

**EFFECTS OF INDUCED MAGNETIC FIELD, HALL CURRENT AND  
RADIATIVE HEAT ON 3-D UNSTEADY HYDROMAGNETIC  
STAGNATION FLOW OF A CASSON FLUID**

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award of the Degree of Master of Science (Applied Mathematics) in the School  
of Pure and Applied Sciences of Kenyatta University**

**March, 2023**

**Declaration**

I hereby solemnly affirm that this project presented is my original work and has never been presented before any university for a degree or diploma award.

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We confirm that the work embodied in this project was as a result of candidate's effort under our supervision.

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## **DEDICATION**

I whole heartedly dedicate this work to God, my parents, my wife Veronicah Ndanu, my son Myles Mali and daughter Amelia Amani for their love and support.

## ACKNOWLEDGEMENT

First I wish to recognize the immense academic support my supervisor Dr. Kimathi gave me and his tireless guidance and dedication towards the completion of this undertaking. To Dr. Mutuku, I thank you for the direction and assistance you gave me during the times I got stuck.

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Above all I acknowledge the almighty God for gracing me with good health and sober mind to pursue this experience to its fruition.

## NOMENCLATURE

$B(t)$	Time dependent magnetic field
$C_p$	Specific heat capacity of the fluid.
$\alpha^*$	Rosseland mean absorption coefficient
$\sigma^*$	Stefan-Boltzmann constant
$E$	Electric field
$b$	Induced magnetic field
$J$	Induced current density
$k$	Thermal conductivity of the fluid
$m$	Hall parameter
$M$	Magnetic parameter
$t$	Time variable
$P$	Momentum of the fluid
$q$	Volume flow rate
$f', g$	Dimensionless velocity in x and z direction
$\theta$	Dimensionless temperature of the fluid
$\eta$	Dimensionless similarity variable
$h'$	Dimensionless induced magnetic field
$u, v, w$	Velocity components in x, y and z directions
$U_w(x, t)$	Time dependent velocity vector
$\beta$	Casson fluid parameter
$\gamma$	Coefficient of viscosity of the fluid
$\rho$	Liquid density
$\nu$	Kinematic coefficient of viscosity
$\mu_e$	Magnetic permeability
$\sigma$	Electrical conductivity of the fluid
$T_\infty$	Temperature of fluid at infinity
$T_w$	Temperature of fluid at surface
$A$	Unsteadiness parameter

$tr$	Temperature ratio parameter
$Nr$	Radiation parameter
$Pr$	Prandtl number
$Pr_m$	Magnetic Prandtl number
$Re_x$	Local Reynolds number

### **Abbreviations**

<b>MHD</b>	Magnetohydrodynamics
<b>ODE</b>	Ordinary differential equation
<b>PDE</b>	Partial differential equations
<b>3-D</b>	Three dimensional

## **ABSTRACT**

In this study, 3-D stagnation flow of an MHD viscous electrically conducting Casson fluid including effects of induced magnetic field, Hall current and radiative heat is analyzed. The unsteady state model governing the flow is analyzed by first coming up with non-linear partial differential equations. Secondly, similarity transformation is done to change the nonlinear partial differential equations which are non-linear to ordinary differential equations, in order to account for the boundary layer and ease the computation. The ordinary differential equations obtained are then solved numerically using the collocation method via the MATLAB software. Analysis is then done to investigate effects of flow parameters like Hall current, magnetic parameters, temperature field, the Casson parameter and unsteadiness parameter.

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## CHAPTER ONE

### INTRODUCTION.

In this chapter, the key terminologies used are defined. The statement of the problem, the objective of the study and the significance are also stated.

#### 1.10 Casson fluid.

A Casson fluid is a type of a non-Newtonian fluid. A non-Newtonian fluid is fluid that does not

obey Newton's law of viscosity;  $\tau_{xy} = \eta \frac{\partial u}{\partial y}$  where  $\tau_{xy}$  is the viscosity co-efficient and  $u$  the

velocity in the x-direction. Casson fluid is a shear-thinning liquid which is assumed to have an infinite viscosity at zero rate of shear, a yield stress below which no flow occurs and a zero viscosity at infinite rate of shear. Simply put, a casson fluid acts as solid when yield stresses are more than shear stresses and starts flowing when shear stress is more significant than yield stress. Such fluids include, honey, ketchup, jelly and condensed milk. Blood is also a casson fluid.

#### 1.12 Induced magnetic field.

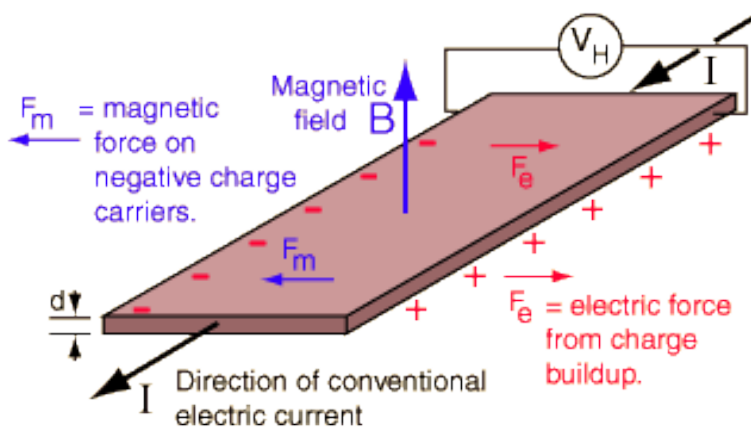
Induced magnetic field arises due to strong magnetic field in a fluid flow. In order to know or predict the velocity distribution of the fluid, it is important to investigate the behavior of the generated electro-motive force (E.M.F). Cooling rate and stretching rate of a material are two of the most important things in manufacturing process. To regulate these two factors, magnetic field is subjected to the flow field externally, perpendicular to the flow. The magnetic field however generates a force that inhibits flow field. This creates a problem of management by acting as stabilizing agent making boundary layer separation take a longer time. Cooling also needs to be controlled. To control this cooling rate, applied magnetic field is subjected to the flow field externally with induced magnetic field in the transverse direction. Induced field will help in

processing by aligning the elementary crystals of a material. Most studies have been conducted however considering a small magnetic Reynolds number thereby neglecting magnetic induction.

### 1.13 Hall current.

Hall current arises from interaction of additional potential difference between the opposite surfaces of the fluid where current is flowing perpendicular to magnetic field and electric field. It was first put forward in 1879 by Edwin Hall. Hall is characterized by current since the effect is produced by movement of small charge carriers, for example, electrons. In the **figure 1.1** the direction of current is the conventional current direction implying electrons are moving in the opposite direction. In the presence of magnetic field ( $B$ ), the charge carriers experience a force known as the Lorentz force I.e.  $F_m$  and  $F_e$ . This force acts non-uniformly causing curved paths due to the magnetic force. This implies in most cases, there is accumulation of Hall element on the face of the material. In fluid flow, Hall parameter can take any value. In this study, Hall

parameter  $m = \frac{eB}{m_e v}$  Where is the  $e$  =electric charge,  $B$  is the magnetic field,  $m_e$  is the mass of an electron and  $v$  the electron frequency.



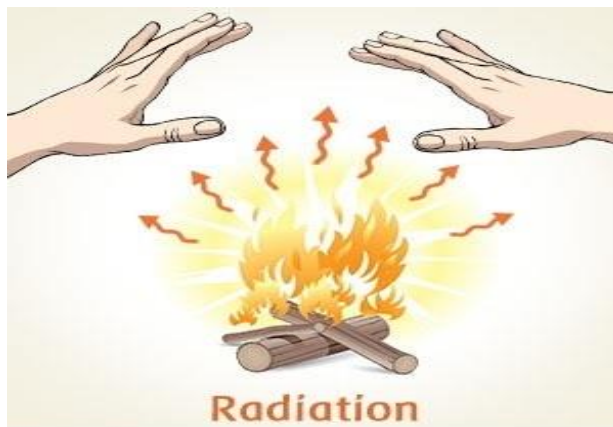
**Figure 1.1** Generation of Hall element. Image source.(hyperphysics)

### 1.14 Thermal Radiation

Thermal radiation is the transfer of heat energy from a heated surface in form of electromagnetic radiation. The rate at which a body emits or absorbs heat energy depends on the nature of the object. Dull objects are considered good absorbers and emitters of radiant heat while polished surfaces are poor emitters and poor absorbers of radiant heat. The emitted energy depends on the temperature of the emitted surface and is proportional to the fourth power of its absolute temperature (Stefan-Boltzmann law). In the assumption that the fluid is optically thick, the

radiative heat flux vector is given by the Roseland approximation as  $q_r = -\frac{4\sigma^*}{3\alpha^*} \cdot \frac{\partial T^4}{\partial y}$

Due to high need to process materials in engineering at high temperature such as high temperature plasmas, Casson fluids and even nuclear reactors, radiative heat transfer has been an area that has attracted research of late. In some cases where radiative heat goes through a fluid, the fluid can be ionized and electrically conduct due to high temperatures. However, little is known about radiation effects on boundary layer.



<https://www.buzzle.com/images/diagrams/example-of-radiation.jpg>

**Figure 1:1 radiative heat transfer**

### 1.15 Magnetohydrodynamics (MHD)

Also known as magneto-fluid dynamics or hydromagnetics is the physical-mathematical phenomenon that deals with magnetic properties and behavior of electrically conducting fluids. Such fluids are known as magneto-fluids. They include liquid metals, electrolytes and plasmas. The word magnetohydrodynamics is derived from 3 words; *magneto*-meaning magnetic, *hydro*-meaning liquid and *dynamic*-meaning movement of a body due to application of force.

The concept of Magnetohydrodynamics was first put forward by Hannes Alfvén after studying behavior of moving conducting fluids. He found that in presence of magnetic field, a moving electrically conducting fluid induces a current which in turn creates a force (Lorentz force) on the fluid and changes the magnetic field itself. The set of equations that describe MHD are the Navier-Stokes equation of fluid dynamics and the Maxwell's equation of electro-magnetism.

### 1.16 Boundary layer or shear layer.

A boundary layer is considered as the immediate layer of fluid next to a solid boundary where viscosity of a fluid only plays a role. Outside of that region for most part the fluid can be treated as being inviscid. This was brought forward in 1904 by Ludwig Prandtl. There are two types of boundary layer flow: laminar and turbulent flow. Most studies involve flow whose velocity is low and therefore a laminar boundary layer. In fluid dynamics, the flow of a fluid particle along the main stream is a case of transfer of mass and heat. In laminar layer the dimensionless characteristic is the Reynolds's number  $Re_x = \frac{Ux}{\nu}$ . The upward distance from the plate at which the velocity is 99% of the free stream velocity is the boundary layer thickness. There are two types: displacement thickness and momentum thickness. The velocity boundary layer is the vertical distance from the plate underlying the fluid where the viscous flow is 99% of free stream velocity. The thermal boundary layer is the vertical distance from the plate underlying the fluid

where the temperature is 99% of the free stream temperature. The dimensionless characteristic describing both velocity boundary layer and thermal boundary layer is the Prandtl number.

### 1.17 Prandtl number

Prandtl number is a non dimensionless number that shows the ratio of kinematic viscosity and thermal diffusivity. It was first put forward by Ludwig Prandtl.  $p_r = \frac{\nu}{\alpha_m}$ . If  $p_r \ll 1$ , then thermal diffusivity is dominant over kinematic viscosity. If  $p_r \gg 1$  then kinematic viscosity dominates over thermal diffusivity. Normally the Prandtl number controls the thickness of the thermal and momentum boundary layers and therefore if  $p_r$  is small then heat diffuses faster than the rate at which the fluid is moving.

### 1.18 Local Reynolds number ( $Re_x$ )

This is a dimensionless parameter expressed as a ratio of inertia forces to viscous forces.

$Re_x = \frac{Ux}{\nu}$  Where,  $U$  is the flow velocity,  $\nu$  is the kinematic viscosity and  $x$  is the characteristic

linear dimension. This parameter is used to determine whether a flow is laminar or turbulent. If  $Re_x$  is small, then the flow is laminar since the fluid particles are in line. A large  $Re_x$  shows that the flow is turbulent and fluid velocity is high. Normally if  $Re_x < 200$  then the flow is lamina and if  $Re > 400$  then the flow is turbulent.

### 1.19 Magnetic Reynolds number ( $R_m$ )

This is a non-dimensionless quantity that indicates dynamic behavior of a magneto fluid. Usually

represented as  $R_m = \frac{Ux}{\eta}$  where  $\eta$  is the magnetic diffusivity,  $U$  is the flow velocity and  $x$  is the

length scale. Magnetic Reynolds number gives an estimate of the effect of ratio of magnetic induction of flux by motion of the magneto fluid to the magnetic diffusion.

There are two types of behavior of the magnetic field of a moving fluid depending on magnetic Reynolds number. If  $R_m \gg 1$ , then the magnetic field lines tend to be ‘frozen’ into the plasma and moves along with the flow of the plasma. If  $R_m \ll 1$ , then the magnetic field will diffuse away and non-uniformity of the fluid is smoothed.

### 1.2.0 Magnetic Prandtl number

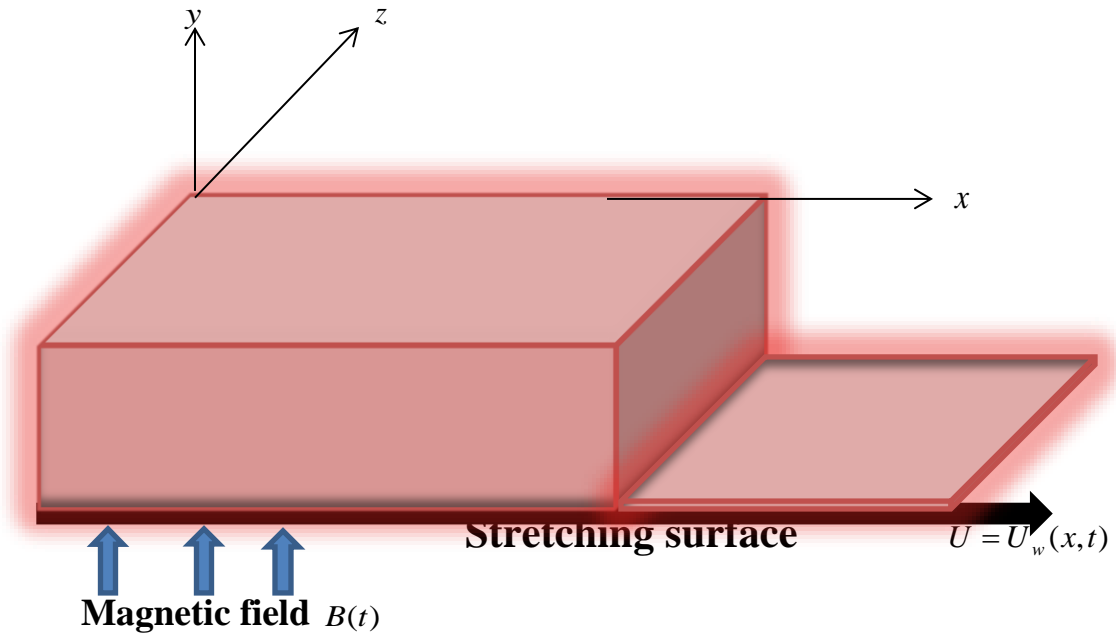
This is the magnetic analogous of the Prandtl number. It is a ratio of magnetic Reynolds number to local Reynolds number. It shows the ratio of momentum diffusivity ( kinematic viscosity) to magnetic viscosity

$$\text{Pr}_m = \frac{R_m}{\text{Re}} = \frac{Ux}{\eta} \cdot \frac{\nu}{Ux} = \frac{\nu}{\eta}$$

### 1.3 Problem statement

We consider a 3-Dimensional non-Newtonian MHD Casson fluid flow over a stretching sheet including effects of induced magnetic field, radiative heat and Hall current. The flow is along  $x$  –  $axis$  with a velocity vector  $U = U_w(x, t)$ . The stretching of the sheet is in the  $x$ -direction only. The magnetic field  $B(t) = B_0(1 - \gamma t)^{-1/2}$  is perpendicular to the stretching sheet. This field is able to produce Hall effects due to its strength.

Below are the coordinate system and a sketch of the graphical model



**Figure 1.2 three-Dimensional physical model of the problem**

## 1.4 Objectives

### 1.4.1 General objective

To determine effects of induced magnetic field, Hall current and heat transfer on an unsteady hydro-magnetic Casson fluid flow along a linearly stretching sheet.

### 1.4.2 Specific objectives

1. To create a mathematical model regulating the flow of unsteady hydro magnetic strongly conducting Casson fluid along a stretching surface.
2. To analyse the effects of Hall current parameter, magnetic strength parameter, Casson parameter and radiation parameter on profiles of fluid velocity.
3. To determine the effects of induced magnetic field on the velocity for a Casson fluid flow along a stretching sheet.

### **1.5 Significance of the study**

The rate at which a surface is stretched and the rate of cooling of a liquid are two of the most important considerations in manufacturing. A Casson fluid is basically a fluid which acts as solid when yield stresses are more than shear stresses and starts flowing when shear stress is more significant than yield stress. Such fluids include, honey, blood, ketchup, jelly and condensed milk. For fast transport of such fluids we need to limit the yield stress. In this case, the viscosity of the fluid is low and therefore velocity is relatively high. Hall current effect helps in increasing the velocity profile and reduces the temperature within the region of the boundary layer. Induced magnetic field for an electrically conducting fluid has an effect on the fluid velocity. For a MHD, if induced magnetic field is high, the fluid velocity increases. It is these combined effects of Hall and induced magnetic field that are considered in this study with an aim of reducing production time and increasing the final product of the manufacturing process in Engineering and industry.

## CHAPTER TWO

### 2.0 LITERATURE REVIEW

The flow of an electrically conducting fluid along a stretched sheet is considered important in the printing industry, silicon suspension, polymer engineering and blood flow. Denno[1967]was the first to look into the effects of induced magnetic field on the inviscid MHD channel flow. He concluded that with magnetic Reynolds number of the order of 0.1 then induced magnetic field must be in cooperated during treatment of inviscid magneto-hydrodynamic flow. Takhar *et al*[2003] discussed the boundary layer flow and heat transfer on a stretching surface in rotational fluid with magnetic field. They reported that magnetic field increases skin friction along the  $x$ -direction while the Nusselt number decreased. Basant *et al* [2017], investigated the effects of induced magnetic field on MHD natural convection flow in a vertical annular micro-channel in the presence of radial magnetic field. They inferred that skin friction numerical value at the micro-channel surfaces decreased with increase in Hartman number. Reddy *et al* [2016] discussed a MHD flow of a Casson fluid over an inclined permeable stretching sheet with heat radiation and chemical reaction and inferred that increasing in the Casson parameter led to a decrease in the thickness of the boundary layer. Vasundhara *et al* [2016] investigated effects of hall current, thermal radiation and heat generation due to exponentially stretching sheet with sanction and convective boundary conditions. They found that temperature increased with increase in magnetic field parameter and decreased with Hall parameter. Afikuzzaman *et al* [2015] analyzed the unsteady MHD casson fluid flow through a parallel plate with Hall current. He observed that increasing the Hall current increased the time taken for two velocity components to reach the steady state. Aziz and Nabil [2012] presented the effect of thermal radiation on steady MHD mixed convection flow past an exponentially stretching sheet with Hall currents assigning wall temperature and stretching velocity to vary according to specific

exponential form. A three-dimensional problem of MHD Casson fluid flow along an unsteady stretching surface embedded into a porous media was studied by Butt *et al* [2016]. Hassan, *et al* [2017] investigated the unsteady MHD flow of Casson fluid through porous media over a permeable shrinking sheet. They reported that an increase in the radiation parameter caused an increase in temperature distribution. Pushpalatha *et al.*[2016] studied Heat and mass transfer in unsteady MHD Casson fluid flow with convective boundary conditions . They found that rise in Casson parameter reduces the velocity profiles of the flow. Mukopadhyay *et al*[2014] analyzed the boundary layer flow of a Casson fluid over an exponentially stretching sheet under the influence of a magnetic field, thermal radiation and suction/injection. Bhattacharyya[2014] studied MHD stagnation-point flow of Casson fluid and heat transfer over a stretching sheet with thermal radiation. In their study they concluded that the boundary layer thickens with magnetic parameter and velocity ratio parameter. Also, due to thermal radiation , they found that the temperature inside the boundary layer decreases. Shah, *et al* [2019] analyzed effects of Hall current and thermal radiation and mass transfer of Casson fluid between two rotating parallel plates. They found that Hall current decreases the rate of conduction of the fluid and subsequently increased the velocity field. Prashu[2018] analysed the effects of Hall current and heat radiation on a 3-D Casson fluid flowing along a stretching surface. He found that Hall current decreases with skin friction coefficient in  $x$  -direction and increased the skin friction coefficient in the  $z$  - direction. In view of these previous studies, there exists a research gap especially on effect of induced magnetic field on flow variables of a Casson fluid. This research aims at bridging this gap by extensively analyzing effects of induced magnetic field, Hall current and radiative heat on hydro-magnetic flow of a Casson fluid along a stretching surface.

## CHAPTER 3

In this chapter, we will come up with a mathematical model of the problem by forming highly non-linear partial differential equations. The equations will be subject to some assumptions however.

### 3.1 Assumptions

The assumptions made for the research problem are as follows

1. The fluid is incompressible
2. The flow is unsteady
3. The fluid is electrically conducting with a magnetic field perpendicular to the x-axis
4. The flow is along the plane  $x - axis$
5. The applied magnetic field intensity is so strong to produce Hall current.
6. The wall temperature  $T_w$  is maintained constant and is higher than ambient temperature  $T_\infty$

### 3.1 Flow field equations

The equations governing the flow considering the assumptions outlined earlier are the continuity equation, the momentum equation, the energy equation and induced magnetic field equation

#### 3.1.0 .The mass continuity equation

In fluid flow, continuity equation states that mass influx (rate of flow of mass entering a system) in a system is equal to mass efflux (rate of flow of mass leaving a system) plus the accumulated mass in the system after flow.

The mathematical equation for conservation of mass is

$$\frac{\partial \rho}{\partial t} + \bar{\nabla} \cdot (\rho \bar{q}) = 0 \tag{3.01}$$

Where  $\bar{q} = u\hat{i} + v\hat{j} + w\hat{k}$

Equation 3.01 can also be expressed as

$$\frac{\partial \rho}{\partial t} + \rho \vec{\nabla} \cdot \vec{q} + \vec{q} \cdot \vec{\nabla} \rho = 0 \quad 3.02$$

For an incompressible fluid,  $\vec{q} \cdot \vec{\nabla} \rho = 0$

Also since density is not changing with time,  $\frac{\partial \rho}{\partial t} = 0$

Hence equation 3.02 becomes

$$\vec{\nabla} \cdot \vec{q} = 0 \quad 3.03$$

$$\vec{q} = u(x, y, z, t) = (ui + v\hat{j} + w\hat{k}) \quad \text{and} \quad \vec{\nabla} = \hat{i} \frac{\partial}{\partial x} + \hat{j} \frac{\partial}{\partial y} + \hat{k} \frac{\partial}{\partial z} \quad 3.04$$

$$\nabla \cdot \vec{q} = \left( \hat{i} \frac{\partial}{\partial x} + \hat{j} \frac{\partial}{\partial y} + \hat{k} \frac{\partial}{\partial z} \right) \cdot (u\hat{i} + v\hat{j} + w\hat{k})$$

Thus the continuity equation for a three dimensional flow is

$$\vec{\nabla} \cdot \vec{q} = \frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} + \frac{\partial w}{\partial z} = 0 \quad 3.05$$

### 3.1.2 Momentum equation or Equation of motion

Momentum equation is a dynamic equation analogous to Newton's second law,  $F = ma$ .

In any MHD system, there are two types of forces that act on a fluid.

- i) External force: Acts on all directions from outside of the fluid. They are the pressure and viscous effect. Most considered forces is the gravitational force  $\rho g$ .
- ii) Internal forces: They form the tensor forces which are net force of all forces acting from within due to interaction with other elements.

In static equilibrium, these forces must balance each other and the net force should be zero.

For an incompressible flow where  $\nabla \cdot \vec{q} = 0$ , we have the incompressible Navier-Stoke equation.

$$\rho \left( \frac{\partial \vec{q}}{\partial t} + \vec{q} \cdot \nabla \vec{q} \right) = -\nabla P + \rho g + \mu \nabla^2 \vec{q} \quad 3.06$$

where

$\rho \mathbf{g}$  is the mass force or the body force

$\nabla P$  is the surface force

$\mu \nabla^2 \vec{q}$  is the viscous force

The body force or the Lorentz force  $F$  can be expressed as

$$F = \vec{J} \times \vec{B}$$

$$\text{Therefore, } \rho \left( \frac{\partial \vec{q}}{\partial t} + \vec{q} \cdot \nabla \vec{q} \right) = -\nabla P + \mu \nabla^2 \vec{q} + \vec{J} \times \vec{B}$$

From equation 3.05, we have

$$(\vec{q} \cdot \nabla) = (u\hat{i} + v\hat{j} + w\hat{k}) \cdot \left( \hat{i} \frac{\partial}{\partial x} + \hat{j} \frac{\partial}{\partial y} + \hat{k} \frac{\partial}{\partial z} \right) = u \frac{\partial}{\partial x} + v \frac{\partial}{\partial y} + w \frac{\partial}{\partial z} \quad 3.08$$

$$(\vec{q} \cdot \nabla) \vec{q} = \left( u \frac{\partial}{\partial x} + v \frac{\partial}{\partial y} + w \frac{\partial}{\partial z} \right) \vec{q} \quad 3.09$$

Along the  $x$ -direction,

$$(\vec{q} \cdot \nabla) \vec{q} = u \frac{\partial u}{\partial x} + v \frac{\partial u}{\partial y} + w \frac{\partial u}{\partial z} \quad 3.10$$

$$\frac{\partial \vec{q}}{\partial t} = \frac{\partial u}{\partial t} \text{ and}$$

$$\text{Equation 3.07 becomes } \rho \left( \frac{\partial u}{\partial t} + u \frac{\partial u}{\partial x} + v \frac{\partial u}{\partial y} + w \frac{\partial u}{\partial z} \right) = -\nabla P + \mu \nabla^2 \vec{q} + \vec{J} \times \vec{B} \quad 3.11$$

$$\text{Also, } \mu \nabla^2 \vec{q} = \mu \left( \frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} + \frac{\partial^2 u}{\partial z^2} \right) \quad 3.12$$

$$\text{But since } \frac{\partial^2 u}{\partial x^2} = 0, \frac{\partial u}{\partial z} = 0, \frac{\partial^2 w}{\partial x} = 0 \text{ and } \frac{\partial w}{\partial z} = 0$$

Then  $\mu \nabla^2 \vec{q} = \mu \frac{\partial^2 u}{\partial y^2}$  in the x-direction and  $\mu \nabla^2 \vec{q} = \mu \frac{\partial^2 w}{\partial y^2}$  in the z-direction

Considering  $\frac{\mu}{\rho} = \left(1 + \frac{1}{\beta}\right) \nu$  and since  $\frac{\rho}{\mu}$  is a constant. Equation 3.11 becomes

$$\left(\frac{\partial u}{\partial t} + u \frac{\partial u}{\partial x} + v \frac{\partial u}{\partial y} + w \frac{\partial u}{\partial z}\right) = \nu \left(1 + \frac{1}{\beta}\right) \frac{\partial^2 u}{\partial y^2} + \frac{1}{\rho} (\vec{J} \times \vec{B}) \quad \text{in the x-direction and}$$

$$\left(\frac{\partial w}{\partial t} + u \frac{\partial w}{\partial x} + v \frac{\partial w}{\partial y} + w \frac{\partial w}{\partial z}\right) = \nu \left(1 + \frac{1}{\beta}\right) \frac{\partial^2 w}{\partial y^2} + \frac{1}{\rho} (\vec{J} \times \vec{B}) \quad \text{in the z-direction} \quad 3.13$$

The general Ohm's law with Hall current is

$$\vec{J} = \sigma \left( E + \vec{q} \times \vec{B} - \frac{1}{en_e} \vec{J} \times \vec{B} + \frac{1}{en_e} \nabla P_e \right) \quad 3.14$$

where  $J = (J_x, J_y, J_z)$

Since no applied electric field and  $\nabla \cdot J = 0$  (conservation of charge) then it implies

$$J_y = \text{constant. Equation 3.14 can be expressed as } J + \frac{m}{B} (\vec{J} \times \vec{B}) = \sigma (\vec{q} \times \vec{B}) \quad 3.15$$

if we consider the electron pressure gradient  $\nabla P_e$  to be negligible.

Where  $\vec{B} = (b, B(t), 0)$

Therefore,

$$\vec{q} \times \vec{B} = \begin{bmatrix} \hat{i} & \hat{j} & \hat{k} \\ u & 0 & w \\ b & B(t) & 0 \end{bmatrix} = -B(t)w\hat{i} + bw\hat{j} + B(t)u\hat{k} \quad 3.16$$

$$\vec{J} \times \vec{B} = \begin{bmatrix} \hat{i} & \hat{j} & \hat{k} \\ J_x & J_y & J_z \\ b & B(t) & 0 \end{bmatrix} = -B(t)J_z\hat{i} + bJ_z\hat{j} + (B(t)J_x - bJ_y)\hat{k} \quad 3.17$$

inserting equations 3.16 and 3.17 in 3.15 we have

$$J_x - mJ_z = \sigma(B(t)w) \quad 3.18$$

$$J_y + \frac{m}{B(t)}bJ_z = \sigma bw \quad 3.19$$

$$J_z + \frac{m}{B(t)}(B(t)J_x - bJ_y) = \sigma B(t)u \quad 3.20$$

To find  $J_z$  we make  $J_x$  subject of the formula in equation 3.18 and  $J_y$  in 3.19 and substitute it in 3.20

$$J_z + m(mJ_z - \sigma B(t)w) - \frac{mb}{B(t)}\left(\sigma bw - \frac{mb}{B(t)}J_z\right) = \sigma B(t)u \quad 3.21$$

$$J_z + m^2J_z - \sigma B(t)mw - \frac{mw\sigma b^2}{B(t)} + \frac{m^2b^2}{B(t)}J_z = \sigma B(t)u \quad 3.22$$

Making  $J_z$  the subject gives

$$J_z = \frac{\sigma B(t)\left[u + mw + \frac{mw b^2}{B^2(t)}\right]}{B^2(t)[1 + m^2] + m^2 b^2} = \quad 3.23$$

To find  $J_x$  we substitute equation 3.23 in 3.18

$$J_x = \left( \frac{\sigma B(t) \left[ u + mw + \frac{mwb^2}{B^2(t)} \right]}{B^2(t)[1+m^2] + m^2b^2} \right) - \sigma B(t)w \quad 3.24$$

$$J_x = \frac{\sigma B^3(t)mu + \sigma B^3(t)m^2w + \sigma B(t)m^2wb^2 - \sigma wB^3(t) - \sigma m^2wB^3(t) - \sigma wm^2b^2B(t)}{B^2(t)[1+m^2] + m^2b^2} \quad 3.25$$

this simplifies to

$$J_x = \frac{\sigma B^3(t)(mu - w)}{B^2(t)[1+m^2] + m^2b^2} \quad 3.26$$

To get  $J_y$  we substitute equation 3.23 in 3.19

$$J_y = \sigma bw - \frac{mb}{B(t)} \left( \frac{\sigma B^3(t) \left( u + mw + \frac{mwb^2}{B^2(t)} \right)}{B^2(t)(1+m^2) + m^2b^2} \right) \quad 3.27$$

$$J_y = \frac{\sigma bwB(t)^2 + \sigma bwB(t)^2m^2 + \sigma b^3wm^2 - \sigma mbB(t)^2u - \sigma m^2bwB(t)^2 - \sigma m^2wb^3}{B(t)^2(1+m^2) + m^2b^2} \quad 3.28$$

$$J_y = \frac{\sigma B(t)^2(bw - mbu)}{B(t)^2(1+m^2) + m^2b^2} \quad 3.29$$

Inserting equations 3.23, 3.26 and 3.29 into equation 3.17 we get

$$\vec{J} \times \vec{B} = \frac{-\sigma B(t)^2[B(t)^2(u + mw) + mwb^2]}{B(t)^2(1+m^2) + m^2b^2} \hat{i} + \frac{\sigma B(t)^3b \left[ u + mw + \frac{mwb^2}{B(t)^2} \right]}{B(t)^2(1+m^2) + m^2b^2} \hat{j} + \left[ \frac{\sigma B(t)^4(mu - w)}{B(t)^2(1+m^2) + m^2b^2} - \frac{B(t)^2(b^2w - mb^2u)}{B(t)^2(1+m^2) + m^2b^2} \right] \hat{k} \quad 3.30$$

Momentum equation 3.13 becomes

$$\frac{\partial u}{\partial t} + u \frac{\partial u}{\partial x} + v \frac{\partial u}{\partial y} + w \frac{\partial u}{\partial z} = \nu \left( 1 + \frac{1}{\beta} \right) \frac{\partial^2 u}{\partial y^2} - \frac{\sigma B^4(t) \left[ u + mw + \frac{mwb^2}{B^2(t)} \right]}{\rho B^2(t)(1+m^2) + m^2 b^2} \quad 3.31$$

$$\frac{\partial w}{\partial t} + u \frac{\partial w}{\partial x} + v \frac{\partial w}{\partial y} + w \frac{\partial w}{\partial z} = \nu \left( 1 + \frac{1}{\beta} \right) \frac{\partial^2 w}{\partial y^2} + \frac{\sigma}{\rho} \left[ \frac{B^4(t)(mu - w)}{B^2(t)(1+m^2) + m^2 b^2} - \frac{B^2(t)(b^2 w - mb^2 u)}{B^2(t)(1+m^2) + m^2 b^2} \right] \quad 3.32$$

Where  $m = (\omega_e t_e)$  is the Hall parameter,  $\omega_e$  is the electron frequency and  $t_e$  the electron collision time.

### 3.1.3 The energy equation

From the thermodynamics law,

$$\delta Q = dE + \delta W \quad 3.33$$

Where  $\delta Q$  is the rate of heat influx,  $\delta W$  is of doing work and  $dE$  is energy change of the system.

For an optically thick incompressible fluid, the governing equation is

$$\rho \left( \frac{\partial T}{\partial t} + (\vec{q} \cdot \nabla) T \right) = \frac{k}{C_p} \nabla^2 T + \Phi \quad 3.34$$

The radiative flux  $\phi = -\frac{1}{C_p} \frac{\partial q_r}{\partial y}$ .

Equation 3.34 therefore becomes

$$\left( \frac{\partial T}{\partial t} + (\vec{q} \cdot \nabla) T \right) = \frac{k}{\rho C_p} \nabla^2 T - \frac{1}{\rho C_p} \frac{\partial q_r}{\partial y} \quad 3.35$$

Taking  $\vec{q} \cdot \nabla T = u \frac{\partial T}{\partial x} + v \frac{\partial T}{\partial y} + w \frac{\partial T}{\partial z}$  and  $\nabla^2 T = \frac{\partial^2 T}{\partial x^2} + \frac{\partial^2 T}{\partial y^2} + \frac{\partial^2 T}{\partial z^2}$  equation 3.34 becomes

$$\frac{\partial T}{\partial t} + u \frac{\partial T}{\partial x} + v \frac{\partial T}{\partial y} + w \frac{\partial T}{\partial z} = \frac{k}{\rho C_p} \left( \frac{\partial^2 T}{\partial x^2} + \frac{\partial^2 T}{\partial y^2} + \frac{\partial^2 T}{\partial z^2} \right) - \frac{1}{\rho C_p} \frac{\partial q_r}{\partial y} \quad 3.36$$

Considering the radiation is in the y-direction, equation 3.36 becomes

$$\frac{\partial T}{\partial t} + u \frac{\partial T}{\partial x} + v \frac{\partial T}{\partial y} + w \frac{\partial T}{\partial z} = \alpha_m \frac{\partial^2 T}{\partial y^2} - \frac{1}{\rho C_p} \frac{\partial q_r}{\partial y} \quad 3.37$$

This is the energy equation for a Casson fluid with Hall current.

### 3.1.4. Induction equation for MHD

Considering the Maxwell's equations

$$\text{i.} \quad \nabla \cdot \vec{B} = 0 \quad (\text{Gauss' law for magnetism}) \quad 3.38$$

$$\text{ii.} \quad \nabla \cdot \vec{E} = \frac{q}{\epsilon_0} \quad (\text{Gauss' law for electricity}) \quad 3.39$$

$$\text{iii.} \quad \nabla \times \vec{B} = \mu_e j + \frac{1}{C^2} \frac{\partial \vec{E}}{\partial t} \quad (\text{Ampere's law}) \quad 3.40$$

$$\text{iv.} \quad \nabla \times \vec{E} = -\frac{\partial \vec{B}}{\partial t} \quad (\text{Faraday's law}) \quad 3.41$$

and Ohm's law;

$$j = \sigma(\vec{E} + q \times \vec{B}) \quad 3.42$$

We can derive the induction equation as follows

For non-relativistic limit, we will ignore the 2<sup>nd</sup> term  $\left(\frac{1}{C^2} \frac{\partial \vec{E}}{\partial t}\right)$  in Maxwell's third equation and

combining it with equation 3.42 we get

$$\nabla \times \vec{B} = j = \sigma(\vec{E} + q \times \vec{B}) \quad 3.43$$

Taking the curl of equation 3.43 we have

$$\nabla \times \nabla \times \vec{B} = \sigma(\nabla \times \vec{E} + \nabla \times (q \times \vec{B})) \quad 3.44$$

From the left hand side of the above equation, we find

$$\nabla \times \nabla \times \vec{B} = -\nabla^2 \vec{B} - \nabla(\nabla \cdot \vec{B}) = -\nabla^2 \vec{B} \text{ Since } \nabla \cdot \vec{B} = 0, \text{ from equation 3.38}$$

then equation 3.44 becomes

$$-\nabla^2 \vec{B} = \sigma \left( -\frac{\partial \vec{B}}{\partial t} + \nabla \times (q \times \vec{B}) \right) \quad 3.45$$

Rewriting equation 3.45 gives

$$\frac{\partial \vec{B}}{\partial t} = \nabla \times (q \times \vec{B}) + \eta \nabla^2 \vec{B} \text{ Is the induction equation} \quad 3.46$$

$$\text{Where } \vec{B} = (b, B(t), 0), \vec{q} = (u\vec{i} + v\vec{j} + w\vec{k}) \text{ and } \sigma = \frac{1}{\eta} \quad 3.47$$

Now we convert equation 3.46 to first order pde as follows

$$\vec{q} \times \vec{B} = \begin{bmatrix} \hat{i} & \hat{j} & \hat{k} \\ u & 0 & w \\ b & B(t) & 0 \end{bmatrix} = -B(t)w\hat{i} + bw\hat{j} + B(t)u\hat{k} \quad 3.48$$

$$\nabla \times (\vec{q} \times \vec{B}) = \begin{bmatrix} \hat{i} & \hat{j} & \hat{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ -B(t)w & bw & B(t)u \end{bmatrix} = \left[ B(t) \frac{\partial u}{\partial y} - \frac{\partial(bw)}{\partial z} \right] \hat{i} - \left[ B(t) \frac{\partial u}{\partial x} + B(t) \frac{\partial w}{\partial z} \right] \hat{j} + \left[ \frac{\partial(bw)}{\partial z} + B(t) \frac{\partial w}{\partial y} \right] \hat{k}$$

$$3.49$$

$$\nabla^2 \vec{B} = \left( \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial z^2} \right) (b\hat{i} + B(t)\hat{j} + 0\hat{k}) = \left( \frac{\partial^2 b}{\partial x^2} + \frac{\partial^2 b}{\partial y^2} + \frac{\partial^2 b}{\partial z^2} \right) \hat{i} + \left( \frac{\partial^2 B(t)}{\partial x^2} + \frac{\partial^2 B(t)}{\partial y^2} + \frac{\partial^2 B(t)}{\partial z^2} \right) \hat{j} + 0\hat{k}$$

3.50

Substituting 3.49 and 3.50 in equation 3.46 we get the following equations.

$$\text{In the x-direction we have } \frac{\partial b}{\partial t} = B(t) \frac{\partial u}{\partial y} - \frac{\partial(bw)}{\partial z} + \eta \left( \frac{\partial^2 b}{\partial x^2} + \frac{\partial^2 b}{\partial y^2} + \frac{\partial^2 b}{\partial z^2} \right)$$

3.51

$$\text{In the y-direction we have } \frac{\partial B(t)}{\partial t} = -B(t) \frac{\partial u}{\partial x} - B(t) \frac{\partial w}{\partial z} + \eta \left( \frac{\partial^2 B(t)}{\partial x^2} + \frac{\partial^2 B(t)}{\partial y^2} + \frac{\partial^2 B(t)}{\partial z^2} \right)$$

3.52

$$\text{In the z-direction we have } 0 = \frac{\partial(bw)}{\partial x} + B(t) \frac{\partial w}{\partial y}$$

3.53

We can simplify the above equations using the following arguments

Since b is a function of y only and the fluid on x-y plate then equation 3.51 becomes,

$$\frac{\partial b}{\partial t} = B(t) \frac{\partial u}{\partial y} - \eta \frac{\partial^2 b}{\partial y^2}$$

3.54

Also applied magnetic field  $B(t)$  is a function of time only and therefore equation 3.52 becomes

$$\frac{\partial B(t)}{\partial t} = -B(t) \frac{\partial u}{\partial x}$$

3.55

Equation 3.53 can be simplified as follows using Gauss's law of magnetism (equation 3.38)

$$0 = \frac{\partial(bw)}{\partial x} + B(t) \frac{\partial w}{\partial y} = b \frac{\partial w}{\partial x} + B(t) \frac{\partial w}{\partial y} \equiv \nabla \cdot (\vec{B}w) \equiv w(\nabla \cdot \vec{B}) = 0$$

In summary the equations governing the 3-D flow are the continuity equation, the momentum equation, the energy equation and the induction equation

$$\vec{\nabla} \cdot \vec{q} = \frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} + \frac{\partial w}{\partial z} = 0$$

$$\frac{\partial u}{\partial t} + u \frac{\partial u}{\partial x} + v \frac{\partial u}{\partial y} + w \frac{\partial u}{\partial z} = \nu \left( 1 + \frac{1}{\beta} \right) \frac{\partial^2 u}{\partial y^2} - \frac{\sigma B^4(t) \left[ u + mw + \frac{mwb^2}{B^2(t)} \right]}{\rho B^2(t)(1+m^2) + m^2 b^2}$$

$$\frac{\partial w}{\partial t} + u \frac{\partial w}{\partial x} + v \frac{\partial w}{\partial y} + w \frac{\partial w}{\partial z} = \nu \left( 1 + \frac{1}{\beta} \right) \frac{\partial^2 w}{\partial y^2} + \frac{\sigma}{\rho} \left[ \frac{B^4(t)(mu - w)}{B^2(t)(1+m^2) + m^2 b^2} - \frac{B^2(t)(b^2 w - mb^2 u)}{B^2(t)(1+m^2) + m^2 b^2} \right]$$

$$\frac{\partial T}{\partial t} + u \frac{\partial T}{\partial x} + v \frac{\partial T}{\partial y} + w \frac{\partial T}{\partial z} = \alpha_m \frac{\partial^2 T}{\partial y^2} - \frac{1}{\rho C_p} \frac{\partial q_r}{\partial y}$$

$$\frac{\partial b}{\partial t} = B(t) \frac{\partial u}{\partial y} - \eta \frac{\partial^2 b}{\partial y^2}$$

The associated boundary conditions of the flow problem are

$$\text{At } y = 0 : u = u_w, v = 0, w = 0, T = T_w, b = 0 \quad 3.56a$$

$$\text{As } y \rightarrow \infty : u \rightarrow 0, w \rightarrow 0, T \rightarrow T_w, b = 0 \quad 3.56b$$

## CHAPTER FOUR

### 4.1 Non-dimensionalization.

The partial differential equations governing the flow derived in chapter three are highly nonlinear. In this chapter we will form ordinary differential equations by using suitable similarity functions.

We employ the following set of transformation equations to transform the partial differential equations.

$$u = \frac{ax}{(1-\gamma)} f'(\eta) \quad v = -\sqrt{\frac{av}{1-\gamma}} f(\eta) \quad w = -\frac{ax}{(1-\gamma)} g(\eta) \quad 4.10$$

$$b = \frac{B_0}{\sqrt{1-rt}} h'(\eta) \quad 4.11$$

$$\eta = y \sqrt{\frac{a}{v(1-\gamma)}} \quad \theta(\eta) = \frac{T - T_\infty}{T_w - T_\infty} \quad 4.12$$

### 4.2 satisfaction of conservation of mass equation

We show that condition for conservation of mass equation (continuity equation 3.05) is satisfied using equations 4.10 and 4.12 we show

$$\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} + \frac{\partial w}{\partial z} = 0$$

$$\frac{\partial u}{\partial x} = \frac{\partial}{\partial x} \left( \frac{ax}{1-rt} f'(\eta) \right) = f' \frac{\partial}{\partial x} \left( \frac{ax}{1-\gamma} \right) + \frac{ax}{1-\gamma} \frac{\partial f'}{\partial x}$$

$$\begin{aligned}
&= f' \frac{\partial}{\partial x} \left( \frac{ax}{1-\kappa} \right) + \frac{ax}{1-\kappa} \frac{\partial f'}{\partial \eta} \cdot \frac{\partial \eta}{\partial x} \\
&= f' \frac{\partial}{\partial x} \left( \frac{ax}{1-\kappa} \right) \text{ Since } \frac{\partial \eta}{\partial x} = 0 \text{ ,} \\
&= \frac{af'}{(1-\kappa)} \tag{4.13}
\end{aligned}$$

$$\begin{aligned}
\frac{\partial v}{\partial y} &= \frac{\partial}{\partial y} \left( -\sqrt{\frac{av}{1-\kappa}} f(\eta) \right) = -\sqrt{\frac{av}{1-\kappa}} \frac{\partial}{\partial y} f(\eta) + 0 \\
&= -\sqrt{\frac{av}{1-\kappa}} \frac{\partial f}{\partial \eta} \cdot \frac{\partial \eta}{\partial y} \\
&= -\sqrt{\frac{av}{1-\kappa}} f' \cdot \sqrt{\frac{a}{v(1-\kappa)}} \\
&= -\sqrt{\frac{va^2}{v(1-\kappa)^2}} f' = -\frac{a}{(1-\kappa)} f' \tag{4.14}
\end{aligned}$$

$$\frac{\partial w}{\partial z} = \frac{\partial}{\partial z} \left( \frac{ax}{1-\kappa} g(\eta) \right) = 0 \tag{4.15}$$

Therefore  $\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} + \frac{\partial w}{\partial z} = \frac{a}{(1-\kappa)} f' - \frac{a}{(1-\kappa)} f' + 0 = 0$  hence condition for continuity is

satisfied

### 4.3 momentum equations

*1st momentum equation (i-direction)*

$$\frac{\partial u}{\partial t} + u \frac{\partial u}{\partial x} + v \frac{\partial u}{\partial y} + w \frac{\partial u}{\partial z} = \nu \left( 1 + \frac{1}{\beta} \right) \frac{\partial^2 u}{\partial y^2} - \frac{\sigma B^4(t) \left[ u + mw + \frac{mwb^2}{B^2(t)} \right]}{\rho B^2(t)(1+m^2) + m^2 b^2}$$

**L.H.S**

$$\begin{aligned} \frac{\partial u}{\partial t} &= \frac{\partial}{\partial t} \left( \frac{ax}{1-\eta} f'(\eta) \right) = f' \frac{\partial}{\partial t} \frac{ax}{(1-\eta)} + \frac{ax}{(1-\eta)} \frac{\partial f'}{\partial \eta} \cdot \frac{\partial \eta}{\partial t} \\ &= f' \frac{axy}{(1-\eta)^2} + \frac{ax}{1-\eta} f'' \left[ y \sqrt{\frac{a}{\nu}} \left( -\frac{1}{2} (1-\eta)^{-\frac{3}{2}} \right) \cdot -\gamma \right] \\ &= f' \frac{axy}{(1-\eta)^2} + \frac{ax}{(1-\eta)} f'' \left( y \sqrt{\frac{a}{\nu}} (1-\eta)^{-1} (1-\eta)^{-\frac{1}{2}} \cdot \frac{\gamma}{2} \right) \\ &= \frac{axy}{(1-\eta)^2} f' + \frac{ax}{(1-\eta)} \left( y \sqrt{\frac{a}{\nu(1-\eta)}} \cdot \frac{\gamma}{2(1-\eta)} \right) f'' \\ &= \frac{axy}{(1-\eta)^2} f' + \frac{axy}{2(1-\eta)^2} \eta f'' \end{aligned} \tag{4.16}$$

$u \frac{\partial u}{\partial x}$  can be solved using equation 4.13

$$u \frac{\partial u}{\partial x} = \frac{ax}{(1-\eta)} f' \cdot \frac{a}{(1-\eta)} f' = \frac{a^2 x}{(1-\eta)^2} f'^2 \tag{4.17}$$

$$\frac{\partial u}{\partial y} = \frac{\partial}{\partial y} \left( \frac{ax}{(1-\eta)} f' \right) = \frac{ax}{(1-\eta)} \frac{\partial f'}{\partial y} + 0$$

$$\begin{aligned}
&= \frac{ax}{(1-\nu)} \frac{\partial f'}{\partial \eta} \cdot \frac{\partial \eta}{\partial y} \\
&= \frac{ax}{(1-\nu)} \sqrt{\frac{a}{\nu(1-\nu)}} f''
\end{aligned} \tag{4.18}$$

$$\text{Thus } \nu \frac{\partial u}{\partial y} = -\sqrt{\frac{a\nu}{1-\nu}} f \cdot \left( \frac{ax}{(1-\nu)} \sqrt{\frac{a}{\nu(1-\nu)}} f'' \right) = -\frac{a^2 x}{(1-\nu)^2} f f'' \tag{4.19}$$

$$\frac{\partial u}{\partial z} = \frac{\partial}{\partial z} \left( \frac{ax}{(1-\nu)} f'(\eta) \right) = 0 \tag{4.20}$$

$$\text{Thus } w \frac{\partial u}{\partial z} = 0 \tag{4.21}$$

### R.H.S

In the 1<sup>st</sup> term, we will find the second derivative of equation 4.18

$$\begin{aligned}
\frac{\partial^2 u}{\partial y^2} &= \frac{\partial}{\partial y} \left( \frac{ax}{(1-\nu)} \sqrt{\frac{a}{\nu(1-\nu)}} f'' \right) = \frac{ax}{(1-\nu)} \cdot \sqrt{\frac{a}{\nu(1-\nu)}} \frac{\partial}{\partial y} f'' \\
&= \frac{ax}{(1-\nu)} \sqrt{\frac{a}{\nu(1-\nu)}} \frac{\partial f''}{\partial \eta} \frac{\partial \eta}{\partial y} \\
&= \frac{ax}{(1-\nu)} \sqrt{\frac{a}{\nu(1-\nu)}} f''' \sqrt{\frac{a}{\nu(1-\nu)}} \\
&= \frac{a^2 x}{\nu(1-\nu)^2} f'''
\end{aligned} \tag{4.22}$$

$$\nu \left(1 - \frac{1}{\beta}\right) \frac{\partial^2 u}{\partial y^2} = \nu \left(1 - \frac{1}{\beta}\right) \frac{a^2 x}{\nu(1-\gamma)^2} f''' \quad 4.23$$

For the second term, we substitute the normal time dependent magnetic field  $B(t)$  with

$$B^2(t) = \frac{B_0^2}{(1-\gamma)} \text{ and induced magnetic field } b^2 = \frac{B_0^2}{(1-\gamma)} h'^2(\eta)$$

$$\begin{aligned} \frac{\sigma B^4(t) \left[ u + mw + \frac{mwb^2}{B^2(t)} \right]}{\rho B^2(t)(1+m^2) + m^2 b^2} &= \frac{\sigma \frac{B_0^2}{(1-\gamma)} \cdot \frac{B_0^2}{(1-\gamma)} \left[ u + mw + mw \frac{B_0^2(1-\gamma)h'^2}{B_0^2(1-\gamma)} \right]}{\rho \frac{B_0^2}{(1-\gamma)} (1+m^2) + m^2 \frac{B_0^2}{(1-\gamma)} h'^2} \\ &= \frac{\frac{\sigma B_0^2}{(1-\gamma)} \cdot \frac{B_0^2}{(1-\gamma)} \left[ \frac{axf'}{(1-\gamma)} + m \frac{axg}{(1-\gamma)} + m \frac{axgh'^2}{(1-\gamma)} \right]}{\frac{\rho B_0^2}{(1-\gamma)} [1+m^2+m^2h'^2]} \\ &= \frac{\frac{\sigma B_0^2}{(1-\gamma)^2} \cdot ax(f' + mg + mgh'^2)}{\rho(1+m^2+m^2h'^2)} \end{aligned} \quad 4.24$$

Combining equations 4.23 and 4.24 gives the R.H.S equation as follows

$$R.H.S = \frac{ax}{(1-\gamma)^2} \left( \left(1 + \frac{1}{\beta}\right) af''' - \frac{\sigma B_0^2 (f' + mg + mgh'^2)}{\rho(1+m^2+m^2h'^2)} \right) \quad 4.25$$

Combining equations 4.16, 4.17, 4.19, 4.21 and 4.25 gives us the 1<sup>st</sup> momentum equation

$$\frac{axy}{(1-\gamma)^2} f' + \frac{axy}{2(1-\gamma)^2} \eta f'' + \frac{a^2 x}{(1-\gamma)^2} f'^2 - \frac{a^2 x}{(1-\gamma)^2} ff'' = \frac{ax}{(1-\gamma)^2} \left( \left(1 + \frac{1}{\beta}\right) af''' - \frac{\sigma B_0^2 (f' + mg + mgh'^2)}{\rho(1+m^2+m^2h'^2)} \right)$$

Factoring out the common term we get

$$\frac{ax}{(1-\gamma)^2} \left( \gamma f' + \frac{\gamma \eta f''}{2} + af'^2 - aff'' \right) = \frac{ax}{(1-\gamma)^2} \left( \left( 1 + \frac{1}{\beta} \right) af''' - \frac{\sigma B_0 (f' + mg + mgh'^2)}{\rho(1+m^2+m^2h'^2)} \right) \quad 4.26$$

$$\left( \gamma f' + \frac{\gamma \eta f''}{2} + af'^2 - aff'' \right) = \left( \left( 1 + \frac{1}{\beta} \right) af''' - \frac{\sigma B_0 (f' + mg + mgh'^2)}{\rho(1+m^2+m^2h'^2)} \right) \quad 4.27$$

Dividing through by  $a$  and Re-arranging 4.27 yields,

$$\left( 1 + \frac{1}{\beta} \right) f''' + ff'' - f'^2 - A \left( f' + \frac{\eta f''}{2} \right) - M \left( \frac{f' + mg + mgh'^2}{1+m^2+mh'^2} \right) = 0$$

4.28

Where  $A = \frac{\gamma}{a}$  is the unsteadiness parameter,  $M = \frac{\sigma B_0^2}{\rho a}$  is the magnetic parameter

**2<sup>nd</sup> momentum equation**

$$\frac{\partial w}{\partial t} + u \frac{\partial w}{\partial x} + v \frac{\partial w}{\partial y} + w \frac{\partial w}{\partial z} = \nu \left( 1 + \frac{1}{\beta} \right) \frac{\partial^2 w}{\partial y^2} + \frac{\sigma}{\rho} \left[ \frac{B^4(t)(mu - w)}{B^2(t)(1+m^2)+m^2b^2} - \frac{B^2(t)(b^2w - mb^2u)}{B^2(t)(1+m^2)+m^2b^2} \right]$$

**L.H.S**

$$\begin{aligned} \frac{\partial w}{\partial t} &= \frac{\partial}{\partial t} \left( \frac{ax}{(1-\gamma)} g(\eta) \right) = g \frac{\partial}{\partial t} \left( \frac{ax}{(1-\gamma)} \right) + \frac{ax}{(1-\gamma)} \frac{\partial g}{\partial \eta} \frac{\partial \eta}{\partial t} \\ &= g \frac{ax\gamma}{(1-\gamma)^2} + \frac{ax}{1-\gamma} g' \left[ y \sqrt{\frac{a}{\nu}} \left( -\frac{1}{2} (1-\gamma)^{-\frac{3}{2}} \right) \cdot -\gamma \right] \\ &= g \frac{ax\gamma}{(1-\gamma)^2} + \frac{ax}{(1-\gamma)} g' \left( y \sqrt{\frac{a}{\nu}} (1-\gamma)^{-1} (1-\gamma)^{-\frac{1}{2}} \cdot \frac{\gamma}{2} \right) \end{aligned}$$

$$\begin{aligned}
&= \frac{axy}{(1-\eta)^2} g + \frac{ax}{(1-\eta)} \left( y \sqrt{\frac{a}{\nu(1-\eta)}} \cdot \frac{\gamma}{2(1-\eta)} \right) g' \\
&= \frac{axy}{(1-\eta)^2} g + \frac{axy}{2(1-\eta)^2} \eta g'
\end{aligned} \tag{4.29}$$

$$\frac{\partial w}{\partial x} = g \frac{\partial}{\partial x} \left( \frac{ax}{(1-\eta)} \right) = \frac{a}{(1-\eta)} g$$

$$u \frac{\partial w}{\partial x} = \frac{axf'}{(1-\eta)} \cdot \frac{ag}{(1-\eta)} = \frac{a^2 xf'g}{(1-\eta)^2} \tag{4.30}$$

$$\begin{aligned}
\frac{\partial w}{\partial y} &= \frac{\partial}{\partial y} \left( \frac{ax}{(1-\eta)} g \right) = \frac{ax}{(1-\eta)} \frac{\partial g}{\partial y} + 0 \\
&= \frac{ax}{(1-\eta)} \frac{\partial g}{\partial \eta} \cdot \frac{\partial \eta}{\partial y} = \frac{ax}{(1-\eta)} \sqrt{\frac{a}{\nu(1-\eta)}} g' \\
&= \frac{ax}{(1-\eta)} \sqrt{\frac{a}{\nu(1-\eta)}} g'
\end{aligned}$$

Therefore

$$\nu \frac{\partial w}{\partial y} = -\sqrt{\frac{a\nu}{1-\eta}} f \cdot \left( \frac{ax}{1-\eta} \sqrt{\frac{a}{\nu(1-\eta)}} g' \right) = -\frac{a^2 x}{(1-\eta)^2} fg' \tag{4.31}$$

$$\frac{\partial w}{\partial z} = \frac{\partial}{\partial z} \left( \frac{ax}{(1-\eta)} g(\eta) \right) = 0. \text{ and therefore } w \frac{\partial w}{\partial z} = 0 \tag{4.32}$$

Combining equations 4.29, 4.30 4.31 and 4.32 gives

$$\begin{aligned}
L.H.S &= \frac{axy}{(1-\kappa)^2} g + \frac{axy}{2(1-\kappa)^2} \eta g' + \frac{a^2 x f' g}{(1-\kappa)^2} - \frac{a^2 x f g'}{(1-\kappa)^2} \\
&= \frac{ax}{(1-\kappa)^2} \left( \gamma g + \frac{\gamma}{2} \eta g + a f' g - a f g' \right)
\end{aligned} \tag{4.33}$$

**R.H.S**

**1<sup>st</sup> term**

$$\begin{aligned}
\frac{\partial^2 w}{\partial y^2} &= \frac{\partial}{\partial y} \left( \frac{ax}{(1-\kappa)} \sqrt{\frac{a}{\nu(1-\kappa)}} g' \right) = \frac{ax}{(1-\kappa)} \cdot \sqrt{\frac{a}{\nu(1-\kappa)}} \frac{\partial}{\partial y} g' \\
&= \frac{ax}{(1-\kappa)} \sqrt{\frac{a}{\nu(1-\kappa)}} \frac{\partial g'}{\partial \eta} \frac{\partial \eta}{\partial y} \\
&= \frac{ax}{(1-\kappa)} \sqrt{\frac{a}{\nu(1-\kappa)}} g'' \sqrt{\frac{a}{\nu(1-\kappa)}} \\
&= \frac{a^2 x}{\nu(1-\kappa)^2} g''
\end{aligned} \tag{4.34}$$

$$\text{therefore } \nu \left( 1 - \frac{1}{\beta} \right) \frac{\partial^2 w}{\partial y^2} = \nu \left( 1 - \frac{1}{\beta} \right) \frac{a^2 x}{\nu(1-\kappa)^2} g'' \tag{4.35}$$

**2<sup>nd</sup> term**

$$\frac{\sigma}{\rho} \left[ \frac{B^4(t)(mu - w)}{B^2(t)(1+m^2) + m^2 b^2} - \frac{B^2(t)(b^2 w - mb^2 u)}{B^2(t)(1+m^2) + m^2 b^2} \right] =$$

$$\begin{aligned}
&= \frac{\sigma}{\rho} \left[ \frac{\frac{B_0^4}{(1-\mathcal{N})^2} \left( m \frac{ax}{(1-\mathcal{N})} f' - \frac{axg}{(1-\mathcal{N})} \right)}{\frac{B_0^2}{(1-\mathcal{N})} \left( 1+m^2 + \frac{B_0^2}{(1-\mathcal{N})} h'^2 m^2 \right)} - \frac{\frac{B_0^2}{(1-\mathcal{N})} \left( \frac{B_0^2}{(1-\mathcal{N})} h'^2 \right) \left( \frac{axg}{(1-\mathcal{N})} - m \frac{axf'}{(1-\mathcal{N})} \right)}{\frac{B_0^2}{(1-\mathcal{N})} \left( 1+m^2 + \frac{B_0^2}{(1-\mathcal{N})} h'^2 m^2 \right)} \right] \\
&= \frac{\sigma}{\rho} \left[ \frac{\frac{B_0^2}{(1-\mathcal{N})} \frac{B_0^2}{(1-\mathcal{N})} \left( m \frac{ax}{(1-\mathcal{N})} f' - \frac{axg}{(1-\mathcal{N})} \right)}{\frac{B_0^2}{(1-\mathcal{N})} \left( 1+m^2 + \frac{B_0^2}{(1-\mathcal{N})} h'^2 m^2 \right)} - \frac{\frac{B_0^2}{(1-\mathcal{N})} \left( \frac{B_0^2}{(1-\mathcal{N})} h'^2 \right) \left( \frac{axg}{(1-\mathcal{N})} - m \frac{axf'}{(1-\mathcal{N})} \right)}{\frac{B_0^2}{(1-\mathcal{N})} \left( 1+m^2 + \frac{B_0^2}{(1-\mathcal{N})} h'^2 m^2 \right)} \right] \\
&= \frac{\sigma}{\rho} \left[ \frac{\frac{B_0^2}{(1-\mathcal{N})} \left( m \frac{ax}{(1-\mathcal{N})} f' - \frac{axg}{(1-\mathcal{N})} \right)}{\left( 1+m^2 + h'^2 m^2 \right)} - \frac{\left( \frac{B_0^2}{(1-\mathcal{N})} h'^2 \right) \left( \frac{axg}{(1-\mathcal{N})} - m \frac{axf'}{(1-\mathcal{N})} \right)}{\left( 1+m^2 + h'^2 m^2 \right)} \right] \\
&= \frac{\sigma \left( \frac{B_0^2}{(1-\mathcal{N})} \right)}{\rho} \left[ \frac{\left( m \frac{ax}{(1-\mathcal{N})} f' - \frac{axg}{(1-\mathcal{N})} \right)}{\left( 1+m^2 + h'^2 m^2 \right)} - \frac{\left( h'^2 \right) \left( \frac{axg}{(1-\mathcal{N})} - m \frac{axf'}{(1-\mathcal{N})} \right)}{\left( 1+m^2 + h'^2 m^2 \right)} \right] \\
&= \frac{\sigma \left( \frac{B_0^2}{(1-\mathcal{N})} \right)}{\rho} \left[ \frac{\left( m \frac{ax}{(1-\mathcal{N})} f' - \frac{axg}{(1-\mathcal{N})} \right)}{\left( 1+m^2 + h'^2 m^2 \right)} - \frac{\left( h'^2 \right) \left( \frac{axg}{(1-\mathcal{N})} - m \frac{axf'}{(1-\mathcal{N})} \right)}{\left( 1+m^2 + h'^2 m^2 \right)} \right] \\
&= \frac{\sigma \left( \frac{B_0^2}{(1-\mathcal{N})^2} \right) ax (mf' - g - h'^2 g + mh'^2 f')}{\rho (1+m^2 + m^2 h'^2)}
\end{aligned}$$

4.36

Combining 4.35 with 4.36 gives

$$R.H.S = v \left( 1 - \frac{1}{\beta} \right) \frac{a^2 x}{v(1-\mathcal{N})^2} g'' + \frac{\sigma B_0^2 ax}{\rho(1-\mathcal{N})} \left( \frac{mf' - g - h'g + mh'^2 f'}{1+m^2 + m^2 h'^2} \right)$$

Which simplifies to

$$R.H.S = \frac{ax}{(1-\pi)^2} \left( \left( 1 - \frac{1}{\beta} \right) ag'' + \frac{\sigma B^2_0}{\rho} \left( \frac{mf' - g - h'g + mh'^2 f'}{1 + m^2 + m^2 h'^2} \right) \right) \quad 4.37$$

Combining equation 4.33 and 4.37 we have the 2<sup>nd</sup> momentum equation

$$\frac{ax}{(1-\pi)^2} \left( \gamma g + \frac{\gamma \eta g'}{2} + af'g - afg' \right) = \frac{ax}{(1-\pi)^2} \left( \left( 1 - \frac{1}{\beta} \right) ag'' + \frac{\sigma B^2_0}{\rho} \left( \frac{mf' - g - h'g + mh'^2 f'}{1 + m^2 + m^2 h'^2} \right) \right)$$

$$\left( \gamma g + \frac{\gamma \eta g'}{2} + af'g - afg' \right) = \left( \left( 1 - \frac{1}{\beta} \right) ag'' + \frac{\sigma B^2_0}{\rho} \left( \frac{mf' - g - h'g + mh'^2 f'}{1 + m^2 + m^2 h'^2} \right) \right)$$

Dividing through by  $a$  and Re-arranging yields

$$\left( 1 + \frac{1}{\beta} \right) g'' - f'g + fg' - A \left( g + \frac{\eta}{2} g' \right) + M \left( \frac{mf' - g - gh' + mh'^2 f'}{1 + m^2 + m^2 h'^2} \right) = 0 \quad 4.38$$

#### 4.4 energy equation

We now transform the energy equation 3.37 using the set of transformations defined in equation

4.10 and 4.12

$$\frac{\partial T}{\partial t} + u \frac{\partial T}{\partial x} + v \frac{\partial T}{\partial y} + w \frac{\partial T}{\partial z} = \alpha_m \frac{\partial^2 T}{\partial y^2} - \frac{1}{\rho C_p} \frac{\partial q_r}{\partial y}$$

**L.H.S**

First we make  $T$  the subject from  $\theta(\eta) = \frac{T - T_\infty}{T_w - T_\infty}$  and find its first derivative with respect to time

$$T = \theta(T_w - T_\infty) + T_\infty$$

$$\begin{aligned}
\frac{\partial T}{\partial t} &= \frac{\partial}{\partial t} (\theta(T_w - T_\infty) + T_\infty) = \frac{\partial(\theta(T_w - T_\infty) + T_\infty)}{\partial \eta} \cdot \frac{\partial \eta}{\partial t} \\
&= (T_w - T_\infty) \theta' \cdot \left( y \sqrt{\frac{a}{\nu(1-\kappa)}} \cdot \frac{\gamma}{2(1-\kappa)} \right) \\
&= (T_w - T_\infty) \theta' \cdot \left( \eta \frac{\gamma}{2(1-\kappa)} \right) \\
&= \left( \frac{\eta \gamma}{2(1-\kappa)} \right) (T_w - T_\infty) \theta'
\end{aligned} \tag{4.39}$$

$$\frac{\partial T}{\partial x} = \frac{\partial T}{\partial \eta} \cdot \frac{\partial \eta}{\partial x} = 0 \text{ since } \frac{\partial \eta}{\partial x} = 0 \tag{4.40}$$

$$\begin{aligned}
\frac{\partial T}{\partial y} &= \frac{\partial((T_w - T_\infty)\theta + T_\infty)}{\partial \eta} \cdot \frac{\partial}{\partial y} \left( y \sqrt{\frac{a}{\nu(1-\kappa)}} \right) = \sqrt{\frac{a}{\nu(1-\kappa)}} (T_w - T_\infty) \theta' \\
\nu \frac{\partial T}{\partial y} &= - \sqrt{\frac{a\nu}{(1-\kappa)}} f \sqrt{\frac{a}{\nu(1-\kappa)}} (T_w - T_\infty) \theta' = - \frac{a}{(1-\kappa)} (T_w - T_\infty) f \theta'
\end{aligned} \tag{4.41}$$

$$\frac{\partial T}{\partial z} = \frac{\partial T}{\partial \eta} \cdot \frac{\partial \eta}{\partial z} = 0 \text{ since } \frac{\partial \eta}{\partial z} = 0 \tag{4.42}$$

Combining equations 4.39, 4.40, 4.41 and 4.42 gives the similarity form on the L.H.S

$$L.H.S = \frac{(T_w - T_\infty)}{(1-\kappa)} \left( \frac{\eta \gamma \theta'}{2} - a f \theta' \right) \tag{4.43}$$

To transform the R.H.S we first change the second term by expressing the radiative heat flux vector  $q_r$  in terms of  $T$

Using Rosseland approximation,  $q_r = -\frac{4\sigma^*}{3\alpha^*} \frac{\partial T^4}{\partial y}$ , where  $T^4 \approx 4T_\infty^3 T - 3T_\infty^4$  is the Taylor series expansion ignoring higher order terms.

Following Pantokratoras and Fang (2014), *Blasius flow with non-linear Roseland thermal*

$$\text{radiation, } q_r \text{ can further be simplified as } q_r = \frac{16\sigma^*}{3\alpha^*} T^3 \frac{\partial T}{\partial y} \quad 4.44$$

R.H.S of Equation 3.37 therefore can be expressed as

$$\begin{aligned} \alpha_m \frac{\partial^2 T}{\partial y^2} - \frac{1}{\rho C_p} \frac{\partial q_r}{\partial y} &= \alpha_m \frac{\partial^2 T}{\partial y^2} + \frac{1}{\rho C_p} \frac{\partial}{\partial y} \left( \frac{16\sigma^*}{3\alpha^*} T^3 \frac{\partial T}{\partial y} \right) \\ &= \frac{\partial}{\partial y} \left( \left( \alpha_m + \frac{16\sigma^* T^3}{3\alpha^* \rho C_p} \right) \frac{\partial T}{\partial y} \right) \end{aligned} \quad 4.45$$

Since  $T = (T_w - T_\infty)\theta + T_\infty$  we can divide both sides with  $T_\infty$  to get

$$\frac{T}{T_\infty} = \left( \frac{T_w}{T_\infty} - 1 \right) \theta + 1. \text{ Taking the cube of both sides gives}$$

$$\left( \frac{T}{T_\infty} \right)^3 = \left( 1 + \left( \frac{T_w}{T_\infty} - 1 \right) \theta \right)^3$$

$$T^3 = \left( 1 + \left( \frac{T_w}{T_\infty} - 1 \right) \theta \right)^3 \cdot T_\infty^3 \quad 4.46$$

From equation 4.45, we get  $\frac{\partial T}{\partial y} = \frac{\partial}{\partial y} ((T_w - T_\infty)\theta(\eta) + T_\infty)$

$$= \frac{\partial T}{\partial \eta} \cdot \frac{\partial \eta}{\partial y} = (T_w - T_\infty) \theta' \sqrt{\frac{a}{\nu(1-\gamma)}} \quad 4.47$$

Equation 4.45 now becomes

$$= \frac{\partial}{\partial y} \left( \left( \alpha_m + \frac{16\sigma^* T^3}{3\alpha^* \rho C_p} \right) \theta' (T_w - T_\infty) \sqrt{\frac{a}{\nu(1-\gamma)}} \right)$$

$$= \frac{\partial}{\partial y} \left( \left( \alpha_m \theta' + \frac{16\sigma^* T^3 \theta'}{3\alpha^* \rho C_p} \right) (T_w - T_\infty) \sqrt{\frac{a}{\nu(1-\gamma)}} \right)$$

$$= \left( (T_w - T_\infty) \sqrt{\frac{a}{\nu(1-\gamma)}} \right) \frac{\partial}{\partial y} \left( \alpha_m \theta' + \frac{16\sigma^* T^3 \theta'}{3\alpha^* \rho C_p} \right) \quad 4.48$$

$$\frac{\partial}{\partial y} \left( \alpha_m \theta' + \frac{16\sigma^* T^3 \theta'}{3\alpha^* \rho C_p} \right) = \alpha_m \frac{\partial \theta'}{\partial \eta} \frac{\partial \eta}{\partial y} + \frac{16\sigma^*}{3\alpha^* \rho C_p} \frac{\partial}{\partial y} (T^3 \theta')$$
4.49

$$= \alpha_m \theta'' \sqrt{\frac{a}{\nu(1-\gamma)}} + \frac{16\sigma^* T_\infty^3}{3\alpha^* \rho C_p} \frac{\partial}{\partial y} \left( 1 + \left( \frac{T_w}{T_\infty} - 1 \right) \theta \right)^3 \theta' \quad 4.50$$

Using product rule, we differentiate  $\frac{\partial}{\partial y} \left( 1 + \left( \frac{T_w}{T_\infty} - 1 \right) \theta \right)^3 \theta'$  to get

$$\frac{\partial}{\partial y} \left( 1 + \left( \frac{T_w}{T_\infty} - 1 \right) \theta \right)^3 \theta' = \left( 1 + \left( \frac{T_w}{T_\infty} - 1 \right) \theta \right)^3 \theta'' \cdot \sqrt{\frac{a}{\nu(1-\gamma)}} + \theta' \cdot 3 \left( \left( \frac{T_w}{T_\infty} - 1 \right) \theta' \left( 1 + \left( \frac{T_w}{T_\infty} - 1 \right) \theta \right)^2 \right) \sqrt{\frac{a}{\nu(1-\gamma)}}$$

$$\frac{\partial}{\partial y} \left( 1 + \left( \frac{T_w}{T_\infty} - 1 \right) \theta \right)^3 \theta' = \sqrt{\frac{a}{\nu(1-\gamma)}} \left( \left( 1 + \left( \frac{T_w}{T_\infty} - 1 \right) \theta \right)^3 \theta'' + 3\theta'^2 \left( \frac{T_w}{T_\infty} - 1 \right) \left( 1 + \left( \frac{T_w}{T_\infty} - 1 \right) \theta \right)^2 \right) \quad 4.52$$

Equation 4.48 now becomes

$$\frac{(T_w - T_\infty)a}{\nu(1-\gamma)} \left( \alpha_m \theta'' + \frac{16\sigma^* T_\infty^3}{3\alpha^* \rho C_p} \left( \left( 1 + \left( \frac{T_w}{T_\infty} - 1 \right) \theta \right)^3 \right) \theta'' + 3\theta'^2 \left( \frac{T_w}{T_\infty} - 1 \right) \left( 1 + \left( \frac{T_w}{T_\infty} - 1 \right) \theta \right)^2 \right) \quad 4.53$$

Equating equations 4.43 and equation 4.53 gives the similarity form of the energy equation

Which simplifies to

$$\left( \frac{\eta\gamma\theta'}{2} - a\theta f' \right) = \frac{a}{\nu} \left( \alpha_m \theta'' + \frac{16\sigma^* T_\infty^3}{3\alpha^* \rho C_p} \left( \left( 1 + \left( \frac{T_w}{T_\infty} - 1 \right) \theta \right)^3 \right) \theta'' + 3\theta'^2 \left( \frac{T_w}{T_\infty} - 1 \right) \left( 1 + \left( \frac{T_w}{T_\infty} - 1 \right) \theta \right)^2 \right)$$

Multiplying both sides with  $\frac{\nu}{a\alpha_m}$  we get

$$\frac{\nu\eta\gamma\theta'}{2a\alpha_m} - \frac{\nu f\theta'}{\alpha_m} = \theta'' + \frac{16\sigma^* T_\infty^3}{3\alpha^* k} \left( 1 + \left( \frac{T_w}{T_\infty} - 1 \right) \theta \right)^3 \theta'' + \frac{16\sigma^* T_\infty^3}{3\alpha^* k} \left( \frac{T_w}{T_\infty} - 1 \right) \left( 1 + \left( \frac{T_w}{T_\infty} - 1 \right) \theta \right)^2 \theta'^2 \quad 4.54$$

Where  $\alpha_m = \frac{k}{\rho C_p}$

$$prA \frac{\eta\theta'}{2} - prf\theta' = \theta'' + \frac{4}{3Nr} (1 + (tr-1)\theta)^3 \theta'' + \frac{4}{3Nr} (tr-1)(1 + (tr-1)\theta)^2 \theta'^2 = 0$$

Grouping equal terms together 4.54 becomes

$$\left( 1 + \frac{4}{3N_r} (1 + (tr-1)\theta)^3 \right) \theta'' - prA \frac{\eta}{2} \theta' + prf\theta' + \frac{4}{3N_r} (tr-1)(1 + (tr-1)\theta)^2 \theta'^2 = 0 \quad 4.55$$

Where  $pr = \frac{\nu}{\alpha_m}$  is the Prandtl number,  $N_r = \frac{k\alpha^*}{4\sigma^*T_\infty^3}$  is the radiation parameter  $tr = \frac{T_w}{T_\infty}$  is the temperature ratio parameter.

#### 4.5 transforming induced magnetic equation

The equation to be transformed is equation 3.54 given as  $\frac{\partial b}{\partial t} = B(t) \frac{\partial u}{\partial y} - \eta \frac{\partial^2 b}{\partial y^2}$

Where the transformation equation 4.11 is used

$$l.H.S = \frac{\partial b}{\partial t} = \frac{\partial}{\partial t} \left( \frac{B_0}{\sqrt{1-\gamma}} h'(\eta) \right) = \frac{B_0 \gamma h'}{2(1-\gamma)^{\frac{3}{2}}} + \frac{B_0}{\sqrt{1-\gamma}} \frac{\partial h'}{\partial \eta} \frac{\partial \eta}{\partial t} \quad 4.56$$

$$= \frac{B_0 \gamma h'}{2(1-\gamma)^{\frac{3}{2}}} + \frac{B_0 h''}{\sqrt{1-\gamma}} \cdot y \sqrt{\frac{a}{\nu(1-\gamma)}} \cdot \frac{\gamma}{2(1-\gamma)}$$

$$= \frac{B_0 \gamma h'}{2(1-\gamma)^{\frac{3}{2}}} + \frac{B_0 h''}{\sqrt{1-\gamma}} \cdot \eta \cdot \frac{\gamma}{2(1-\gamma)}$$

$$= \frac{B_0 \gamma h'}{2(1-\gamma)^{\frac{3}{2}}} + \frac{B_0 h'' \eta \gamma}{2(1-\gamma)^{\frac{3}{2}}}$$

$$= \frac{B_0 \gamma}{2(1-\gamma)^{\frac{3}{2}}} (h' + \eta h'') \quad 4.56$$

$$R.H.S = B(t) \frac{\partial u}{\partial y} + \eta \frac{\partial^2 b}{\partial y^2}$$

$$\begin{aligned}
B(t) \frac{\partial u}{\partial y} &= \frac{B_0}{(1-\gamma)^{\frac{1}{2}}} \cdot \frac{axf''}{(1-\gamma)} \sqrt{\frac{a}{\nu(1-\gamma)}} \\
&= \frac{B_0 axf''}{(1-\gamma)^{\frac{3}{2}}} \sqrt{\frac{a}{\nu(1-\gamma)}}
\end{aligned} \tag{4.57}$$

$$\frac{\partial^2 b}{\partial y^2} = \frac{\partial}{\partial y} \left( \frac{\partial b}{\partial y} \right) = \frac{\partial}{\partial y} \left( \frac{B_0 h'}{(1-\gamma)^{\frac{1}{2}}} \right) = \frac{\partial}{\partial y} \left( \frac{B_0}{(1-\gamma)^{\frac{1}{2}}} h'' \frac{\partial \eta}{\partial y} \right) = \frac{\partial}{\partial y} \left( \frac{B_0}{(1-\gamma)^{\frac{1}{2}}} h'' \cdot \sqrt{\frac{a}{\nu(1-\gamma)}} \right) \tag{4.58}$$

$$\begin{aligned}
&= \frac{B_0}{(1-\gamma)^{\frac{1}{2}}} \sqrt{\frac{a}{\nu(1-\gamma)}} \frac{\partial h''}{\partial y} \\
&= \frac{B_0}{(1-\gamma)^{\frac{1}{2}}} \sqrt{\frac{a}{\nu(1-\gamma)}} \cdot \sqrt{\frac{a}{\nu(1-\gamma)}} h''' \\
&= \frac{B_0 ah'''}{\nu(1-\gamma)^{\frac{3}{2}}}
\end{aligned} \tag{4.59}$$

$$R.H.S = \frac{B_0 axf''}{(1-\gamma)^{\frac{3}{2}}} \sqrt{\frac{a}{\nu(1-\gamma)}} + \eta \frac{B_0 ah'''}{\nu(1-\gamma)^{\frac{3}{2}}} \tag{4.60}$$

Equating equations 4.56 and equation 4.60 yields

$$\frac{B_0 \gamma}{2(1-\gamma)^{\frac{3}{2}}} (h' + \eta h'') = \frac{B_0 a}{(1-\gamma)^{\frac{3}{2}}} \left( xf'' \sqrt{\frac{a}{\nu(1-\gamma)}} + \frac{\eta h'''}{\nu} \right)$$

$$\frac{\gamma}{2}(h' + \eta h'') = a \left( x f'' \sqrt{\frac{a}{\nu(1-\gamma)}} + \frac{\eta h'''}{\nu} \right)$$

$$\frac{\gamma}{2a}(h' + \eta h'') = \left( f'' \sqrt{\frac{ax^2}{\nu(1-\gamma)}} + \frac{\eta h'''}{\nu} \right)$$

$$\frac{\gamma}{2a}(h' + \eta h'') = \sqrt{\frac{x}{\nu}} \cdot \sqrt{\frac{ax}{(1-\gamma)}} f'' + \frac{\eta h'''}{\nu}$$

$$\frac{\gamma}{2a}(h' + \eta h'') = \sqrt{\frac{x}{\nu}} \cdot \sqrt{\frac{ax}{(1-\gamma)}} f'' + \frac{\eta h'''}{\nu}$$

$$\frac{\gamma}{2a}(h' + \eta h'') = \sqrt{\frac{U_w x}{\nu}} \cdot f'' + \frac{\eta h'''}{\nu}$$

Since

$$U_w = \frac{ax}{(1-\gamma)}$$

$$\frac{A}{2}(h' + \eta h'') = \text{Re}_x^{\frac{1}{2}} f'' + \frac{h'''}{\text{Pr}_m}$$

4.61

Equation 4.61 can also be written as

$$\frac{h'''}{\text{Pr}_m} - \frac{A\eta h''}{2} - \frac{Ah'}{2} + \text{Re}_x^{\frac{1}{2}} f'' = 0$$

4.62

Where ,  $\text{Pr}_m = \frac{\nu}{\eta}$  is the magnetic Prandtl number,  $\text{Re}_x = \frac{U_w x}{\nu}$  is the local Reynolds number

In summary the four ordinary differential equations are as follows

$$\left(1 + \frac{1}{\beta}\right) f''' + ff'' - f'^2 - A\left(f' + \frac{\eta f''}{2}\right) - M\left(\frac{f' + mg + mgh'^2}{1 + m^2 + mh'^2}\right) = 0$$

$$\left(1 + \frac{1}{\beta}\right) g'' - f'g + fg' - A\left(g + \frac{\eta}{2}g'\right) + M\left(\frac{mf' - g - gh' + mh'^2f'}{1 + m^2 + m^2h'^2}\right) = 0$$

$$\left(1 + \frac{4}{3N_r}(1 + (tr-1)\theta)^3\right)\theta'' - prA\frac{\eta}{2}\theta' + prf\theta' + \frac{4}{3N_r}(tr-1)(1 + (tr-1)\theta)^2\theta'^2 = 0$$

$$\frac{h'''}{pr_m} - \frac{A\eta h''}{2} - \frac{Ah'}{2} + \text{Re}_x^{\frac{1}{2}} f'' = 0$$

The boundary conditions on using the transformation equations 4.10,4.11 and 4.12 reduces to

$$\text{At } \eta = 0 : f' = 1, f = 0, g = 0, \theta = 1, h' = 0 \quad 4.63a$$

$$\text{As } \eta \rightarrow \infty, f' \rightarrow 0, g \rightarrow 0, \theta \rightarrow 0, h' \rightarrow 0 \quad 4.63b$$

### 4.6 Reducing the equations to first order

The above equations will now be reduced to 10 equivalent first order differential equations as follows.

We let

$$f = x_1, f' = x_2, f'' = x_3, g = x_4, g' = x_5, h = x_6, h' = x_7, h'' = x_8, \theta = x_9, \theta' = x_{10}$$

And use the above equations to obtain the following system of equations

$$x_1' = x_2$$

$$x_2' = x_3$$

$$x_3' = -\left(\frac{\beta}{\beta+1}\right)x_1x_3 + \left(\frac{\beta}{\beta+1}\right)x_2^2 + A\left(\frac{\beta}{\beta+1}\right)\left(x_2 + \frac{\eta}{2}x_3\right) + M\left(\frac{\beta}{\beta+1}\right)\left(\frac{x_2 + mx_4 + mx_4x_7^2}{1 + m^2 + mx_7^2}\right)$$

$$x_4' = x_5$$

$$x_5' = -\left(\frac{\beta}{\beta+1}\right)x_1x_5 + \left(\frac{\beta}{\beta+1}\right)x_2x_4 + A\left(\frac{\beta}{\beta+1}\right)\left(x_4 + \frac{\eta}{2}x_5\right) - M\left(\frac{\beta}{\beta+1}\right)\left(\frac{mx_2 - x_4 - x_4x_7 + mx_2x_7^2}{1 + m^2 + m^2x_7^2}\right)$$

$$x_6' = x_7$$

$$x_7' = x_8$$

$$x_8' = pr_m A \frac{\eta}{2} x_8 + \frac{1}{2} pr_m A x_7 - pr_m \text{Re}_x^{\frac{1}{2}} x_3$$

$$x_9' = x_{10}$$

$$x_{10}' = \frac{3N_r pr A \eta}{2(3N_r + 4(1 + (tr-1)x_9)^3)} x_{10} - \frac{3N_r pr}{3N_r + 4(1 + (tr-1)x_9)^3} x_1 x_{10} - \frac{4(tr-1)(1 + (tr-1)x_9)^2}{3N_r + 4(1 + (tr-1)x_9)^3} x_{10}^2$$

## CHAPTER 5

### 5.0 Results and discussion

In this chapter, we analyse the effects of important parameters under study on velocity profiles in  $x$  – direction and in  $z$  – direction, on temperature and on induced magnetic field. These results are presented graphically in figures 5.11 to 5.42 . First, numerical results are presented and a detailed discussion done on impact of flow parameters such as magnetic parameter  $M$  , unsteadiness parameter  $A$  , Hall current parameter  $m$  Casson fluid parameter  $\beta$  ,radiation parameter  $Nr$  , Prandtl number  $Pr$  ,temperature ratio parameter  $tr$  ,magnetic parameter number  $Pr_m$  and local Reynolds number  $Re_x$  . The results are obtained using the following parameter values;

$$A = 1, M = 6, m = 0.1, \beta = 0.3, Nr = 2, Pr = 10, tr = 1, Pr_m = 0.5, Re_x = 3$$

#### 5.1 Effects of variation of magnetic parameter $M$ on velocity, temperature and induced magnetic field profiles.

The effect of magnetic parameter on fluid velocity in  $x$  – direction is presented in **figure 5.11**.

As evident from the graph, an increase in the magnetic parameter leads to a decrease in the fluid flow in the  $x$  – direction within the boundary layer region. This is because applied magnetic field generates resistance inside the flow field hence the higher the value of  $M$  the less the velocity  $f'(\eta)$ .

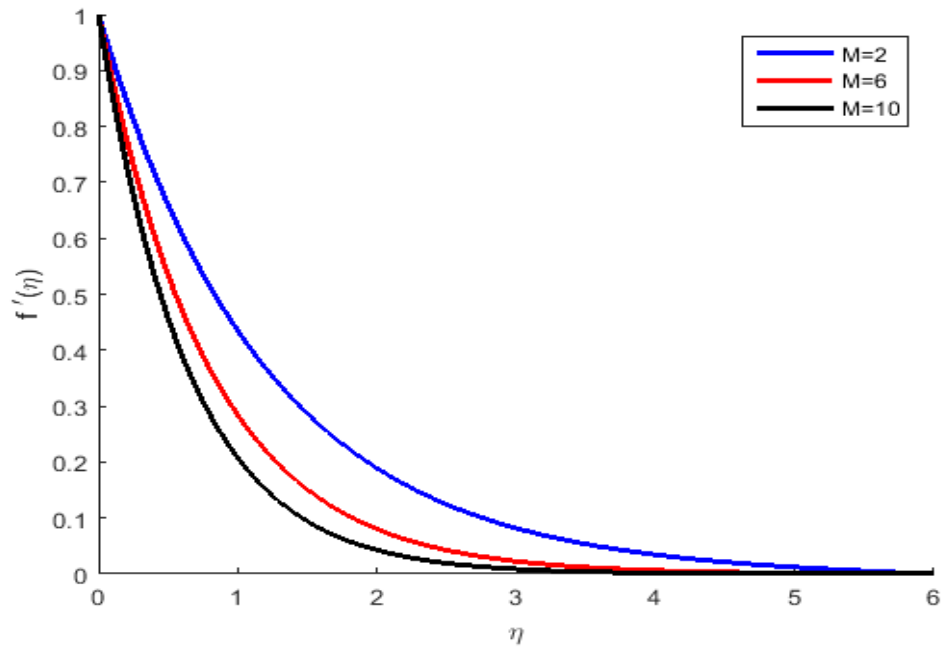
**Figure 5.12** shows the effects of magnetic parameter  $M$  on profile of the fluid in  $z$  – direction.

Two distinct behaviors are observed on increasing the magnetic parameter  $M$  . Initially, near the surface inside the boundary layer region, the secondary velocity profile first increases with increase in magnetic parameter. This is because the fluid flow is induced by the stretching of the sheet and the fluid being viscous, it will move with the stretching sheet until the Lorentz force

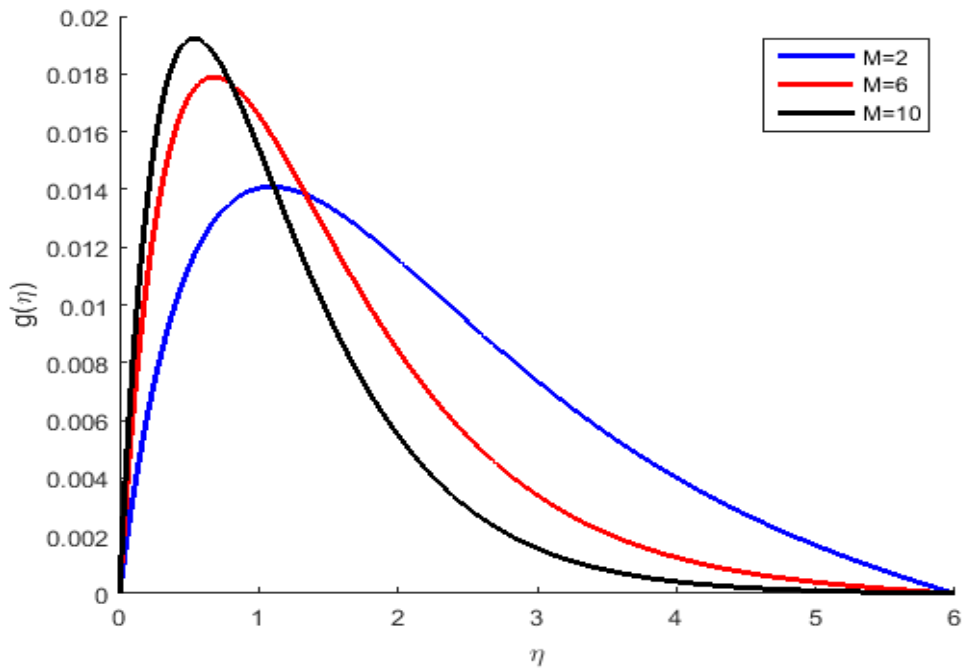
sufficiently increases to cause a decline in the fluid velocity. Further away from the boundary layer, the secondary velocity profile decelerates as magnetic parameter  $M$  increases. As magnetic parameter increases, Lorentz force also increases. This shows that eventually, resistance is generated leading to decreased in velocity  $g(\eta)$  as magnetic parameter increases.

The effects of magnetic parameter  $M$  on temperature profile  $\theta$  is presented on **figure 5.13**. It shows that increasing the magnetic parameter  $M$  leads to corresponding increase in fluid temperature profile  $\theta$ . An increase of magnetic parameter leads to decrease in fluid velocity which in turn compromises heat transfer, hence the temperature rises in the boundary layer as magnetic parameter increases.

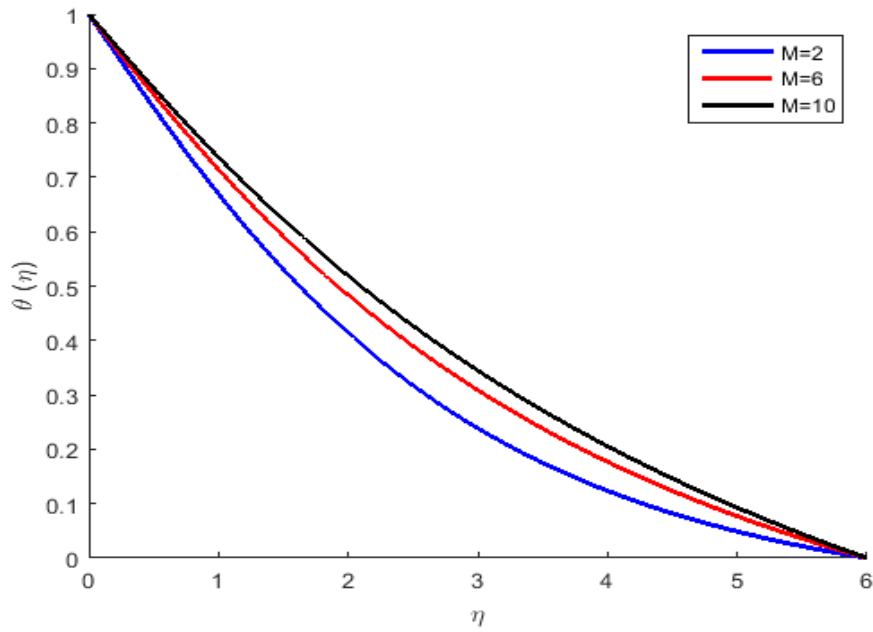
**Figure 5.14** shows the effect of magnetic parameter  $M$  on induced magnetic field profile  $h'(\eta)$ . It is evident that near the surface inside the boundary layer region, induced magnetic profile does not change with increase in magnetic parameter  $M$ . Further away from the surface induced magnetic field profile increased with magnetic parameter  $M$ . As the primary and secondary velocity profiles decelerates, for an electrically conducting fluid then as the field stretches more magnetic field was induced as magnetic parameter  $M$  increased.



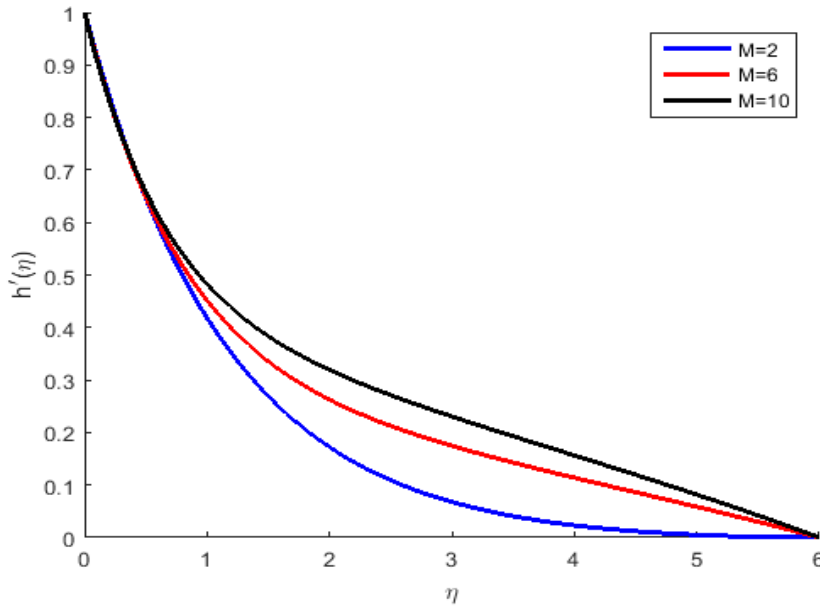
**Figure 5.11:** Effect of magnetic parameter  $M$  on velocity profile in  $x$ -direction.



**Figure 5.12:** Effect of magnetic parameter  $M$  on velocity profile in  $z$ -direction.



**Figure 5.13 Effect of magnetic parameter  $M$  on Temperature profile.**



**Figure 5.14 effect of magnetic parameter  $M$  on induced magnetic field profile**

$M = 1,4,7$      $A = 0,1,2$      $Nr = 1,2,4$      $B = 0.1,0.2,0.4$      $m = 1,2,4$   
 $pr = 0.71$      $tr = 0,1,2$      $pr_m = 0.1,0.2,0.4$      $Re = 1,5,10$

## 5.2 Effects of variation of Casson parameter $\beta$ on velocity, temperature $\theta$ and induced magnetic field profiles.

Effects of Casson parameter on profile of fluid primary velocity are shown in **figure 5.15**. It is evident from the graph that increasing the value of the Casson parameter  $\beta$  in the  $x$ -direction decelerates primary velocity within the boundary layer region. In turn, the boundary layer thickness decreases. This shows an increase in the Casson parameter (decreasing yield stress) makes the fluid behave as a Newtonian fluid since  $\beta$  measures the yield stress. This behavior causes stabilization effect.

**Figure 5.16** shows how variation of Casson parameter  $\beta$  affects the secondary velocity profile of the fluid. Initially, inside the boundary layer region near the stretching surface, increasing the Casson parameter causes an increase in the profile of the secondary velocity. Away from the stretching surface, increasing the Casson parameter  $\beta$  causes a decrease in secondary velocity. This shows that the boundary layer thickness decreases. Decrease in secondary velocity therefore causes stabilization effect in the flow of the fluid due to the resistance generated.

**Figure 5.17** shows the effects of Casson parameter  $\beta$  on the fluid temperature  $\theta$ . The temperature of the fluid increases with increase in casson parameter  $\beta$ . Due to decrease in primary and secondary velocity, then the fluid temperature increases as the casson parameter increases.

The variation of Casson parameter  $\beta$  on induced magnetic field  $h'$  is shown in **figure 5.18**. It shows that increase in Casson parameter accelerates induced magnetic field.

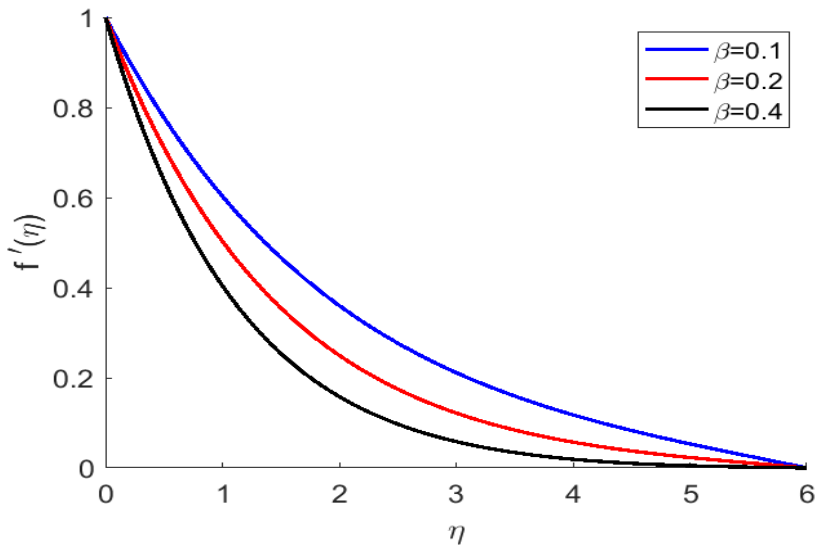


Figure 5.15 Effects of Casson parameter  $\beta$  on fluid velocity profile in  $x$ -direction.

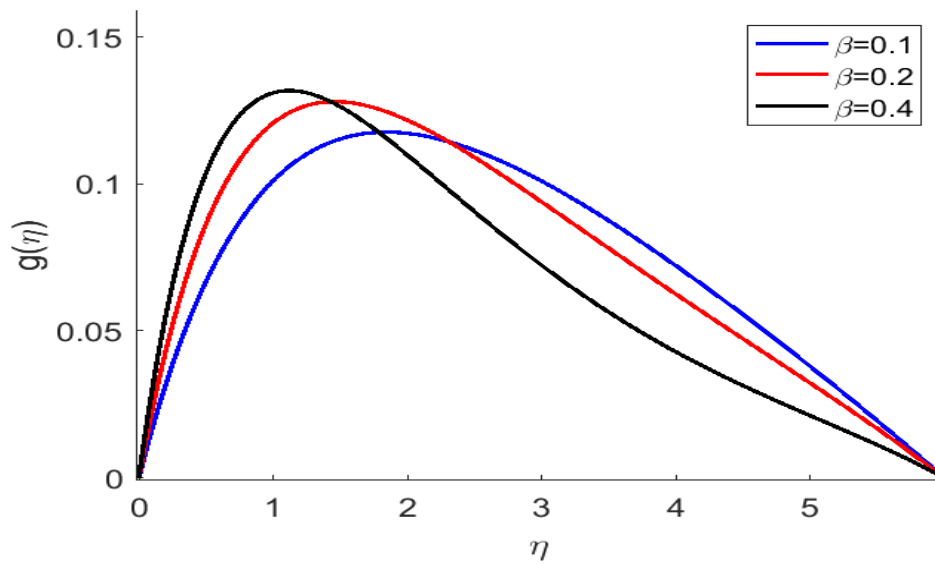


Figure 5.16: effects of casson parameter  $\beta$  on fluid velocity profile in  $z$  - direction.

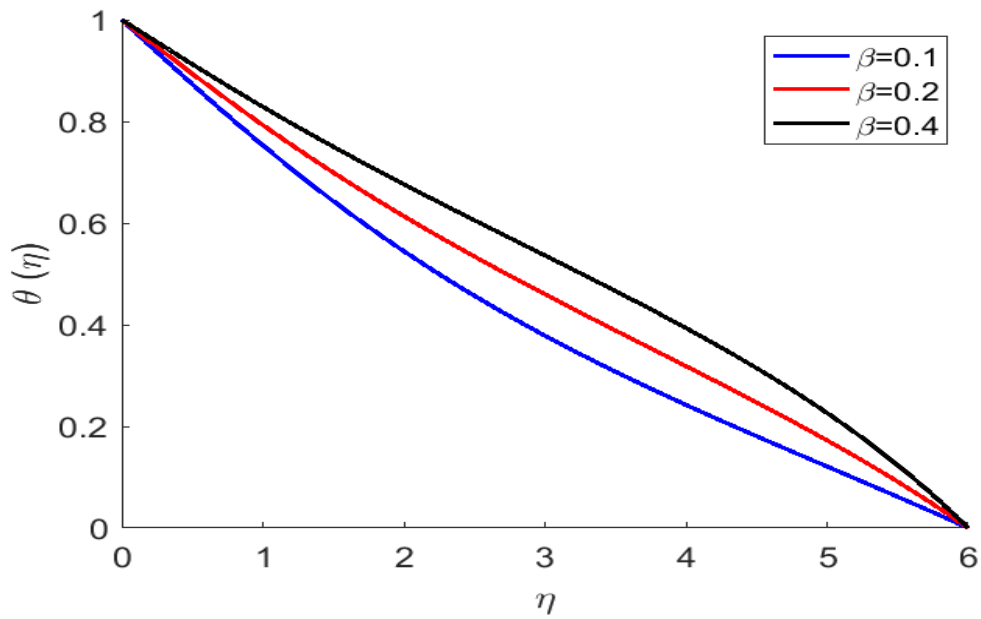


Figure 5.17 effects of casson parameter  $\beta$  on fluid temperature  $\theta$

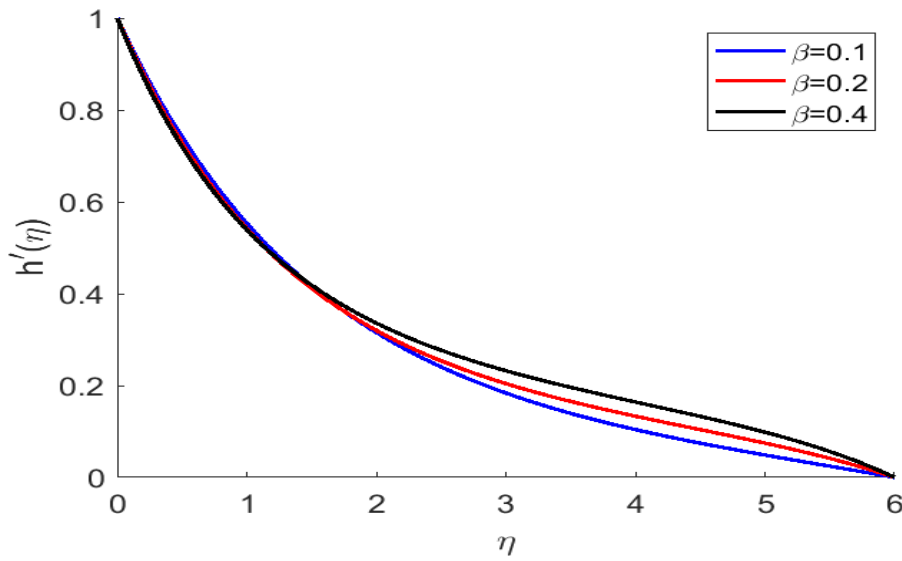


Figure 5.18: effects of casson parameter  $\beta$  on induced magnetic field  $h'$

$M = 1,4,7$      $A = 0,1,2$      $Nr = 1,2,4$      $B = 0.1,0.2,0.4$      $m = 1,2,4$

$pr = 0.71$      $tr = 0,1,2$      $pr_m = 0.1,0.2,0.4$      $Re = 1,5,10$

### 5.3 Effects of Hall current parameter $m$ on fluid velocity profile in $x$ -and $z$ -directions, fluid temperature $\theta$ and induced magnetic field $h'$

The **figure 5.19** shows effects of variation of Hall current parameter  $m$  on fluid velocity profile in  $x$ -direction. Evidently, changes in the Hall current parameter have little effect on fluid primary velocity. Increase in the Hall current parameter  $m$  leads to increase in primary velocity. Notably, Hall current is caused by presence of magnetic field  $B_0$  for electrically conducting Casson fluid.

Effects of variation of Hall current parameter  $m$  on secondary velocity is shown in **figure 5.20**. Increase in Hall current parameter  $m$  causes a significant increase in fluid velocity in  $z$ -direction. Since in this case Hall current arises due to MHD casson fluid flowing normally in a magnetic field, then increase in magnetic field leads to increase in Hall effect.

**Figure 5.21** shows effect of variation of Hall current parameter  $m$  on the fluid temperature  $\theta$ .

From the graph, the fluid temperature  $\theta$  decreases with increase in Hall current.

**Figure 5.22** shows effects of hall current  $m$  on induced magnetic field  $h'$ .

For a fixed  $\eta$ , increase in Hall current causes a decrease in induced magnetic field.

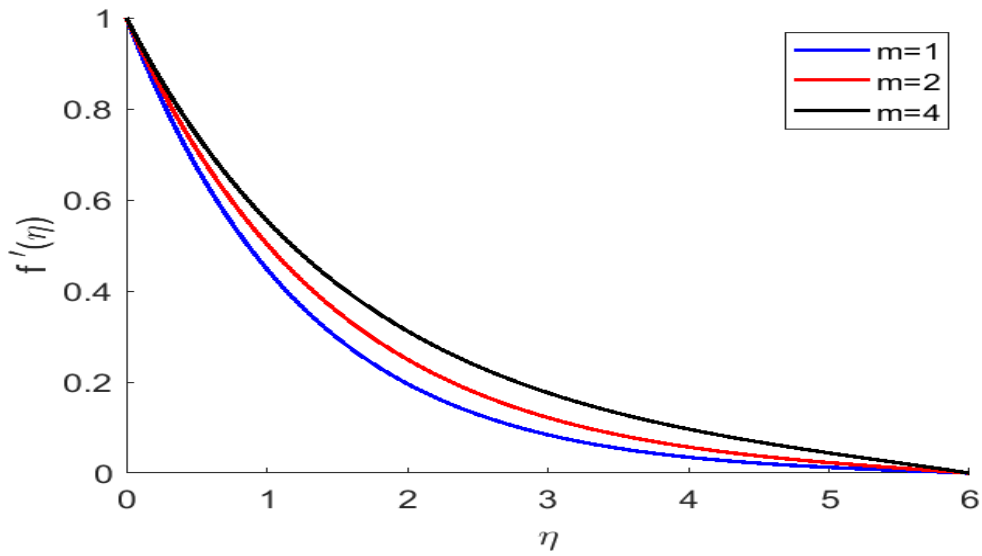


Figure 5.19: Effects of Hall current parameter  $m$  on fluid velocity profile in  $x$ -direction.

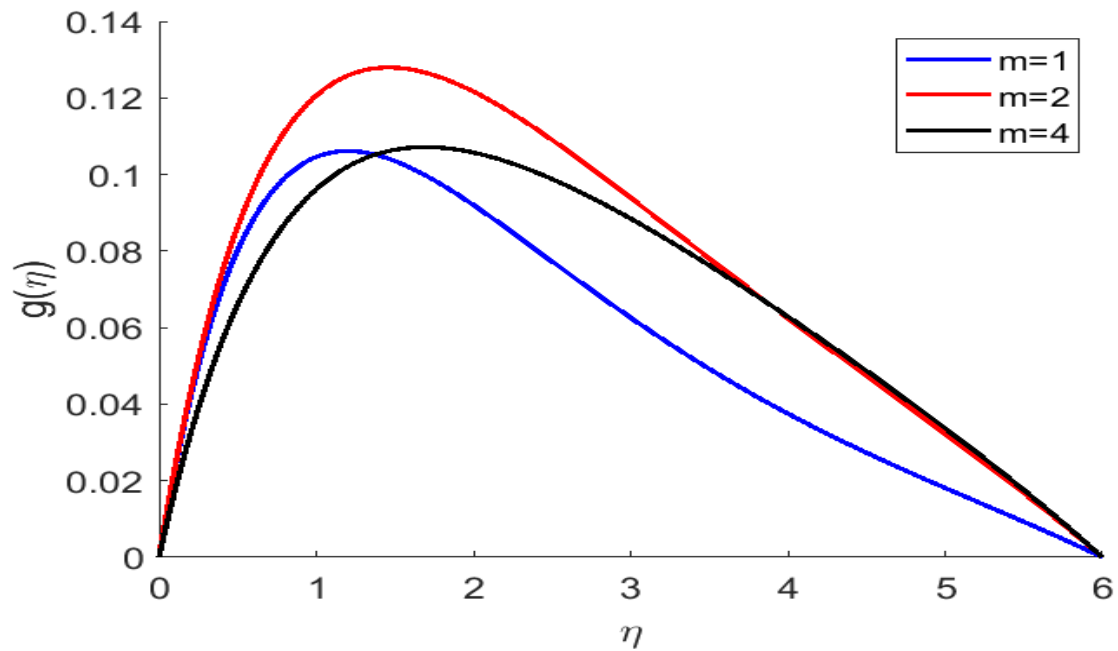
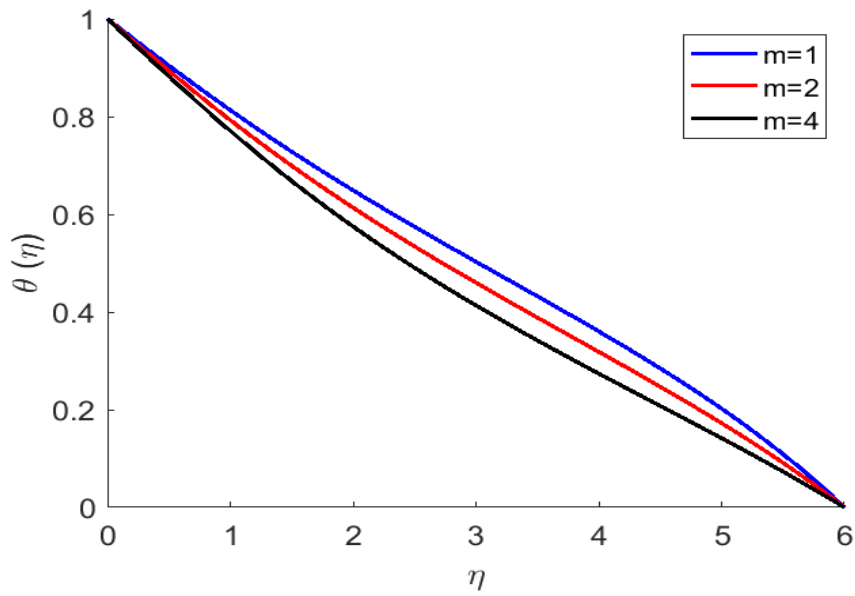
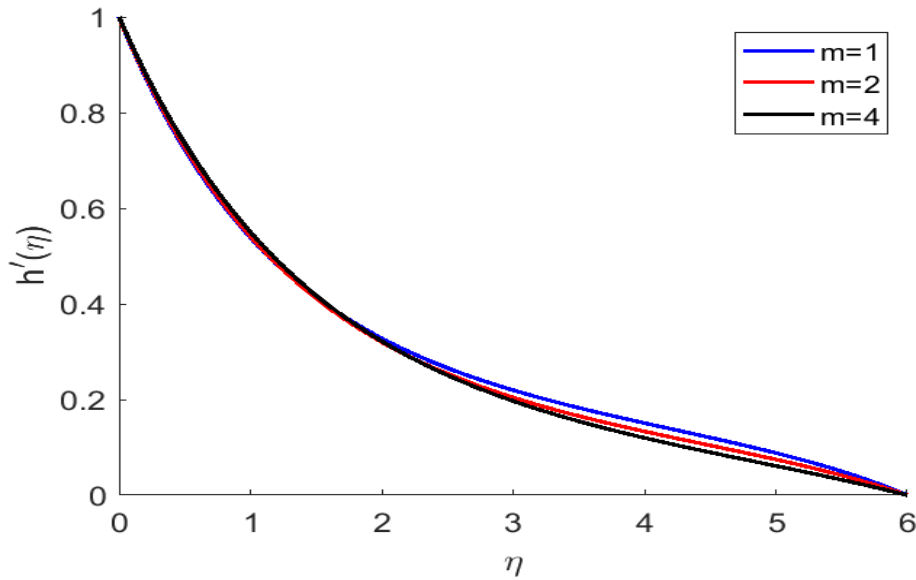


Figure 5.20: effects of Hall current parameter on fluid velocity in  $z$ -direction.



**Figure 5.21: Effects of Hall current parameter  $m$  on fluid temperature  $\theta$**



**Figure 5.22: Effects of variation of Hall current parameter  $m$  on induced magnetic field  $h'$**

$M = 1,4,7$      $A = 0,1,2$      $Nr = 1,2,4$      $B = 0.1,0.2,0.4$      $m = 1,2,4$

$pr = 0.71$      $tr = 0,1,2$      $pr_m = 0.1,0.2,0.4$      $Re = 1,5,10$

#### 5.4 Effects of unsteadiness parameter $A$ on fluid velocity profile in $x$ -and $z$ -directions, fluid temperature $\theta$ and induced magnetic field $h'$

Effects of variation of the unsteadiness parameter  $A$  on velocity profile in the  $x$  – direction is shown in **figure 5.23**. As seen in the figure, the unsteadiness parameter  $A$  has very marginal effect on velocity profile in the  $x$  – direction. This marginal effect is an initial decrease in velocity as steadiness parameter increases implying reduction of thickness of the momentum boundary layer near the wall in the  $x$  – direction. At  $\eta = 3$ , away from the wall, increase in unsteadiness parameter causes increase in primary velocity. This shows steady flow is eventually achieved at  $A = 0$ .

**Figure 5.24** shows effect of the unsteadiness parameter in the  $z$  – direction. Like in the  $x$  – direction, the unsteadiness parameter  $A$  has very marginal effect on secondary velocity. The secondary velocity decrease marginally with increase in unsteadiness parameter  $A$ . This shows there is also decrease in momentum boundary layer thickness in this direction near the wall. At around  $\eta = 5$  away from the wall, secondary velocity increases with increase in unsteadiness parameter implying both secondary field and corresponding boundary layer also increases with increase in unsteadiness parameter.

The effect of unsteadiness parameter  $A$  on fluid temperature  $\theta$  is depicted in **figure 5.25**. As seen in the figure, temperature  $\theta$  increases with increase in unsteadiness parameter  $A$ . As the unsteadiness parameter increases, more heat is transferred from the sheet to the plate. This causes increase in thermal boundary layer thickness.

**Figure 5.26** shows the effect of the unsteadiness parameter  $A$  on induced magnetic field  $h'$ .

It shows that the induced magnetic field  $h'$  increases with the unsteadiness parameter  $A$ .

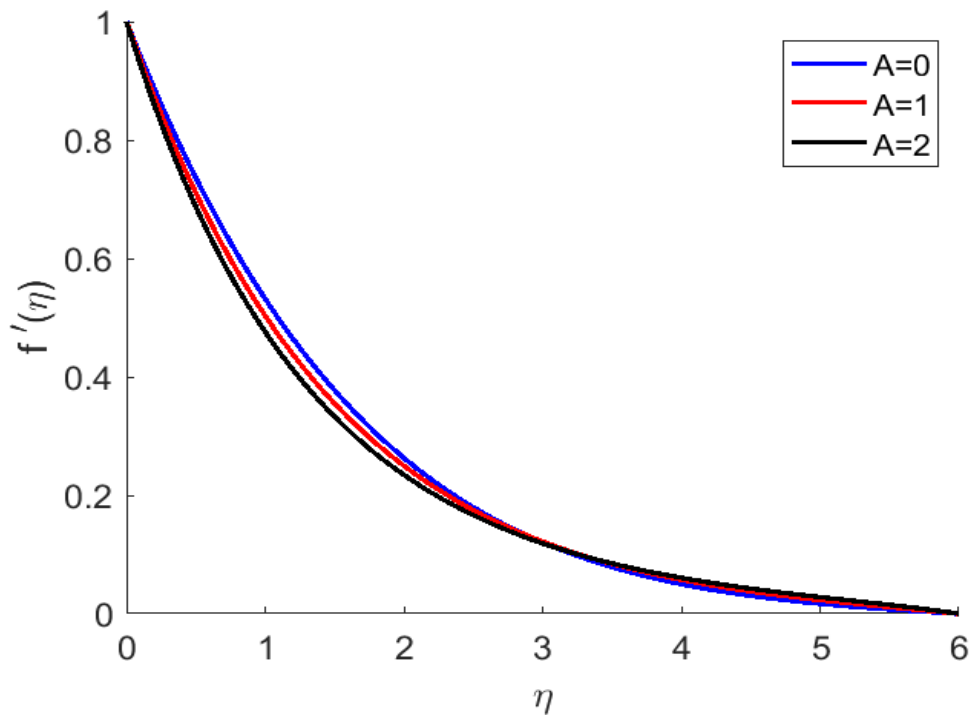


Figure 5.23: Effect of unsteadiness parameter  $A$  on fluid velocity profile in  $x$ -direction

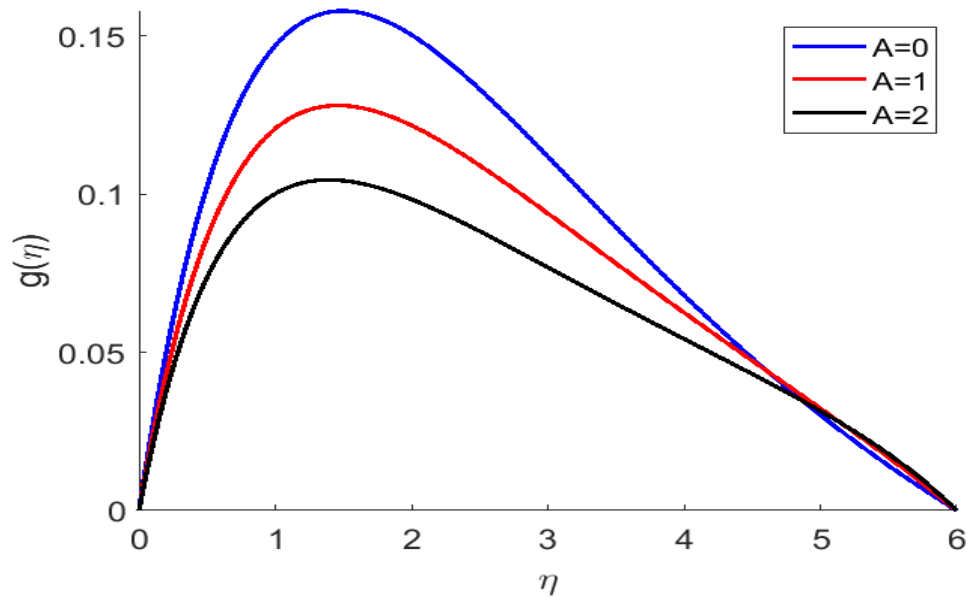
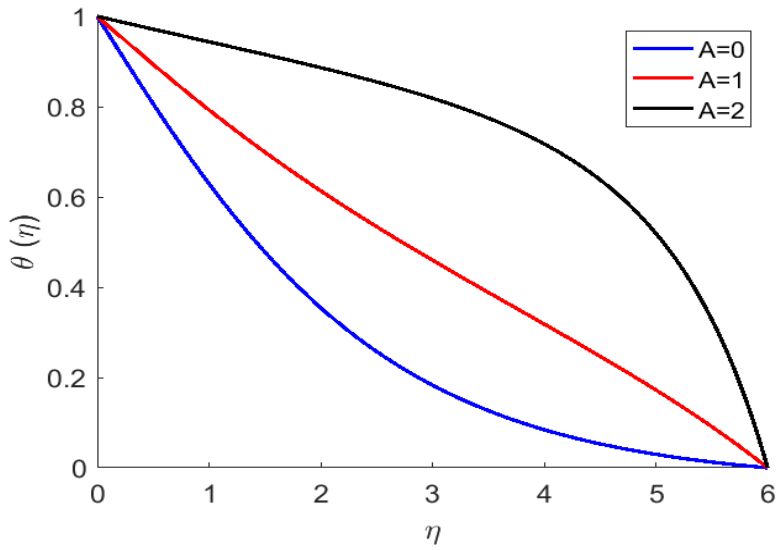
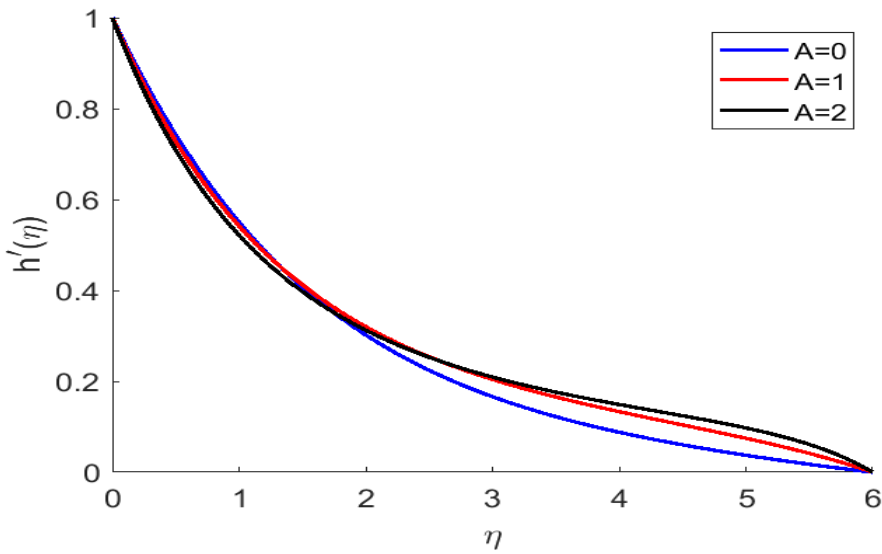


Figure 5.24: Effect of unsteadiness parameter  $A$  on the fluid velocity profile in  $z$ -direction



**Figure 5.25: Effect of unsteadiness parameter  $A$  on the fluid temperature  $\theta$ .**



**Figure 5.26: Effect of unsteadiness parameter  $A$  on the induced magnetic field  $h'$**

$M = 1,4,7$      $A = 0,1,2$      $Nr = 1,2,4$      $B = 0.1,0.2,0.4$      $m = 1,2,4$

$pr = 0.71$      $tr = 0,1,2$      $pr_m = 0.1,0.2,0.4$      $Re = 1,5,10$

### 5.5 Effects of radiation parameter $N_r$ , fluid on velocity profiles $f'$ and $g$ , fluid temperature $\theta$ and induced magnetic field $h'$ .

Figures 5.27 and 5.28 shows the radiation parameter  $N_r$  has no effect on both primary and secondary velocity profiles and therefore is negligible.

The effect of radiation parameter on fluid temperature  $\theta$  is captured in figure 5.29.

Notably, as the radiation parameter  $N_r$  is increased, the temperature of the fluid is found to

decrease within the boundary layer region. This agrees by its definition,  $Nr = \frac{k\alpha^*}{4\sigma^*T_\alpha^3}$  such that

there will be increase in radiation parameter when ambient temperature decreases. Furthermore

Since impact of radiation is inversely proportional to radiation parameter  $N_r$ , then we can

conclude that increase in thermal radiation causes increase in temperature of the fluid given that

decrease in radiation parameter  $N_r$  corresponds to increase in radiation and vice versa.

Figure 5.30 shows that radiation parameter has no effect on induced magnetic field  $h'$  and therefore is negligible.

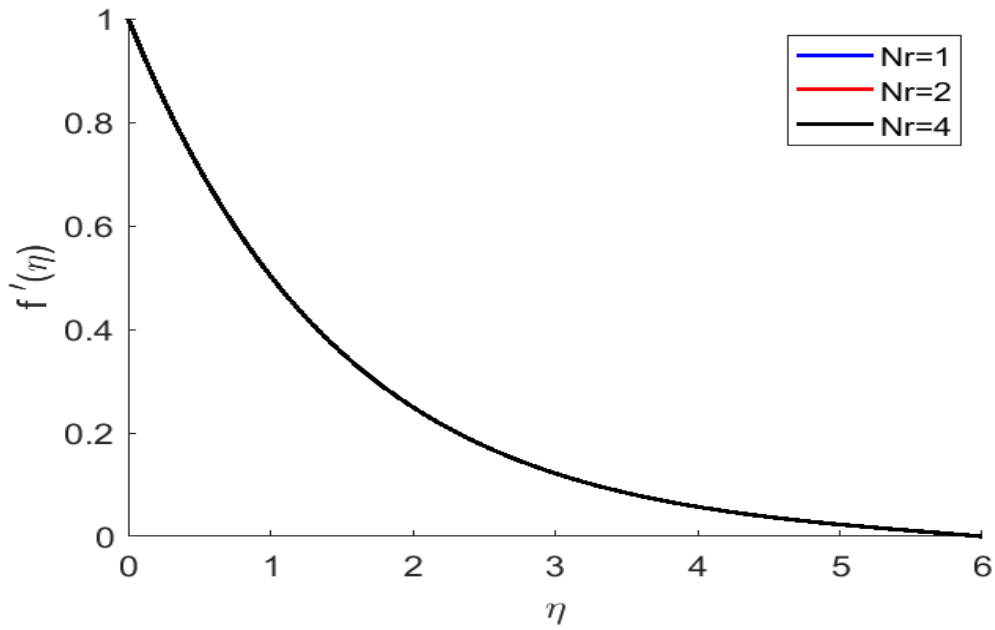


Figure.5.27 effects of radiation parameter  $N_r$  on velocity profile in the  $x$  – direction.

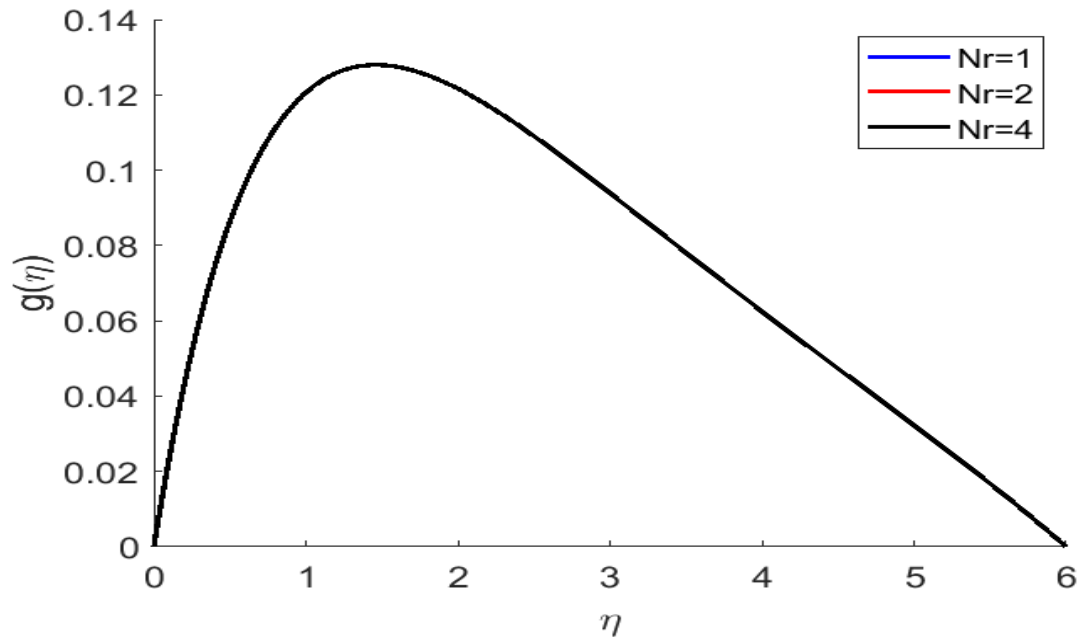
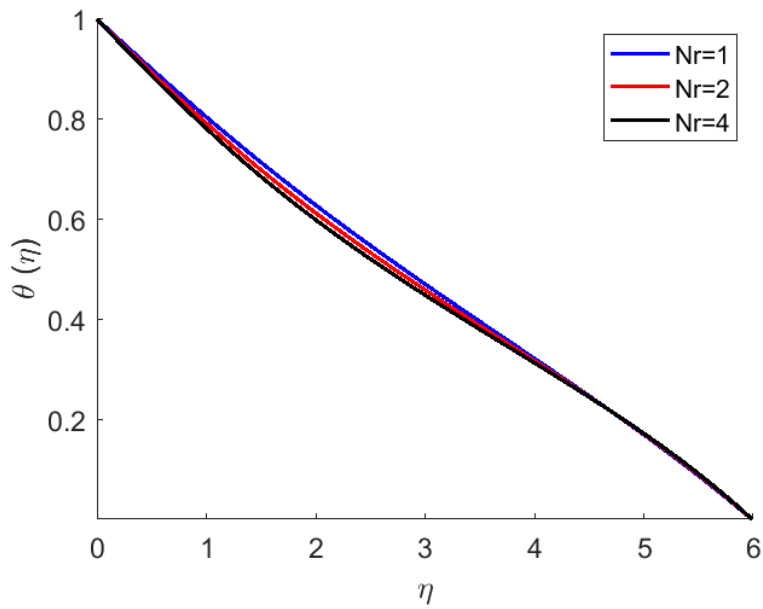
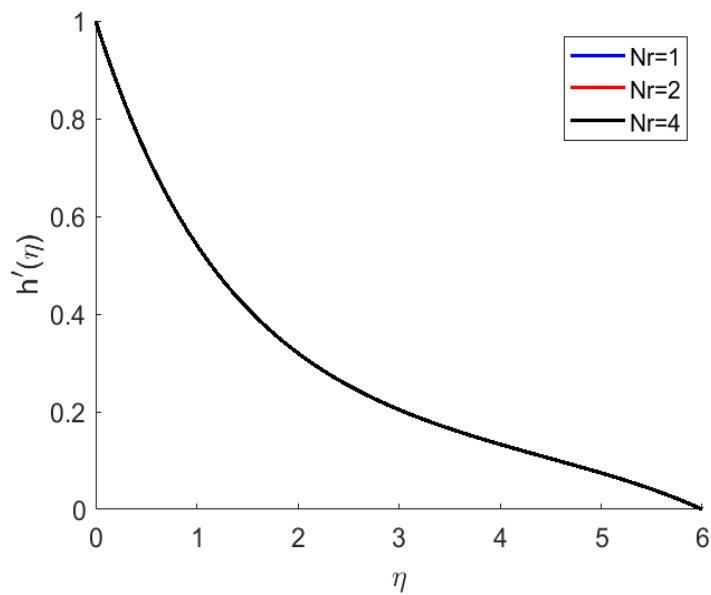


Figure.5.28 effects of radiation parameter  $N_r$  on velocity profile in the  $z$  – direction.



### 5.29 effects of radiation parameter $N_r$ on fluid temperature $\theta$ .



### 5.30 Effects of radiation parameter $N_r$ on induced magnetic field $h'$

$M = 1,4,7$      $A = 0,1,2$      $Nr = 1,2,4$      $B = 0.1,0.2,0.4$      $m = 1,2,4$

$pr = 0.71$      $tr = 0,1,2$      $pr_m = 0.1,0.2,0.4$      $Re = 1,5,10$

### 5.6 Effects of temperature ratio parameter $tr$ on velocity profiles $f'$ and $g$ , fluid temperature $\theta$ and induced magnetic field $h'$

Figures 5.31 and 5.32 show that the temperature ratio parameter  $tr$  has no effect on both primary and secondary velocity and are negligible.

From figure 5.33, it is evident that increase in the temperature parameter  $tr$  causes the fluid temperature to increase. By definition, Temperature ratio parameter  $tr$  is the ratio of fluid

temperature at the surface to fluid temperature beyond boundary layer region; that is  $tr = \frac{T_w}{T_\infty}$ .

This shows that temperature of the fluid becomes less as you move from the surface into the fluid beyond boundary layer region.

Figure 5.34 shows that the temperature ratio parameter  $tr$  has no effect on induced magnetic field and can be considered negligible.

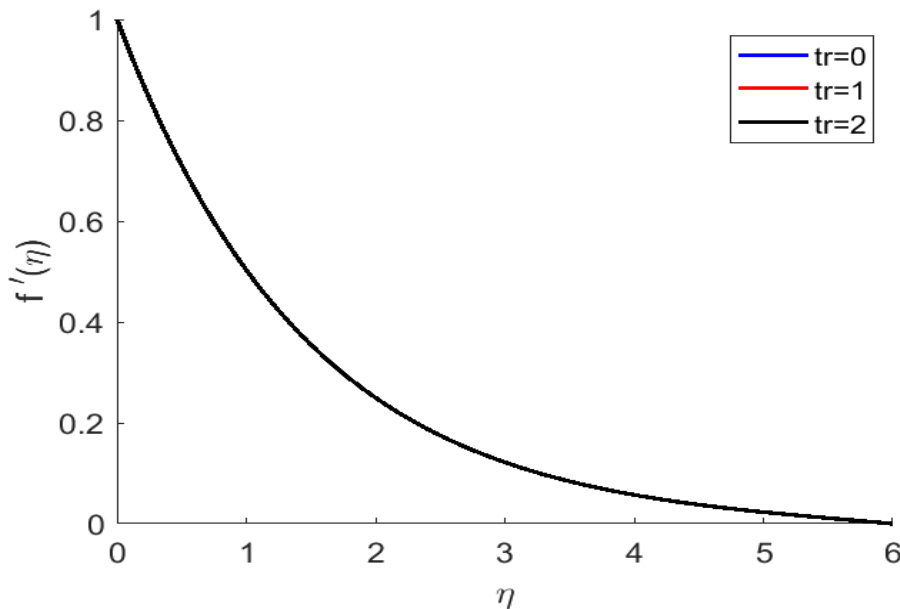
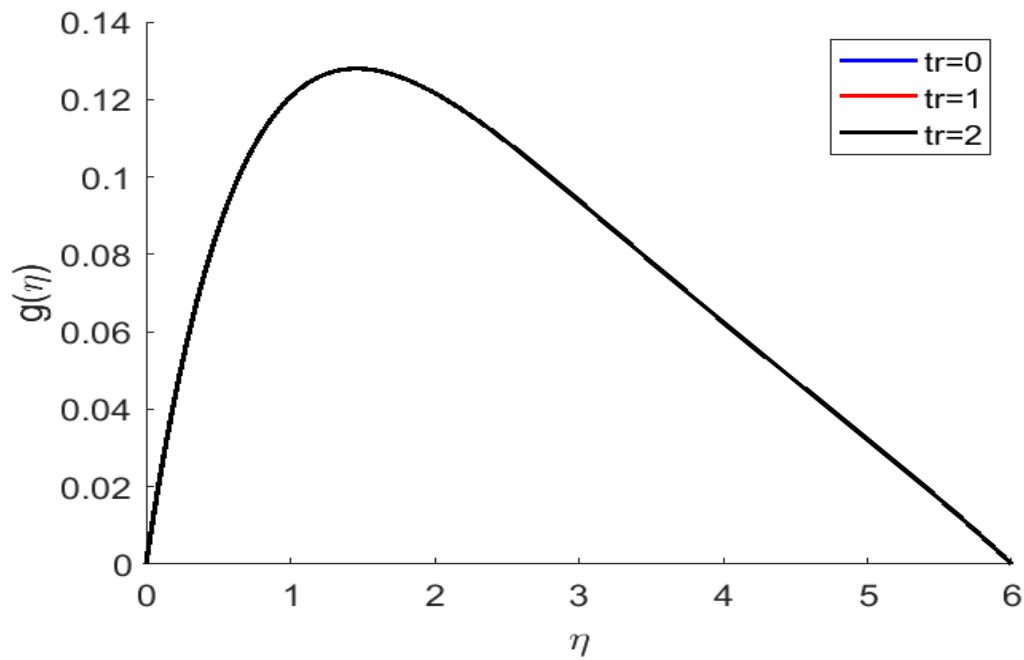
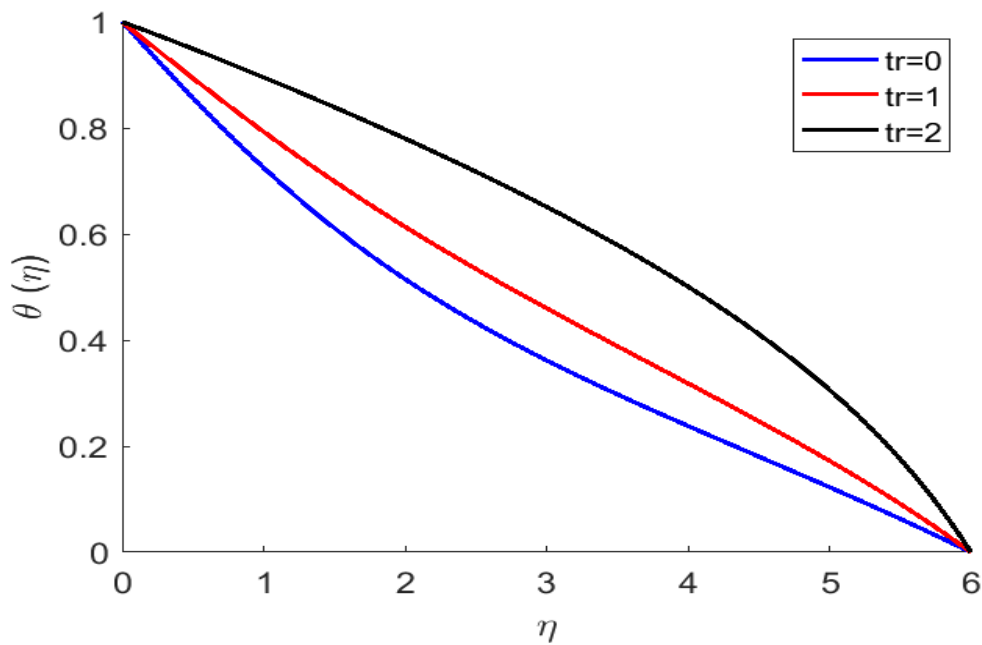


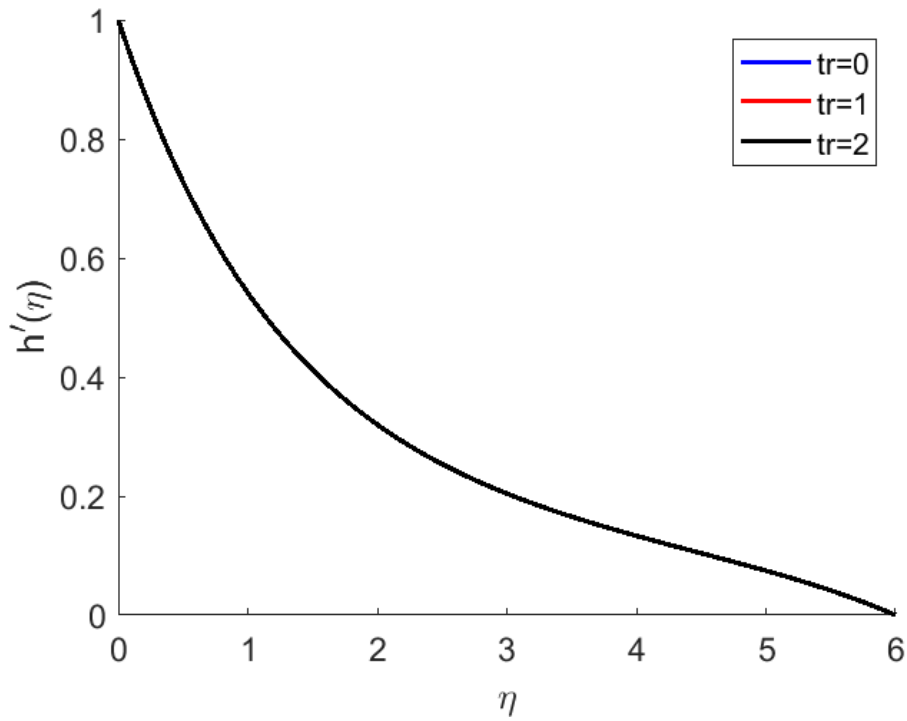
Figure 5.31. Effects of temperature ratio parameter  $tr$  on velocity profile in the  $x$  – direction



**Figure 5.32** Effects of temperature ratio parameter  $tr$  on velocity profile in the  $z$  – direction.



**Figure 5.33:** Effects of temperature ratio  $tr$  parameter on fluid temperature  $\theta$ .



**Figure 5.34. Effects of temperature ratio  $tr$  parameter on induced magnetic field  $h'$**

$M = 1,4,7$        $A = 0,1,2$        $Nr = 1,2,4$        $B = 0.1,0.2,0.4$        $m = 1,2,4$

$pr = 0.71$        $tr = 0,1,2$        $pr_m = 0.1,0.2,0.4$        $Re = 1,5,10$

### 5.7 Effects of magnetic Prandtl number $pr_m$ on velocity profiles $f'$ and $g$ , fluid temperature $\theta$ and induced magnetic field $h'$ .

Magnetic Prandtl number  $pr_m$  has no effect on primary velocity profile as shown in **figure 5.35**.

It is however seen to reduce the secondary velocity of the fluid in near boundary layer region as depicted in **figure 5.36**. This is attributed to the limited magnetic diffusion rate which in turn causes the viscous effects to dominate the flow.

As shown in **figure 5.37**, magnetic Prandtl number  $pr_m$  is found to have no effect on the fluid temperature  $\theta$  and is therefore considered negligible.

**Figure 5.38** shows there is decrease in induced magnetic field as magnetic Prandtl number  $pr_m$  increases. This may indicate that induced magnetic flux arises in the  $z$  – direction into the boundary layer and traverses into the plate. Notably, the values picked for magnetic Prandtl number  $pr_m$  are less than unity and since by definition,  $pr_m = \frac{\nu}{\eta}$  then it implies that the magnetic diffusion rate is higher than viscous diffusion rate.

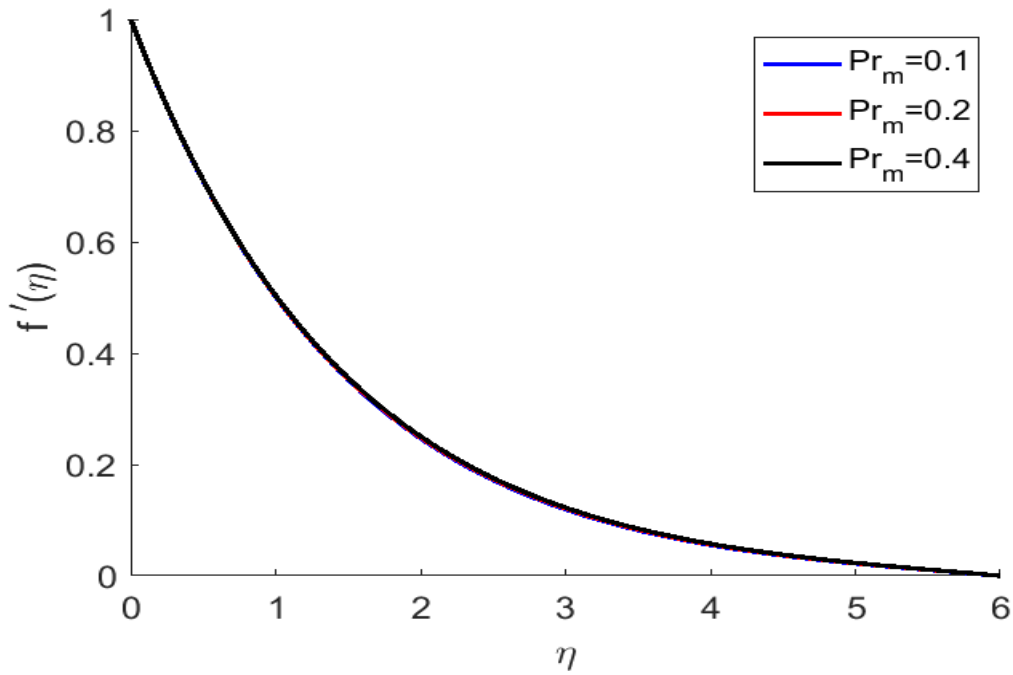


Figure 5.35. Effects of magnetic Prandtl number  $Pr_m$  on velocity profiles in  $x$  – direction

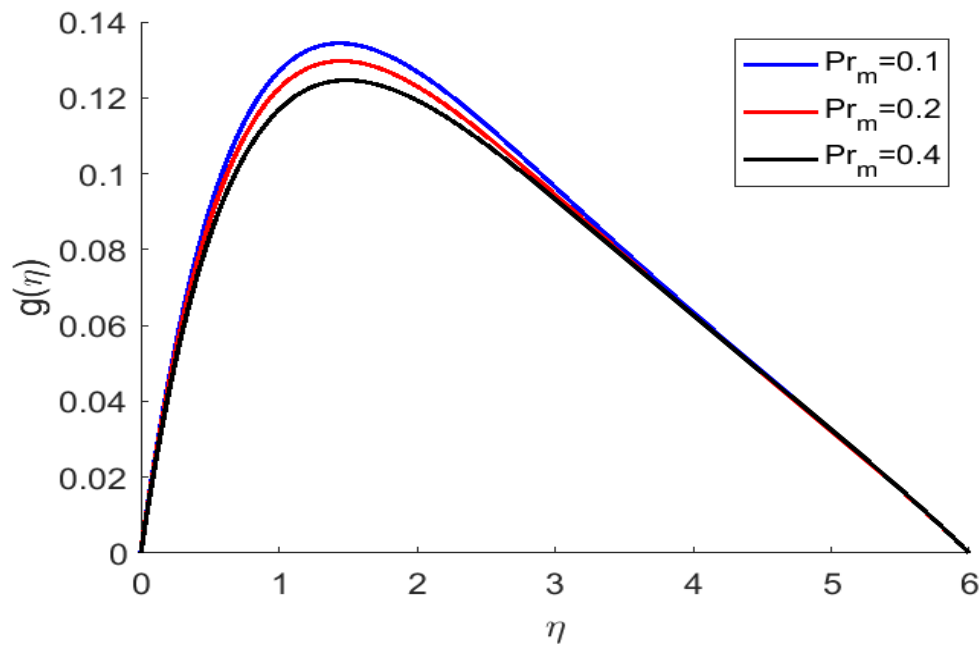
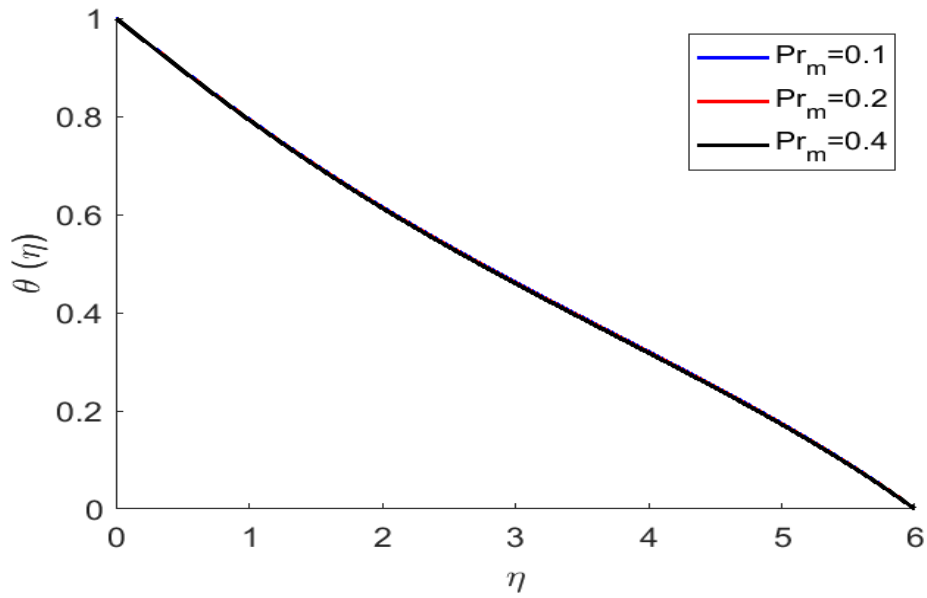
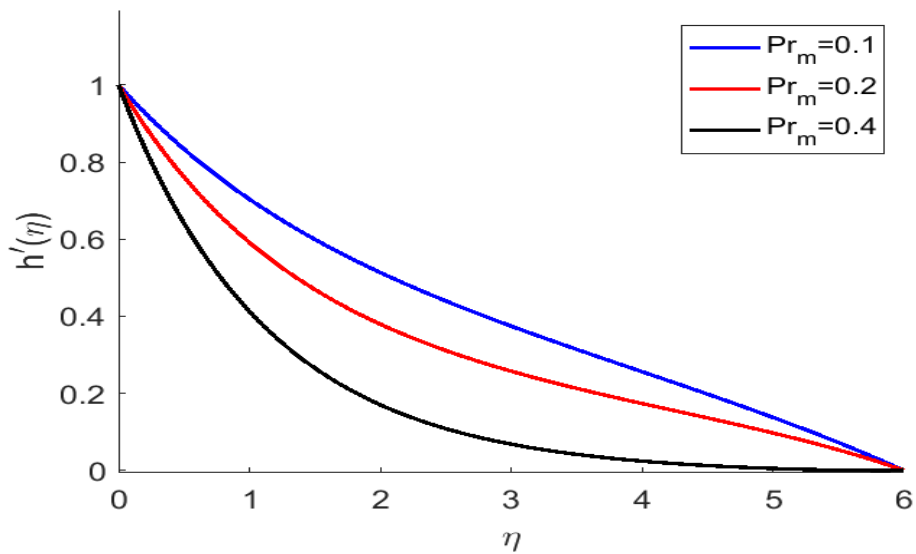


Figure 5.36. Effects of magnetic Prandtl number  $Pr_m$  on velocity profiles in  $z$  – direction



**Figure 5.37.** Effects of magnetic Prandtl number  $pr_m$  on fluid temperature  $\theta$



**5.38:** Effects of magnetic Prandtl number  $pr_m$  on induced magnetic field  $h'$

$M = 1,4,7$      $A = 0,1,2$      $Nr = 1,2,4$      $B = 0.1,0.2,0.4$      $m = 1,2,4$

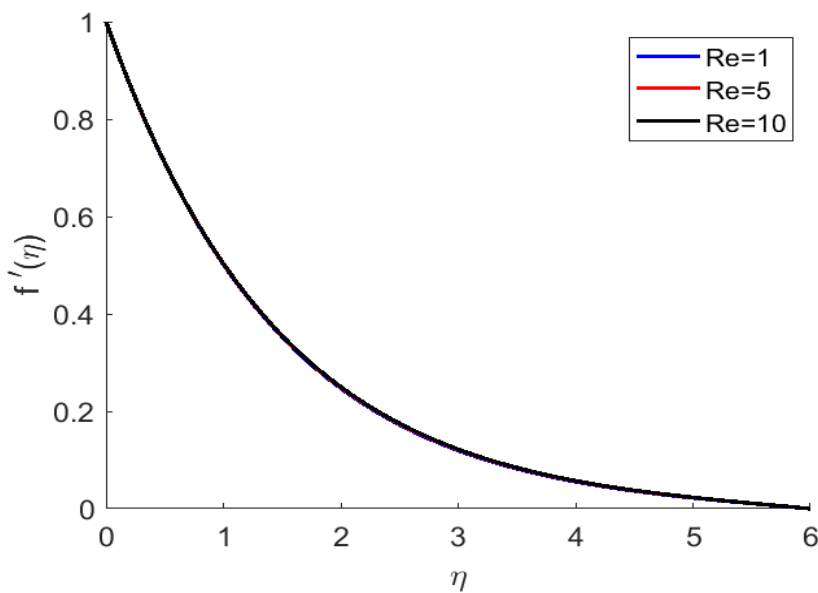
$pr = 0.71$      $tr = 0,1,2$      $pr_m = 0.1,0.2,0.4$      $Re = 1,5,10$

### 5.8 Effects of local Reynolds number $Re_x$ on fluid velocity profiles $f'$ and $g$ , fluid temperature $\theta$ , and induced magnetic field $h'$ .

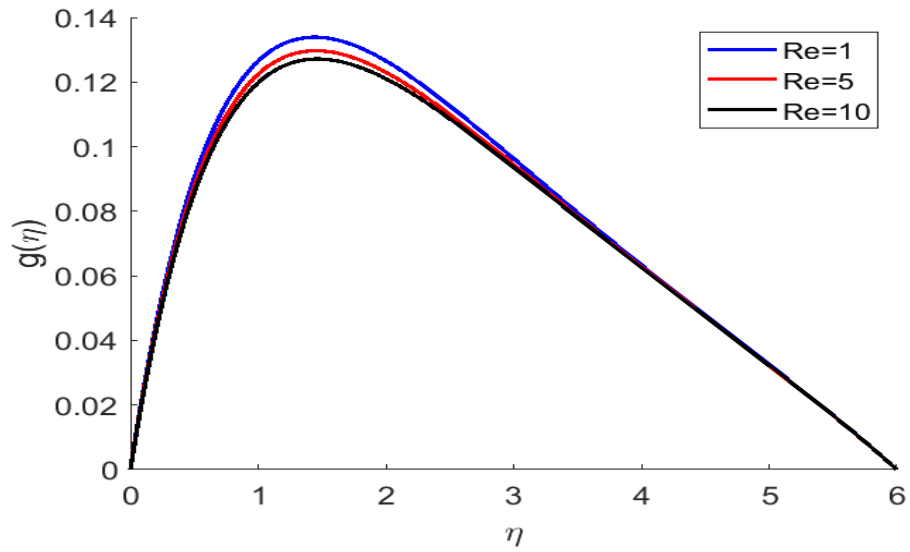
From **figure 5.39**, Reynolds number  $Re_x$  has no effect on primary velocity but is found to have a decreasing effect on the secondary velocity. An increase in Reynolds number causes the secondary velocity to decrease as seen in **figure 5.40**. Due to the transverse magnetic field  $B_0$  applied to the fluid, there is generation of resistive Lorentz force than reduces the secondary momentum boundary layer thickness. This therefore reduces the secondary velocity of the fluid near the wall.

The local Reynolds number is found to have no effect on the fluid temperature profile  $\theta$  as seen in **figure 5.41**.

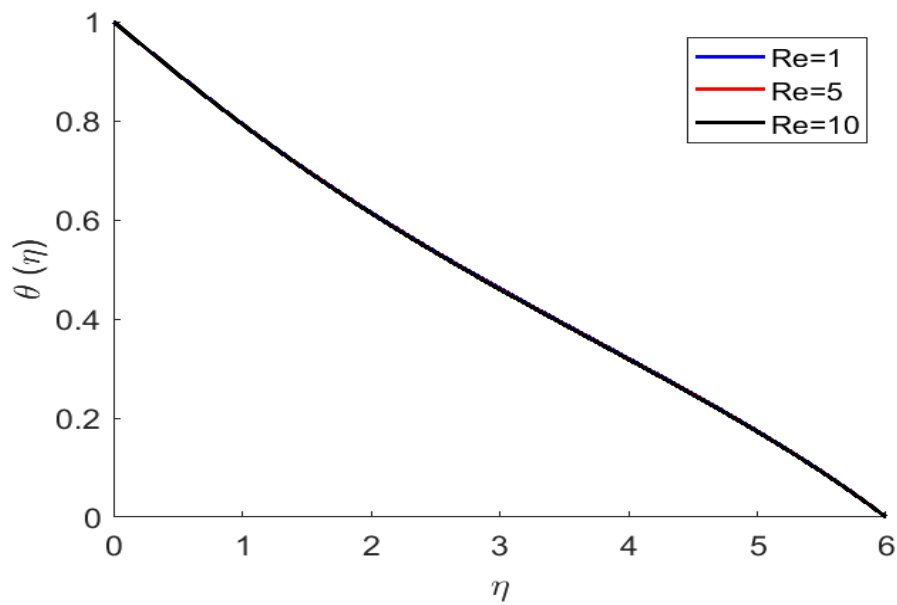
From **figure 5.42**, increasing local Reynolds number  $Re_x$  leads to decrease in induced magnetic field  $h'$ .



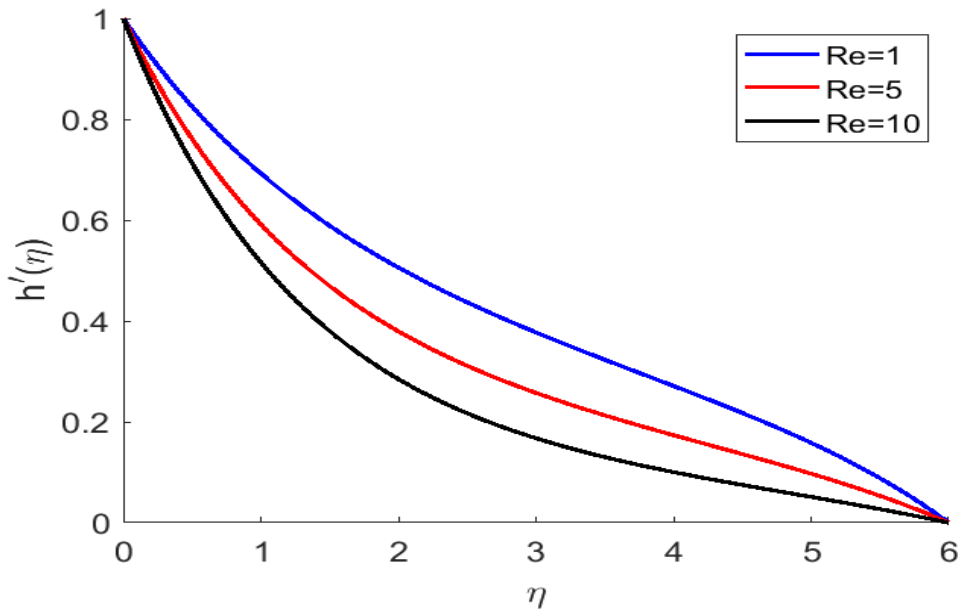
**Figure 5.39.** Effects of local Reynolds number  $Re_x$  fluid on velocity profiles in  $x$  – direction



**Figure 5.40** Effects of local Reynolds number  $Re_x$  on fluid velocity profiles in  $z$  – direction



**Figure 5.41** Effects of Reynolds number  $Re_x$  on fluid temperature  $\theta$



**Figure 5.42** Effects of local Reynolds number  $Re_x$  on induced magnetic field  $h'$

$M = 1,4,7$        $A = 0,1,2$        $Nr = 1,2,4$        $B = 0.1,0.2,0.4$        $m = 1,2,4$

$pr = 0.71$        $tr = 0,1,2$        $pr_m = 0.1,0.2,0.4$        $Re = 1,5,10$

### 5.90. Conclusion

Combined effects of induced magnetic field, Hall current, and radiative heat on a 3-D time dependent unsteady hydro-magnetic Casson fluid flow along a stretching sheet are investigated.

The flow model is comprised of highly non-linear PDEs. Similarity transformation is done to convert the PDEs to ODEs in order to adequately represent the boundary layer. Lastly, a numerical approach known as collocation method is used to solve these ODEs. Analysis of effects of parameters such as Hall current parameter, Magnetic parameter, Casson parameter, Unsteadiness parameter, Radiation parameter, Magnetic parameter, Reynolds number and magnetic Prandtl number on velocity fluid profiles, temperature and induced magnetic field is done and presented graphically. The following conclusions are therefore drawn from the numerical investigation done.

- i) The momentum boundary layer thickness increases with increase in Hall current parameter but decreased with increase in Casson parameter and magnetic parameter.
- ii) Magnetic Prandtl number and Reynolds number are decreasing functions of secondary velocity but have no impact on primary velocity
- iii) Increase in magnetic parameter and Casson parameter increases induced magnetic field whereas Prandtl number and Reynolds number decreases induced magnetic field.
- iv) Thermal boundary layer thickness is enhanced by Casson parameter and magnetic parameter but is reduced by Hall current parameter.

### **5.91 recommendations for further research**

1. In future, a study on effects of Hall current, thermal radiation and induced magnetic field on MHD flow of a Casson fluid along a stretching/shrinking wedge can be studied.
2. In this study, fluid of interest is non-Newtonian Casson fluid. In future, effects of Hall current, thermal radiation and induced magnetic field on unsteady micropolar fluid can be studied.

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