}\right)=\frac{\partial}{\partial x_j}\overline{u_i}$$

$$\overline{\frac{\partial(\overline{u_j}+\acute{u}_j)}{\partial x_i}}=\frac{\partial}{\partial x_i}\left(\overline{\overline{u_j}+\acute{u}_j}\right)=\frac{\partial}{\partial x_i}\left(\overline{u_j}+\overline{\acute{u}_j}\right)=\frac{\partial}{\partial x_i}\overline{u_j}$$

$$\overline{\frac{\partial(\overline{u_k}+\acute{u}_k)}{\partial x_k}}=\frac{\partial}{\partial x_k}\left(\overline{\overline{u_k}+\acute{u}_k}\right)=\frac{\partial}{\partial x_k}\left(\overline{u_k}+\overline{\acute{u}_k}\right)=\frac{\partial}{\partial x_k}\overline{u_k}$$

$$\overline{\frac{\partial(\bar{\rho}+\acute{\rho})(\overline{u_i}+\acute{u}_i)}{\partial t}}=\frac{\partial}{\partial t}\left(\overline{\bar{\rho}\overline{u_i}}+\overline{\bar{\rho}\acute{u}_i}+\overline{\overline{u_i}\acute{\rho}}+\overline{\acute{\rho}\acute{u}_i}\right)=\frac{\partial}{\partial t}\left(\bar{\rho}\overline{u_i}+\bar{\rho}\overline{\acute{u}_i}+\bar{\acute{\rho}}\overline{u_i}+\overline{\acute{\rho}\acute{u}_i}\right)=\frac{\partial}{\partial t}(\bar{\rho}\overline{u_i}+\overline{\acute{\rho}\acute{u}_i})$$

$$\overline{\frac{\partial}{\partial x_j}(\bar{\rho}+\acute{\rho})(\overline{u_i}+\acute{u}_i)(\overline{u_j}+\acute{u}_j)}=\frac{\partial}{\partial x_j}\left(\overline{\left(\bar{\rho}\overline{u_i}+\bar{\rho}\overline{\acute{u}_i}+\bar{\acute{\rho}}\overline{u_i}+\overline{\acute{\rho}\acute{u}_i}\right)(\overline{u_j}+\acute{u}_j)}\right)=\frac{\partial}{\partial x_j}(\bar{\rho}\overline{u_i}\overline{u_j}+\bar{\rho}\overline{\acute{u}_i}\overline{u_j}+$$

$$\bar{\acute{\rho}}\overline{u_i}\overline{u_j}+\overline{\acute{\rho}\acute{u}_i}\overline{u_j}+\bar{\rho}\overline{u_i}\overline{\acute{u}_j}+\bar{\rho}\overline{\acute{u}_i\acute{u}_j}+\overline{\acute{\rho}\acute{u}_j}\overline{u_i}+\overline{\acute{\rho}\acute{u}_i\overline{u_j}})$$

$$= \frac{\partial}{\partial x_j}(\bar{\rho}\overline{u_\iota}\overline{u_J} + \overline{\acute{\rho}\acute{u}_\iota}\overline{u_J} + \overline{\acute{\rho}\acute{u}_J}\overline{u_\iota} + \bar{\rho}\overline{\acute{u}_\iota\acute{u}_J} + \overline{\acute{\rho}\acute{u}_\iota\overline{u_J}})$$

The momentum equation becomes;

$$-\frac{\partial\bar{P}}{\partial x_i} + \bar{\rho}g_i + \frac{\partial}{\partial x_j}\left[\mu\left(\frac{\partial\overline{u_\iota}}{\partial x_j} + \frac{\partial\overline{u_J}}{\partial x_i}\right) + \mu_s\frac{\partial\overline{u_k}}{\partial x_k}\right] = \frac{\partial}{\partial t}\left(\bar{\rho}\overline{u_\iota} + \overline{\acute{\rho}\acute{u}_\iota}\right) + \frac{\partial}{\partial x_j}(\bar{\rho}\overline{u_\iota}\overline{u_J} + \overline{\acute{\rho}\acute{u}_\iota}\overline{u_J} + \overline{\acute{\rho}\acute{u}_J}\overline{u_\iota} +$$

$$\bar{\rho}\overline{\acute{u}_\iota\acute{u}_J} + \overline{\acute{\rho}\acute{u}_\iota\overline{u_J}})$$

Or

$$-\frac{\partial\bar{P}}{\partial x_i} + \bar{\rho}g_i + \frac{\partial}{\partial x_j}\overline{\tau_{\iota J}} = \frac{\partial}{\partial t}\left(\bar{\rho}\overline{u_\iota} + \overline{\acute{\rho}\acute{u}_\iota}\right) + \frac{\partial}{\partial x_j}(\bar{\rho}\overline{u_\iota}\overline{u_J} + \overline{\acute{\rho}\acute{u}_\iota}\overline{u_J} + \overline{\acute{\rho}\acute{u}_J}\overline{u_\iota} + \bar{\rho}\overline{\acute{u}_\iota\acute{u}_J} + \overline{\acute{\rho}\acute{u}_\iota\acute{u}_J}) \quad (3.24)$$

Where;

$$\overline{\tau_{\iota J}} = \mu\left(\frac{\partial\overline{u_\iota}}{\partial x_j} + \frac{\partial\overline{u_J}}{\partial x_i}\right) + \mu_s\delta_{ij}\frac{\partial\overline{u_k}}{\partial x_k} \quad (3.25)$$

3.5.3 Energy Equation

Similarly, the decomposition and time averaging process of the energy equation will yield;

$$\frac{\partial}{\partial t}\left(c_p\bar{\rho}\bar{T} + c_p\overline{\acute{\rho}\acute{T}}\right) + \frac{\partial}{\partial x_j}\left(c_p\bar{\rho}\overline{u_J}\bar{T}\right) = \frac{\partial\bar{P}}{\partial t} + \frac{\partial\bar{P}}{\partial x_j} + \overline{\acute{u}_\iota\frac{\partial\acute{P}}{\partial x_J}} + \frac{\partial}{\partial x_j}\left(\lambda\frac{\partial\bar{T}}{\partial x_j} - c_p\bar{\rho}\overline{\acute{u}_\iota\acute{T}} - c_p\overline{\acute{\rho}u_\iota\acute{T}}\right) + \bar{\Phi}$$

(3.26)

With $\bar{\Phi}$being given by

$$\bar{\Phi} = \overline{\tau_{\iota J}}\frac{\partial\overline{u_\iota}}{\partial x_j} + \overline{\acute{\tau}_{\iota J}\frac{\partial\acute{u}_\iota}{\partial x_J}} \quad (3.27)$$

In the mean and the continuity and momentum equations governing natural turbulence there is an addition of unknown quantities. The quantities are a representation of mean effect of turbulence. Specifically, the terms given by $\overline{\acute{\rho}\acute{u}_\iota\acute{u}_J}$ represents the turbulent transport term while the terms given by $\overline{\acute{\rho}\acute{u}_J}$ and $\overline{\acute{\rho}\acute{u}_\iota}$ are the density generated terms. On time averaging the mean energy equation there is an introduction of turbulent correlation between the fluctuations of temperature. The terms are $\overline{\acute{u}_\iota\acute{T}}$ and $\overline{\acute{u}_J\acute{T}}$. The product of these terms and the density ρ yields the transport of heat mass resulting from the fluctuation motion. The transport in the x_i direction or the turbulent heat flux is represented by $-\rho\overline{\acute{u}_\iota\acute{T}}$. The mean mass term given by $\overline{\acute{\rho}\acute{u}_J}$ and the momentum transport terms including $\rho\overline{\acute{u}_\iota\acute{u}_J}$ are eliminated by the mass weighted averages across a mean streamline as espoused by Cebeci and Smith (1974).

Equations (3.23), (3.24), (3.26) represent the continuity, momentum, and the energy equations that are accepted as governing the mean flow quantities $\bar{u}_\iota$, the fluid properties given by $\bar{\rho}, \bar{p}$ as well as the temperature $\bar{T}$. There are models that have been proposed for use in the approximation of the correlation of a given order in regard to the lower correlation and the mean flow quantities. The commonly used models are the Reynolds-Stress models and the eddy viscosity models.

3.5. 4 Reynolds-Stress Equations

The derivation of an exact equation for the Reynolds-Stress $\overline{\acute{u}_\iota\acute{u}_J}$ can be made from the momentum equation given in (3.7) and the decomposed equation called the Reynolds equation given in (3.24). In the process of determination of the exact Reynolds stress there is a need for first subtracting the Reynolds equation from the momentum equation (Navier-Stokes) for the mean momentum in the x_i direction and the x_j direction. Secondly, the component i is multiplied with fluctuating velocity $\acute{u}_j$ while fluctuating velocity $\acute{u}_i$ is multiplied with the j component. The third step is to add the resulting equations and time averaging is applied to the exact equation for Reynolds stress $\overline{\acute{u}_\iota\acute{u}_J}$ obtained. This results in the equation given below.

$$\frac{\partial}{\partial t}\left(\bar{\rho}\overline{\acute{u}_\iota\acute{u}_J}\right)+\frac{\partial}{\partial x_k}\left(\overline{u_k\rho}\;\overline{\acute{u}_\iota\acute{u}_J}\right)= -\frac{\partial}{\partial x_k}\left(\bar{\rho}\;\overline{\acute{u}_\iota\acute{u}_J\acute{u}_k}\right)-\overline{\acute{u}_J\frac{\partial\acute{p}}{\partial x_\iota}}-\overline{\acute{u}_\iota\frac{\partial\acute{p}}{\partial x_J}}-\overline{\acute{\rho}\acute{u}_\iota\acute{u}_k}\frac{\partial\overline{u_J}}{\partial x_k}-\overline{\acute{\rho}\acute{u}_J\acute{u}_k}\frac{\partial\overline{u_\iota}}{\partial x_k}-$$

$$\left(g_i\overline{\acute{\rho}\acute{u}_J}-g_i\overline{\acute{\rho}\acute{u}_\iota}+\overline{\acute{u}_J\frac{\partial}{\partial x_k}\left(\acute{\tau}_{\iota J}\right)}+\overline{\acute{u}_\iota\frac{\partial}{\partial x_k}\left(\acute{\tau}_{\iota J}\right)}\right) \tag{3.28}$$

3.5.5. Eddy viscosity concepts

In general flow situation the eddy viscosity concept shows that the turbulent transport terms $\overline{\rho \acute{u}_i \acute{u}_j}$ is expressed as;

$$\overline{\rho \acute{u}_i \acute{u}_j} = -\mu_t \left(\frac{\partial \overline{u_i}}{\partial x_j} + \frac{\partial \overline{u_j}}{\partial x_i} \right) + \frac{2}{3} \bar{\rho} k \delta_{ij} \tag{3.29}$$

Where,

μ_t- Turbulent viscosity representing the proportionality parameter between the Reynolds stress and the mean strain

$k = \overline{\acute{u}_i \acute{u}_i}$- is the kinetic energy of the turbulent velocity fluctuation

3.6 $k - \varepsilon$ equation turbulence model

In the $k - \varepsilon$ turbulence model the eddy viscosity is taken as the products of characteristic velocity scale and the approximate length scale of turbulence eddies. The turbulence velocity scale and turbulence length scale in the $k - \varepsilon$ turbulence model is taken as $k^{\frac{1}{2}}$ and $\frac{k^{\frac{3}{2}}}{\varepsilon}$ respectively. Eddy viscosity is given as

$$\mu_t = c_\mu \rho \frac{k^2}{\varepsilon} \tag{3.30}$$

In this case μ_t is the constant of proportionality that should be determined empirically. In their book Tennekes and Lumley (1972) affirm that the contraction of the three normal stress equations, that is for the case where i=j= 1, 2, and 3 the resulting exact turbulent kinetic energy is as below;

$$\frac{\partial}{\partial t}(\bar{\rho} k) + \frac{\partial}{\partial x_j}\left(\overline{\rho u_j} k\right) = \overline{\acute{u}_j \frac{\partial}{\partial x_j} \tau_{ij}} - \frac{1}{2} \frac{\partial}{\partial x_j}\left(\overline{\rho \acute{u}_i \acute{u}_i \acute{u}_j}\right) - \overline{\rho \acute{u}_i \acute{u}_j} \frac{\partial \overline{u_i}}{\partial x_j} + \overline{\acute{\rho} \acute{u}_i} g_i - \overline{\acute{u}_j \frac{\partial p}{\partial x_i}} \tag{3.31}$$

Tennekes and Lumley (1972) added that in the $k - \varepsilon$ model, the derivation of an exact equation can be derived from using the Navier Stokes equation for the fluctuating vorticity and hence getting the dissipation of ε. The derivation of the dissipation ε is given as below;

$$\frac{\partial}{\partial t}(\bar{\rho}\varepsilon)+\frac{\partial}{\partial x_j}\left(\overline{\rho u_j}\varepsilon\right)= -\frac{\partial}{\partial x_k}\left(\overline{\mu\acute{\mu}_k\frac{\partial \acute{u}_l}{\partial x_j}\frac{\partial \acute{u}_l}{\partial x_j}}+2v\frac{\partial \acute{u}_k}{\partial x_i}\frac{\partial \acute{p}}{\partial x_i}\right)-2\mu\overline{\frac{\partial \acute{u}_l}{\partial x_k}\frac{\partial \acute{u}_l}{\partial x_j}\frac{\partial \acute{u}_j}{\partial x_j}}-2\bar{\rho}\left[v\overline{\frac{\partial^2 \acute{u}_l}{\partial x_k \partial x_j}}\right]^2+$$

$$2v\overline{\frac{\partial \acute{u}_l}{\partial x_j}\frac{\partial \acute{\rho}}{\partial x_j}}g_i-2\mu\frac{\partial \bar{u}_i}{\partial x_k}\left(\frac{\partial \acute{u}_i}{\partial x_i}\frac{\partial \acute{u}_k}{\partial x_j}+\frac{\partial \acute{u}_j}{\partial x_i}\frac{\partial \acute{u}_j}{\partial x_k}\right)-2\mu\overline{\frac{\partial^2 \acute{u}_l}{\partial x_j \partial x_k}}\overline{\acute{u}_k\frac{\partial u_l}{\partial x_j}} \tag{3.32}$$

3.7 Final set of Equations

For simplicity, upper case letters will be used to replace the over-bar for the time mean values of the variables while lower case letters will be used to replace the prime that indicates the fluctuating quantities. The final set of equations for turbulent natural convection is;

$$\frac{\partial \rho}{\partial t}+\frac{\partial}{\partial x_j}\left(\rho U_j+\overline{\rho u_l}\right)=0 \tag{3.33}$$

$$\frac{\partial}{\partial t}(\rho U_i+\overline{\rho u_l})+\frac{\partial}{\partial x_j}\left(\rho U_i U_j+U_i\overline{\rho u_l}\right)=-\frac{\partial p}{\partial x_i}+pg_i+\frac{\partial}{\partial x_j}(\tau_{ij}+U_i\overline{\rho u_l}-\rho\overline{u_l u_j}-\overline{\rho u_l u_j}) \tag{3.34}$$

$$\frac{\partial}{\partial t}\left(C_p\rho T+C_p\overline{\rho T}\right)+\frac{\partial}{\partial x_j}\left(C_p\overline{\rho U_j T}\right)=\frac{\partial p}{\partial t}+U_j\frac{\partial p}{\partial x_j}+\overline{u_j\frac{\partial p}{\partial x_j}}+\frac{\partial}{\partial x_j}\left(\lambda\frac{\partial T}{\partial x_j}-C_p\overline{U_t T}-C_p\overline{U_l T}\right)+\Phi \tag{3.35}$$

$$\frac{\partial}{\partial t}(\rho k)+\frac{\partial}{\partial x_j}\left(\rho U_j k\right)=\overline{u_j\frac{\partial}{\partial x_j}\mu\left(\frac{\partial u_l}{\partial x_j}+\frac{\partial u_j}{\partial x_l}\right)}-\frac{1}{2}\frac{\partial}{\partial x_j}\left(\overline{\rho u_l u_l u_j}\right)-\overline{\rho u_l u_j}\frac{\partial U_i}{\partial x_j}+\overline{\rho u_l}g_i-\overline{u_j\frac{\partial p}{\partial x_l}} \tag{3.36}$$

Where,

$$\tau_{ij}=\mu\left(\frac{\partial U_i}{\partial x_j}+\frac{\partial U_j}{\partial x_i}\right)+\mu_s\delta_{ij}\frac{\partial U_k}{\partial x_k} \tag{3.37}$$

$$\Phi=\tau_{ij}+\frac{\partial U_i}{\partial x_j}+\overline{\mu\left(\frac{\partial u_l}{\partial x_j}+\frac{\partial u_j}{\partial x_l}\right)}\frac{\partial u_i}{\partial x_j} \tag{3.38}$$

$$\frac{\partial}{\partial t}(\rho\varepsilon)+\frac{\partial}{\partial x_j}\left(\rho U_j\varepsilon\right)= -\frac{\partial}{\partial x_k}\left(\overline{\mu\mu_k\frac{\partial u_l}{\partial x_j}\frac{\partial u_l}{\partial x_j}}+2v\frac{\partial u_k}{\partial x_i}\frac{\partial p}{\partial x_i}+\mu\frac{\partial \varepsilon}{\partial x_k}\right)-2\mu\overline{\frac{\partial u_l}{\partial x_k}\frac{\partial u_l}{\partial x_j}\frac{\partial u_k}{\partial x_j}}-$$

$$2\rho\left[v\overline{\frac{\partial^2 u_l}{\partial x_k \partial x_j}}\right]^2+2v\overline{\frac{\partial u_l}{\partial x_j}\frac{\partial \rho}{\partial x_j}}g_i-2\mu\frac{\partial u_i}{\partial x_k}\left(\frac{\partial u_i}{\partial x_i}\frac{\partial u_k}{\partial x_j}+\frac{\partial u_j}{\partial x_i}\frac{\partial u_j}{\partial x_k}\right)-2\mu\overline{\frac{\partial^2 u_l}{\partial x_j \partial x_k}}\overline{u_k\frac{\partial u_l}{\partial x_j}} \tag{3.39}$$

3.8 Non-dimensionalization

This involves the conversion of the governing equations to a non-dimensional form to ease the problem analysis and for the reduction of the number of free parameters. In addition, the process of non-dminensionalization is useful for making the solutions bounded. Non-dimensionalization can be done by dividing the dependent and the independent flow variables by meaningful and relevant quantities. Precisely, the length variables can be divided by the width of the enclosure L_0 which is the characteristic length, the velocities can as well be divided by U_* as the reference velocity which in this case is the inlet velocity. In addition, the pressure can be divided by twice the dynamic pressure for the channel that is given by $P_O = \rho {U_*}^2$ while the temperature can be divided by a suitable difference in the temperature say $\frac{(T-T_*)}{\Delta T_*}$ for the enclosure. The same scaling variables are found in the mean variables and the fluctuating quantities. As such, the process of non-dimensionalization can be carried out on the basis of the following general scaling variables;

$$X_j = \acute{X}_j L_o, \quad U_j = \acute{U}_j U_*, P = \acute{P} P_O, \Theta = \frac{(T-T_*)}{\Delta T_*}, K = \acute{K} U_*,\ \varepsilon = \acute{\varepsilon} \frac{{U_*}^3}{L_O}, t = \acute{t} \frac{L_O}{U_*}, \mu = \acute{\mu} \mu_O,$$

$$\mu_s = \acute{\mu}_s \mu_o, V = \acute{V} \mu_O\ , \rho = \acute{\rho} \rho_O,\ C_p = C_O C_{po} \tag{3.40}$$

Where;

L_O is the characteristic length, ΔT_* is the characteristic temperature difference, T_* is the convenient temperature that could result in the value of Θ being bounded in the solution region.

In the general scaling variables given by equation (3.40), the prime represents the variable that can be defined arbitrarily for the specification of the non-dimensional scheme. On applying the scaling variables on the mean continuity equation (3.33), we have;

$$\frac{\partial \rho}{\partial t} + \frac{\partial}{\partial x_j}\left(\rho U_j\right) + \frac{\partial}{\partial x_j}\left(\overline{\rho u_j}\right) = 0$$

The first part in the left of this equation becomes;

$\frac{\partial\rho}{\partial\acute{\rho}}\frac{\partial\acute{\rho}}{\partial\acute{t}}\frac{\partial\acute{t}}{\partial t}$, from equation (3.40) we have $\rho = \acute{\rho}\rho_o$ and $t = \acute{t}\frac{L_o}{U_*}$ or $\frac{\acute{t}}{t} = \frac{U_*}{L_o}$

Therefore; $\frac{\partial\rho}{\partial\acute{\rho}}\frac{\partial\acute{\rho}}{\partial\acute{t}}\frac{\partial\acute{t}}{\partial t} = \rho_o\frac{U_*}{L_o}\frac{\partial\acute{\rho}}{\partial\acute{t}}$

For the second part of the mean continuity equation we have;

$$\frac{\partial}{\partial x_j}(\rho U_j) = \rho\frac{\partial U_j}{\partial x_j} + U_j\frac{\partial\rho}{\partial x_j}$$

Here $\rho\frac{\partial U_j}{\partial x_j} = \rho\left(\frac{\partial U_j}{\partial\acute{U}_j}\frac{\partial\acute{U}_j}{\partial\acute{x}_j}\frac{\partial\acute{x}_j}{\partial x_j}\right) = \acute{\rho}\rho\left(\frac{U_*}{L_o}\frac{\partial\acute{U}_j}{\partial\acute{x}_j}\right)$ and $U_j\frac{\partial\rho}{\partial x_j} = U_j\left(\frac{\partial\rho}{\partial\acute{\rho}}\frac{\partial\acute{\rho}}{\partial\acute{x}_j}\frac{\partial\acute{x}_j}{\partial x_j}\right) = \acute{U}_j U_*\left(\frac{\rho_o}{L_o}\frac{\partial\acute{\rho}}{\partial\acute{x}_j}\right)$

Therefore, the second part of the mean continuity equation is given as;

$$\frac{\partial}{\partial x_j}(\rho U_j) = \acute{\rho}\rho\left(\frac{U_*}{L_o}\frac{\partial\acute{U}_j}{\partial\acute{x}_j}\right) + \acute{U}_j U_*\left(\frac{\rho_o}{L_o}\frac{\partial\acute{\rho}}{\partial\acute{x}_j}\right) = \frac{\rho_o U_*}{L_o}\left(\acute{\rho}\frac{\partial\acute{U}_j}{\partial\acute{x}_j} + \acute{U}_j\frac{\partial\acute{\rho}}{\partial\acute{x}_j}\right) = \frac{\rho_o U_*}{L_o}\frac{\partial(\acute{U}_j\acute{\rho})}{\partial\acute{x}_j}$$

For the last part of the mean continuity equation we have;

$$\frac{\partial}{\partial x_j}(\overline{\rho u_j}) = \frac{\rho_o U_*}{L_o}\frac{\partial(\overline{\acute{u}_j\acute{\rho}})}{\partial\acute{x}_j}$$

Therefore the non-dimensionalized continuity equation is given as;

$$\rho_o\frac{U_*}{L_o}\frac{\partial\acute{\rho}}{\partial\acute{t}} + \frac{\rho_o U_*}{L_o}\frac{\partial(\acute{U}_j\acute{\rho})}{\partial\acute{x}_j} + \frac{\rho_o U_*}{L_o}\frac{\partial(\overline{\acute{u}_j\acute{\rho}})}{\partial\acute{x}_j} = 0$$

On dividing through by $\rho_o\frac{U_*}{L_o}$ we get;

$$\frac{\partial\acute{\rho}}{\partial\acute{t}} + \frac{\partial(\acute{U}_j\acute{\rho})}{\partial\acute{x}_j} + \frac{\partial(\overline{\acute{u}_j\acute{\rho}})}{\partial\acute{x}_j} = 0$$

Or

$$\frac{\partial\acute{\rho}}{\partial\acute{t}} + \frac{\partial}{\partial\acute{x}_j}\left(\acute{U}_j\acute{\rho} + \overline{\acute{u}_j\acute{\rho}}\right) = 0 \qquad (3.41)$$

On application of a similar procedure to the mean momentum equation, mean energy equation, the kinetic equation, and the equation of rate of dissipation of turbulent kinetic energy (Equations 3.34, 3.35, 3.36, and 3.39) respectively it results in the following non-dimensional forms of the equations;

$$\frac{\partial}{\partial t}\left(\acute{\rho}\acute{U}_i+\overline{\acute{\rho}\acute{u}_\iota}\right)+\frac{\partial}{\partial \acute{x}_i}\left(\acute{\rho}\acute{U}_i\acute{U}_j+\acute{U}_i\acute{\rho}\acute{u}_j\right)=-\left\{\frac{p_o}{\rho_0 U^2{}_*}\right\}\frac{\partial \acute{p}}{\partial \acute{x}_i}+\left\{\frac{gL_0}{U^2{}_*}\right\}\rho\acute{g}_i\frac{\partial}{\partial x_i}\left(\left\{\frac{\mu_o}{\rho_o U_* L_o}\right\}\tau_{ij}-\acute{U}_j\overline{\acute{\rho}\acute{u}_J}-\overline{\acute{\rho}\acute{u}_\iota\acute{u}_J}\right) \quad (3.42)$$

$$\frac{\partial}{\partial \acute{t}}\left(\acute{C}_p\acute{\rho}\Theta+\acute{C}_p\overline{\acute{\rho}\theta}\right)+\frac{\partial}{\partial \acute{x}_j}\left(C_p\acute{\rho}\acute{U}_J\Theta\right)=\left\{\frac{p_o}{C_{po}\rho_0\Delta T_*}\right\}\left[\frac{\partial \acute{p}}{\partial \acute{t}}+\acute{U}_j\frac{\partial \acute{p}}{\partial \acute{x}_J}+\overline{\acute{u}_J\frac{\partial \acute{p}}{\partial x_J}}\right]+\frac{\partial}{\partial \acute{x}_j}\left(\left\{\frac{\lambda_o}{C_{po}\rho_0\Delta T_* L_o}\right\}\lambda\frac{\partial\Theta}{\partial \acute{x}_j}-C_p\acute{\rho}\overline{u_\iota\theta}-C_p\overline{\acute{\rho}\acute{u}_\iota\theta}\right)+\left\{\frac{\mu_o U_*}{C_{po}\rho_0\Delta T_* L_o}\right\}\Phi \quad (3.43)$$

$$\frac{\partial}{\partial \acute{t}}\acute{\rho}\acute{k}+\frac{\partial}{\partial x_j}\rho\acute{U}_J\acute{k}=\left\{\frac{\mu_o}{\rho_o U_* L_o}\right\}\overline{\acute{u}_J\frac{\partial}{\partial \acute{x}_J}\acute{u}_J\left(\frac{\partial \acute{u}_J}{\partial \acute{x}_J}+\frac{\partial \acute{u}_J}{\partial \acute{x}_J}\right)}-\acute{\rho}\overline{\acute{u}_\iota\acute{u}_J}\frac{\partial \acute{U}_i}{\partial x_j}-\frac{1}{2}\frac{\partial}{\partial x_j}\overline{\acute{p}\acute{u}_\iota\acute{u}_J}+\left\{\frac{gL_o}{U^2{}_*}\right\}\acute{p}\acute{U}_i-\left\{\frac{P_o}{p_o U_*{}^2}\right\}\overline{\acute{u}_J\frac{\partial \acute{p}}{\partial \acute{x}_\iota}}$$

(3.44)

$$\frac{\partial}{\partial t}\acute{p}\acute{\varepsilon}+\frac{\partial}{\partial \acute{x}_j}\acute{p}\acute{U}_j\acute{\varepsilon}=-\left\{\frac{\mu_o}{p_o U_* L_o}\right\}\frac{\partial}{\partial \acute{x}_k}\left(\overline{\acute{\mu}\acute{\mu}_k\frac{\partial \acute{u}_\iota}{\partial x_J}\frac{\partial \acute{u}_J}{\partial x_\iota}}+2\left\{\frac{P_o}{p_o U_*{}^2}\right\}\acute{v}\overline{\frac{\partial \acute{u}_k}{\partial \acute{x}_\iota}\frac{\partial \acute{p}}{\partial \acute{x}_\iota}-\acute{\mu}\frac{\partial \acute{\varepsilon}}{\partial \acute{x}_k}}\right)-2\left\{\frac{\mu_o}{p_o U_*{}^2}\right\}\acute{\mu}\overline{\frac{\partial \acute{u}_J}{\partial \acute{x}_k}\frac{\partial \acute{u}_\iota}{\partial \acute{x}_J}\frac{\partial u_k}{\partial \acute{x}_J}}-2\left\{\frac{u_o{}^2}{p_o{}^2 U_*{}^2 L_o{}^2}\right\}\bar{p}\left(\acute{v}\frac{\partial^2 u_i}{\partial \acute{x}_k\partial \acute{x}_j}\right)+2\left\{\frac{\mu_o\acute{g}}{p_o{}^2 U_*{}^3}\right\}\acute{v}\frac{\partial \acute{u}_i}{\partial \acute{x}_j}\frac{\partial \acute{p}}{\partial \acute{x}_j}\acute{g}-2\left\{\frac{\mu_o}{P_o U_* L_o}\right\}\acute{\mu}\frac{\partial \acute{u}_j}{\partial \acute{x}_k}\left(\frac{\partial \acute{u}_i}{\partial \acute{x}_j}\frac{\partial \acute{u}_k}{\partial \acute{x}_j}+\frac{\partial \acute{u}_i}{\partial \acute{x}_i}\frac{\partial \acute{u}_j}{\partial \acute{x}_k}\right)-2\left\{\frac{\mu_o}{\rho_o U_* L_o}\right\}\acute{\mu}\frac{\partial^2 u_i}{\partial \acute{x}_j\partial \acute{x}_k}\overline{\acute{u}_k\frac{\partial \acute{u}_\iota}{\partial \acute{x}_J}} \quad (3.45)$$

To avoid any ambiguities any equation or variable will be non-dimensional and the prime which has been used to represent the non-dimensionality will be avoided henceforth. The equations (3.41), (3.42), (3.43), (3.44), and (3.45) will become;

$$\frac{\partial\rho}{\partial t}+\frac{\partial}{\partial x_j}\left(\rho U_j+\overline{\rho u_J}\right)=0 \quad (3.46)$$

$$\frac{\partial}{\partial t}\left(\rho U_i+\overline{\rho u_\iota}\right)+\frac{\partial}{\partial x_i}\left(\rho U_i U_j+U_i\overline{\rho u_J}\right)=-A_1\frac{\partial p}{\partial x_i}+A_2\rho g_i\frac{\partial}{\partial x_i}\left(A_3\tau_{ij}-U_j\overline{\rho u_J}-\overline{\rho u_\iota u_J}\right) \quad (3.47)$$

$$\frac{\partial}{\partial t}\left(C_p\rho\Theta+C_p\overline{\rho\Theta}\right)+\frac{\partial}{\partial x_j}\left(C_p\rho U_j\Theta\right)=B_1\left[\frac{\partial p}{\partial t}+U_j\frac{\partial p}{\partial x_j}+\overline{u_J\frac{\partial p}{\partial x_J}}\right]+\frac{\partial}{\partial x_j}\left(B_2\lambda\frac{\partial\Theta}{\partial x_j}-C_p\rho\overline{u_\iota\Theta}-C_p\rho u_i\Theta\right)+B_3\Phi \quad (3.48)$$

$$\rho k + \frac{\partial}{\partial x_j}\rho U_j k = E_1 \overline{u_j \frac{\partial}{\partial x_j} u_j \left(\frac{\partial u_j}{\partial x_j} + \frac{\partial u_j}{\partial x_j}\right)} - \frac{1}{2}\frac{\partial}{\partial x_j}\rho \overline{u_i u_j}\frac{\partial U_i}{\partial x_j} + E_2 \overline{\rho u_i} g_i - E_3 \overline{u_j \frac{\partial p}{\partial x_i}} \qquad (3.49)$$

And

$$\frac{\partial}{\partial t}\rho\varepsilon + \frac{\partial}{\partial x_j}\rho U_j \varepsilon = -\frac{\partial}{\partial x_k}\left(F_1 \overline{\mu u_k \frac{\partial u_i}{\partial x_j}\frac{\partial u_j}{\partial x_i}} + 2F_2 v \overline{\frac{\partial u_k}{\partial x_i}\frac{\partial \rho}{\partial x_i}} - F_1 \mu \frac{\partial \varepsilon}{\partial x_k}\right) -$$

$$2F_1 \mu \frac{\partial U_i}{\partial x_j}\left(\overline{\frac{\partial u_i}{\partial x_j}\frac{\partial u_k}{\partial x_j} + \frac{\partial u_i}{\partial x_j}\frac{\partial u_j}{\partial x_k}}\right) - 2F_1 \mu \frac{\partial^2 U_i}{\partial x_j \partial x_k}\overline{\mu_k \frac{\partial u_i}{\partial x_j}} \qquad (3.50)$$

Where the coefficients A_1, A_2 , A_3, B_1, B_2 , B_3, E_1, E_2 , E_3, F_1, F_2 , F_3, F_4, are given in the table 3.1 below;

Symbol	Coefficient
U_*	
A_1	$\frac{p_o}{\rho_0 U^2{}_*}$
A_2	$\frac{gL_0}{U^2{}_*}$
A_3	$\frac{\mu_o}{\rho_o U_* L_o}$
B_1	$\frac{p_o}{C_{po}\rho_0 \Delta T_*}$
B_2	$\frac{\lambda_o}{C_{po}\rho_0 \Delta T_* L_o}$
B_3	$\frac{\mu_o U_*}{C_{po}\rho_0 \Delta T_* L_o}$
E_1	$\frac{\mu_o}{\rho_o U_* L_o}$
E_2	$\frac{gL_o}{U^2{}_*}$

E_3	$\frac{P_o}{p_o {U_*}^2}$
F_1	$\frac{\mu_o}{p_o U_* L_o}$
F_2	$\frac{\mu_o p_o}{{\rho_o}^2 {U_*}^2 {L_o}^2}$
F_3	$\left(\frac{\mu_o}{p_o U_* L_o}\right)^2$
F_4	$\frac{g\mu_o}{p_o U^3{}_*}$

Table 3.1 Table for Non-Dimentional Governing Equations (3.47-3.50)

The set of governing equations that have been presented have more unknowns as compared to the equations that are present. In this regard, there is a need for more assumptions to be used for the reduction of the number of the unknowns. Additionally, there is a need for the use of a model for the unknown non-linear term ($\overline{u_\iota u_J}$) for the solution of equation (3.47).

CHAPTER FOUR

TURBULENCE MODELING

4.1 Introduction

In the averaging process of the governing equations in the previous chapter, there is an introduction of an unknown turbulent correlations into the mean flow equations representing the turbulent transport of momentum, heat, and mass. This necessitates for the determination of a set of relations and equations to be used in finding the unknown turbulent correlation. The two equation turbulence model that include two extra transport equations for the representation of the turbulent properties of flow. The common transport component is the kinetic energy k. The second transport variable varies depending on the two equation model that is used. The turbulence dissipation (ε) and the specific dissipation (ω) are the common choices that are used. The variable k can be thought of as a variable for the determination of energy in the turbulence while the second variable is used to determine the scale of turbulence

4.2 Model Description

In this project there is numerical investigation of turbulent natural convection in 3-dimension. The geometry of the problem is as illustrated in figure 4.1 below. The heating and cooling of the rectangular enclosure is done on the face-wall. The lower part of the face-wall is heated (painted red) while the upper half of the face-wall is cooled. The Ampofo and Karayiannis (2003) measurements were used because they carried out the experiment under high accuracy. Precisely, the walls of the rectangular enclosure measure 0.75m by 0.75m by 1.5m wide. The cold and the hot parts of the enclosure were isothermal at 323k and 283k respectively. This gives a Rayleigh number of 1.58×10^9. All the boundaries of the rectangular enclosure are rigid, non-permeable, and have no slip. The other walls of the enclosure are adiabatic.

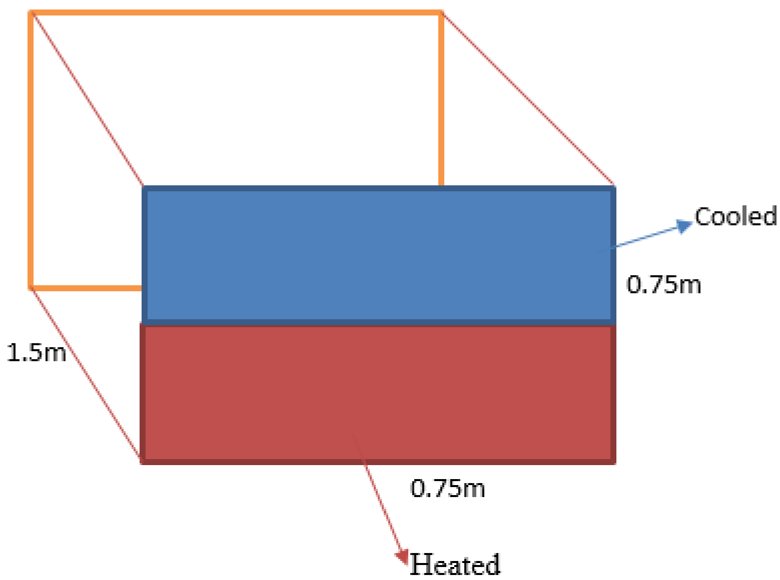


Figure 4.1: Geometry of the Model

The Boussinesq approximation is expected to help in reducing the governing equations further in this research. In this project, we will zero in on the $k - \varepsilon$ model and the study of variables as used by Ampofo and Karayiannis (2003).

4.3 Boussinesq Approximation

This is a useful approximation for natural convection or in buoyancy-driven flows in enclosures. The Boussinesq approximation assumptions are as follows;

i) All the physical properties except the density of the fluid are constant

ii) The fluid under consideration are Newtonian and there is an internal source of heat

iii) The variation of density is negligible because it is small

iv) The density varies proportionately to the temperature difference (ΔT) due to the buoyancy forces

v) The effects due to viscous dissipation is neglected

4.3.1 Simplification of Governing Equations Using Boussinesq Approximation

As indicated in the third assumption above, the variation of density is proportional to temperature and the change is small. This can be shown as follows;

$$\rho = \rho_o + (T + T)\left(\frac{\partial \rho}{\partial T}\right)_{T_o} \qquad (4.1)$$

At a constant pressure the coefficient of thermal expansion is given as;

$$\beta_o = \frac{-1}{\rho_o}\left(\frac{\partial \rho}{\partial T}\right)_{T_o} \qquad (4.2)$$

On substituting equation (4.2) into equation (4.1) we get;

From (4.1) $\left(\frac{\partial \rho}{\partial T}\right)_{T_o} = \frac{(\rho - \rho_o)}{T - T_o}$ and

From 4.2 $\left(\frac{\partial \rho}{\partial T}\right)_{T_o} = -\rho_o \beta_o$

$$\rho = \rho_o(1 - \beta_o(T - T_o)) \qquad (4.3)$$

By making use of Boussinesq approximation and equation (4.3) in the governing equations, the presentation of non-dimensional form for an incompressible flow equations that govern natural convection in an enclosure is as follows;

The continuity equation (3.46) becomes;

$$\frac{\partial U_j}{\partial x_j} = 0 \qquad (4.4)$$

The momentum equation (3.47) becomes

$$\frac{\partial U_j}{\partial t} + \frac{\partial U_i U_j}{\partial x_j} = \frac{A_1}{\rho_o}\frac{\partial p}{\partial x_i} - A_2 \Theta g_i + \frac{\partial}{\partial x_j}\left(A_3\left(\frac{\partial U_i}{\partial x_j} + \frac{\partial U_j}{\partial x_i}\right) - \overline{u_i u_j}\right) \qquad (4.5)$$

Where $A_2 = (A_2)_{old}\beta_o \Delta T_*$ and the $(A_2)_{old}$ is given in table (3.1)

The energy equation (3.48) becomes;

$$\frac{\partial \Theta}{\partial t} + \frac{\partial U_j}{\partial x_j}\Theta = \frac{\partial}{\partial x_j}\left(B_2 \frac{\partial \Theta}{\partial x_j} - \overline{u_j \theta}\right) \qquad (4.6)$$

The turbulent kinetic energy equation (3.49) simplifies to;

$$\frac{\partial k}{\partial t}+\frac{\partial}{\partial x_j}U_jk = E_1\overline{u_J\frac{\partial}{\partial x_J}v\left(\frac{\partial u_\iota}{\partial x_J}+\frac{\partial u_J}{\partial x_\iota}\right)}-\frac{\partial}{\partial x_j}\overline{u_J\left(\frac{u_\iota u_J}{2}+\frac{p}{\rho}\right)}-\overline{u_\iota u_J}\frac{\partial U_i}{\partial x_j}+E_2\overline{\rho u_\iota}\frac{g_i}{\rho}$$

Or

$$\frac{\partial k}{\partial t}+\frac{\partial}{\partial x_j}U_jk = E_1\overline{u_J\frac{\partial}{\partial x_J}v\left(\frac{\partial u_\iota}{\partial x_J}+\frac{\partial u_J}{\partial x_\iota}\right)}-\frac{\partial}{\partial x_j}\overline{u_J\left(\frac{u_\iota u_J}{2}+\frac{p}{\rho}\right)}-P_k+G_k \tag{4.7}$$

Where

$$P_k=\overline{u_\iota u_J}\frac{\partial U_i}{\partial x_j}\text{ , }G_k=\overline{\rho u_\iota}\frac{g_i}{\rho} \tag{4.8}$$

The specific dissipation equation given by equation (3.50) simplifies to;

$$\frac{\partial \varepsilon}{\partial t}+\frac{\partial}{\partial x_j}U_j\varepsilon=-\frac{\partial}{\partial x_k}\left(F_1\overline{vu_k\frac{\partial u_\iota}{\partial x_J}\frac{\partial u_k}{\partial x_J}}+F_2v\overline{\frac{\partial u_k}{\partial x_\iota}\frac{\partial p}{\partial x_\iota}}-F_1v\frac{\partial \varepsilon}{\partial x_k}\right)-2F_1v\overline{\frac{\partial u_\iota}{\partial x_J}\frac{\partial u_\iota}{\partial x_J}\frac{\partial u_k}{\partial x_J}}-$$

$$2F_3\left(\overline{v\frac{\partial^2 u_\iota}{\partial x_k\partial x_J}}\right)^2+2F_4\frac{v}{\rho}\frac{\partial u_i}{\partial x_j}\frac{\partial p}{\partial x_i}g_i-2F_1v\frac{\partial u_i}{\partial x_k}\left(\overline{\frac{\partial u_\iota}{\partial x_J}\frac{\partial u_k}{\partial x_J}}+\overline{\frac{\partial u_J}{\partial x_\iota}\frac{\partial u_J}{\partial x_k}}\right)-2F_1v\frac{\partial^2 u_i}{\partial x_j\partial x_k}U_k\frac{\partial u_i}{\partial x_j} \tag{4.9}$$

In this case the density is neglected everywhere except in the instance that it causes buoyancy forces. For instance, in equation (4.8) the density is not neglected in G_k because it causes buoyancy forces hence have a contribution to the generation of notation as argued by Gatheri *et al*. (1993). In these equations the turbulent stress and the heat flux given by $\overline{u_\iota u_J}$ and $\overline{u_J\theta}$ respectively are given by;

$$\overline{u_\iota u_J}=\frac{2}{3}k\delta_{ij}-V_t\tau_{ij}$$

Or

$$\overline{u_\iota u_J}=\frac{2}{3}k\delta_{ij}-v_t\left(\frac{\partial U_i}{\partial x_j}+\frac{\partial U_j}{\partial x_i}\right) \tag{4.10}$$

And

$$\overline{u_J\theta}=-\frac{V_t}{\sigma T}\frac{\partial \Theta}{\partial x_j} \tag{4.11}$$

V_t In equation (4.10) and (4.11) in the $k-\varepsilon$ model is given as

$$V_t = c_\mu \frac{k^2}{\varepsilon} \tag{4.12}$$

4.4 Eliminating the Pressure Term in the Momentum Equation

The primitive variables in the momentum equation (4.5) are the velocity and the pressure. The equation has been written in velocity pressure formulation and has three velocity components and one pressure component that needs to be solved. The numerical solution of the primitive formulation is possible but there is a challenge in handling the pressure term. For instance, it could bring a challenge in using the same orders of interpolation for both the velocity and pressure. In this project we will use vorticity stream function approach in eliminating the pressure term.

4.4.1 Vorticity Stream function formulation

Equations (4.4) and (4.5) are the non-dimensional forms of the continuity and the momentum equation and may be written in Cartesian coordinate for two dimensional flow as follows;

$$\frac{\partial U}{\partial x} + \frac{\partial V}{\partial y} = 0 \tag{4.13}$$

$$\frac{\partial U}{\partial t} + U\frac{\partial U}{\partial x} + V\frac{\partial U}{\partial y} = -\frac{A_1}{\rho_o}\frac{\partial P}{\partial x} - A_2\Theta\cos\gamma + \frac{\partial}{\partial x}\left((A_3 + v_t)2\frac{\partial U}{\partial x} - \frac{2}{3}k\right) + \frac{\partial}{\partial y}\left((A_3 + v_t)2\frac{\partial U}{\partial y} + \frac{\partial V}{\partial x}\right)$$

(4.14 (a))

$$\frac{\partial V}{\partial t} + U\frac{\partial V}{\partial x} + V\frac{\partial U}{\partial y} = -\frac{A_1}{\rho_o}\frac{\partial P}{\partial x} - A_2\Theta\sin\gamma + \frac{\partial}{\partial y}\left((A_3 + v_t)\frac{\partial V}{\partial x} + \frac{\partial U}{\partial y}\right) + \frac{\partial}{\partial y}\left((A_3 + v_t)2\frac{\partial V}{\partial y} - \frac{2}{3}k\right)$$

(4.14 (b))

In this case, γ is the angle between the gravitational vector and the x-axis. The stream function $\psi(x, y, t)$ can be introduced for the case if a 2 dimensional flow to obtain the solution satisfying the continuity equation.

The stream function is defined as follows;

$$U = \frac{\partial \psi}{\partial y}, V = -\frac{\partial \psi}{\partial x} \tag{4.15}$$

The expanded form of the continuity equation (4.13) is satisfied by any continuous $\psi(x,y)$. This implies that the differential of (4.15) with respect to x and y will yield equation (4.13). One of the aims of defining the stream function $\psi(x,y,t)$ is to enable us to plot the stream lines. In addition, the introduction of the stream function is intended to help in reducing the number of simultaneous equations that are to be solved. For example, the expanded form of the momentum equation (4.14 (a)) and (4.14 (b)) are merged into one equation when the vorticity vector ξ is introduced. Conventionally, the vorticity vector is defined as; $\xi = \nabla \times U$ and U is the velocity vector.

In three dimensions, the components are given as;

$\xi_1 = \frac{\partial W}{\partial y} - \frac{\partial V}{\partial z}$, $\xi_3 = -\left(\frac{\partial W}{\partial x} - \frac{\partial U}{\partial z}\right)$, and $\xi_3 = \left(\frac{\partial V}{\partial x} - \frac{\partial U}{\partial y}\right)$ and in two dimension in x and y we have $\xi_1 = 0 = \xi_2$ and $\xi_3 = \frac{\partial V}{\partial x} - \frac{\partial U}{\partial y}$ using the right hand rule. The vorticity in 2 dimension can therefore be written as;

$$\xi = \frac{\partial V}{\partial x} - \frac{\partial U}{\partial y} \tag{4.16}$$

When equation (4.15) is differentiated with respect to y and x respectively and the results substituted to equation (4.16) we can obtain the relation between the stream function and the vorticity. This yields;

$$\frac{\partial^2 \psi}{\partial x^2} + \frac{\partial^2 \psi}{\partial y^2} = -\xi \tag{4.17}$$

The expanded form of the momentum equation (4.14 (a)) and (4.14 (b)) can be cross differentiated with respect to y and x respectively to eliminate the pressure term. Using the definition of vorticity in equation (4.16) and the difference between the resulting equations, the pressure term is eliminated and the equation obtained is a parabolic vorticity transport equation given as;

$$\frac{D\xi}{Dt} = (A_3 + V_t)\nabla^2\xi + 2\left(\frac{\partial V_t}{\partial x}\frac{\partial \xi}{\partial x} + \frac{\partial V_t}{\partial y}\frac{\partial \xi}{\partial y}\right) - (\nabla^2 V_t)\xi + 2\left(\frac{\partial^2 V_t}{\partial x^2}\frac{\partial V}{\partial x} - \frac{\partial^2 V_t}{\partial y^2}\frac{\partial U}{\partial y} + 2\frac{\partial^2 V_t}{\partial x \partial y}\frac{\partial V}{\partial y}\right) - A_2\left(\frac{\partial \Theta}{\partial x}\sin \Upsilon - \frac{\partial \Theta}{\partial y}\cos \Upsilon\right) \tag{4.18}$$

The equation (4.17) and (4.18) can replace the primitive variable equation (4.13) and (4.14) and the provided boundary condition for the stream function and vorticity are in line with the boundary condition for vorticity. The Dirichlet condition $\psi = constant$ has been customarily been used and the Neumann condition $\frac{\partial \psi}{\partial \eta} = 0$ is incorporated into the boundary condition of vorticity.

An accurate formulation of the wall vorticity ξ has been obtained by Wood (1954) and is given as;

$$\xi_b = \frac{3\psi_{b+1} - \psi_b}{(\Delta n)^2} - \frac{1}{2}\xi_{b+1} \tag{4.19}$$

In this equation, b+1 indicates the position that is one mesh point away from the boundary and gives an assumption that there is a linear variation of the vorticity in the interval. However, when this direct approach is used, the boundary conditions will be complicated for one to solve the Navier-stokes equations. The vorticity stream formulation is more attractive as compared to making use of primitive variable formulation because;

i. The continuity equation is satisfied automatically

ii. There is no need for a stagged finite difference grid

iii. There is a reduction in the number of differential equations that need to be solved

4.4.2 The vector potential formulation for three dimension flow

The vorticity stream function formulation is suited for two dimensional flow problems due to the fact that the stream function does not exist in three dimension and hence it cannot be extended to three dimension. However, a vector potential given as $\psi = \vec{\psi} = \overrightarrow{U_i} + \overrightarrow{V_j} + \overrightarrow{W_k}$ exist for solenoidal vector field. Nonetheless, the analogy has not been used much because the equations to be solved does not reduce as expected but instead increase. In addition, the understanding of the boundary conditions is difficult.

In this account, the implementation has only been restricted to the simple cases where the region is closed and connected. As aforementioned the advantage of making use of the vector potential formulation is that there is an automatic satisfaction of the continuity equation. One of the application of vector potential formulation was carried out by Hiroyuki *et al*. (1985) in the study of three dimensional enclosure in which one of the vertical walls was cooled and heated on the floor. In the study, Hiroyuki *et al*. (1985) considered the range of the Rayleigh was 10^6 and 10^7. The vector potential given by ψ is defined as $U = \nabla \times \psi$ and when this is assumed to be solenoidal we will have $\nabla . \psi = 0$. By making use of this definition one can get the velocity components in terms of the stream function as;

$$U = \frac{\partial \psi_3}{\partial y} - \frac{\partial \psi_2}{\partial z}, V = -\left(\frac{\partial \psi_3}{\partial x} - \frac{\partial \psi_1}{\partial z}\right),\ W = \frac{\partial \psi_2}{\partial x} - \frac{\partial \psi_1}{\partial y} \tag{4.20}$$

The relation of vorticity and the stream function is given as;

$$\xi = -\nabla^2 \psi \tag{4.21}$$

This can be used in the derivation of the vorticity in three dimension to get the following;

$$\frac{\partial^2 \psi_1}{\partial x^2} + \frac{\partial^2 \psi_1}{\partial y^2} + \frac{\partial^2 \psi_1}{\partial z^2} = -\xi_1,\ \frac{\partial^2 \psi_2}{\partial x^2} + \frac{\partial^2 \psi_2}{\partial y^2} + \frac{\partial^2 \psi_2}{\partial z^2} = -\xi_2, \frac{\partial^2 \psi_3}{\partial x^2} + \frac{\partial^2 \psi_3}{\partial y^2} + \frac{\partial^2 \psi_3}{\partial z^2} = -\xi_1 \tag{4.22}$$

On recasting the momentum equation into a vorticity vector potential one can overcome the problem that is seen in the use of primitive variables, pressure and the need for solving the continuity equation. The three components of the vorticity vector ξ are;

$$\xi_1 = \frac{\partial W}{\partial y} - \frac{\partial V}{\partial z},\quad \xi_2 = -\left(\frac{\partial W}{\partial x} - \frac{\partial U}{\partial z}\right),\quad \xi_3 = \frac{\partial V}{\partial x} - \frac{\partial U}{\partial y} \tag{4.23}$$

The vorticity transport equation can be obtained by taking the curl of the momentum equation (4.5) as affirmed by Ozoe *et al*. (1976) and hence one will get the transport equation in component form as;

$$\frac{\partial \xi_1}{\partial t} + U\frac{\partial \xi_1}{\partial x} + V\frac{\partial \xi_1}{\partial y} + W\frac{\partial \xi_1}{\partial z} - \xi_1\frac{\partial U}{\partial x} - \xi_2\frac{\partial U}{\partial y} - \xi_3\frac{\partial U}{\partial z} = (A_3 + v_t)\nabla^2\xi_1 + \frac{\partial v_t}{\partial x}\frac{\partial \xi_1}{\partial x} + 2\frac{\partial v_t}{\partial y}\frac{\partial \xi_1}{\partial y} +$$

$$\frac{\partial v_t}{\partial z}\frac{\partial \xi_1}{\partial z} - \frac{\partial v_t}{\partial y}\frac{\partial \xi_2}{\partial x} - \frac{\partial v_t}{\partial z}\frac{\partial \xi_3}{\partial x} - \left(\frac{\partial^2 v_t}{\partial y^2} + \frac{\partial^2 v_t}{\partial z^2}\right)\xi_1 + \frac{\partial^2 v_t}{\partial x \partial y}\xi_2 + \frac{\partial^2 v_t}{\partial x \partial z}\xi_3 + 2\left[\frac{\partial^2 v_t}{\partial x \partial y}\frac{\partial W}{\partial x} + \frac{\partial^2 v_t}{\partial y^2} + \frac{\partial^2 v_t}{\partial y^2}\frac{\partial W}{\partial y} +\right.$$

$$\left.\frac{\partial^2 v_t}{\partial y \partial z}\frac{\partial W}{\partial z} - \left(\frac{\partial^2 v_t}{\partial x \partial z}\frac{\partial V}{\partial x} + \frac{\partial^2 v_t}{\partial y \partial x}\frac{\partial V}{\partial y} + \frac{\partial^2 v_t}{\partial z^2}\frac{\partial V}{\partial z}\right)\right] \tag{4.24}$$

$$\frac{\partial \xi_2}{\partial t} + U\frac{\partial \xi_2}{\partial x} + V\frac{\partial \xi_2}{\partial y} + W\frac{\partial \xi_2}{\partial z} - \xi_1\frac{\partial V}{\partial x} - \xi_2\frac{\partial V}{\partial y} - \xi_3\frac{\partial V}{\partial z} = (A_3 + v_t)\nabla^2\xi_2 + 2\frac{\partial v_t}{\partial x}\frac{\partial \xi_2}{\partial x} + \frac{\partial v_t}{\partial y}\frac{\partial \xi_2}{\partial y} +$$

$$2\frac{\partial v_t}{\partial z}\frac{\partial \xi_2}{\partial z} - \frac{\partial v_t}{\partial x}\frac{\partial \xi_1}{\partial y} - \frac{\partial v_t}{\partial z}\frac{\partial \xi_3}{\partial y} - \left(\frac{\partial^2 v_t}{\partial x^2} + \frac{\partial^2 v_t}{\partial z^2}\right)\xi_2 + \frac{\partial^2 v_t}{\partial x \partial y}\xi_1 + \frac{\partial^2 v_t}{\partial y \partial z}\xi_3 - A_3\frac{\partial \Theta}{\partial z} + 2\left[\frac{\partial^2 v_t}{\partial x \partial z}\frac{\partial U}{\partial x} +\right.$$

$$\left.\frac{\partial^2 v_t}{\partial y \partial z}\frac{\partial W}{\partial y} + \frac{\partial^2 v_t}{\partial z^2}\frac{\partial U}{\partial z} - \left(\frac{\partial^2 v_t}{\partial x^2}\frac{\partial W}{\partial x} + \frac{\partial^2 v_t}{\partial x \partial y}\frac{\partial W}{\partial y} + \frac{\partial^2 v_t}{\partial x \partial z}\frac{\partial W}{\partial z}\right)\right] \tag{4.25}$$

$$\frac{\partial \xi_3}{\partial t} + U\frac{\partial \xi_3}{\partial x} + V\frac{\partial \xi_3}{\partial y} + W\frac{\partial \xi_3}{\partial z} - \xi_1\frac{\partial W}{\partial x} - \xi_2\frac{\partial W}{\partial y} - \xi_3\frac{\partial W}{\partial z} = (A_3 + v_t)\nabla^2\xi_3 + 2\frac{\partial v_t}{\partial x}\frac{\partial \xi_3}{\partial x} + 2\frac{\partial v_t}{\partial y}\frac{\partial \xi_3}{\partial y} +$$

$$\frac{\partial v_t}{\partial z}\frac{\partial \xi_3}{\partial z} - \frac{\partial v_t}{\partial x}\frac{\partial \xi_1}{\partial z} - \frac{\partial v_t}{\partial y}\frac{\partial \xi_2}{\partial z} - \left(\frac{\partial^2 v_t}{\partial x^2} + \frac{\partial^2 v_t}{\partial y^2}\right)\xi_3 + \frac{\partial^2 v_t}{\partial x \partial z}\xi_1 + \frac{\partial^2 v_t}{\partial y \partial z}\xi_2 - A_2\frac{\partial \Theta}{\partial y} + 2\left[\frac{\partial^2 v_t}{\partial x^2}\frac{\partial V}{\partial x} + \frac{\partial^2 v_t}{\partial x \partial y}\frac{\partial V}{\partial y} +\right.$$

$$\left.\frac{\partial^2 v_t}{\partial x \partial z}\frac{\partial V}{\partial z} - \left(\frac{\partial^2 v_t}{\partial x \partial y}\frac{\partial U}{\partial x} + \frac{\partial^2 v_t}{\partial y^2}\frac{\partial U}{\partial y} + \frac{\partial^2 v_t}{\partial y \partial z}\frac{\partial U}{\partial z}\right)\right] \tag{4.26}$$

The continuity and the momentum equations given by equations (4.4) and (4.5) are replaced by the vorticity transport equations (4.24), (4.25), and (4.26) and equation (4.22).

The variables that are to be solved in this equations are ξ_1, ξ_2, ξ_3, U, V, W, Θ, k, and ε. The variables are obtained by solving equations (4.24), (4.25), and (4.26) for ξ_1, ξ_2, ξ_3 respectively, and U, V, W will be obtained by solving equation (4.20) while k and ε will be obtained by using the turbulent energy equations derived by Ince and Launde (1989) and given as follows;

$$\frac{\partial k}{\partial t} + \frac{\partial}{\partial x_j}U_j k = v_t\left(\frac{\partial U_i}{\partial x_j} + \frac{\partial U_j}{\partial x_i}\right)\frac{\partial U_i}{\partial x_j} - \varepsilon\frac{\partial}{\partial x_j}\left[\left(E_2 + \frac{v_t}{\sigma_k}\right)\frac{\partial k}{\partial x_j}\right] - E_2 g_i\overline{U_i\theta} \tag{4.27}$$

$$\frac{\partial \bar{\varepsilon}}{\partial t} + \frac{\partial}{\partial x_j} U_j \bar{\varepsilon} = C_{\varepsilon 1} \frac{\bar{\varepsilon}}{k} v_t \left(\frac{\partial U_i}{\partial x_j} + \frac{\partial U_j}{\partial x_i} \right) \frac{\partial U_i}{\partial x_j} - C_{\varepsilon 2} \frac{\bar{\varepsilon}^2}{k} + 2F_1^2 v_t \left(\frac{\partial^2 U_j}{\partial x_j \partial x_k} \right)^2 + \frac{\partial}{\partial x_j} \left[\left(F_1 + \frac{v_t}{\sigma_\varepsilon} \right) \frac{\partial \bar{\varepsilon}}{\partial x_i} \right] +$$

$$F_4 g_i \overline{U_\iota \theta} \frac{\bar{\varepsilon}}{k} + 0.83 \left(\frac{k^{\frac{3}{2}}}{\bar{\varepsilon} C_1 x_n} - 1 \right)^2 \frac{\bar{\varepsilon}^2}{k} \quad (4.28)$$

In this equations $\bar{\varepsilon}$ in equation (4.28) is related to the total dissipation ε by the relation;

$$\varepsilon = \bar{\varepsilon} + D \quad (4.29)$$

$D = 2v \left(\frac{\partial k^{\frac{1}{2}}}{\partial x_j} \right)^2$ is the extra rate of destruction in the wall region and far from solid boundary we have $\varepsilon = \bar{\varepsilon}$ because D approaches zero as one goes away from the solid boundary. The empirical coefficient in the study by Launder and Sharma (1974) are given as;

$$C_{\varepsilon 1} = 1.44,\ C_{\varepsilon 2} = 1.92[1 - 0.3exp(-R_t^2)], R_t = \frac{k^2}{F_1 \varepsilon^2}, \sigma_k = 1.0, \sigma_\varepsilon = 1.3 \quad (4.30)$$

4.5 Boundary Conditions

4.5.1 Temperature Boundary Conditions

As defined earlier the non-dimensional temperature was given as $\Theta = \frac{(T - T_*)}{\Delta T_*}$, whereby ΔT_* is the characteristic temperature difference between the cold and the hot surfaces that is $\Delta T_* = T_{hot} - T_{cold}$. The choice of Θ is such that it is bounded between 0 and 1. The isothermal and adiabatic boundary conditions are used. These conditions are given as

$$\Theta = \text{constant and } \frac{\partial \Theta}{\partial \eta} = 0 \quad (4.31)$$

respectively where η represents the direction of the wall. Since the problem at hand involves heating on the lower part of the face-wall and cooling on the upper part of the face wall, the other five walls of the enclosure are kept adiabatic. The Dirichlet boundary conditions are used on the hot part of the wall and the cold part of the wall whereby $\Theta_{hot} = 1$ and $\Theta_{cold} = 0$ are used. On the adiabatic walls, the Neumann boundary condition is used. For each of the adiabatic walls $\frac{\partial\Theta}{\partial\eta} = 0$ is used. For instance, in the x-y plane $\frac{\partial\Theta}{\partial\eta} = 0$.

In the study, the results were obtained for difference Rayleigh numbers and the calculation of the Rayleigh number is carried out as indicated in the formula below;

$$Ra = \frac{g\beta\Delta\rho L^3}{\mu\alpha} = \frac{g\beta\Delta T L^3}{v\alpha} \tag{4.32}$$

Where g is the acceleration due to gravity (-9.81)

ρ is the density

μ is the dynamic viscosity

β is the volume of thermal expansively

ΔT the difference in temperature between the hot and the cold walls

v is the kinematic viscosity,

L is the characteristic length, and

α is the thermal diffusivity

The temperature of the Bottom (hot) wall is kept constant at 313K while the temperature of the top (cold) wall is kept constant at 293K. The aspect ratio is kept constant at 1 and the dimensions of the enclosure are varied for the change in the Rayleigh number. The dimension of the enclosure are 1m by 1 m, 2m by 2m, 4m by 4m, and 18m by 18m to get the Rayleigh numbers of 1.797×10^9, 1.437×10, 1.150×10^{11}, and 1.048×13 respectively. The operating temperature of the enclosure is 303K and the other four walls are kept adiabatic (completely insulated).

4.5.2 Velocity Boundary Conditions

Typically, the conditions in the boundary for a fluid that is in motion depends on the velocity of the fluid. In this case, the boundary conditions that have been used are of no-slip and hence the fluid will have zero velocity in relation to the boundary. In addition, the normal components of velocity at each boundary is zero due to the fact that for a closed cavity the boundary is considered to be impermeable and hence it is capable of motion in its own place only.

4.5.3 Vector Potential Boundary Conditions

In the no-slip boundary, it is difficult to determine the boundary conditions because the components of ψ are not always zero. Only the components that are tangential and those that are normal derivatives are zero. For instance, for the case of the wall along the y-z plane we have $\frac{\partial \psi_1}{\partial x} = 0, \psi_2 = \psi_3 = 0$ at x=0.

Along the x-z plane, $\frac{\partial \psi_2}{\partial y} = 0, \psi_1 = \psi_3 = 0$ at y=0

And along x-y plane, $\frac{\partial \psi_3}{\partial z} = 0, \psi_1 = \psi_2 = 0$ and z=0.

4.5.4 Vorticity Boundary Conditions

The vorticity boundary conditions are obtained from equation (4.21). The vorticity components for the no-slip can be expressed using the fundamental velocities as in equation (4.23). For the wall along y-z, we have $\frac{\partial W}{\partial y} = \frac{\partial V}{\partial z} = \frac{\partial U}{\partial z} = \frac{\partial U}{\partial y} = 0$ using the right hand rule and hence the boundary condition will be given as $\xi_1 = 0, \xi_2 = -\frac{\partial W}{\partial x}, \xi_3 = \frac{\partial V}{\partial x}$.

For the wall along x-y we have $\frac{\partial U}{\partial y} = \frac{\partial V}{\partial x} = \frac{\partial W}{\partial x} = \frac{\partial W}{\partial y} = 0$. this will give the boundary condition as,

$\xi_1 = -\frac{\partial V}{\partial z}, \xi_2 = \frac{\partial U}{\partial z}, \xi_3 = 0$

Along x-z wall we have $\frac{\partial W}{\partial x} = \frac{\partial U}{\partial z} = \frac{\partial V}{\partial z} = \frac{\partial V}{\partial x} = 0$. This will give the boundary condition as

$\xi_1 = \frac{\partial W}{\partial y}, \xi_2 = 0, \xi_3 = -\frac{\partial U}{\partial y}$

The equations (4.22), (4.23), (4.24), (4.25), (4.26), (4.27), and (4.28) together with the boundary conditions will completely give the mathematical model description for the $k - \varepsilon$ model.

CHAPTER FIVE

NUMERICAL METHODS

The coupled non-linear differential equations with the boundary conditions are solved using the finite difference technique. In this technique, the partial differential equations are to be approximated using a set of linear equations that are related to the values of the functions at specific mesh points and thereafter the set of algebraic equations that are formed are to be solved. The non-linearity of the differential equations necessitate for the development of an iteration procedure and hence calling for the use of the false transient method.

5.1 False Transient Method

This involves adding fictitious transient factors to the time derivative of the mean energy equation, (4.6), the vorticity transport equations, (4.24), (4.25), and (4.26), the turbulent kinetic energy equation, (4.27), and to the rate of dissipation of turbulent kinetic energy equation, (4.28). The fictitious transient factors that will be added will be $\frac{1}{\alpha_\Theta}$ for the mean energy equation, $\frac{1}{\alpha_\xi}$ for the vorticity transport equations, $\frac{1}{\alpha_k}$ for the turbulent kinetic energy equation, and $\frac{1}{\alpha_\varepsilon}$ for the rate of dissipation of turbulent kinetic energy equation. These coefficients are useful for the determination of the time step that is suitable for the solution and allows for faster convergence of the solution.

A suitable mesh grid should be determined for the approximation of the differential equations. In this project, the solution region has been sub-divided into mesh points that represent the volume with the integers i, j, and k used to denote the mesh points.

5.2 Finite Difference Approximations

In this project, the forward, central, and backward difference approximations have been used where applicable. For instance, the forward difference and the backward difference have been used near the boundaries and the corner locations. Furthermore, the hybrid scheme has been used in the difference of the convective terms. Non-uniform grids have been used as a result of the dominance of the convective terms. In this regard, there is a change in the grid spacing throughout the solution volume. For instance, there is use of a larger spacing in regions with low flow change and smaller spacing are used in regions with a higher flow gradient.

The mesh points are denoted as

$\Phi = \Phi(i, j, k)$

$$\Phi(\mathrm{i}-1) = \Phi(i-1, j, k) \tag{5.1}$$

The derivatives at the mesh points are;

$$\frac{\partial \Phi}{\partial x} = \frac{h_i^2}{h_i^2 h_{i+1} + h_i h_{i+1}^2} \Phi_{i+1} + \frac{-(h_i^2 + h_{i+1}^2)}{h_i^2 h_{i+1} + h_i h_{i+1}^2} \Phi + \frac{-h_{i+1}^2}{h_i^2 h_{i+1} + h_i h_{i+1}^2} \Phi_{i-1} \tag{5.2(a)}$$

$$\frac{\partial \Phi}{\partial y} = \frac{h_j^2}{h_j^2 h_{j+1} + h_j h_{j+1}^2} \Phi_{j+1} + \frac{-(h_j^2 + h_{j+1}^2)}{h_j^2 h_{j+1} + h_j h_{j+1}^2} \Phi + \frac{-h_{j+1}^2}{h_j^2 h_{j+1} + h_j h_{j+1}^2} \Phi_{j-1} \tag{5.2(b)}$$

$$\frac{\partial \Phi}{\partial z} = \frac{h_k^2}{h_k^2 h_{k+1} + h_k h_{k+1}^2} \Phi_{k+1} + \frac{-(h_k^2 + h_{k+1}^2)}{h_k^2 h_{k+1} + h_k h_{k+1}^2} \Phi + \frac{-h_{k+1}^2}{h_k^2 h_{k+1} + h_k h_{k+1}^2} \Phi_{k-1} \tag{5.2(c)}$$

The second derivative is given as;

$$\frac{\partial^2 \Phi}{\partial x^2} = \frac{2h_i}{h_i^2 h_{i+1} + h_i h_{i+1}^2} \Phi_{i+1} + \frac{-2(h_i + h_{i+1})}{h_i^2 h_{i+1} + h_i h_{i+1}^2} \Phi + \frac{2h_{i+1}}{h_i^2 h_{i+1} + h_i h_{i+1}^2} \Phi_{i-1} \tag{5.3(a)}$$

$$\frac{\partial^2 \Phi}{\partial y^2} = \frac{2h_j}{h_j^2 h_{j+1} + h_j h_{j+1}^2} \Phi_{j+1} + \frac{-2(h_j + h_{j+1})}{h_j^2 h_{j+1} + h_j h_{j+1}^2} \Phi + \frac{2h_{j+1}}{h_j^2 h_{j+1} + h_j h_{j+1}^2} \Phi_{j-1} \tag{5.3(b)}$$

$$\frac{\partial^2 \Phi}{\partial z^2} = \frac{2h_k}{h_k^2 h_{k+1} + h_k h_{k+1}^2} \Phi_{k+1} + \frac{-2(h_k + h_{k+1})}{h_k^2 h_{k+1} + h_k h_{k+1}^2} \Phi + \frac{2h_{k+1}}{h_k^2 h_{k+1} + h_k h_{k+1}^2} \Phi_{k-1} \tag{5.3(c)}$$

For simplification let;

$$\frac{-h_{i+1}^2}{h_i^2 h_{i+1} + h_i h_{i+1}^2} = AX1, \quad \frac{-(h_i^2 + h_{i+1}^2)}{h_i^2 h_{i+1} + h_i h_{i+1}^2} = AX2, \quad \frac{h_i^2}{h_i^2 h_{i+1} + h_i h_{i+1}^2} = AX3$$

$$\frac{2h_{i+1}}{h_i^2 h_{i+1} + h_i h_{i+1}^2} = AX4, \quad \frac{-2(h_i + h_{i+1})}{h_i^2 h_{i+1} + h_i h_{i+1}^2} = AX5 \ \text{ and } \ \frac{2h_i}{h_i^2 h_{i+1} + h_i h_{i+1}^2} = AX6$$

For the x-direction, this can be simplified as well for the y and z direction to reduce equations (5.2) and 5.3). For instance equation (5.2(a)) and (5.3(a)) will reduce to;

$\frac{\partial \Phi}{\partial x} = AX3\Phi_{i+1} + AX2\Phi + AX1\Phi_{i-1}$ and $\frac{\partial^2 \Phi}{\partial x^2} = AX6\Phi_{i+1} + AX5\Phi + AX4\Phi_{i-1}$ respectively. On applying this to the equations (4.6), (4.20), (4.22), (4.24), (4.26), (4.27), and (4.28) we get the finite difference equations as follows;

The mean energy equation given in equation (4.6) becomes;

$$\frac{\Theta^{n+1}-\Theta}{\alpha_\Theta \Delta t} = -U(AX1\Theta_{i-1} + AX2\Theta + AX3\Theta_{i+1}) - V\left(AY1\Theta_{j-1} + AY2\Theta + AY3\Theta_{j+1}\right) - W(AZ1\Theta_{K-1} + AZ2\Theta + AZ3\Theta_{K+1}) + \left(B_2 + \frac{v_t}{\sigma_t}\right)\left(AX4\Theta_{i-1} + AX5\Theta + AX6\Theta_{i+1} + AY4\Theta_{j-1} + AY5\Theta + AY6\Theta_{j+1} + AZ4\Theta_{k-1} + AZ5\Theta + AZ6\Theta_{k+1}\right) + \frac{1}{\sigma_T}\left[\left(AX1v_{t_{i-1}} + AX2v_t + AX3v_{t_{i+1}}\right)(AX1\Theta_{i-1} + AX2\Theta + AX3\Theta_{i+1}) + \left(AY1v_{t_{j-1}} + AY2v_t + AY3v_{t_{j+1}}\right)\left(AY1\Theta_{j-1} + AY2\Theta + AY3\Theta_{j+1}\right) + \left(AZ1v_{t_{k-1}} + AZ2v_t + AZ3v_{t_{k+1}}\right)(AZ1\Theta_{k-1} + AZ2\Theta + AZ3\Theta_{k+1})\right] \quad (5.4)$$

Where $B_2 = \frac{1}{Pr\sqrt{Gr}}$ as given in table (3.1)

The velocity equations that are given by equation (4.20) become;

$$U = AY1\psi_{3j-1} + AY2\psi_3 + AY3\psi_{3j+1} - (AZ1\psi_{2k-1} + AZ2\psi_2 + AZ3\psi_{2k+1}) \quad (5.5)$$

$$V = AZ1\psi_{1k-1} + AZ2\psi_1 + AZ3\psi_{3i+1} - (AX1\psi_{3i-1} + AX2\psi_3 + AX3\psi_{3i+1}) \quad (5.6)$$

$$W = AX1\psi_{2i-1} + AX2\psi_2 + AX3\psi_{2i+1} - \left(AY1\psi_{1j-1} + AY2\psi_1 + AY3\psi_{1j+1}\right) \quad (5.7)$$

The vector potential equation given by equation (4.22) becomes;

$$AX4\psi_{1i-1} + AX5\psi_1 + AX6\psi_{1i+1} + AY4\psi_{1j-1} + AY5\psi_1 + AY6\psi_{1j+1} + AZ4\psi_{1k-1} + AZ5\psi_1 + AZ6\psi_{1k+1} = -\xi_1 \quad (5.8)$$

$$AX4\psi_{2i-1} + AX5\psi_2 + AX6\psi_{2i+1} + AY4\psi_{2j-1} + AY5\psi_2 + AY6\psi_{2j+1} + AZ4\psi_{2k-1} + AZ5\psi_2 + AZ6\psi_{2k+1} = -\xi_2 \quad (5.9)$$

$AX4\psi_{3i-1} + AX5\psi_3 + AX6\psi_{3i+1} + AY4\psi_{3j-1} + AY5\psi_3 + AY6\psi_{3j+1} + AZ4\psi_{3k-1} +$

$$AZ5\psi_3 + AZ6\psi_{3k+1} = -\xi_3 \tag{5.10}$$

The vorticity transport equations in component for as given in equations (4.24), (4.25), and (4.26) for the x direction becomes;

$\frac{\xi_1^{n+1}-\xi_1}{\alpha_\xi \Delta t} = -U(AX1\xi_{1i-1} + AX2\xi_1 + AX3\xi_{1i+1}) - V\big(AY1\xi_{1j-1} + AY2\xi_1 + AY3\xi_{1j+1}\big) -$

$W(AZ1\xi_{1k-1} + AZ2\xi_1 + AZ3\xi_{1k+1}) + \xi_1(AX1U_{i-1} + AX2U + AX3U_{i+1}) + \xi_2\big(AY1U_{j-1} +$

$AY2U + AY3U_{j+1}\big) + \xi_3(AZ1U_{k-1} + AZ2U + AZ3U_{k+1}) + (A_3 + v_t)\big[AX4\xi_{1i-1} + AX5\xi_1 +$

$AX6\xi_{1i+1} + AY4\xi_{1j-1} + AY5\xi_1 + AY6\xi_{1j+1} + AZ4\xi_{1k-1} + AZ5\xi_1 + AZ6\xi_{1k+1}\big] +$

$(AX1v_{ti-1} + AX2v_t + AX3v_{ti+1})(AX1\xi_{1i-1} + AX2\xi_1 + AX3\xi_{1i+1}) + 2\big(AY1v_{tj-1} + AY2v_t +$

$AY3v_{tj+1}\big)\big(AY1\xi_{1j-1} + AY2\xi_1 + AY3\xi_{1j+1}\big) + 2(AZ1v_{tk-1} + AZ2v_t +$

$AZ3v_{tk+1})(AZ1\xi_{1k-1} + AZ2\xi_1 + AZ3\xi_{1k+1}) - \big(AY1v_{tj-1} + AY2v_t + AY3v_{tj+1}\big)(AX1\xi_{2i-1} +$

$AX2\xi_2 + AX3\xi_{2i+1}) - (AZ1v_{tk-1} + AZ2v_t + AZ3v_{tk+1})(AX1\xi_{2i-1} + AX2\xi_2 + AX3\xi_{2i+1}) -$

$\big(AY4v_{tj-1} + AY5v_t + AY6v_{tj+1} + AZ4v_{tk-1} + AZ5v_t + AZ6v_{tk+1}\big)\xi_1 +$

$\big[AY1\big(AX1v_{ti-1,j-1} + AX2v_{t,j-1} + AX3v_{ti+1,j-1}\big) + AY2(AX1v_{ti-1} + AX2v_t + AX3v_{ti+1}) +$

$AY3\big(AX1v_{ti-1,j+1} + AX2v_{t,j+1} + AX3v_{ti+1,j+1}\big)\big]\xi_2 + \big[AZ1\big(AX1v_{ti-1,k-1} + AX2v_{t,k-1} +$

$AX3v_{ti+1,k-1}\big) + AZ2(AX1v_{ti-1} + AX2v_t + AX3v_{ti+1}) + AZ3\big(AX1v_{ti-1,k+1} + AX2v_{t,k+1} +$

$AX3v_{ti+1,k+1}\big)\big]\xi_3 + 2\big\{\big[AY1\big(AX1v_{ti-1,j-1} + AX2v_{t,j-1} + AX3v_{ti+1,j-1}\big) +$

$AY2\big(AX1v_{ti-1,j-1} + AX2v_{t,j-1} + AX3v_{ti+1,j-1}\big)\big](AX1W_{i-1} + AX2W + AXW_{i+1}) +$

$\big(AY4v_{tj-1} + AY5v_t + AY6v_{tj+1}\big)(AX1W_{i-1} + AX2W + AXW_{i+1}) + \big[AZ1\big(AY1v_{tj-1,k-1} +$

$AY2v_{t,k-1} + AY3v_{tj+1,k-1}\big) + AZ2\big(AY1v_{tj-1} + AY2v_t + AY3v_{tj+1}\big) + AZ3\big(AY1v_{tj-1,k+1} +$

$AY2v_{t,k+1} + AY3v_{tj+1,k+1}\big)\big](AX1W_{i-1} + AX2W + AXW_{i+1}) - \big[AZ1\big(AX1v_{ti-1,k-1} +$

$$AX2v_{t,k-1} + AX3v_{ti+1,k-1}\big) + AZ2(AY1v_{ti-1} + AY2v_t + AY3v_{ti+1}) + AZ3\big(AX1v_{ti-1,k+1} +$$

$$AX2v_{t,k+1} + AX3v_{ti+1,k+1}\big)\big](AX1V_{i-1} + AX2V + AXV_{i+1}) + \big[AZ1\big(AY1v_{tj-1,k-1} +$$

$$AY2v_{t,k-1} + AY3v_{tj+1,k-1}\big) + AZ2\big(AY1v_{tj-1} + AY2v_t + AY3v_{tj+1}\big) + AZ3\big(AY1v_{tj-1,k+1} +$$

$$AY2v_{t,k+1} + AY3v_{tj+1,k+1}\big)\big(AY1V_{j-1} + AY2V + AY3V_{j+1}\big)(AY4V_{tk-1} + AY5V_t +$$

$$AY6V_{tk+1})(AY1V_{k-1} + AY2V + AY3V_{k+1})\big]\big\} \quad (5.11)$$

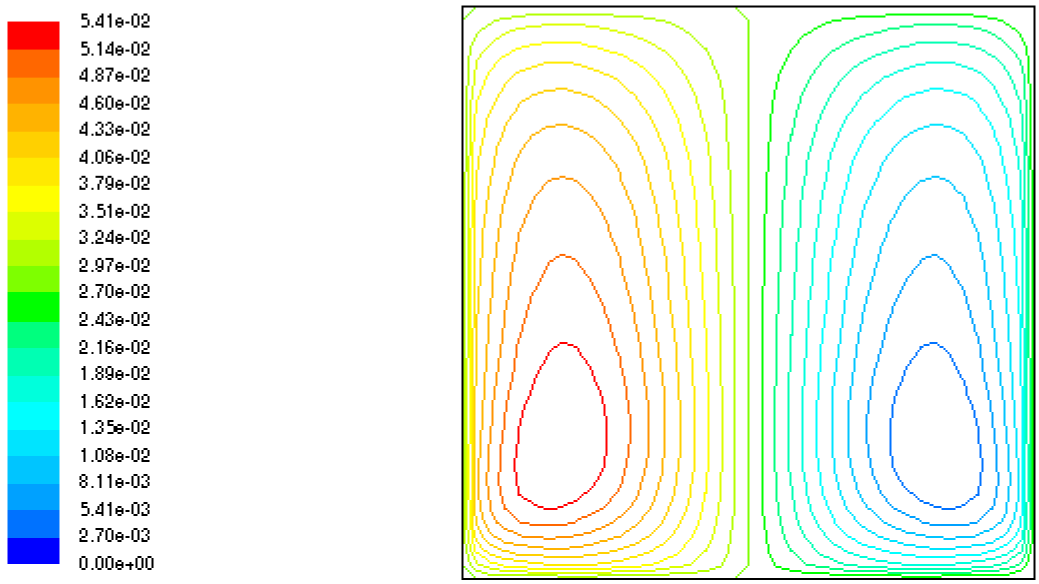
These differential equations are solved using a computer program called Fluent 6.3.26 and the results obtained are discussed in the following chapter.

CHAPTER SIX

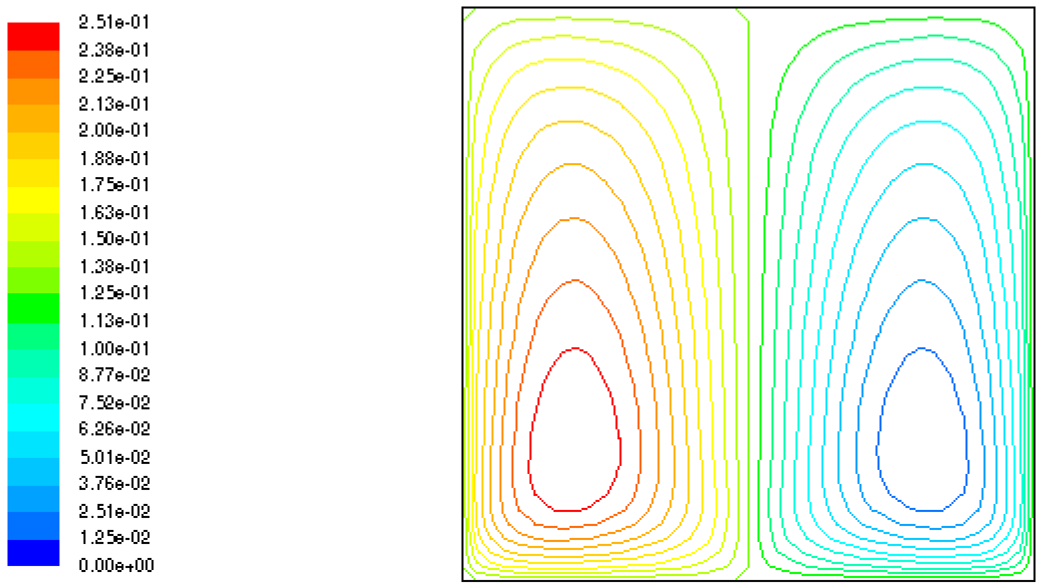
RESULTS AND DISCUSSION

6.1 Distribution of Streamlines

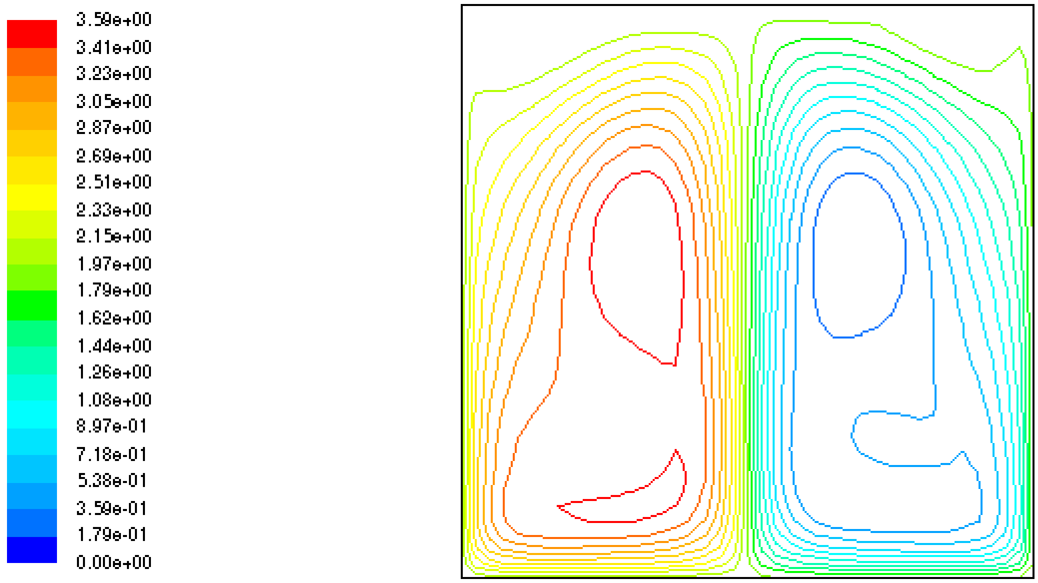
The path that is traced by a massless particle as it moves with the flow is referred to as the streamline. These are tangent lines to the velocity of flow and are designed to minimize the resistance of flow in the fluid such as air. The results of this study were obtained for Rayleigh numbers ranging from 1.797×10^{9} to 1.048×10^{13}. Figures 6.1 (a), (b), (c), and (d) shows the distribution of streamlines. In figure 6.1 (a), there are two circulating vortices. In this case, a vortex is considered to be a whirling fluid motion. With the increase in the Rayleigh number, it is seen that there is an increase in the number of vortices and stream functions. In addition, the increase in the Rayleigh number causes an increase in the velocity with the maximum velocity in the streamlines for b, c, and d below being $2.51 \times 10^{-1} Kg/s$, $3.50 \times 10^{-1}\ Kg/s$ and, $4.35 \times 10^{-1} Kg/s$ as it can be seen in the figures below. The movement of the streamlines is seen to be from the bottom wall (hot) and down from the top (cold) wall. This observation is in line with the principle of heat transfer. The increase in the Rayleigh number causes an increase in buoyancy forces and hence increasing the size of the vortices as well as the strength of the stream function. These results were seen to be in line with the experimental results by Doğan and Doğan (2017) who presented that the change in the Rayleigh number caused a direct change in the velocity of the fluid.



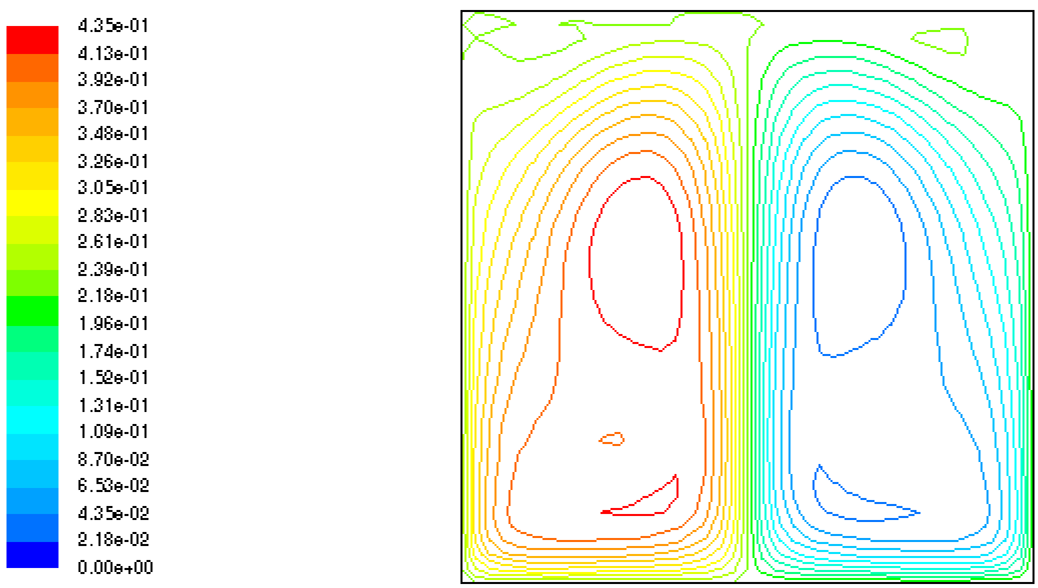
a. $Ra = 1.797 \times 10^9$



b. $Ra = 1.437 \times 10^{10}$



c. $Ra = 1.150 \times 10^{11}$

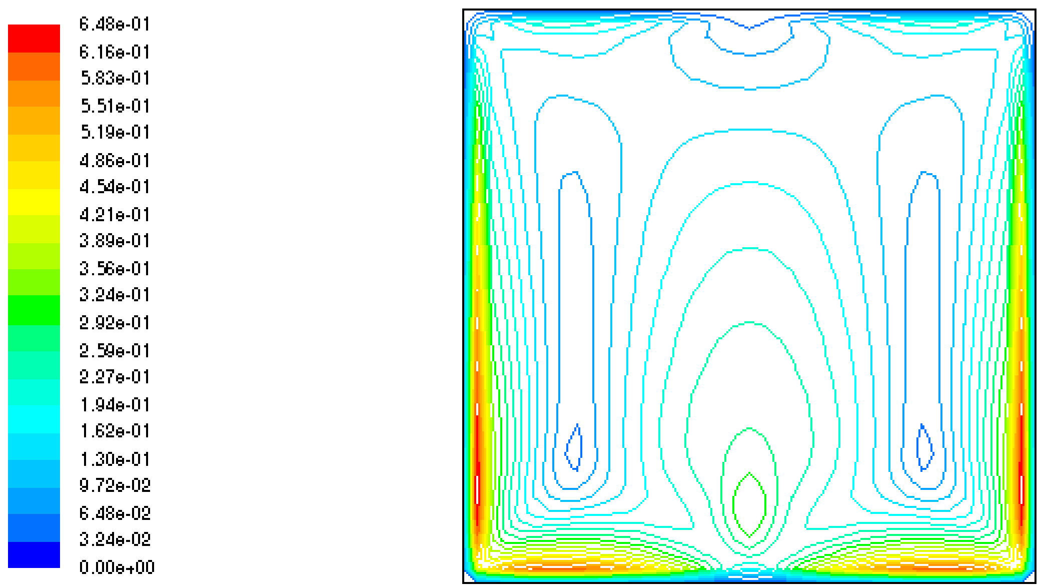


d. $Ra = 1.048 \times 10^{13}$

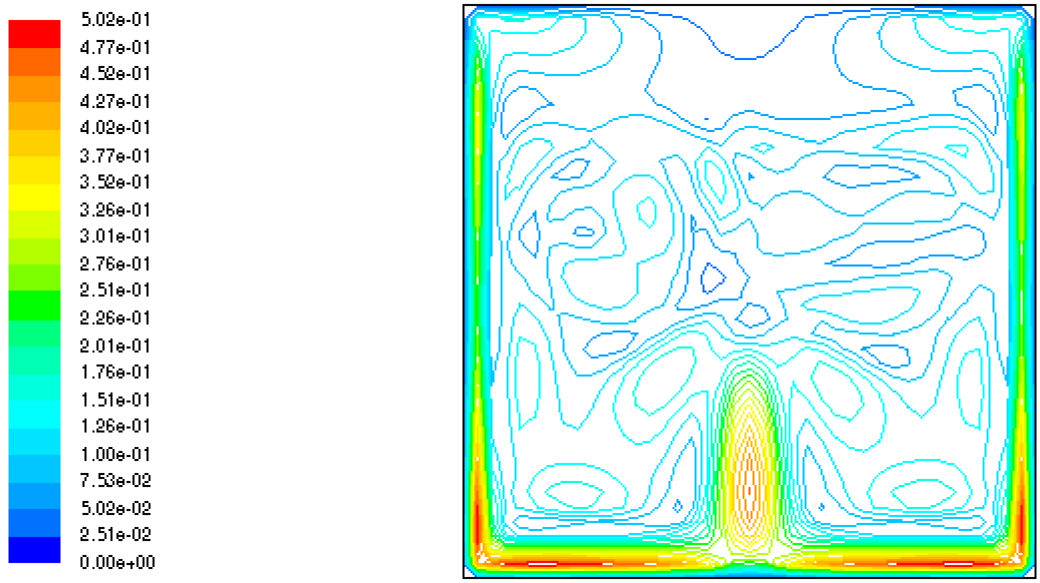
Figure 6.1. Contours of Stream function (Kg/s)

6.2 Contours of Velocity magnitudes

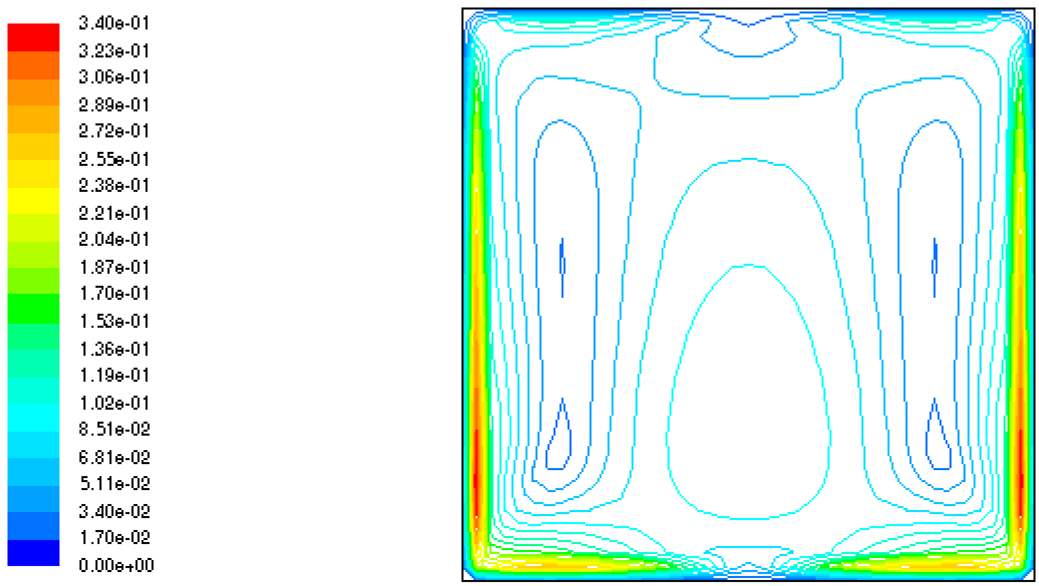
The representation of the contours of velocity magnitudes are shown in Figure 6.2 (a), (b), (c), and (d). With the increase in the Rayleigh number it can be seen that there is an increase in the number of vortices. In addition, the increase in the Rayleigh number results in the increase in the number of streamlines in the bottom (hot) wall. Furthermore, it can be observed that there is an increase in turbulence with the increase in the Rayleigh number because the flow becomes more chaotic with the increase in the Rayleigh number leading to an increase in the velocity magnitude with the minimum velocity magnitude being recorded to be $6.48 \times 10^{-1}\ m/s$ as in figure 6.2 (a) and the maximum velocity being $1.30 \times 10^{0}\ m/s$ in figure 6.2 (a). The results are in line with the experimental results by Doğan and Doğan (2017) who pointed out that an increase in the Rayleigh number results in the increase in the velocity magnitude. These are shown in the figures below.



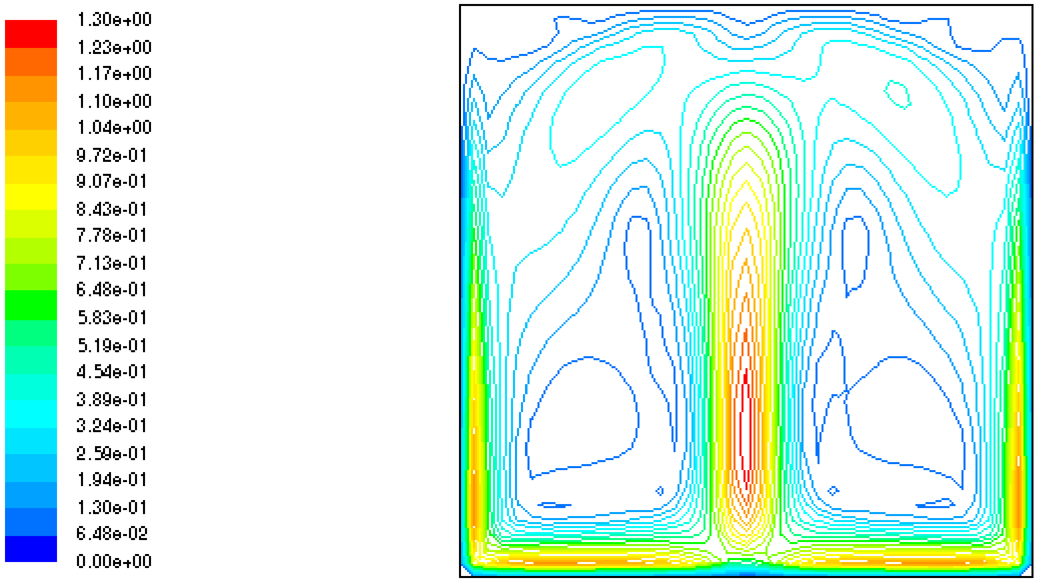
a. $Ra = 1.797 \times 10^{9}$



b. $Ra = 1.437 \times 10^{10}$



c. $Ra = 1.150 \times 10^{11}$

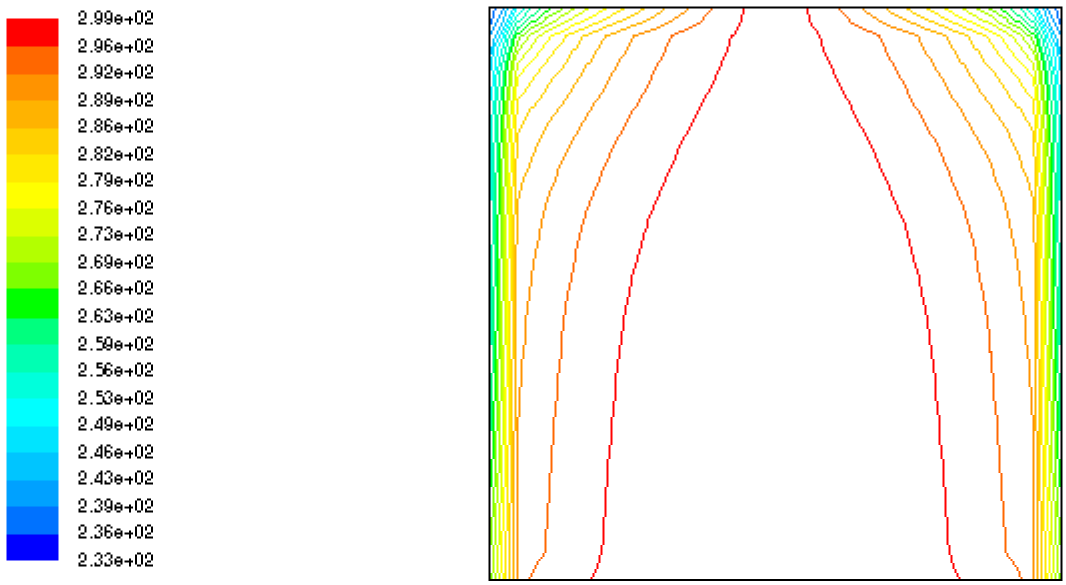


d. $Ra = 1.048 \times 10^{13}$

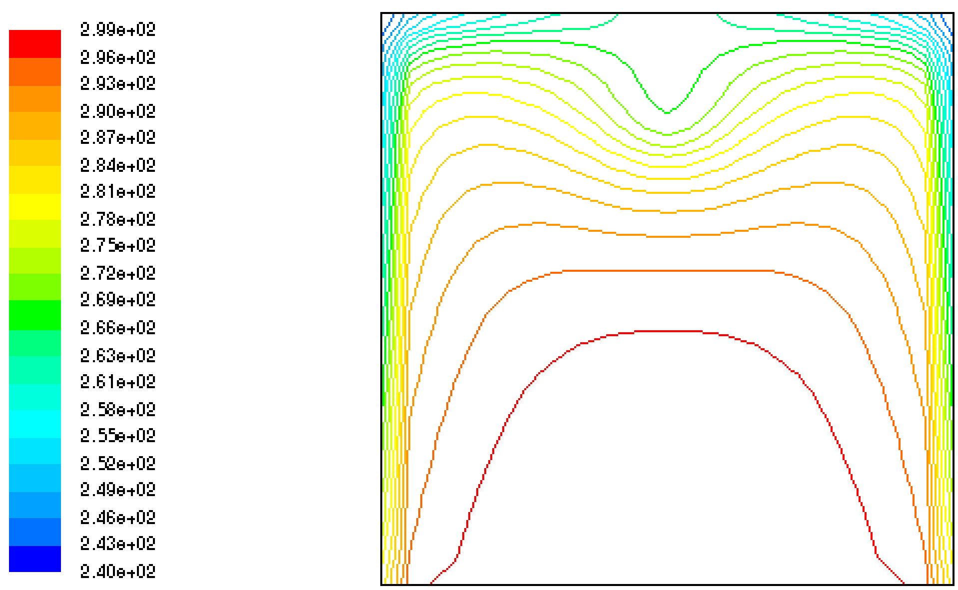
Figure 6.2 Contours of Velocity Magnitude (m/s)

6.3 Distribution of Isotherms

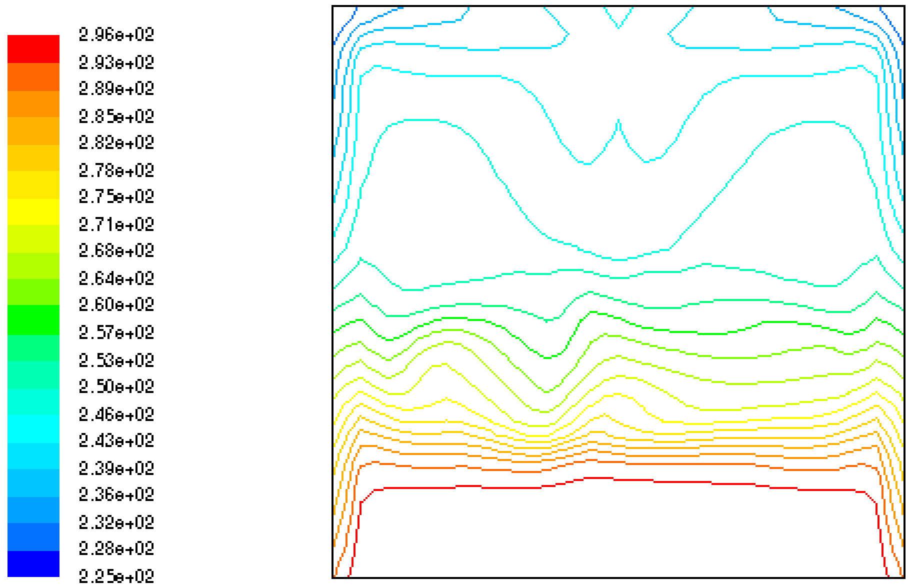
An isotherm is a line on a map or a graph connecting parts that have equal temperatures and every given time. Simultaneous temperature readings in different locations helps in the creation of isotherms. The contours of the isotherms for difference Rayleigh numbers are represented in the figures 6.3 (a), (b), (c), and (d). The increase in Rayleigh number results in the enlargement and rise of the contours of temperature from the bottom wall with the highest temperature being at the middle part of the face. The Rayleigh number decreases with the maximum temperatures for a, b, c, and d below being indicated as 2.99×10^2, 2.99×10^2, 2.96×10^2, and 2.23×10^2. Additionally, the number of contours of temperature near the bottom (hot) wall are more and reduce towards the top wall. The transfer of heat through the fluid (air in this case) in the enclosure starts at the bottom wall to the other parts of the enclosure. The results that are presented in the experimental study by Doğan and Doğan (2017) are in line with the results that have been found in this project.



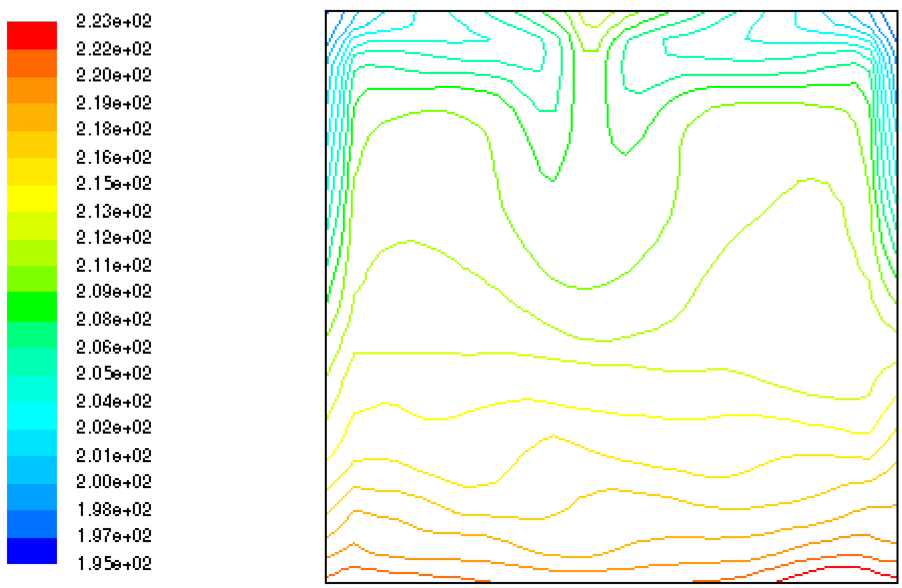
a. $Ra = 1.797 \times 10^{9}$



b. $Ra = 1.437 \times 10^{10}$



c. $Ra = 1.150 \times 10^{11}$



d. $Ra = 1.048 \times 10^{13}$

Figure 6.3 Contours of Isotherms (K)

CHAPTER SEVEN

CONCLUSION AND RECOMMENDATION

7.1 Conclusion

In this study, the velocity of flow in different parts of the enclosure have been obtained for varying Rayleigh numbers. The results indicate that a variation in the Rayleigh number affects the fluid properties such as velocity and temperature. As such, the velocity magnitude rises with the increase in the Rayleigh number. It is found that an increase in the Rayleigh number results in the increase in the size of vortices. In relation to the temperature distribution in different parts of the enclosure, it is found that the temperature of the enclosure is higher at the bottom of the enclosure and the temperature decreases with the increase in the Rayleigh number. The increase in the Rayleigh number as well is seen to results in an increase in the turbulence and hence the flow becomes more chaotic.

7.2. Recommendations

In this work, the determination of flow properties is attained with the change in the dimensions of the enclosure and keeping the aspect ratio constant. Furthermore, the bottom part of the face-wall is heated while the top part of the face-wall is cooled and the other four walls are adiabatic. It is recommended that further investigation is carried out in instances where one makes use of a different turbulence model such as the $k - \omega\, SST$ turbulence model and observe the fluid properties. In addition, it is recommended that further research should be carried out with variation of the aspect ratio and the dimensions of the enclosure and keeping the Rayleigh number constant. Moreover, it is recommended that an investigation of the fluid properties in an enclosure with a heater being introduced at the bottom wall and a window at the top wall is carried out.

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