SOME INVESTIGATIONS ON SINGULAR CAUCHY PROBLEMS

By

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DECLARATION

THIS THESIS IS MY ORIGINAL WORK AND HAS NOT BEEN PRESENTED FOR A DEGREE IN ANY OTHER UNIVERSITY OR ANY OTHER AWARD.

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DEDICATION

To my lovely family.

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I owe a great deal to the late Professor Morris Sika Alala, who introduced me to the Singular Cauchy Problem with characteristics along with both the time and space coordinates.

I register a special note of thanks to Dr. Charles Nyandwi, who first introduced me to the theory of distributions and Sobolev spaces that have made it possible for me to do research. I thank him for his patience to have taken quite a spell of time to supervise me and to proof my work.

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Last but not least I thank my wife Knight and my children for their moral support and forbearance.

ABSTRACT

The purpose of our study is to get a solution to the Cauchy problem of:

(i) The wave equation in n-dimension space $\mathbb{R}^n$ which is effectively a good example of **regular Cauchy problems**

(ii) The Euler Poisson Darboux equation which we call **singular Cauchy problem** by use of Riemann's method.

The Riemann-Green function for each case is calculated, which enables us to evaluate any solution at a point by the Cauchy data on a non-characteristic curve.

In case (i) the Riemann-Green function is in terms of Legendre polynomial and the solution obtained is shown to solve the wave equation as well.
In case (ii) the Riemann-Green function written in terms of the Appell's hyper geometric function of two variables is arrived at, this is of interest and may be a good model for a more general theory.

A discussion of the generalized singular Cauchy problem of Euler-Poisson-Darboux equation is included and found to have solution that is continuous and analytic over the interval that contains the singular point.

TABLE OF CONTENTS

CHAPTER ONE

§ 1.0 **Introduction**

The purpose of this chapter is to define and explain the terminologies that will be used throughout the manuscript.

§ 1.1 **Notations and definitions**

Manifolds: curves, surfaces and hyper surfaces.

Any consideration of a partial deferential equation draws heavily upon geometrical concepts. Thus we explain certain terms related to geometrical structures in a Euclidean n-space.

In the xy-plane a curve is replaced by a parametric equation $x = x(t), y = y(t); a \le t \le b$ where $x(t), y(t)$ are continuous functions of a real parameter t. In a special case the parameter may be arc lengths. The parametric representations of a given curve are not unique.

In $\mathbb{R}^3$ equations $x = x(t), y = y(t), z = z(t)$ denote a space curve. The basic idea of a space curve is that of a twisted wire which can be unwounded into the shape of a linear interval.

In the three -dimensional space $\mathbb{R}^3$, a surface may be represented by two parameters u, v as

$$x = x(u,v), y = y(u,v), z = z(u,v)$$

provided $$\frac{\partial(x,y)}{\partial(u,v)} \neq 0 \;; \qquad \frac{\partial(y,z)}{\partial(u,v)} \neq 0$$

A curve $x_i = x_i(t)(i = 1,2,...,n), a \le t \le b$ in n-space is called a c^n curve if each $x_i(t) \in c^n$.

In general if $m < n$, then for real independent parameters $s_1, s_2, ..., s_m$, the equations

$$\bar{x} = \bar{x}(s_1, s_2, ..., s_n) \quad \text{i.e.} \quad x_i = x_i(s_1, s_2, ..., s_n) \qquad (i = 1,2,...,n) \qquad (1.1.1)$$

represent a geometrical structure called an m-dimensional manifold or surface in the n-space.

If $m = n-1$, the manifold is called a hyper surface. A curve is a one-dimensional manifold.

If, $\bar{x} \in \mathbb{R}^n$, then the equation

$$a(\bar{x},\bar{x}) + 2b(\bar{x}) + c = 0 \tag{1.1.2}$$

is called a quadratic hyper surface, where $a(\bar{x},\bar{x})$ is a quadratic form, $b(\bar{x})$ is a linear form and c is a constant. If the running coordinates of a point $\bar{x}$ are $x_1, x_2, \ldots, x_n$, then (1.1.2) can be written in the form

$$\sum_{i,j}^{n} a_{ij} x_i x_j + 2\sum_{i=1}^{n} b_i x_i + c = 0; \quad i,j = 1 \tag{1.1.3}$$

In our work we shall designate a quadratic hyper surface simply as a surface.

The canonical form

$$x_1^2 + x_2^2 + \ldots + x_n^2 = r^2$$

is the equation of a sphere center origin and radius r. The equation

$$\left|\bar{x} - \bar{a}\right|^2 = (x_1 - a_1)^2 + \ldots + (x_n - a_n) = r^2$$

is the sphere with radius r and center $\bar{a} = (a_1, a_2, \ldots, a_n)$.

An ordinary differential equation is supposed to be defined on interval I. On the same lines solutions of partial differential equations are defined in a region T. For example, T is an n-dimensional region in the n-space $\mathbb{R}^n$ for the Lap lace's equation and $\bar{S} = (x_1, x_2, \ldots, x_n)$. For the wave and diffusion equation T is an $(n+1)$-dimensional region of n-space coordinates $x_1, x_2, \ldots, x_n$ and one time coordinate t.

A point $P(\bar{a})$ is said to be interior to a region T if it id possible to draw a sphere $\sum(x_i - a_i)^2 = r^2$ such that all points $\bar{x} = (x_1, x_2, \dots, x_n)$ for which $\sum(x_i - a_i)^2 < r^2$ belong to T. P is called an exterior point of T if there exists a sphere centered at p none of whose interior points belong to T. If every sphere about p contains both interior and exterior points, then P is called a boundary point of T. The set of all boundary points of T is called the boundary of T and is denoted by ∂T. The set of all points in T and ∂T, denoted by $\bar{T} = T \cup \partial T$ is called the closure of T. A region T is called open if all its points are interior points.

A region T is called bounded if it can be contained in the interior of a ball of finite radius. T is said to be convex (or connected or arc wise connected) if any two points of T can be joined by an arc of a curve all of whose points belong to T. An open connected region is called a domain.

A region is called simply connected if every closed curve lying in the region can be shrunk continuously to a point without leaving the region. As an example, a ball of finite radius is a bounded simply connected region.

§ 1.2 **Measures and Integrals**

The integral of a function f over a subset Ω of $\mathbb{R}^n$ with respect to Lebesgue measure will be denoted by $\int_\Omega f(x)dx$ or simply by $\int_\Omega f$. If no subscript occurs on the integral sign, the region of integration is understood to be $\mathbb{R}^n$. If S is a smooth hyper surface the natural Euclidean surface measure on S will be denoted by $d\sigma$; thus the integral of f over S is $\int_S f(x)d\sigma(x)$.

§ 1.3 **Multi-indices and Derivatives**

An n-tuple $\alpha = (\alpha_1, \dots, \alpha_n)$ of nonnegative integers will be called a multi-index. We define

$$|\alpha| = \sum_{1}^{n} \alpha_{j\Leftarrow}, \alpha ! = \alpha_1 ! \alpha_2 ! ... \alpha_n \text{ and for } x \in \mathbb{R}^n, \quad x^\alpha = x_1^{\alpha_1} x_2^{\alpha_2} ... x_n^{\alpha_n}.$$

We will generally use the short hand $\partial_j = \partial / \partial x_j$ for derivatives on $\mathbb{R}^n$. Higher-order derivatives are then conveniently expressed by multi-indices

$$\partial^\alpha = \prod_{1}^{n} \left(\frac{\partial}{\partial x_j} \right) \alpha_j = \frac{\partial^{|\alpha|}}{\partial x_1^{\alpha_1} ... \partial x_n^{\alpha_n}}.$$

In particular if $\alpha = 0, \partial^\alpha$ is the identity operator. We denote by ∇u the n-tuple of functions ($\partial_1 u, ..., \partial_n u$) when u is a differential function.

§ 1.4 **Results from Advanced Calculus**

Every $x \in \mathbb{R}^n \setminus \{0\}$ can be written uniquely as $x = ry$ with $r > 0$ and $y \in S_1(0)$-namely, $r = |x|$ And $y = x / |x|$. The formula $x = ry$ is called the polar coordinate representation of x. Lebesgue measure is given in polar coordinates by $dx = r^{n-1} dr d\sigma(y)$, where $d\sigma$ is surface measure on $S_1(0)$.

For example if $0 < a < b < \infty$ and $\lambda \in \mathbb{R}$, we have

$$\int_{a<|x|<b} |x|^\lambda dx = \int_{S_1(0)} \int_a^b r^{n-1+\lambda} dr = \begin{cases} \omega_n \dfrac{b^{n+\lambda} - a^{n+\lambda}}{n+\lambda}, & if \lambda \neq -n \\ \omega_n \log\left(\dfrac{b}{a}\right) if \lambda = -n \end{cases}$$

where ω_n is the area of $S_1(0)$ (which we shall compute shortly).

Proposition

$$\int e^{-\pi|x|^2}dx = 1.$$

Proof; Let $I_n = \int_{\mathbb{R}^n} e^{-\pi|x|^2}dx$. Since $e^{-\pi|x|^2} = \Pi_1^n e^{-\pi|x_j|^2}$,

Fubini's theorem shows that $I_n = (I_1)^n$ or equivalently that $I_n = (I_2)^{n/2}$. But in polar coordinates

$$I_2 = \int_0^{2\pi}\int_0^{\infty} e^{-\pi r^2} r\,dr\,d\theta = 2\pi\int_0^{\infty} re^{-r^2}dr = \pi\int_0^{\infty} e^{-\pi s}ds = 1.$$

Proposition

The area of $S_1(0)$ in $\mathbb{R}^n$ is

$$\omega_n = 2\pi^{n/2}/\Gamma\left(n/2\right)$$

***Proof*:** We integrate $e^{-\pi|x|^2}$ in polar coordinates and set $s = \pi r^2$

$$1 = \int e^{-\pi|x|^2}dx = \int_{S_{1(0)}}\int_0^{\infty} e^{-\pi r^2} r^{n-1}dr$$

$$= \omega_n \int_0^{\infty} e^{-\pi r^2} r^{n-1}dr$$

$$= \frac{\omega_n}{2\pi^{n/2}}\int_0^{\infty} e^{-s}s^{(n/2)-1}ds$$

$$= \frac{\omega_n \Gamma\left(n/2\right)}{2\pi^{n/2}}$$

where

$$\Gamma(s) = \int_0^{\infty} e^{-t}t^{s-1}dt \text{ (Gamma function); } \operatorname{Re} s > 0 \text{ and}$$

$$\Gamma(s+1) = s\Gamma(s), \quad \Gamma(1) = 1, \quad \Gamma\left(1/2\right) = \sqrt{\pi}$$

§ 1.5 **Basic Concepts on Partial Differential Equations of Order-k**

A partial differential equation of order k is an equation of the form

$$F\left(x,u,\partial_1 u,\ldots,\partial_n u,\partial_1^2 u,\ldots,\partial_n^k u\right)=0$$

Relating a function u of the variable $x\in\mathbb{R}^n$ and its derivatives of order $\leq k$. To write the equation a more compact form, we order the set of multi-indices by saying that α comes before β if $|\alpha|<|\beta|$ or $|\alpha|=|\beta|$ and $\alpha_i<\beta_i$ where i is the largest number with $\alpha_i\neq\beta_i$. Given complex numbers $a_\alpha\left(|x|\leq k\right)$; we denote by $\left(a_\alpha\right)_{|\alpha|\leq k}$ the element of $\mathbb{C}^{N(k)}$ given by ordering the $\alpha's$ in this fashion, where $N(k)$ is the cardinality of $\{\alpha:|\alpha|\leq k\}$. Similarly, if $S\subset\{\alpha:|\alpha|\leq k\}$, we can consider the ordered (Cards)-tuple $\left(a_\alpha\right)_{\alpha\in s.}$

Now let Ω be an open set in $\mathbb{R}^n$ and let F be a function of the variables $x\in\Omega$ and $\left(u_\alpha\right)_{|\alpha|\leq k}\in\mathbb{C}^{N|k|}$. Then we can form the partial differential equation

$$F\left\{x,\left(\partial^\alpha u\right)_{|\varepsilon|\leq k}\right\}=0 \qquad (1.5.1)$$

A (complex-valued) function $u=u(x)$ on Ω is a classical solution of this equation if the derivatives $\partial^\alpha u$ occurring in the F exist on Ω, and $F\left\{x,\left(\partial^\alpha u\right)_{|\varepsilon|\leq k}\right\}=0$ for all $x\in\Omega$..

The equation (1.4) is called linear if F is a linear function of the vector variable $\left(u_\alpha\right)_{|\alpha|\leq k}$ i.e. if (1.5.1) can be re- written as

$$\sum_{|\alpha|\leq k} a_{|\alpha|}(x)\partial^\alpha u=f(x) \qquad (1.5.2)$$

In this case we speak of the differential operator $L=\sum_{|\alpha|\leq k} a_\alpha \partial^\alpha$ and write (1.5.2) simply as $Lu=f$. More general than the linear equations are the quasi-linear equations, those equations (1.1.1) where F is a linear function of $(u_\alpha)_{|\alpha|=k}$. Such equations can be written as

$$\sum_{|\alpha|=k} a_\alpha \left\{x,\left(\partial^\beta u\right)_{|\beta|\leq k-1}\right\}\partial^\alpha u = b\left\{x,\left(\partial^\beta u\right)_{|\beta|\leq k-1}\right\} \tag{1.5.3}$$

In the linear case a simple measure of the 'strength' of a differential operator in a certain direction is provided by the notion of the characteristics. If $L=\sum_{|\alpha|\leq k} a_\alpha \partial^\alpha$ is a linear differential operator of order K $\Omega \subset \mathbb{R}^n$, its characteristic form at $x\in\Omega$ is the homogeneous polynomials of degree k on $\mathbb{R}^n$ defined by

$$\chi L[x,\xi]=\sum_{|\beta|=k} a_\alpha(x)\xi^\alpha \qquad \left(\xi\in\mathbb{R}^n\right)$$

A nonzero vector ξ is called characteristic for L at x if $\chi L(x,\xi)=0$ and the set of all such ξ is called the characteristic variety of L at x and is denoted by $Char_x(L)$:

$$Char_x(L)=\{\xi\neq 0 : \chi L(x,\xi)=0\}$$

A hyper surface S is called characteristic for L at $x\in S$ if the normal vector $\nu(x)$ to S at x is in $Char_x(L)$, and S is called non-characteristic if it is not characteristic at any point.

CHAPTER TWO

§ 2.0 LITERATURE REVIEW

The solution of the Cauchy problem

$$\Delta u = \partial_t^2 u + \frac{k}{t}\partial_t u \tag{2.0.1}$$

$$u(x_1, x_2, \ldots, x_m, 0) = f(x_1, x_2, \ldots, x_m), u(x_1, x_2, \ldots, x_m, 0) = 0 \tag{2.0.2}$$

is partially motivated by problem in physics, geometry, applied mathematics etc. Equation (2.0.1) for special values of k and m, occurs in many important and classical problems since the time of Euler (1770). He considered $m = 1$ and a partial differential equation {denoted by $E(\beta, \beta')$ by Darboux (1914-15) which is equivalent to (2.0.1) when $\beta = \beta' = k/2$. The equation (2.0.1) with $m = 1$ was later treated by Poisson (1823). An exposition of the theory of Euler (1770) and Poison (1823) is given by Darboux (1914-15).These treatments were not considered with the singular initial values (2.0.2). The important special case $m = 1,\ k = \frac{1}{3}$ of (2.0.1) - (2.0.2) plays an important role in the work of Tricomi (1923).

Poisson (1823), in solving the equation of the propagation of sound waves in 3-dimensional space considered the case $m = 3,\ \ k = 2$ in (2.0.1). Asgeirsson (1937) gave a solution of (2.0.1) - (2.0.2) for all positive integers m and $k = m - 1$. Related questions were treated by John (1934, 1935). Equation (2.0.1) for $m = 1,\ k = -1, -2, -3, \ldots, \ldots$ appears in the work of Martin (1951) and Diaz and Martin (1952).

Kapilevic (1952) has given solutions of (2.0.1) - (2.0.2) for $m = 1, 2$ and $0 < k < 1$. The most frequently discussed special case of (2.0.1) is, of course $k = 0$, the wave equation. All these various cases were treated by special methods. A unified solution of (2.0.1) - (2.0.2) for all values of k was given by Weinstein (1954), by a combination of generalized method of descent with a recurrence formula. For the cases $k=-1,-3,\ldots$, Weinstein assumed that $u(x,t)$ satisfies certain differentiability conditions; he found that a solution exists only if the initial value function $f(x)$ is a polyharmonic function of order $(1-k)/2$.

The paper of Diaz and Weinberger (1952) contains another solution of the problem of (2.0.1) - (2.0.2), for all values of k. They found a solution of arbitrary $f(x)$ for the exceptional values $k=-1,-3,\ldots,$

The t derivatives of order $1-k$ of the solution is logarithmic at $t=0$ when $f(x)$ is not polyharmonic of order $(1-k)/2$. Their attention was drawn to this behavior by the example $u=x^2+t^2\log t$ For $k=-1$, $m=1$.

Diaz and Weinberger (1953) employed Hadamard's (1923) "methods of descent" to obtain solutions of

(2.0.1) - (2.0.2), for $k=m,m+1,m+2,\ldots,$ from the known solution for $k=m-1$. They directly verified that the resulting formula gives a solution of the problem for any k with $\operatorname{Re} k>m-1$. This part is in common with Weistein (1952). However the definite integral, in terms of which the solution is expressed for $\operatorname{Re} k>m-1$, is divergent for $k\le m-1$. They analytically continued the integral in k and verified that the resulting formula did indeed furnished a solution of (2.0.1) - (2.0.2) for $k\le m-1$ with the exception of $k=-1,-3,\ldots$. The results of the equations (2.0.1) - (2.0.2) may be summarized as follows;

(i) If $k=m-1$, the solutions as given explicitly by Arsgeirsson(1936) is

$$u(x,t)=\frac{1}{\omega_n}\int_{\sum_1^m \alpha_j^2=1}\cdots\int f(x+\alpha t)\,d\omega_n \tag{2.0.3}$$

where $u(x,t)=u(x_1,x_2,\ldots,x_m,t)$, $\alpha=\alpha_1+\alpha_2+\ldots+\alpha_m$

$\omega_n=2\pi^{m/2}/\Gamma(m/2)$ is the case of the m-dimensional unit sphere.

(ii) If $k>m-1$, the solution was obtained by Weinstein (1952) and is given as

$$u(x,t)=\frac{\omega_{k+1-m}}{\omega_{k+1}}\int_{\sum_1^m \alpha_j^2\le 1}\cdots\int f(x+\alpha t)\left(1-\alpha^2\right)^{(k-m-1)/2}d\alpha \tag{2.0.4}$$

where $\alpha=\alpha_1+\alpha_2+\ldots+\alpha_n$, $d\alpha=d\alpha_1,d\alpha_2,\ldots,\ldots,d\alpha_m$.

(iii) For $k<m-1$ but $k\neq-1,-3,-5,\ldots$ Weinstein (1954) improving on his results of 1952, obtained the solution

$$u(x,t)=t^{1-k}\left(\frac{\partial}{t\partial t}\right)^n\left[t^{k+2n-1}u^{(k+2n)}(x,t)\right] \tag{2.0.5}$$

where n is a positive integer chosen such that $k+2n\geq m-1$ and $u^{(k+2n)}$ is given by (2.0.3) or (2.0.4) with f replaced by $f/(k+1)(k+3)\ldots(k+2n-1)$.

The solution of the Cauchy problem is unique for $k>0$ where as for $k<0$ it is not unique as indicated by Weinstein (1952). In particular, the solution for the exceptional values is not unique.

Blum (1954) obtained by essentially different methods another solution for the exceptional cases. His solution differs from the solution of Weinstein (1954) in that it is given as an explicit formula and has further advantage that the function f is required to have fewer continuous derivatives namely, it is sufficient for f to have derivatives of order at least $(m-k+3)/2$.

$$u(x,t)=\frac{1}{\omega_n}\int_{\sum_1^m \alpha_j^2=1}\cdots\int f(x+\alpha t)\,d\omega_n \tag{2.0.3}$$

where $u(x,t)=u(x_1,x_2,\ldots,x_m,t)$, $\alpha=\alpha_1+\alpha_2+\ldots+\alpha_m$

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where $\alpha=\alpha_1+\alpha_2+\ldots+\alpha_n$, $d\alpha=d\alpha_1,d\alpha_2,\ldots,\ldots,d\alpha_m$.

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Weinstein (1954) studied the mixed type partial differential equations in the theory of transonic flow. The best known equation of this kind is the Tricomi equation which occurs in hodograph method for plane flows of comprehensible fluids. In its normal form, Tricomi's equation is related to the generalized axially symmetrical potential theory (GASPT), and to the hyperbolic (EPD) equation.

An exposition of GASPT has been given by Weinstein (1953) and Gilbert (1969) amongst others.

Using Chaplygin's notation, the Tricomi equation may be written in the form

$$\sigma\Psi_{\phi\phi} + \Psi_{\sigma\sigma} = 0 \tag{2.0.6}$$

where Ψ denote the stream function. For $\sigma > 0$ the equation is elliptic. For $\sigma < 0$, it is hyperbolic.

A problem of great importance in connection with Tricomi's equation is the determination of a fundamental solution in the large, that is a solution defined in the entire θ,σ-plane and having a logarithmic singularity at one point in the elliptic half-plane $(\sigma > 0)$, say $\theta = 0$, $\sigma = \beta > 0$.

Weinstein (1955) determined such as solution by first obtaining a closed formula for $\sigma \geq 0$. Then he used the data given by the formula on $\sigma = 0$ as Cauchy data to determine the continuation of his entire hyperbolic half-plane $(\sigma < 0)$.

Weinstein (1955) introduced the variables

$$x = \theta, \quad g = \tfrac{2}{3}\sigma^{2/3}; \quad \sigma > 0$$

that reduces (2.0.6) to the normal form

$$\partial_x^2 u + \partial_y^2 u + \frac{1}{3y}\partial_y u = 0 \tag{2.0.7}$$

whose generalization is the GASPT equation

$$\sum_{i=1}^{m} \partial_{x_i}^2 u + \partial_y^2 u + \frac{k}{y}\partial_y u = 0 \tag{2.0.8}$$

For $\sigma < 0$, introducing the variables

$$x = \theta, \quad t = \tfrac{2}{3}(-\sigma)^{3/2} \tag{2.0.9}$$

he obtained the normal form

$$\partial_x^2 u - \partial_t^2 u - \frac{1}{3t}\partial_t u = 0 \tag{2.0.10}$$

whose generalization is the EPD equation

$$\sum_{i=1}^{n} \partial_{x_i}^2 u - \partial_t^2 u - \frac{k}{t}\partial_t u = 0 \tag{2.0.11}$$

for $u(x_1,...,x_m,y)$ and $u(x_1,...,x_m,t)$ respectively in (2.0.8) and (2.0.11).

Weinstein(1957) made an observation that mean value of a sufficiently smooth function is the solution of a singular Cauchy problem for the hyperbolic EPD equation with a special positive value of the index.

Motivated by this observation he solved the singular Cauchy problem for an EPD equation with non-negative index. He investigated the properties of solution of the Cauchy problem under the assumption that the initial data satisfies a differential equation of the Helmoholtz type.

To obtain the minimum and convexity properties of the solution he assumed the initial data to be subharmonic. The main tool of investigation was the transformation of the EPD equation into a generalized Tricomi equation in several variables. He noted that while the original Tricomi equation had found its main applications in gas dynamics the same equation and its generalizations to several variables can be used in the theory of subharmonic functions.

Fox (1959) studied a Singular Cauchy Problem for a hyperbolic partial differential equation in $m+1$ variables. The classical problem in this domain is that for the EPD equation, which in its full generality has been investigated extensively since the appearance of Weinstein (1952).

His equation

$$\partial_t^2 u + \frac{k}{t}\partial_t u - \sum_{i=1}^{m}\left(\partial_{x_i}^2 u + \frac{\lambda_i}{x_i^2}u\right) \equiv L(u) = 0 \tag{2.0.12}$$

where k and λ_i are real or complex parameters restricted to two variables, goes back to the Euler and Darboux. Stellmacher (1955), without connection with earlier studies, investigating Huygens principle, found for (2.0.12) the fundamental solution and thus; in principle, solved the regular Cauchy problem.

In (2.0.12) Fox (1959) excludes the exceptional values $k = -1, -3, -5, \ldots$ the Singular Cauchy Problem in (2.0.12) corresponds to the initial data (2.0.2).

The differential equation in the form considered by Stellmacher (1955), is obtained from (2.0.12) by putting, $u = vt^{-\frac{k}{2}}$. For $\lambda_i = 0$ for every i (2.0.12)-(2.0.2) reduces to EPD equation. Assuming that for each, i, $|x_i| > |t| \geq 0$,the right hand sum is a regular analytic elliptic differential operator $X(u)$,and therefore (2.0.12) belongs to the class of singular hyperbolic equations considered by Bureau (1955). Bureau (1955) developed a solution for the singular Cauchy problem that generalizes Weinstein (1952) for the EPD equation, but it is valid only in the small and had the character of an existence theorem rather than that of an explicit solution. He did not examine features such as differentiability conditions, uniqueness, and Huygens's principle.

Fox (1959) gives an explicit solution in the large of the Singular Cauchy Problem (2.0.12)-(2.0.2) in the domain $\left|x_i > |t| \geq 0\right|$. He determines the exact range of the parameter k for which the solution is given by an integral operator. For all other non-exceptional values of k he obtained the solution by means of a recursive procedure which gives the analytic continuation in k of the integral. The question of uniqueness was extensively treated by means of techniques of Zaremba (1915), Fredrichs' and Lewy (1928), the generality achieved being the same as that obtained by Walter (1957) for the EPD equation. Fox (1959) shows that the equation (2.0.12) admits several useful transformations associated with Weinstein's (1957) correspondence principle.

Carroll (1961) proves the existence and uniqueness for some generalized EPD equations and growth and convexity properties of the solutions were studied for multiply, subharmonic, initial values. Solutions of EPD equatins due to Carroll case presented in a separated locally convex space E.

Diaz and Kiwan (1966) remarked on the publication of Diaz and Ludford (1956), for obtaining a solution for all values of the time, to the Singular Cauchy Problem for the EPD equation for values of the parameter $k > m-1$. They verified directly that a certain multiple integral involving the initial value function $f(x_1, x_2, \ldots, x_n)$ provides a solution to the problem. Their treatment is a modification of an argument due to Weinstein (1953). Weinstein's proof requires that f be of class C^3 while Diaz and Ludford (1955) discussion requires only that f be of class C^2.

Carroll (1976) developed in the context of general harmonic analysis on certain symmetric spaces that had a far reaching extension of the classical theory of EPD equations. Fusaro (1966) obtained a series solution of the singular, mixed EPD problem (2.0.1)-(2.0.2) with $u(x,t)=u(\pi,t)=0; 0\le t$, by elementary means and series solution reduced to a closed form. He showed that the solution obtained is unique. He expressed the solution as the mean value of the initial datum f, taking the form of the known solution of the initial value problem (2.0.1)-(2.0.2). He finally discussed the physical interpretation and a regular mixed problem.

Fujiie' (1993a, 1993b, and 1997) discussed the equation of the type

$$\partial_t^2 u-\partial_x^2 u+\frac{p+q}{t}\partial_t u-\frac{p-q}{t}\partial_x u=0 \qquad (2.0.14)$$

which referred to as Fuchsian equation (Baouendi, 1973, Tahara 1979) with respect to the hypersurface $\{t=0\}$. This equation may be reduced to the EPD equation

$$\partial_{\xi\eta}^2 u-\frac{q}{\xi-\eta}\partial_\xi u+\frac{p}{\xi-\eta}\partial_\eta u=0 \qquad (2.0.15)$$

if we let $x=\xi+\eta$ and $t=\xi-\eta$. We note that $u=(\xi-a)^{-p}(\eta-a)^{-q}$ is a solution of (2.0.15), where 'a' is a parameter. Fujiie' (1993a, 1993b, 1997) studies were in the space of holomorphic functions.

His aim was to determine the singularities of the solution obtained by the holomorphic data. The problem has no analogy in the theory of ordinary differential equations and in the theory of partial differential equations it plays a fundamental role because the singularities do not depend on data.

Dernek (2002) determined the solution of the generalized and equation

$$\left(\partial_t^2+\left(mt+\frac{n}{t}\right)\partial_t\right)u-\Delta_x u=0 \qquad (2.0.16)$$

subject to (2.0.2 by determining a series that is continuous and absolutely convergent over some suitable interval.

Iyaya (2004) discussed the problem

$$\left(\partial_t^2 + \frac{n}{t}\partial_t - \partial_x^2 - \frac{n}{x}\partial_x\right)u = 0 \qquad (2.0.17)$$

that is characteristic on the hyper planes $t = 0$ and $x = 0$.

§ 2.1 **STATEMENT OF THE PROBLEM**

We wish to solve (i) a Cauchy Problem for wave equation (ii) Singular Cauchy Problem of EPD equation by the Riemann's method that has not featured in any of the literature cited. We shall as well discuss the generalized singular Cauchy problem due to Dernek (2002) first by taking the Fourier transform in the space coordinate where certain properties of the Fourier transform are assumed. An integral differential equation is obtained on some suitable interval and a series solution is obtained which is continuous and analytic over the interval that contains the singularity.

§ 2.2 **OBJECTIVES OF THE STUDY**

(i) To solve Euler-Poisson-Darboux equations by Riemann's Method.

(ii) Get a generalized solution of EPD equation in the space of tempered distributions where *Trace Theorem* has to be employed along the boundary. By using such a fine space, the model represents almost an ideal state of the phenomena under study.

CHAPTER THREE

§ 3.0 THE GENERAL CAUCHY PROBLEM

§ 3.1 Introduction

In this section we discuss the general Cauchy problem, existence and uniqueness of solution. We determine the solution of the general Cauchy problem which is the wave equation we obtain the D$^{/}$Alemberts solution for one-dimension homogeneous case which is unique and depends continuously on the data prescribed. For dimension $n>1$, we apply method of Spherical means to get solution for n odd is got by reducing the n–dimensional wave equation to 1-dimensional so that solution obtained is the D$^{/}$ Alemberts and on unraveling the transformations used a solution is written. For $n>1$ and even, a ``method of descend" is deployed to solve Cauchy problem.

We look at the general k th order equation

$$F\left\{x,\left(\partial^{\alpha}u\right)_{|\alpha|\le k}\right\}=0 \tag{3.1.1}$$

where F is always assumed to be (at least) C^1.

Let S be a hypersurface of class C^k. If u is a C^{k-1} function defined near S, the quantities $u,\partial_v u,\dots,\partial_v^{k-1}u$ on S are called the Cauchy data of u on s; where ∂_v is the normal derivative on a neighbourhood of S. The Cauchy problem is to solve (3.1) when the Cauchy data of u on S are pre-assigned.

We shall assume all our considerations to be restricted to a neighbourhood of a given point on S .We thus assume that a change of coordinates has been made so that S contains the origin and, near the origin, coincides with the hyperplane $x_n=0$. We can then make a slight change in the notation. We shall consider $\mathbb{R}^n$ as $\mathbb{R}^{n-1}\times\mathbb{R}$ and denote the coordinate by (x,t) where $x=(x_1,\ldots,x_{n-1})$. Derivatives with respect to x variables will be denoted by ∂_x^α, where $\alpha=(\alpha_1,\ldots,\alpha_{n-1})$, and derivatives with respect to t will be denoted by ∂_t^j. We can then restate Cauchy problem as follows: Given functions $\phi_0,\ldots,\phi_{k-1}$ of x, solve

$$F\left(x,t,\left(\partial_x^\alpha\partial_t^j u\right)|\alpha|\, j\le k\right)=0 \tag{3.1.2}$$

$$\partial_t^j u(x,0)=\phi_j(x) \qquad\qquad (0\le j<k). \tag{3.1.3}$$

We observe that if u is a function of class C^r with $r\ge k$, then the Cauchy data $\{\phi_j\}$ determine all derivatives $\partial_x^\alpha\partial_t^j u$ on S with $j<k$ and $|\alpha|+j\le r$; in fact

$$\partial_x^\alpha\partial_t^j u(x,o)=\partial_x^\alpha\phi_j(x)$$

Hence the only quantity in the differential equation (3.1.2-3.1.3) which is unknown on S is $\partial_t^k u$

In order for the Cauchy problem to be well posed, we must assume that the equation

$F=0$ can be solved for $\partial_t^k u$.

In the linear case,

$$F\left\{x,t,\left(\partial_x^\alpha\partial_t^j u\right)_{|\alpha|+j\le k}\right\}=\sum_{|\alpha|+j\le k} a_{\alpha j}(x,t)\partial_x^\alpha\partial_t^j u-f(x,t),$$

this assumption means that S is non-characteristic. Indeed, "S is non-characteristic" means that $a_{0k}(x,0)\ne 0$, hence by continuity $a_{0k}(x,t)\ne 0$, for small t, and we can solve for $\partial_t^k u$:

$$\partial_t^k u = (a_{0k})^{-1}\left[\sum_{|\alpha|_j \le k, j<k} a_{\alpha j}\partial_x^\alpha \partial_t^j u - f\right]$$

If this condition is not satisfied then the solution obtained fails to be unique as by the examples mentioned below.

(i) The line $t=0$ is characteristic for the equation $\partial_x \partial_t u = 0$ in $\mathbb{R}^2$. If u is a solution of this equation with $u(x,0)=\phi_0(x)$ and $\partial_t u(x,0)=\phi_1(x)$, we must have $\partial_x \phi_1 = 0$ i.e. ϕ_1 is a constant. Thus the Cauchy problem is not solvable in general. On the other hand, if ϕ_1 is a constant, then there is no uniqueness! We can take $u(x,t)=\phi_0(x)+f(t)$ where f is any function with $f'(0)=\phi_1$.

(ii) The line $t=0$ is characteristic for the equation $\partial_x^2 u - \partial_t u = 0$ in $\mathbb{R}^2$. Here if we are given that u is a solution with $u(x,0)=\phi_0(x)$, then $\partial_t u(x,0)$ is already completely determined; $\partial_t u(x,0)=\phi_0''(x)$

In the quasi – linear case,

$$F\left\{x,t,\left(\partial_x^\alpha \partial_t^j u\right)_{|\alpha|+j\le k}\right\} = \sum_{|\alpha|+j=k} a_{\alpha j}\left\{x,t,\left(\partial_x^\beta \partial_t^i u\right)_{|\beta|+i\le k-1}\right\}\partial_\alpha^x \partial_t^i u$$

$$-b\left\{x,t,\left(\partial_x^\beta \partial_t^i u\right)_{|\beta|+i\le k-1}\right\},$$

We say that the Cauchy problem (3.1.2-3.1.3) is non-characteristic if

$$a_{0k}\left[x,0,\left\{\partial_x^\beta \phi_i(x)\right\}_{|\beta|+i\le k-1}\right] \ne 0$$

for all x ; again, this allows us to solve for the derivative ∂_t^k.

In the general case, the equation

$$F\left[x,0,\left\{\partial_x^\alpha \phi_j(x)\right\}_{|\alpha|+j\le k,\, j<k,}\, u_{0k}(x)\right]=0,$$

will usually not determine u_{0k} uniquely as a function of x on S. Therefore, we phrase the non-characteristic condition as follows: the quantity u_{0k} can be determined as C^1 a function of x on S so that

$$F\left[x,0,\left\{\partial_x^\alpha \phi_j(x)\right\}_{|\alpha|+j\le k,\, j<k,}\, u_{0k}(x)\right]=0$$

$$\frac{\partial F}{\partial u_{0k}}\left[x,0,\left\{\partial_x^\alpha \phi_j(x)\right\}_{|\alpha|+j\le k,\, j<k,}\, u_{0k}(x)\right]\neq 0,$$

for all x. In this case, we can solve the equation $F=0$ for u_{0k} as a C^1 function G of the remaining variables near S, by the implicit function theorem, and write the differential equation in the normal form!

$$\partial_t^k u = G\left[x,t,\left\{\partial_x^\alpha \partial_t^j u\right\}_{|\alpha|+j\le k,\, j<k}\right] \tag{3.1.4}$$

The Cauchy data $\{\phi_j\}$ together with (3.1.4) determine all derivatives of u of order $\le k$ on S. If G is sufficiently smooth, we can also determine higher derivatives of u. Namely, differentiating (3.1.4) with respect to,

$$\partial_t^{k+1}u=\frac{\partial G}{\partial t}+\sum_{|\alpha|+j\le k,\, j<k}\frac{\partial G}{\partial u_{\alpha j}}\partial_x^\alpha \partial_t^{j+1}u.$$

All the quantities on the right are known on S, so $\partial_t^{k+1}u$ is also; hence we know all derivatives of u of order $\le k+1$ on S. Applying ∂_t more times, we obtain higher derivatives .In particular, and we have:

Proposition 3.1.

Suppose that G, $\phi_0,\ldots,\phi_{k-1}$ are analytic functions. Then there is at most one analytic function u satisfying (3.1.4) such that $\partial_t^j u(x,0)=\phi_j(x)$ for $0\le j<k$.

Proof; By Taylor's formula, an analytic function is completely determined by the values of its derivatives at one point.

Proposition 3.2.

Suppose G is continuous and there is a constant $C>0$ such that for all $x,t\in\mathbb{R}^n$ and all vectors $(u_{\alpha j}),(v_{\alpha j})(0\le|\alpha|+j\le k, j<k)$,

$$\left|G\{x,t,(u_{\alpha j})\}-G\{x,t,(v_{\alpha j})\}\right|\le c\sum_{\alpha,j}\left|u_{\alpha j}-v_{\alpha j}\right|$$

If u and v are two solutions of (3.4) with the same Cauchy data on S, and the derivatives $\partial_x^\alpha\partial_t^j u$ and $\partial_x^\alpha\partial_t^j v$ exist for $|\alpha|\le q$ and $j\le r(q,r\ge k)$, then these derivatives agree on S.

Proof. Let $\omega=u-v$. It suffices to show that $\partial_t^m\omega=0$ on S for $m\le r$, as then the x-derivatives of these functions also vanish on S. We proceed by induction on m, the case $m<k$, being true by assumption. Suppose then that $m\ge k$ and $\partial_t^i\omega=0$ on S for $i<m$. By Taylor's theorem,

$$\partial_t^j\omega(x,t)=\frac{t^{m-j}}{(m-j)!}\partial_x^\alpha\partial_t^m\omega(x,0)+0\left(t^{m-j}\right)=0\left(t^{m-k}\right)\qquad(t\to 0)\qquad\ldots\ldots(3.1.5)$$

and for j<k and $|\alpha|+j\le k$,

$$\partial_x^\alpha\partial_t^j\omega(x,t)=\frac{t^{m-j}}{(m-j)!}\partial_x^\alpha\partial_t^m\omega(x,0)+0\left(t^{m-j}\right)=0\left(t^{m-k}\right)\quad(t\to 0)$$

Folland, (1979), John (1982), Lawrence (1998), Garabedian (1964) , et al). Thus by assumption on G,

$$\left|\partial_t^k \omega(x,t)\right| = \left|G\left[x,t,\left\{\partial_x^\alpha \partial_t^j u(x,t)\right\}\right] - G\left[x,t,\left\{\partial_x^\alpha \partial_t^j v(x,t)\right\}\right]\right|$$

$$\leq c\sum\left|\partial_x^\alpha \partial_t^j u(x,t) - \partial_x^\alpha \partial_t^j v(x,t)\right|$$

$$= c\sum\left|\partial_x^\alpha \partial_t^j \omega(x,t) = 0\left(t^{m-k}\right)\right| \qquad (t \to 0)$$

Hence by (3.1.5)

$$\frac{t^{m-k}}{(m-k)!}\partial_t^m \omega(x,0) = 0\left(t^{m-k}\right) \qquad (t \to 0),$$

which force $\partial_t^m \omega(x,0)$ to be 0.

Although the problem of Cauchy is good from the point of view of determining a unique solution, existence is another matter, especially if we want a solution in a specified domain and not just in some neighbourhood of the initial hyper surface S. The Cauchy problem tends to be overdetermined except in certain special situations. The appropriate boundary conditions for a differential equation depend strongly on the particular form of the equation.

The discussion on existence is summarized in the theorem:

Theorem 3.1. (The Cauchy – Kowaleskia theorem)

If G, $\phi_0,\ldots,\phi_{k-1}$ are analytic near the origin, there is a neighbourhood of the origin on which the Cauchy problem

$$\partial_t^k u = G\left\{x,t,\left(\partial_x^\alpha \partial_t^j u\right)_{|\alpha|+j\leq k,\, j<k}\right\},$$

$$\partial_t^j u(x,0) = \phi_j(x) \qquad (0 \leq j < k)$$

has a unique analytic solution.

Now for the solution of a boundary or initial value problem to be meaningful, it must conform to the physical situation. i.e. well-posed.

Boundary value problem for the elliptic equation and Cauchy problems for a hyperbolic equation are well posed. A Cauchy problem for the elliptic equation and boundary value problem for the hyperbolic equation are not well posed.

Solution of the Cauchy Problems

In this section, we apply method of spherical means in solution of wave equation.

The wave operator is the prototype of the class hyperbolic operators. We shall construct the solution of the Cauchy problem

$$\partial_t^2 u - \Delta u = 0 \tag{3.1.6}$$

$$u(x,0) = f(x,) \qquad \partial_t u(x,0) = g(x). \tag{3.1.7}$$

For the one dimensional case the classical solution (D'Alembert's) to (3.1.6-3.1.7) is

$$u(x,t) = \frac{1}{2}\left[f(x+t) + f(x-t)\right] + \frac{1}{2}\int_{x-t}^{x+t} g(s)\,ds \tag{3.1.8}$$

If f is C^2 and g is c^1 on $\mathbb{R}^n$, then u on $\mathbb{R}^n \times \mathbb{R}$ and satisfies (3.1.6-3.1.7) in the classical sense.

For $n > 1$, we construct the solution when n is odd, where we can reduce the problem to one-dimensional case, and then obtain the even-dimensional solutions by modifying the odd-dimensional ones.

We have to discuss some results that prepare us to solve the problem.

Theorem 3.2.

Suppose L is a partial differential operator on $\mathbb{R}^n$. Then L commutes with translations and rotations if and only if L is a polynomial in Δ-that is, $L = \sum a_j \Delta^j$ for some constants a_j.

Since the Laplaceian commutes with rotations, it preserves the class of radial functions, on which it reduces to an ordinary differential operator called the radial part of the Laplacian. We compute it explicitly.

Proposition 3.3.

If $f(x)=\phi(r)$ where $x\in\mathbb{R}^n$ and $r=|x|$, then

$$\Delta f(x)=\phi''(r)+\frac{n-1}{r}\phi'(r.) \tag{3.1.9}$$

Proof. Since, $\partial r/\partial x_j = x_j/r$, we have

$$\Delta f(x)=\sum_1^n \partial_j\left[\frac{x_j}{r}\phi'(r)\right] \quad =\sum_1^n\left[\frac{x_j^2}{r^2}\phi''(r)+\frac{1}{r}\phi'(r)-\frac{x_j^2}{r^3}\phi'(r)\right]$$

$$=\phi''(r)+\frac{n}{r}\phi'(r)-\frac{1}{r}\phi'(r).$$

Theorem 3.3. **The mean value theorem**

Suppose u is harmonic on an open set Ω. If $x\in\Omega$ and r>0 is small enough so that $\overline{B_r(x)}\subset\Omega$, then

$$.u(x)=\frac{1}{r^{n-1}\omega_n}\int_{S_r(x)}u(y)\,d\sigma(y)=\frac{1}{\omega_n}\int_{S_1(0)}u(x+ry)\,d\sigma(y) \tag{3.1.10}$$

Proof. The second equality follows from the change of variable $y\to x+ry$ and by composing with a translation we may assume that $x=0$. To prove the first equality, we use Green's identity

$$\int_S\left(v\partial_\eta u-u\partial_\eta v\right)d\sigma=\int_\Omega\left(v\Delta u-u\Delta v\right)dx \tag{3.1.11}$$

Where Ω is a bounded domain with smooth boundary S and u, v are C^1 functions on $\overline{\Omega}$. We take u to be our harmonic function, $v(y) = |y|^{2-n}$ if $n \neq 2$ or $v(y) = \log|y|$ if $n = 2$, and $\Omega = B_r(0) \setminus \overline{B_\varepsilon}(0)$ where $0 < \varepsilon < r$. With this specification of v, it is harmonic in Ω and $\partial_v v$ is the constant $(2-n)r^{1-n}$ is on $S_\varepsilon(0)$ and the constant $-(2-n)\varepsilon^{1-n}$ on $S_\varepsilon(0)$(The minus sign is there because of the orientation of. $\overline{S_\varepsilon}(0)$ is the opposite of the usual one and the factor $(2-n)$ should be omitted when $n = 2$.) Thus by (3.11)

$$0 = \int_{S_r(0)} (v\partial_\eta u - u\partial_\eta v)\, d\sigma - \int_{S_\varepsilon(0)} (v\partial_\eta u - u\partial_\eta v)\, d\sigma$$

$$= r^{2-n} \int_{S_r(0)} \partial_\eta u\, d\sigma + \varepsilon^{2-n} \int_{S_\varepsilon(0)} \partial_\eta\, d\sigma$$

$$-(2-n)r^{1-n} \int_{S_r(0)} u\, d\sigma + (2-n)\varepsilon^{1-n} \int_{S_\varepsilon(0)} u\, d\sigma$$

Now if u is harmonic on Ω then $\int_S \partial_\eta u\, d\sigma = 0$. This comes as a result of the Green's identity with $v = 1$. Hence the first two terms in the last sum vanish, so

$$\frac{1}{r^{n-1}\omega_n} \int_{S_r(0)} u\, d\sigma = \frac{1}{\varepsilon^{n-1}\omega_n} \int_{S_\varepsilon(0)} u\, d\sigma$$

But u is continuous, so the right hand side, being the mean value of u on $S_\varepsilon(0)$, converges to $u(0)$ as $\varepsilon \to 0$

Definition:

If ϕ is a continuous function on $\mathbb{R}^n$, $x \in \mathbb{R}^n$, and $r > 0$, we define the spherical mean $M_\phi(x,r)$ to be the average value of ϕ on $S_r(x)$:

$$M_\phi(x,r)=\frac{1}{r^{n-1}\omega_n}\int_{|z-x|=r}\phi(z)d\sigma(z)$$

The substitution $z=x+ry$ turns this into

$$M_\phi(x,r)=\frac{1}{\omega_n}\int_{|y|=1}\phi(x+ry)d\sigma(y) \tag{3.1.12}$$

for $r\in\mathbb{R}$. We regard M_ϕ as the function on $\mathbb{R}^n\times\mathbb{R}$ defined by (3.1.12). It is even in r and C^k in both x and r if ϕ is C^k. Moreover $M_\phi(.,0)=\phi$.

Now if T is any rotation on $\mathbb{R}^n$, by theorem (3.2) we have

$$\Delta_x\left[\phi(x+Ty)\right]=\left[\Delta\phi\right](x+Ty)=\Delta_y\left[\phi(x+Ty)\right]$$

where Δ_x and Δ_y denote the Laplacian acting in the variables x and y. Since the average of $\phi(x+Ty)$ over all relations is $M_\phi(x,|y|)$, we obtain

$$\Delta_x M_\phi(x,|y|)=\Delta_y M_\phi(x,|y|).$$

Therefore by (3.1.8),

$$\Delta_x M_\phi(x,r)=\left(\partial_r^2+\frac{n-1}{r}\partial_r\right)M_\phi(x,r) \tag{3.1.13}$$

We explain what we mean by (3.1.13).

Proposition 3. 4.

If ϕ is a C^2 function on $\mathbb{R}^n$, then M_ϕ satisfies (3.1.13) on $\mathbb{R}^n\times\mathbb{R}$.

Proof. It suffices to consider $r>0$, since both sides of (3.1.13) are even functions of r. Also if u is a differentiable function defined near S, we can define the normal derivatives of u on S by

$$\partial_\eta u=\eta.\nabla u.$$

We used the derivative on the sphere $s_r(y)$. since lines through the centre of the sphere are perpendicular to the sphere, we have

$$\eta(x)=\frac{x-y}{r}, \qquad \partial_\eta=\frac{1}{r}\sum_1^n(x_j-y_j)\partial_j \quad ; \quad r=|x-y|.$$

Hence $\partial_r M_\phi(x,r)=\frac{1}{\omega_n}\int_{|y|=1}\sum y_k\partial_{y_k}\phi(x+ry)\,d\sigma(y)$

and by divergence theorem which states;

$$\int_S F(y).\eta(y)\,d\sigma(y)=\int_\Omega \nabla.F(x)\,dx \tag{3.1.14}$$

where $\Omega\in\mathbb{R}^n$ is a bounded domain with C^1 boundary $S=\partial\Omega$ and F a C^1 vector field on $\overline{\Omega}$; we have

$$=\frac{1}{\omega_n}\int_{|y|\le 1} r\Delta\phi(x+ry)\,dy$$

$$=\frac{1}{r^{n-1}\omega_n}\int_{|z|\le r}\Delta\phi(x+z)\,dz.$$

or $$r^{n-i}\partial_r M_\phi(x,r)=\frac{1}{\omega_n}\int_0^r\int_{|y|=1}\Delta\phi(x+\rho y)\rho^{n-1}\,d\sigma(y)\,d\rho,$$

so $$\partial_r\left[r^{n-1}\partial_r M_\phi(x,r)\right]=\frac{1}{\omega_n}\int_{|y|=1}\Delta\phi(x+ry)r^{n-1}\,d\sigma(y)=r^{n-1}\Delta_x M_\phi(x,r)$$

Differentiating the left and dividing by r^{n-1} completes our proof.

Corollary 3.1.

Suppose $u(x,t)$ is a c^2 function on $\mathbb{R}^n\times\mathbb{R}$, and let $M_u(x,r,t)$ Denote the spherical mean of the function $x\to u(x,t)$. Then u satisfies the wave equation if and only if M_u satisfies

$$\left[\partial_r^2+\frac{n-2}{r}\partial_r\right]M_u(x,r,t)=\partial_t^2 M_u(x,r,t) \qquad (3.1.15)$$

for each $x\in\mathbb{R}^n$.

The proof is immediate by proposition (3.4).

When n is odd the differential equation (3.1.15) can be reduced to the one-dimensional wave equation by means of the following identities.

Lemma 3.1.

If $k\geq 1$ and $\phi\in C^{k+1}(\mathbb{R})$, then

$$\partial_r^2\left(r^{-1}\partial_r\right)^{k-1}\left[r^{2k-1}\phi(r)\right]=\left(r^{-1}\partial_r\right)^k\left[r^{2k}\phi'(r)\right] \qquad (3.1.16)$$

The right side of (3.1.16) equals

$$\left(r^{-1}\partial_r\right)^{k-1}\left[r^{2k-1}\phi''(r)+2kr^{2k-1}\phi'(r)\right].$$

Thus if we define the differential operator T_k by

$$T_k\phi(r)=\left(r^{-1}\partial_r\right)^{k-1}\left[r^{2k-1}\phi(r)\right],$$

(3.1.16) says that

$$\partial_r^2 T_k\phi=T_k\left[\left(\partial_r^2+2kr^{-1}\partial r\quad\right)\phi\right].$$

Thus if $n=2k+1$, T_k converts (3.1.15) into the one-dimensional wave equation.

Now in $T_k\phi$ there are $2k-1$ powers of r in the numerator and $k-1$ in the denominator, and $k-1$ derivatives. Expand $T_k\phi$ by the product rule; if j derivatives act on ϕ, then $k-1-j$ derivatives must act on the powers of r, leaving the factor of r to the power

$$(2k-1)-(k-1)-(k-1-j)=j+1.$$

Thus

$$T_k\phi(r)=\sum_0^{k-1} C_j r^{j+1}\phi^{(j)}(r), \tag{3.1.17}$$

where

$$C_o r=\left(r^{-1}\partial_r\right)^{k-1} r^{2k-1}=1.3...(2k-1)r$$

We are now ready to solve the Cauchy problem (3.1.1-3.1.2) when the space dimension n is odd and >1. Suppose that u satisfies (3.1.1-3.1.2), and suppose for the moment that u, f, and g are smooth – at least of class $C^{(n+3)/2}$. By *corollary (1)*, the spherical mean M_u satisfies the differential equation (3.1.15) with initial conditions.

$$M_u(x,r,0)=M_f(x,r), \partial_t M_u(x,r,0)=M_g(x,r)$$

Thus if we set

$\tilde{u}=TM_u, \tilde{f}=TM_f, \tilde{g}=TM_g$, where

$$T(.)=T_{(n-1)/2}(.)=\left(r^{-1}\partial_r\right)^{(n-3)/2}\left[r^{n-2}(.)\right]$$

The remark following *lemma (3.1)* show that

$$\partial_r^2\tilde{u}(x,r,t)=\partial_t^2\tilde{u}(x,r,t)$$

$$\tilde{u}(x,r,0)=\tilde{f}(x,r,),\partial_t\tilde{u}(x,r,0)=\tilde{g}(x,r).$$

The solution to this problem is given by

$$\tilde{u}(x,r,t)=\frac{1}{2}\left[\ \tilde{f}(x,r+t)+\tilde{f}(x,r-t)\ \right]+\frac{1}{2}\int_{r-t}^{r+t}\tilde{g}(x,s)ds,$$

so it remains to recover u from $\tilde{u}$. M_u is obtained from $\tilde{u}$ by undoing the operator T, i.e, by successive integrations, and then u is obtained from M_u by setting $r=0$. By using (3.1.17) with $\phi(r)=M_u(x,r,t)$ and $k=\frac{1}{2}(n-1)$:

$$u(x,t)=M_u(x,0,t)=\lim_{r\to 0}\frac{\tilde{u}(x,r,t)}{C_o r}$$

where $C_o=1.3...(n-2)$. Since M_f and M_g are even functions of r, $\tilde{f}$ and $\tilde{g}$ are odd, and $\partial_r\tilde{f}$ is even; hence by L'Hospital's rule,

$$u(x,t)=\lim_{x\to 0}\frac{1}{2C_o r}\left[\tilde{f}(x,r+t)+\tilde{f}(x,r-t)+\int_{r-t}^{r+t}\tilde{g}(x,s)\,ds\right]$$

$$=\frac{1}{2C_o}\left[(\partial_r\tilde{f})(x,r)\big|_{r=t}+(\partial_r\tilde{f})(x,r)\big|_{r=-t}+\tilde{g}(x,t)-\tilde{g}(x,-t)\right].$$

$$=\frac{1}{C_o}\left[(\partial_r\tilde{f})(x,r)\big|_{r=t}+\tilde{g}(x,t)\right]$$

If we unravel the derivative of $\tilde{f},\tilde{g}$, and C_o we obtain the formula for u interns of f and g as:

Theorem 3.3.

Suppose n is odd and $n\geq 3$. If $f\in C^{n-3/2}(\mathbb{R}^n)$, and $g\in C^{(n+1)/2}(\mathbb{R}^n)$ the function

$$u(x,t)=\frac{1}{1.3...(n-2)\omega_n}\left[\partial_t\left(t^{-1}\partial_t\right)^{(n-3)/2}\left\{t^{n-2}\int_{|y|=1}f(x+ty)\,d\sigma(y)\right\}\right.$$

$$+\left(t^{n-1}\partial_t\right)^{(n-3)/2}\left\{t^{n-2}\int_{|y|=1} g(x+ty)\,d\sigma(y)\right\}\Bigg]\Bigg] \tag{3.1.18}$$

solves the Cauchy problem (3.1.1-3.1.2).

Proof: Up to a constant factor, the second term on the right of (3.1.18) is

$$v(x,t)=\left(t^{-1}\partial_t\right)^{(n-3)/2}\left[t^{n-2}M_g(x,t)\right],$$

and by *corollary (3.1)* and *lemma (3.1)* we have

$$\Delta_x v(x,t)=\left(t^{-1}\partial_t\right)^{(n-3)/2}\left[t^{n-2}\Delta_x M_g(x,t)\right]$$

$$=\left(t^{-1}\partial_t\right)^{(n-3)/2}\left[t^{n-2}\partial_t^2 M_g(x,t)+(n-1)t^{n-3}\partial_t M_g(x,t)\right]$$

$$=\left(t^{-1}\partial_t\right)^{(n-1)/2}\left[t^{n-1}\partial_t M_g(x,t)\right]$$

$$=\partial_t^2\left(t^{-1}\partial_t\right)^{(n-3)/2}\left[t^{n-2}M_g(x,t)\right]==\partial_t^2 v(x,t)$$

so this term satisfies the wave equation. Likewise, the function

$$\omega(x,t)=\left(t^{-1}\partial_2\right)^{(n-3)/2}\left[t^{n-2}M_f(x,t)\right]$$

satisfies the wave equation, and hence so does $=\partial_t\omega(x,t)$, which is the first term on the right of (3.18). As for the initial conditions, by (3.17) we have

$$u(x,t)=\partial_t\left[tM_f(x,t)+\frac{C_1}{C_0}t^2\partial_t M_f(x,t)+0\left(t^3\right)\right]+tM_g(x,t)+0\left(t^2\right)$$

$$=M_f(x,t)+\frac{C_0+2C_1}{C_0}t\partial_t M_f(x,t)+tM_g(x,t)+0\left(t^2\right)$$

Hence $u(x,0)=M_f(x,0)=f(x)$

and,

$$\partial_t u(x,0) = 2\frac{(C_o + C_1)}{C_o}\partial_t M_f(x,0) + M_g(x,0) = g(x)$$

which completes the proof.

The solution of the Cauchy problem (3.1.1-3.1.2) for even n is readily derived from the solution for odd

n by the " method of descent"(See Carabedian, 1986; P.204). We observe that if u is a solution of the wave equation in $\mathbb{R}^{n+1} \times \mathbb{R}$ that does not depend on $x_n + 1$, and then u satisfies the wave equation in $\mathbb{R}^n \times \mathbb{R}$. Thus we state the result as a theorem:

Theorem 3.4

Suppose n is even. If $f \in C^{(n+4)/2}(\mathbb{R}^n)$ and, $g \in C^{(n+2)/2}(\mathbb{R}^n)$, the function

$$\mathrm{u(x,t)} = \frac{2}{1.3\ldots(n-1)\omega_{n+1}}\left[\partial_t\left(t^{-1}\partial_t\right)^{(n-2)/2}\left\{t^{n-1}\int_{|y|\le 1}\frac{f(x+ty)}{\sqrt{1-|y|^2}}dy\right\} + \left(t^{-1}\partial_t\right)^{(n-2)/2}\left\{t^{n-1}\int_{|y|\le 1}\frac{g(x+ty)}{\sqrt{1-|y|^2}}dy\right\}\right]$$

(3.1.19)

solves the Cauchy problem (3.1.1-3.1.2).

The next hurdle is to solve the n-dimension wave equation by Riemann's method.

CHAPTER FOUR

§ 4.0 Solution of n-dimension Wave equation by Riemann's method

We shall construct the solution of the Cauchy problem (n-dimension wave equation)

$$\partial^2 u - \Delta u = 0 \tag{4.0.1}$$

$$u(x,0) = f(x); \qquad \partial_t u(x,0) = g(x) \tag{4.0.2}$$

The method devised by Riemann to solve the problem of Cauchy applies to linear hyperbolic, partial differential equations of second order for one unknown function u of the independent variables. For homogeneous equation the essential points in the method are:

(i) The introduction of the characteristic as coordinate lines

(ii) The construction of the line integral

$I = \int (Bdx - Ady)$ which vanishes around closed paths where A and B are certain bilinear forms in u, u_x, u_y and v, u_x, v_y where v (Riemann's) function is a properly chosen two parameter family of solution of a second order linear partial differential equations, called the adjoint equation.

There is one advantage to be gained from solving for the Riemann's function if this is possible. Once this is determined the differential equation under question can be solved for Cauchy data on any other non-characteristic curve.

Since the Laplacian commutes with rotations, it preserves the class of radial functions, on which it reduces to an ordinary differential operator called the radial part of the Laplacian.

For $x \in \mathbb{R}^n$ and $r = |x|$, the radial part is written

$$\Delta_r = \partial_r^2 + \frac{n-1}{r}\partial r.$$

Hence we write (4.0.1-4.0.2) as:

$$\partial_t^2 u = \partial_r^2 u + \frac{n-1}{r}\partial_r u \tag{4.0.3}$$

$$u(r,0) = f(r); \qquad \partial_t u(r,0) = g(r) \tag{4.0.4}$$

The principal tool we shall use to derive an integral formula for the solution of the Cauchy problem (4.0.1) is Green's theorem

$$\iint_D (u_x + v_y)\,dxdy = \int_c (udy - vdx) \tag{4.0.5}$$

Here the line integral on the right is evaluated in the counter clockwise direction over the closed contour C bounding the region of integration D. The integrand on the left is recognized to be divergence of the vector (u,v) . Our objective is to set up such a divergence of expression $(u_x + v_y)$ involving the linear differential operator

$$L(u) = \partial_t^2 u - \partial_r^2 u - \frac{n-1}{r}\partial_r u \tag{4.0.6}$$

that appears in the partial differential equation (4.0.3).

For the purpose at hand we introduce an adjoint operator $M(v)$ on a new unknown v . The operator $M(v)$ is defined so that the combination of terms

$$vL(u) - uM(v) = u_x + v_y. \tag{4.0.7}$$

To find $M(v)$, we set ourselves the task of integrating the product $vL(u)$ by parts so as to remove differentiations from the function u . Examining the terms in $L(u)$, we are lead to write

$$v\partial_t^2 u = \partial_t(v\partial_t u - u\partial_t v) + u\partial_t^2 v$$

$$v\partial_r^2 u = \partial_r\left(v\partial_r u - u\partial_r v\right) + u\partial_r^2 v$$

$$\frac{(n-1)}{r} v\partial_r u = \partial_r\left\{\frac{(n-1)uv}{r}\right\} - u\partial_r\left\{\frac{(n-1)v}{r}\right\}$$

$$VL(u) - u\left[\partial_t^2 v - \partial_r^2 v + \partial v\left\{\left(\frac{n-1}{r}\right)v\right\}\right]$$

$$= \partial_t\left(v\partial_t u - u\partial_t v\right) - \partial_r\left\{v\partial_r u - u\partial_r v + \frac{(n-1)uv}{r}\right\}$$

$$VL(u) - uM(v) = \partial_t\left(v\partial_t u - u\partial_t v\right) - \partial_r\left\{v\partial_r u - u\partial_r v + \frac{(n-1)uv}{r}\right\} \tag{4.0.8}$$

where

$$M(v) = \partial_t^2 v - \partial_r^2 v + \partial_r\left\{\left(\frac{n-1}{r}\right)v\right\} \tag{4.0.9}$$

The equation (4.0.8) is called the Lagrange's Identity.

Solutions of the adjoint equation

$$M(v) = 0 \tag{4.0.10}$$

play an important role in the theory of linear partial differential equation (4.0.3).

Our efforts will now be directed towards expressing the general solution of (4.0.3) in terms of particular solutions of the adjoint equation (4.0.10). Inserting (4.0.7) into Green's theorem (4.0.5) to establish the fundamental formula

$$\iint_D \left\{vL(u) - uM(v)\right\} drdt = \int_C \left[\left(v\partial_t u - u\partial_t v\right) dr + \left(v\partial_r u - u\partial_r v + \frac{(n-1)}{r} uv\right) dt\right]. \tag{4.0.11}$$

Let R denote the point (ξ,η) .The characteristics through R play an important role in formula (4.0.11) since they serve to define the limits of integration here. The triangular region D bounded by the initial curve and a pair of characteristics through R, together with the Cauchy data along the arc fro P to Q of the initial curve which is cut out by these characteristics are sufficient to determine the solution u of the Cauchy problem at the point R.

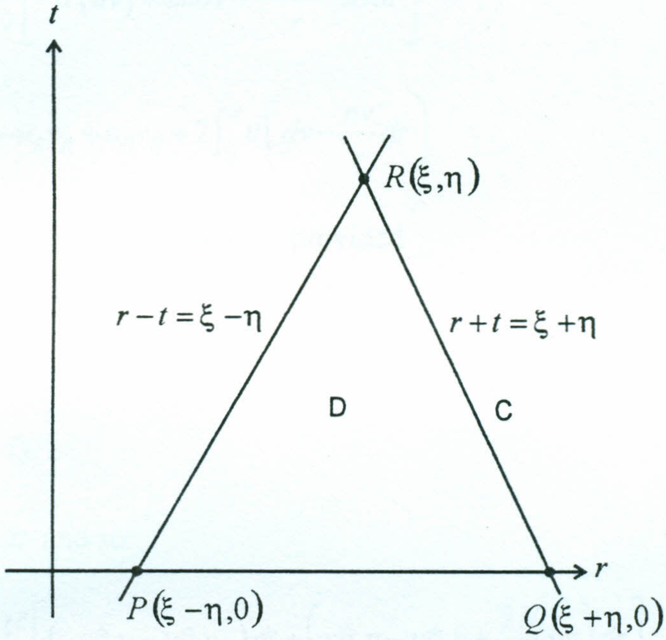

Fig. 4.1 The Characteristic Initial Value Problem

We integrate the right-hand side of (4.0.11) with (4.0.10) in mind by parts along the characteristic segments QR and RP to eliminate partial derivatives of u there.

We explicitly discuss the derivation:

Since $L(u)=0$ (4.0.11) reduces to

$$0=\int_C\left[\left(v\partial_t u-u\partial_t v\right)dr+\left(v\partial_r u-u\partial_r v+\frac{n-1}{r}uv\right)dt\right].$$

Along QR, $dr = -dt$

$$\text{r.h.s} = \int_Q^R (v\partial_t u - u\partial_t v)(-dt) + \left(v\partial_r u - u\partial_r v + \frac{n-1}{r}uv\right)(-dr) \;]$$

$$= \int_Q^R \left[-vdu + udv - \frac{(n-1)}{r}uvdr\right]$$

$$= \int_Q^R \left[-d(uv) + 2udv - \frac{(n-1)}{r}uvdr\right]$$

$$= -u_R v_R + u_Q v_Q + 2\int_Q^R u\left(dv - \frac{nv}{r}dr\right)$$

$$= -u_R v_R + u_Q v_Q + 0 \quad \text{provided}$$

$$dv = \frac{Nv}{r}dr \quad \text{on} \quad r + t = \xi + \eta \tag{4.0.12}$$

where $N = (n-1)/2$

Along RP; $dr = dt$ and so

$$\text{r.h.s.} = \int_R^P \left[(\; v\partial_t u - u\partial_t v \;)dt + \left(v\partial_r u - u\partial_r v + \frac{2N}{r}uv\right)dr\right]$$

$$= \int_R^P \left[vdu - udv + \frac{2Nuv}{r}dr\right]$$

$$= \int_R^P \left[d(uv) - 2u\left(dv - \frac{Nv}{r}dr\right)\right]$$

$$= u_P v_P - u_R v_R - 2\int_R^P u\left(dv - \frac{Nv}{r}dr\right)$$

$$= u_P v_P - u_R v_R + 0 \quad \text{provided}$$

$$dv = \frac{Nv}{r} dr \qquad \text{on} \qquad r - t = \xi - \eta \tag{4.0.13}$$

In addition we ask that

$$v(R) = 1 \tag{4.0.14}$$

With the four restrictions (4.0.10), (4.0.12), (4.0.13), and (4.0.14), we have

$$0 = -2u_R + u_Q v_Q + u_P v_P + \int_P^Q (v\partial_t u - u\partial_t v) dr \quad \text{at} \quad t = 0$$

or

$$u(\xi,\eta) = \frac{f(\xi+\eta)v_Q + f(\xi-\eta)v_P}{2} + \frac{1}{2}\int_{\xi-\eta}^{\xi+\eta} \left[v(r,0;\xi,\eta)g(r) - f(r)\partial_t(r,0;\xi,\eta)dr \right] \tag{4.0.15}$$

By integration we find that

$$V = \left(r/\xi \right)^N$$

which satisfies the conditions (4.0.12-4.0.14) but not (4.0.10). Hence we try

$$v(r,t;\xi,\eta) = \left(r/\xi \right)^N F(\mu) \quad ; \mu = -\frac{(r-\xi)^2 - (t-\eta)^2}{2r\xi}$$

For simplicity we change notations as $\partial_r \mu = \mu_r$ and $\frac{dF}{d\mu} = F'$ etc.

We have

$$\partial_t v = \left(r/\xi \right)^N F'(\mu)\mu_t$$

$$\partial_t^2 v = \left(r/\xi \right)^N \left[F''(\mu)\mu_t^2 + F'(\mu)\mu_{tt} \right]$$

$$\partial_r v = -\frac{N}{\xi}\left(\frac{r}{\xi}\right)^{N+1} F(\mu) + \left(\frac{\xi}{r}\right)^{N} F'(\mu)\mu_r$$

$$\partial_r^2 v = \frac{N(N+1)}{\xi^2}\left(r/\xi\right)^{N+2} F(\mu) - \frac{2N}{\xi}\left(r/\xi\right)^{N+1} F'(\mu)\mu_r$$

$$+\left(r/\xi\right)^{N} F''(\mu)(\mu_r)^2 + \left(r/\xi\right)^{N} F'(\mu)\mu_{rr}$$

$$\frac{N}{\xi}\partial\left[\left(\frac{\xi}{r}\right)^{N+1} F\right] = \frac{N}{\xi}\left[-\left(\frac{N+1}{\xi}\right)\left(\frac{\xi}{r}\right)^{N+2} F + \left(\frac{\xi}{r}\right)^{N+1} F'(\mu)\mu_r\right]$$

$$= \frac{-N(N+1)}{\xi^2}\left(\frac{\xi}{r}\right)^{N+2} F(\mu) + \frac{N}{\xi}\left(\frac{\xi}{r}\right)^{N+1} F'(\mu)\mu_r$$

The adjoint equation (4.0.10) becomes;

$$\left(\mu_t^2 - \mu_r^2\right)F''(\mu) + \left(\mu_{tt} - \mu_{rr} + \frac{3N}{r}\mu_r\right)F'(\mu) - \frac{2N(N+1)}{r^2}F(\mu) = 0 \qquad (4.0.16)$$

Now $\qquad \mu_r = -\frac{r-\xi}{2r\xi} - \frac{\mu}{r}$

$$\therefore \quad r\mu_r = -\mu - \frac{r-\xi}{2\xi_0}$$

$$\mu_t = \frac{t-\eta}{2r\xi} \qquad \therefore \qquad r\mu_t = \frac{t-\eta}{2\xi}$$

$$r^2\left(\mu_r^2 - \mu_t^2\right) = \left(\mu + \frac{r-\xi}{2\xi}\right)^2 - \frac{(t-\eta)^2}{4\xi^2}$$

$$= \mu^2 + \frac{\mu(r-\xi)}{\xi} + \frac{(r-\xi)^2}{4\xi^2} - \frac{(t-\eta)^2}{4\xi^2}$$

$$= \mu^2 + \frac{\mu r}{\xi} - \mu + \frac{(r-\xi)^2 - (t-\eta)^2}{4\xi^2}$$

$$= \mu^2 - \mu + \frac{\mu r}{\xi} + \frac{r}{\xi} + \frac{(r-\xi)^2 - (t-\eta)^2}{4r\xi}$$

$$= \mu^2 - \mu + \frac{\mu r}{\xi} - \frac{\mu r}{\xi}$$

$$= \mu^2 - \mu$$

Hence $F(\mu)$ satisfies

$$(\mu^2 - \mu)F''(\mu) + (2\mu - 1)F'(\mu) - N(N+1)F(\mu) = 0 \tag{4.0.17}$$

On letting $w = 1 - 2\mu$, then $F(w)$ satisfies

$$(1-w^2)F''(w) - 2wF'(w) + N(N+1)F(w) = 0$$

which is the Legendre differential equation of order N.

Thus $F(w) = P_N(w) = P_N\left\{1 + \frac{(r-\xi)^2 - (t-\eta)^2}{2r\xi}\right\}$

$$= P_N\left\{\frac{r^2 + \xi^2 - (t-\eta)^2}{2r\xi}\right\} \tag{4.0.18}$$

Hence the Riemann's function for the equation (4.1.3) is

$$v(r,t;\xi,\eta) = \left(\xi/_r\right)^N P_N\left\{\frac{r^2 + \xi^2 - (t-\eta)^2}{2r\xi}\right\}$$

Thus $v(x,0;\xi,\eta) = \left(\xi/_r\right)^N P_N\left\{\frac{r^2 + \xi^2 - \eta^2}{2r\xi}\right\}$

$$\partial_t v = \left(\xi/_r\right) P_N'\left\{\frac{r^2 + \xi^2 - (t-\eta)^2}{2r\xi}\right\}\left\{-\frac{(t-\eta)}{r\xi}\right\}$$

$$\therefore \quad v_t(x,0;\xi,\eta) = -\frac{\eta}{r\xi}\left(\frac{\xi}{r}\right)^N P_N'\left(\frac{r^2+\xi^2-\eta^2}{2r\xi}\right)$$

where dashes denote differentiation with respect to argument.

Hence

$$u(\xi,\eta) = \frac{u_P v_P + u_Q v_Q}{2} + \frac{1}{2}\int_{\xi-\eta}^{\xi+\eta}\left[v(r,0;\xi,\eta)\partial_t u - u(x,0)\partial_t v\big|_{t=0}\right]dr \qquad (4.0.19)$$

Now $\qquad v_P(r,0;\xi,\eta) = \dfrac{\xi^N}{(\xi-\eta)^N} P_N\left[\dfrac{(\xi-\eta)^2+\xi^2-\eta^2}{2\xi(\xi-\eta)}\right]$

$$= \frac{\xi^N}{(\xi-\eta)^N} P_N(1) = \frac{\xi^N}{(\xi-\eta)^N} \qquad \text{provided} \quad \xi \neq \eta$$

and $\quad P_N(1) = 1 \quad \forall \quad N$.

Similarly

$$v_B(r,0;\xi,\eta) = \xi^N/(\xi+\mu)^N$$

Hence the solution

$$u(\xi,\eta) = \frac{f(\xi+\eta)\left(\frac{\xi}{\xi+\eta}\right)^N + f(\xi-\eta)\left(\frac{\xi}{\xi-\eta}\right)^N}{2} + \frac{1}{2}\int_{\xi-\eta}^{\xi+\eta}\left\{g(\tau)\left(\frac{\xi}{\tau}\right)^N P_N\left(\frac{\tau^2+\xi^2-\eta^2}{2\tau\xi}\right)\right.$$

$$\left. - f(\tau)\left[\frac{\eta\xi^{N-1}}{\tau^{N-1}} P_N'\left(\frac{\tau^2+\xi^2-\eta^2}{2\tau\xi}\right)\right]\right\} d\tau$$

In the variables (r,t) we have

$$u(r,t) = \frac{r^N\left[\ f(r-t)(r+t)^N + (r-t)^N f(r+t)\ \right]}{2\left(r^2-t^2\right)^N} \quad + \frac{r^N}{2}\int_{r-t}^{r+t}\tau^{-N} g(\tau) P_N\left(\frac{\tau^2+r^2-t^2}{2r\tau}\right)d\tau$$

$$-\frac{r^{N-1}}{2}t\int_{x-t}^{x+t}\tau^{-N-1}f(\tau)P_N'\left(\frac{\tau^2+r^2-t^2}{2r\tau}\right)d\tau \tag{4.0.20}$$

For $N=0 \Rightarrow$ n=1, (4.0.3) reduces to one-dimension equation whose solution by (4.0.20) is

$$\text{u}(\text{r,t})=\frac{f(r+t)+f(r-t)}{2}+\frac{1}{2}\int_{r-t}^{r+t}g(\tau)P_0\left(\frac{\tau^2+r^2-t^2}{2\tau r}\right)d\tau$$

$$-\frac{t}{2r}\int_{r-t}^{r+t}\tau^{-1}f(\tau)P_0'\left(\frac{\tau^2+r^2-t^2}{2\tau r}\right)d\tau$$

$$=\frac{f(r+t)+f(r-t)}{2}+\frac{1}{2}\int_{r-t}^{r+t}g(\tau)d\tau \text{ ; since } P_0'(x)=0$$

This is D'Alemberts formula.

Theorem 4.1.

If f is C^2 and g is C^1 on $\mathbb{R}^n$, then u is C^2 on $\mathbb{R}^n\times\mathbb{R}$ and satisfies (4.0.20) in the classical sense. We can afford a partial check by taking

$N=1,\quad f(x)=r^3,\quad g(x)=r^2$. Then

$$u(r,t)=\frac{r\left[(r-t)(r+t)^3+(r+t)(r-t)^3\right]}{2(r^2-t^2)}$$

$$+\frac{1}{4}\int_{r-t}^{r+t}\left(\tau^2+r^2-t^2\right)d\tau-\frac{t}{2}\int_{r-t}^{r+t}\tau d\tau \tag{4.0.21}$$

$$=x^3+x^2t-\frac{1}{3}t^3$$

Hence the unique solution of the equation

$$\partial_t^2u-\partial_r^2-\frac{2}{r}\partial_r u=0 \tag{4.0.22}$$

which satisfies the initial conditions $u(r,0)=r^3, \quad \partial_r u(r,0)=r^2$ is given by (4.0.21).

Verification :

$$u(r,0)=r^3$$
$$\partial_t u(r,t)=r^2-t^2$$
$$\partial_t u(r,0)=r^2$$

Also

$$\partial_r u = 3r^2+2rt$$
$$\partial_r^2 u = 6r+2t$$
$$\partial_t u = r^2-t^2$$
$$\partial_t^2 u = -2t$$

$$\left(\partial_r^2-\partial_t^2-\frac{2}{r}\partial_r\right)u = 6r+2t+2t-\frac{2}{r}\left(3r^2+2rt\right)$$

$$= 6r+4t-6r-4t=0 \quad \square$$

CHAPTER FIVE

§ 5.0 SINGULAR CAUCHY PROBLEM OF EULER POISSON DARBOUX EQUATION

§ 5.1 Introduction

The singular Cauchy problem for Euler Poison Darboux (EPD) equation can be formulated as follows: Let $f(x) = f(x_1,...,x_n)$ be an arbitrary function is differentiable continuously. It is required to find a function $u(x)$ which satisfies the conditions;

$$\Delta u = \partial_t^2 u + \frac{k}{t}\partial_t u \tag{5.1.1}$$

$$u(x,0) = f(x), \quad \partial_t u(x,0) = 0 \tag{5.1.2}$$

where in the EPD equation (5.1.1) it is understood that Δ is n-dimensional Laplace operator, $x = (x_1, x_2,..., x_n)$ is a point in $\mathbb{R}^n$, k is a real parameter and t is the time variable.

§ 5.2 CONSTRUCTION OF SOLUTION

Now for $x \in \mathbb{R}^n$ and $r = |x|$, then

$$\Delta\phi(r) = \partial_r^2 + \frac{n-1}{r}\partial_r\phi .$$

Thus we write (5.1.1) as

$$\partial_t^2 u + \frac{k}{t}\partial_t u = \partial_r^2 u + \frac{n-1}{r}\partial_r u \tag{5.2.1}$$

We solve this problem for $k = n-1$ which we rename as N.

The characteristics of (5.2.1) being $r \pm t$, we reduce (5.2.1) to canonical form:

$$4\partial_{XY}U + \frac{N}{X+Y}(\ \partial_X U + \partial_Y U\ ) - \frac{N}{X-Y}(\partial_X U - \partial_Y U) = 0$$

where $X = r + t, \quad Y = r - t$. Further, the substitution $r = X^2$ and $s = -Y^2$ leads to

$$(r-s)\partial_{rs}^2 u - N(\partial_r u - \partial_s u) = 0 \tag{5.2.2}$$

We next determine the adjoint equation of (5.2.2) whose solution is a two parameter function denoted by

$v(r,s;\bar{r},\bar{s})$ known as Riemann's function.

Let

$$L(u) = (r-s)\partial_{rs}^2 u - N(\partial_r u - \partial_s u) \tag{5.2.3}$$

Then

$$vL(u) = (r-s)\partial_{rs}^2 u - Nv(\partial_r u - \partial_s u) \tag{5.2.4}$$

$$(r-s)v\partial_{rs}^2 u = \partial_r\{(r-s)v\partial_s u\} - \partial_r\{(r-s)v\}\partial_s u$$

$$= \partial_r\{(r-s)v\partial_s u\} - \partial_s\{u\partial_r(r-s)v\} + u\partial_{rs}^2\{(r-s)v\}$$

$$= \partial_r\{(r-s)v\partial_s u\} - \partial_s\{uv + (r-s)u\partial_r v\}$$

$$+u\{-\partial_r v + \partial_s v + (r-s)\partial_{rs}^2 v\}. \tag{5.2.5}$$

For geometry; we also have

$$(r-s)v\partial_{rs}^2 u = \partial_s\{(r-s)v\partial_r u\} - \partial_r\{-uv + (r-s)u\partial_s v\}$$

$$+u\{-\partial_r v + \partial_s v + (r-s)\partial_{rs}^2\}. \tag{5.2.6}$$

$$-Nv\partial_r u = -\partial_r(Nuv) + u\partial_r(Nv) \tag{5.2.7}$$

$$Nv\partial_s u = \partial_s(Nuv) - u\partial_s(Nv) \tag{5.2.8}$$

Using (5.2.5-5.2.8); results

$$vL(u)-uM(v)=\partial_r\left[\frac{r-s}{2}(v\partial_s u-u\partial_s v)-\left(N-\frac{1}{2}\right)uv\right]$$

$$+\partial_s\left[\frac{r-s}{2}(v\partial_r u-u\partial_r v)-\left(N-\frac{1}{2}\right)uv\right] \tag{5.2.9}$$

where

$$M(v)=(r-s)\partial_{rs}^2 v+(N-1)(\partial_r v-\partial_s v) \tag{5.2.10}$$

is the adjoint equation of (5.2.3).

We take recourse of the figure5.1 below to write the solution of (5.2.1)

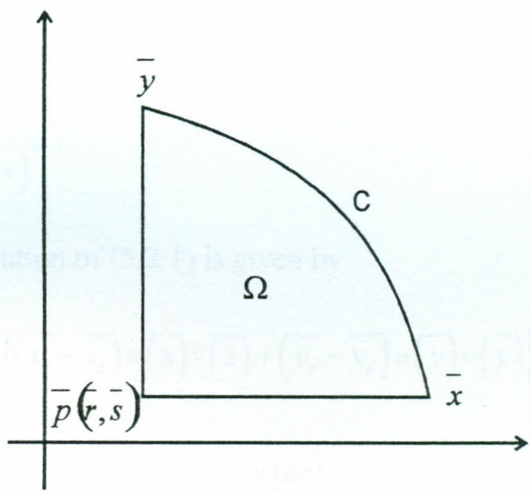

Fig. 5.1 Singular Cauchy Problem

Applying Green's theorem to the identity (5.2.9);

$$\iint_\Omega\{vL(u)-uM(v)\}drds=\frac{1}{2}\int_C\left\{\left[(r-s)(v\partial_s u-u\partial_s v)-\left(N-\frac{1}{2}\right)uv\right]ds\right.$$

$$\left.-\left[(r-s)(v\partial_r u-u\partial_r v)-\left(N-\frac{1}{2}\right)uv\right]dr\right\} \tag{5.2.11}$$

We require that $M(v)=0$; then, along $\overline{yp}$ we have $dr=0$ and

$$0=\frac{1}{2}\int_{\bar{y}}^{\bar{p}}\left[(r-s)(v\partial_s u-u\partial_s v)-\left(N-\frac{1}{2}\right)uvds\right]$$

$$=\frac{1}{2}\int_{\bar{y}}^{\bar{p}}\left\{(r-s)\left[(v\partial_s u+u\partial_s v)-2u\partial_s v\right]-\left(N-\frac{1}{2}\right)uv\right\}ds.$$

On integrating by parts we get

$$0=-\frac{1}{2}(r-s)\left[u(\bar{p})v(\bar{p})+(r-s)u(\bar{y})v(\bar{y})\right]+0 \quad \text{provided}$$

$$v(\bar{r},s;\bar{r},\bar{s},)=(\bar{r}-s)^{N-1}$$

and along $\overline{px}$;

$$0=-\frac{1}{2}(r-s)\left[u(\bar{x})v(\bar{x})-u(\bar{p})v(\bar{p})\right]+0$$

$$v(r,\bar{s};\bar{r},\bar{s},)=(r-\bar{s})^{N-1}$$

The required Riemann's solution of (5.2.1) is given by

$$(\bar{r},\bar{s},)u(\bar{p})v(\bar{p})=\frac{1}{2}\left\{(\overline{x_r}-\overline{x_s})u(\bar{x})v(\bar{x})+(\overline{y_r}-\overline{y_s})u(\bar{y})v(\bar{y})\right\}$$

$$-\int_{\bar{x}}^{\bar{y}}\left\{p(u,v)ds+Q(u,v)dr\right\} \tag{5.2.12}$$

where $\bar{x}=(\overline{x_r}-\overline{x_s})$, $\bar{y}=(\overline{y_r}-\overline{y_s})$,

$$P(u,v)=\frac{1}{2}(r-s)(v\partial_s u-u\partial_s v)-\left(N-\frac{1}{2}\right)uv,$$

$$Q(u,v)=\frac{1}{2}(r-s)(v\partial_r u-u\partial_r v)-\left(N-\frac{1}{2}\right)uv.$$

To this far what only remains to be determined is v, the Riemann's function. It is a two parameter family $v = v\left(r, s; \bar{r}, \bar{s}\right)$ of solutions of adjoint equation (5.2.10); which satisfies the conditions

$$v\left(\bar{r}, s; \bar{r}, \bar{s}\right) = \left(\bar{r} - s\right)^{N-1}, \qquad v\left(r, \bar{s}; \bar{r}, \bar{s}\right) = \left(r - \bar{s}\right)^{N-1} \tag{5.2.13}$$

A one parameter family of solutions of (5.2.10) was given by Darboux (1915) as

$$v\left(r, s\right) = \left(r - a\right)^{N-1} \left(a - s\right)^{N-1} \tag{5.2.14}$$

a being the parameter of the family. The one parameter (5.2.14) offers the starting point for our construction of the Riemann function. The form of the Riemann function usually predicted by what Copson (1958) calls an "inspired guess" rather than following a formal procedure which requires no intuition. We may accordingly confine our attention to the half plane lying under the line $r = s$.

We construct the Riemann function for (5.2.4) under the following restrictions:

$$r > \bar{r} > 0, \quad s < \bar{s}, \quad N = -\lambda, \quad \left(0 < \lambda < 1\right) \tag{5.2.15}$$

By forming the solution (cf .Darboux, op. cit; pp-66-68);

$$V = \int_{s} \phi\left(a\right)\left(r - a\right)^{-\lambda} \left(a - s\right)^{-\lambda} da,$$

where $\phi\left(a\right)$ is an arbitrary function, we gain a solution of (5.2.10) containing an arbitrary function. Taking $\phi\left(a\right) = 0$ for $\bar{s} \le a \le \bar{r}$, we write a two parameter family of solutions

$$v=\int_{\bar{r}}^{r}\phi(a)(r-a)^{-\lambda}(a-s)^{-\lambda}\,da+\int_{s}^{\bar{s}}\phi(a)(r-a)^{-\lambda}(a-s)^{-\lambda}\,da \qquad (5.2.16)$$

involving an arbitrary function.

We determine this arbitrary function $\phi(a)$ such that the conditions (5.2.13) are met. We consider the first of the integrals. We define

$$\rho(t)=G(t)=0, 0\le t\le \bar{r},$$

$$\rho(t)=\phi(t)\left(t-\bar{s}\right)^{-\lambda}(r-s)^{-\lambda}\left(r-\bar{r}\right),$$

$$G(t)=t-\bar{r}; \qquad t\ge \bar{r}$$

Then by use of the fact that the solution of Abel's integral equation; Bucher (1929)

$$f(x)=\int_{a}^{x}\frac{u(\xi)\,d\xi}{(x-\xi)^{\lambda}}\ ; \qquad 0<\operatorname{Re}\lambda<1$$

is

$$u(x)=\frac{\sin\pi\lambda}{\pi}\frac{d}{dx}\int_{a}^{x}\frac{f(t)\,dt}{(x-t)^{1-\lambda}}$$

we have

$$\int_{0}^{r}\rho(t)(r-t)^{-\lambda}\,dt=G(t)\ ;$$

The solution of this integral is

$$\rho(a)=\frac{\sin\pi\lambda}{\pi}\int_{0}^{a}G'(t)(a-t)^{\lambda-1}\,dt=\frac{\sin\pi\lambda}{\pi}\int_{\bar{r}}^{a}(a-t)^{\lambda-1}\,dt,$$

from which we obtain

$$\frac{\sin\pi\lambda}{\pi\lambda}\frac{\left(a-\bar{r}\right)^{\lambda}\left(a-\bar{s}\right)^{\lambda}(r-s)^{\lambda}}{r-\bar{r}} \qquad (5.2.17)$$

Substituting $\bar{\phi}(a)$ in (5.2.16), we obtain

$$v=\int_{\bar{r}}^{r}\frac{\sin\pi\lambda}{\pi\lambda}\frac{\left(a-\bar{r}\right)^{\lambda}\left(a-\bar{s}\right)^{\lambda}\left(r-s\right)^{\lambda}}{\left(r-\bar{r}\right)\left(r-a\right)^{\lambda}\left(a-s\right)^{\lambda}}da+\int_{s}^{\bar{s}}\frac{\sin\pi\lambda}{\pi\lambda}\frac{\left(a-\bar{r}\right)^{\lambda}\left(a-\bar{s}\right)^{\lambda}\left(r-s\right)^{\lambda}}{\left(s-\bar{s}\right)\left(r-a\right)^{\lambda}\left(a-s\right)^{\lambda}}da\,. \quad (5.2.18)$$

On introducing a new variable t by setting

$$a=\bar{r}+t\left(r-\bar{r}\right);\qquad a=\bar{s}+t\left(s-\bar{s}\right) \quad (5.2.19)$$

in the first and second integrals respectively, we obtain;

$$V=\frac{\sin\pi\lambda}{\pi\lambda}\left\{\int_0^1\frac{t^{\lambda}\left(r-\bar{r}\right)^{\lambda}\left[\bar{r}-\bar{s}+t\left(r-\bar{r}\right)\right]^{\lambda}\left(r-s\right)^{\lambda}}{\left(r-\bar{r}\right)^{\lambda}\left(1-t\right)^{\lambda}\left[\bar{r}-s+t\left(r-\bar{r}\right)\right]^{\lambda}}dt\right.$$

$$\left.+\int_0^1\frac{\left[\left(\bar{s}-\bar{r}\right)+t\left(s-\bar{s}\right)\right]^{\lambda}t^{\lambda}\left(s-\bar{s}\right)^{\lambda}\left(r-s\right)^{\lambda}}{\left[\left(r-\bar{s}\right)-t\left(s-\bar{s}\right)\right]^{\lambda}\left(s-\bar{s}\right)^{\lambda}\left(1-t\right)^{\lambda}}dt\right\}.$$

Hence the Riemann function v is obtained in the form

$$v=\frac{\sin\pi\lambda}{\pi\lambda}\left\{\left(r-s\right)^{\lambda}\int_0^1\frac{t^{\lambda}\left[\bar{r}-\bar{s}+t\left(r-\bar{r}\right)\right]^{\lambda}}{\left(1-t\right)^{\lambda}\left[\bar{r}-s+t\left(r-\bar{r}\right)\right]^{\lambda}}dt\right.$$

$$\left.+\left(r-s\right)^{\lambda}\int_0^1\frac{t^{\lambda}\left[\bar{r}-\bar{s}-t\left(s-\bar{s}\right)\right]^{\lambda}}{\left(1-t\right)^{\lambda}\left[r-\bar{s}-t\left(s-\bar{s}\right)\right]^{\lambda}}dt\right\} \quad (5.2.20)$$

This Riemann function can be expressed in terms of the Appell's hypergoemetric functions of two variables, A. Erdélyi (1950)

$$F_1\left(\alpha,\beta,\beta';\gamma;x,y\right)=\sum_{m,n=0}^{\infty}\frac{\left(\alpha,m+n\right)\left(\beta,m\right)\left(\beta',n\right)}{\left(\gamma,m+n\right)\left(1,m\right)\left(1,n\right)}x^m y^n, \quad (5.2.21)$$

$$(\alpha,r)=\alpha(\alpha+1)...(\alpha+r-1).$$

Now we have

$$\int_0^1 t^{\alpha-1}(1-t)^{\gamma-\alpha-1}(1-tx)^{-\beta}(1-ty)^{-\beta'}\,dt=\frac{\Gamma(\alpha)\Gamma(\gamma-\alpha)}{\Gamma(\gamma)}F_1$$

where F_1 is given by (5.2.21). $\Gamma(z)$ and $\beta(1+\lambda,1-\lambda)$ are Gamma and Beta functions respectively.

If we put $\alpha=1+\lambda$, $\beta=-\lambda$, $\beta'=\lambda$, $\gamma=2$ in this formula we observe that

$$\frac{\Gamma(\alpha)\Gamma(\gamma-\alpha)}{\Gamma(\gamma)}=\frac{\Gamma(1+\lambda)\Gamma(1-\lambda)}{\Gamma(2)}=\Gamma(1+\lambda)\Gamma(1-\lambda).$$

Theorem 5.1.

$$\Gamma(1+\lambda)\Gamma(1-\lambda)=\frac{\pi\lambda}{\sin\pi\lambda}.$$

Proof. Let $x={}^{t}\!/\!_{(1-t)}$; $dx=\dfrac{dt}{(1-t)^2}$

$$\therefore\qquad \int_0^\infty\frac{x^\lambda}{(1+x)^2\,dx}=\int_0^1 t^\lambda(1-t)^{-\lambda}\,dt$$

$$=B(1+\lambda,1-\lambda)=\frac{\Gamma(1+\lambda)\Gamma(1-\lambda)}{\Gamma(2)}=\Gamma(1+\lambda)\Gamma(1-\lambda)........(a)$$

On the other hand, consider

$$\oint\frac{z^\lambda}{(1+z)^2}\,dz\ ;\qquad |\lambda|<1.$$

Since $z=0$ is a branch point,

choose C as the contour of figure 5.2 where the positive real axis is the branch line and where AB and GH is coincident with the x-axis but are separated for visual purposes.

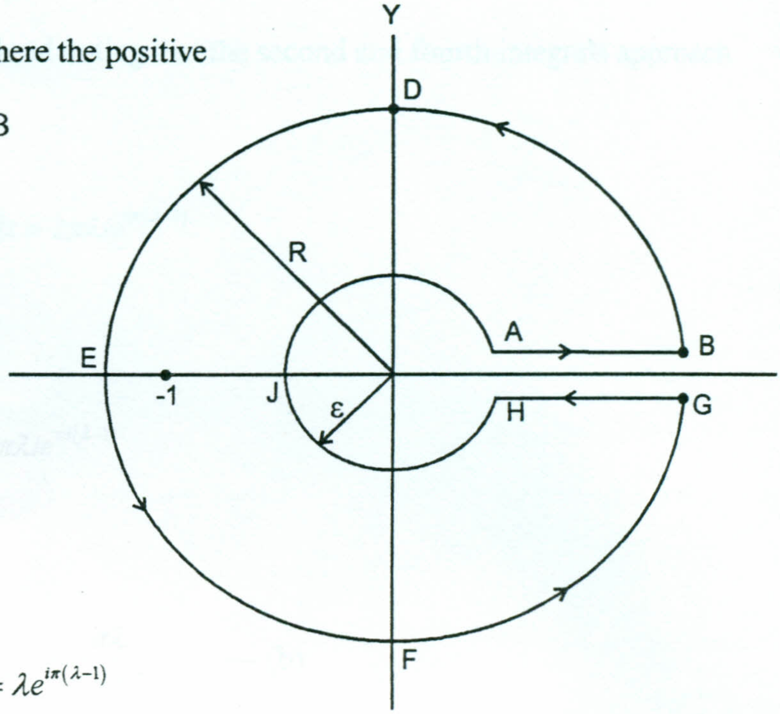

Fig. 5.2 Contour Integration

The integrand has the pole $z=-1$ of multiplicity two inside C.

Residue at $z=-1=e^{\pi i}$ is

$$\lim_{z\to -1}\frac{d}{dz}\left\{(z+1)^2 . \frac{z^{\lambda}}{(z+1)^2}\right\} = \lambda e^{i\pi(\lambda-1)}$$

By residue's theorem we have

$$\oint_C \frac{z^{\lambda}}{(1+z)^2}dz = 2\pi i\lambda e^{i\pi(\lambda-1)},$$ or omitting the integrand,

$$\int_{AB}+\int_{BDEFG}+\int_{GH}+\int_{HJA}=2\pi i\lambda e^{i\pi(\lambda-1)}$$

We thus have

$$\int_{\varepsilon}^{R}\frac{x^{\lambda}}{(1+x)^2}dx+\int_0^{2\pi}\frac{\left(Re^{i\theta}\right)^{\lambda}.i\mathrm{Re}^{i\theta}\,d\theta}{\left(1+\mathrm{Re}^{i\theta}\right)^2}+\int_R^{\varepsilon}\frac{\left(xe^{2\pi i}\right)^{\lambda}}{\left(1+x\mathrm{e}^{2\pi i}\right)^2}+\int_{2\pi}^{0}\frac{\left(\varepsilon e^{i\theta}\right)^{\lambda}i\varepsilon e^{i\theta}d\theta}{\left(1+\varepsilon e^{i\theta}\right)^2}=2\pi\lambda ie^{i\pi(\lambda-1)}$$

where we have used $z=xe^{2\pi i}$ for the integral along GH, since the argument of z is increased by 2π

in going around the circle BDEFG.

Taking the limit as $\varepsilon \to 0$ and $R \to \infty$ and noting that the second and fourth integrals approach zero, we find

$$\int_0^\infty \frac{x^\lambda}{(1+x)^2}dx + \int_\infty^0 \frac{e^{2\pi i\lambda}.x^\lambda}{(1+x)^2}dx = 2\pi\lambda i e^{i\pi(\lambda-1)}$$

or

$$\left(1-e^{2\pi i\lambda}\right)\int_0^\infty \frac{x^\lambda}{(1+x)^2}dx = 2\pi\lambda i e^{\pi i(\lambda-1)}$$

so that

$$\int_0^\infty \frac{x^\lambda}{(1+x)^2}dx = \lambda\pi.\frac{2i}{e^{\pi\pi i}-e^{-\pi\lambda i}} = \frac{\pi\lambda}{\sin\pi\lambda} \qquad \text{---- (b)}$$

From (a) and (b) the result follows □

We find on writing;

$$x=\frac{r-\bar r}{\bar s-\bar r}, \qquad y=\frac{r-\bar r}{s-\bar r}, \quad \text{and} \quad x=\frac{s-\bar s}{\bar r-\bar s}, \qquad y=\frac{s-\bar s}{r-\bar s}$$

that the Riemann function may be expressed as follows:

$$v=\left(r-\bar r\right)\left(\frac{\bar r-\bar s}{\bar r-s}\right)^\lambda F_1\left(1+\lambda;-\lambda;\lambda;2;\frac{r-\bar r}{\bar s-\bar r},\frac{r-\bar r}{s-\bar r}\right)$$

$$-\left(s-\bar s\right)\left(\frac{\bar r-\bar s}{r-\bar s}\right)^\lambda F_1\left(1+\lambda;-\lambda;\lambda;2;\frac{s-\bar s}{\bar r-\bar s},\frac{s-\bar s}{r-\bar s}\right) \qquad (5.2.22)$$

It will be recalled that the Riemann function was derived under the restrictions (5.2.15). These restrictions we now proceed to remove. Availing ourselves of the formula: Bailey, (1935)

$$F_1(\alpha;\beta,\beta';\gamma;x,y)=(1-x)^{-\beta}(1-y)^{-\beta'}F_1\left(\gamma-\alpha;\beta,\beta';\gamma;\frac{x}{x-1},\frac{y}{y-1}\right),$$

We find alternative expressions for the hypergeometric functions in (5.2.22), namely,

$$F_1\left(1+\lambda;-\lambda,\lambda;2;\frac{r-\bar r}{\bar s-\bar r},\frac{r-\bar r}{s-\bar r}\right)=\left(\frac{r-\bar s}{\bar r-\bar s}\right)^{\lambda}\left(\frac{\bar r-s}{r-s}\right)^{\lambda}F_1\left(1-\lambda;-\lambda,\lambda;2;\frac{r-\bar r}{r-\bar s},\frac{r-\bar r}{r-s}\right),$$

$$F_1\left(1-\lambda;-\lambda,\lambda;2;\frac{s-\bar s}{\bar r-\bar s},\frac{s-\bar s}{r-\bar s}\right)=\left(\frac{\bar r-s}{\bar r-\bar s}\right)^{\lambda}\left(\frac{r-\bar s}{r-s}\right)^{\lambda}F_1\left(1-\lambda;-\lambda,\lambda;2;\frac{s-\bar s}{s-\bar r},\frac{s-\bar s}{s-r}\right)$$

and thus we are let to consider three additional forms for v:

$$v=(r-\bar r)\left(\frac{r-\bar s}{r-s}\right)^{\lambda}F_1\left(1-\lambda;-\lambda;\lambda;2;\frac{r-\bar r}{r-\bar s},\frac{r-\bar r}{r-s}\right)$$

$$-(s-\bar s)\left(\frac{\bar r-\bar s}{\bar r-s}\right)^{\lambda}F_1\left(1-\lambda;-\lambda;\lambda;2;\frac{s-\bar s}{\bar r-\bar s},\frac{s-\bar s}{r-\bar s}\right), \tag{5.2.23}$$

$$v=(r-\bar r)\left(\frac{\bar r-\bar s}{\bar r-s}\right)^{\lambda}F_1\left(1+\lambda;-\lambda,\lambda;2;\frac{r-\bar r}{\bar s-\bar r},\frac{r-\bar r}{s-\bar r}\right)$$

$$-(s-\bar s)\left(\frac{\bar r-s}{r-s}\right)^{\lambda}F_1\left(1-\lambda;-\lambda,\lambda;2;\frac{s-\bar s}{s-\bar r},\frac{s-\bar s}{s-r}\right) \tag{5.2.24}$$

$$v=(r-\bar r)\left(\frac{r-\bar s}{r-s}\right)^{\lambda}F_1\left(1-\lambda;-\lambda,\lambda;2;\frac{r-\bar r}{r-\bar s},\frac{r-\bar r}{r-s}\right)$$

$$-\left(s-\bar{s}\right)\left(\frac{\bar{r}-s}{r-s}\right)^{\lambda} F_1\left(1-\lambda;-\lambda,\lambda;2;\frac{s-\bar{s}}{s-\bar{r}},\frac{s-\bar{s}}{s-r}\right). \tag{5.2.25}$$

Appell (1925) has shown that the power series (5.2.21) converges for arbitrary real or complex values of α,β,γ Inside the unit square $|x|<1,\quad |y|<1$, provided of course, γ is not a negative integer.The four forms (5.2.22-.5.225) obtained for v are analytic conditions of each other and taken together suffice to define the Riemann's function throughout the region

$$s<\bar{r},\quad r>s,\quad r>\bar{s},$$

for any real value of λ. Thus v is adequately determined.

The formula (Appell, 1925)

$$F_1\left(\alpha;\beta,\beta';\gamma;x,x\right)=F\left(\alpha,\beta+\beta';\gamma;x\right),$$

reducing the Appell's hypergeometric function to ordinary hypergeometric function affords a partial check on our computation. Taking $r=\bar{r}$ and $s=\bar{s}$ in (5.2.22) yields

$$v\left(\bar{r},s;\bar{r},\bar{s}\right)=\left(\bar{r}-s\right)^{N-1}F\left(1+\lambda,0;2;\frac{s-\bar{s}}{\bar{r}-\bar{s}}\right)=\left(\bar{r}-s\right)^{N-1}$$

$$v\left(r,\bar{s};\bar{r},\bar{s}\right)=\left(r-\bar{s}\right)^{N-1}F\left(1+\lambda,0;2;\frac{r-\bar{r}}{\bar{s}-\bar{r}}\right)=\left(r-\bar{s}\right)^{N-1}$$

which agrees with (5.2.13).

As an example we consider the simplest case $N=1$ so that the partial differential equation (5.2.2) becómes

$$(r-s)\partial_{rs}^{2}u-(\partial_{r}u-\partial_{s}u)=0 \tag{5.2.26}$$

Taking $\lambda=-1$, the Riemann's function of (5.2.26) is calculated with the aid of (5.2.21) as;

$$v=\frac{(r-\bar{r})(r-\bar{s})}{\bar{r}-\bar{s}}F_{1}\left(0;1,-1;2;\frac{r-\bar{r}}{\bar{s}-\bar{r}},\frac{r-\bar{r}}{\bar{s}-\bar{r}}\right)$$

$$-\frac{(s-\bar{s})(r-\bar{s})}{\bar{r}-\bar{s}}F_{1}\left(0;1,-1;2;\frac{s-\bar{s}}{\bar{r}-\bar{s}},\frac{s-\bar{s}}{\bar{r}-\bar{s}}\right)$$

$$=\frac{(r-\bar{r})(r-\bar{s})-(s-\bar{s})(r-\bar{s})}{\bar{r}-\bar{s}}$$

since $F_{1}(0;1-1;2;x,y)=\sum_{m,n=0}^{\infty}\frac{(0,m+n)(1,m)(-1,n)}{(2,m+n)(1,m)(1,n)}x^{m}y^{n}=1$

and a direct computation verifies that this function v is the solution of the adjoint equation to (5.2.26) and fulfills the conditions (5.2.13)

CHAPTER SIX

§ 6.0 A Generalized Singular Cauchy Problem of the Euler Poisson Darboux Equation

In this chapter, a solution is given for the following Singular Cauchy Problem:

$$\Delta u = u_{tt} + \left(mt + \frac{n}{t} \right) u_t \tag{6.0.1}$$

$$u(x,0) = \delta, u_t(x,0) = 0 \tag{6.0.2}$$

The solution is given as an integral equation that is continuous and analytic in the interval containing the singular point. m *and* n are any real numbers while δ is a Dirac delta function.

The value of t in the Cauchy data (6.0.2) must be understood in the limiting sense $t \to 0^+$.

Dernek (2002) studied the generalized singular Cauchy problem for EPD:

$$\Delta u = u_{tt} + \left(mt + {}^{n}\!/_{t} \right) u_t \tag{6.0.3}$$

$$u(x,0) = f(x), u_t(x,0) = 0 \tag{6.0.4}$$

where he obtained a solution of the series form.

In this chapter, we determine the solution to the problem (6.0.3) subject to

$$u(x,0) = \delta,\ u_t(x,0) = 0 \tag{6.0.5}$$

via integral equation that is solved iteratively into a series that is continuous and analytic on some interval that contains the singular point.

§ 6.1 A FORMAL SOLUTION:

On taking Fourier transform of (6.0.3) and (6.0.5); we obtain

$$\bar{u}_{tt}+\left(mt+\tfrac{n}{t}\right)\bar{u}+A(\xi)\bar{u}=0 \tag{6.0.6}$$

$$\bar{u}(\xi,0)=1;\ \bar{u}_t(\xi,0)=0 \tag{6.0.7}$$

where we have taken A as the Fourier transform of Δ.

Take $$\eta(t)=\exp\left(-\int_t^b\left(m\zeta+\frac{n}{\zeta}\right)\right)d\zeta=\left(\frac{t}{b}\right)^n\exp\left[\frac{m\left(t^2-b^2\right)}{2}\right]$$

So that

$$\eta(\tau)/\eta(t)=\left(\frac{\tau}{t}\right)^n\exp\left[\frac{\left(\tau^2-t^2\right)}{2}\right]<1$$

for $0\le\tau\le t\le b<\infty$ *and* $\eta(t)\to 0$ *as* $t\to 0$

With the $\eta(t)$ so chosen (6.0.6) reduces to

$$\frac{d}{dt}\left(\eta(t)\bar{u}_t\right)+\eta(t)A(\xi)\bar{u}=0 \tag{6.0.8}$$

This leads to integral equation $(\tau\le\xi\le\sigma\le t)$ equivalent to (6.0.6) with initial condition $t=\tau$ as

$$\bar{u}(\xi,t,\tau)=1-A(\xi)\int_\tau^t\frac{1}{\eta(\sigma)}\int_\tau^\sigma\eta(\xi)\bar{u}(\xi,\zeta,\tau)\,d\zeta\,d\sigma$$

$$=1-A(\xi)\int_\tau^t\int_\tau^\sigma\left(\frac{\zeta}{\sigma}\right)^n\exp\left[\frac{m\left(\zeta^2-\sigma^2\right)}{2}\right]\bar{u}(\xi,\zeta,\tau)\,d\zeta\,d\sigma \tag{6.0.9}$$

We may write this integral in the form

$$\overline{u_k} = 1 - A(\xi)\int_\tau^t\int_\tau^\sigma\left(\frac{\zeta}{\sigma}\right)^n \exp\left[\frac{-m(\sigma^2-\xi^2)}{2}\right]\overline{u}_{k-1}(\xi,\zeta,\tau)\,d\zeta \qquad (6.0.10)$$

If we let $\overline{u_0}(\xi,t,\tau)=1$; (6.0 10) is an integral representation for solutions of (6.0.6)-(6.0.7). We consider that the integral (6.0.10) are calculated on the domain which is defined as $[0\le\tau\le t\le b]\times k.$ If we transform the variables as follows:

$$\rho = \sigma^2 - \xi^2;\quad r = \frac{\xi}{\sigma}.$$

The functional determinant of this transformation is $\dfrac{\partial(\sigma,\xi)}{\partial(\rho,r)} = \dfrac{1}{2(1-r^2)};\ r\ne 1.$

Under this transformation (6.0.10) takes the form;

$$\overline{u_k}(\xi,t,\tau) = 1 - A(\xi)\int_{r=\xi/\sigma}^{1}\int_{\rho=\tau}^{t^2(1-r^2)} r^n \exp\left[\frac{-m\rho}{2}\right]\overline{u}_{k-1}\left(r\sqrt{\rho/(1-r^2)},\zeta,\tau\right)\frac{dr\,d\rho}{2(1-r^2)} \qquad (6.0.11)$$

A solution to (6.0.11) is given formally by the series;

$$\overline{u_k} = \sum_{k=0}^{\infty}(-1)^k J^k.1$$

Where J^k denotes the k^{th} iterate of the operation J and J^0 is the identity. Writing

$$\overline{u_0}(\xi,t,\tau)=1 \text{ and } \overline{u_p} = 1 - Ju_{p-1} = \sum_{k=0}^{p}(-1)^k J^k.1 \qquad (6.0.11^*)$$

With $n=1$; (6.0.11) reduces to;

$$\overline{u_k}(\xi,t,\tau) = 1 - A(\xi)\int_{r=\xi/\sigma}^{1}\int_{\rho=\tau}^{t^2(1-r^2)} r^n \exp\left[\frac{-m\rho}{2}\right]\frac{dr\,d\rho}{2(1-r^2)} \qquad (6.0.12)$$

If $\xi\subset K$, *with* K compact then $|A(\xi)|_C \le c_k$ and in particular

$$0 \le \left|\overline{u_1} - \overline{u_0}\right| \le 1 - \left|A(\xi)\right|_c \int_{r=\tau/\sigma}^{1} \int_{\rho=\tau}^{t^2(1-r^2)} r^n \exp\left[\frac{-m\rho}{2}\right] \frac{dr\,d\rho}{2(1-r^2)} \tag{6.1.13}$$

$$\le c_k F_0(t,\tau) \le c_k F_0(t,0) = c_k F_0(t);$$

Where $F_0(t) = \int_{r=0}^{1} \int_{\rho=0}^{t^2(1-r^2)} r^n \exp\left[\frac{-m\rho}{2}\right] \frac{dr\,d\rho}{2(1-r^2)}$ (6.1.14)

To find an upper bound to the function $F(t)$ we take $t^2(1-r^2) = s(t,r) > 0$ and integrate (6.0.14) with respect to ρ to obtain

$$F_0(t) \le \frac{t^2}{2} \int_{r=0}^{1} 2\frac{r^n}{as}\left(1 - e^{\frac{-a\rho}{2}}\right) dr$$

Since $1 - e^{\frac{-a\rho}{2}} \le as \quad for \quad as > 0$ and from (6.0.14) it can be written

$$F_0(t) \le \frac{t^2}{2} \int_{r=0}^{1} r^n dr \le \frac{t^2}{4(n+1)}; \quad n+1 > 0$$

Continuing we have, using (6.0.13)

$$0 \le \left|\overline{u_2} - \overline{u_1}\right| \le c_k^2 \int_{r=\tau/\sigma}^{1} \int_{\rho=\tau}^{t^2(1-r^2)} r^n \exp\left[\frac{-m\rho}{2}\right] \frac{F(\rho)\,dr\,d\rho}{2(1-r^2)}$$

$$\le c_k^2 \int_{r=\tau/\sigma}^{1} F_1(r) \int_{\rho=\tau}^{t^2(1-r^2)} r^n \exp\left[\frac{-m\rho}{2}\right] \frac{dr\,d\rho}{2(1-r^2)}$$

$$\le \frac{c_k^2}{2!} \cdot \frac{t^4}{2^2(b+1)(b+3)}$$

$$= c_k^2 F_1(t).$$

By iteration, we obtain

$$\left|\bar{u}_p - \bar{u}_{p-1}\right| \le c_k^p \frac{F_p(b)}{p!} \tag{6.0.15}$$

where

$$F_r(t) = t^2 \sum_{r=0}^{\infty} (-1)^{r+2} \frac{(m)^r t^{2r}}{(2r+2)(m+2r+1)}$$

Hence the series (6.0.11*) is the solution of (6.0.9) and converges absolutely and uniformly on $[0 \le \tau \le t \le b] \times K$ and since the terms $J^p.1$ are continuous in (ξ, t, τ) and analytic in ξ the same is true of $\bar{u}(\xi, t, \tau)$. Hence we state

Theorem 6.1

The series (6.0.11*) represents a solution of (6.0.9) with $\bar{u}(\xi, t, \tau) = 1$ and $\bar{u}_t(\xi, t, \tau) = 0$ $\left(0 \le \tau \le t \le b \text{ and } n > -\frac{1}{2}\right)$. The maps $(\xi, t, \tau) \to \bar{u}(\xi, t, \tau)$ and $(\xi, t, \tau) \to \bar{u}_t^n(\xi, t, \tau)$ are continuous numerical functions while $\bar{u}(., t, \tau)$ and $\bar{u}_t^n(., t, \tau)$ are analytic.

Proof:

Everything has been proved for $\bar{u}_n$ and the statements for $(\bar{u}_n)_t$ follow immediately upon differentiating (6.0.9) in t.

Theorem 6.2.

For $[0 \le \tau \le t \le b]$ the maps $(\xi, t) \to \bar{u}_n(\xi, t, \tau)$ and $(\xi, t) \to \left(mt + \frac{n}{t}\right)\bar{u}_t(\xi, t, \tau)$ are continuous while $\xi \to \bar{u}_n(\xi, t, \tau)$ and $(\xi, t) \to \left(mt + \frac{n}{t}\right)\bar{u}_t(\xi, t, \tau)$ are analytic. $\bar{u}(\xi, t, \tau)$ Satisfies (6.0.6) with $\bar{u}(\xi, t, \tau) = 1$ and $\bar{u}_t(\xi, t, \tau) = 0$

Proof.

Differentiating (6.0.9) with respect to t and multiplying both sides by $\left(mt+\frac{n}{t}\right)$;

$$\left(mt+\frac{n}{t}\right)\bar{u}_t(\xi,t,\tau)=-A(\xi)\left(mt+\frac{n}{t}\right)\int_\tau^t\left(\frac{\zeta}{t}\right)^n\exp\left[\frac{m(\zeta^2-\sigma^2)}{2}\right]d\zeta \tag{6.0.14}$$

We let $h(t,\tau)=\left(mt^2+\frac{n}{t}\right)\int_\tau^t\left(\frac{\zeta}{t}\right)^n\exp\left[\frac{m(\zeta^2-\sigma^2)}{2}\right]d\zeta$

$$\left(mt^2+\frac{n}{t}\right)\frac{1}{t^{m+1}}\sum_{r=0}^{\infty}\frac{(-1)^{r+1}(m)^r}{(n+2r+1)}\left[t^{n+2r+1}e^{\frac{mt^2}{2}}-\tau^{n+2r+1}e^{\frac{mt^2}{2}}\right] \tag{6.0.15}$$

We note that $(t,\tau)\to h(t,\tau)$ is not continuous as $(t,\tau)\to(0,0)$, so there is no hope that $(\xi,t,\tau)\to\left(mt+\frac{n}{t}\right)\bar{u}_t(\xi,t,\tau)$ will be continuous on $[0\le\tau\le t\le b]\times k$. On writing $h(t,0)=h(t)=0$, then from $(6.0.14)-(6.0.15)$, $(\xi,t)\to\left(mt+\frac{n}{t}\right)\bar{u}_t(\xi,t,0)$ is continuous on $[0,b]\times k$ with limit as $(\xi,t)\to(\xi_0,0)$ equal to zero.

Again from (6.0.14) $\xi\to\left(mt+\frac{n}{t}\right)\bar{u}_t(\xi,t,\tau)$ is analytic for $0\le\tau\le t\le b$. These properties are reflected in $\bar{u}_{tt}(\xi,t,\tau)$ after differentiating (6.0.9) twice in t and equation (6.0.1) is satisfied for $0\le\tau\le t\le b$ □

On taking the inverse Fourier transform of (6.1.9) we obtain the solution of (6.0.1)-(6.0.2).

CHAPTER SEVEN

§7.0 CONCLUDING REMARKS

The Singular Cauchy problem we have solved is

$$\Delta u = \partial_t^2 u + \frac{k}{t}\partial_t u \tag{7.0.1}$$

$$u(x,0) = f(x);\ u_t(x,0) = 0 \tag{7.0.2}$$

by Riemann's method.

Our approach to singular Cauchy problem by use of Riemann's method is: First we determine the Riemann's solution to the problem (7.0.1). Then determine the adjoint equation to (7.0.1). The solution to the adjoint equation (7.0.1) is called the Riemann's function and there is one advantage to be gained from solving for the Riemann function if this is possible. Once this is determined the differential equation (7.0.1) can be solved for Cauchy data on any other non-characteristic curve.

We obtained the Riemann's-Green function to (7.0.1) in the form of Appell's hyper geometric function

$$F_1(\alpha;\beta,\beta';\gamma;x,y) = \sum_{mn=0}^{\infty} \frac{(\alpha,m+n)(\beta,m)(\beta',n)}{(\gamma,m+n)(1,m)(1,n)} x^m y^n \tag{7.0.3}$$

where $(\alpha,r) = \alpha(\alpha+1)\ldots(\alpha+r-1)$.

Corresponding analytic continuations were considered. The power series (7.0.3) converges for arbitrary real or complex value of α,β,γ inside the unit squire $|x|<1, |y|<1$ provided γ is not negative.

§7.1 Recommendations

These results imposed us to new interesting problems.

a. First, suppose the operator $L = \Delta_x - \partial_t^2 - \frac{k}{t}\partial_t$ is given a perturbation, is the hyper geometric equation of the Riemann-Green function still valid.

b. According to the theory of ordinary differential equations in the complex domain, $F(\alpha,\beta,\gamma;z)$ is characterized as the solution of Fuchsian equations which have three regular singular points. Could we extend the Fuchs' theory which permits to find the solution of the Riemann's equation (Fuchsian equation with three singular points) in the form of Gauss Hypergeometric functions ${}_2F_1(\alpha,\beta,\gamma,z)$ to the study of the Appell's hypergeometric functions $F_1(\alpha;\beta,\beta';\gamma;x,y)$?

c. Can we have a class of equations whose general solution is in the form $F_1(\alpha;\beta,\beta';\gamma;x,y)$

Being very elegant this approach has nevertheless the disadvantage that it only solves linear hyperbolic equations with constant coefficients in two independent variables.

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