

**MAXIMUM LIKELIHOOD ESTIMATION OF PARAMETERS FOR
POISSON-EXPONENTIAL DISTRIBUTION UNDER PROGRESSIVE
TYPE I INTERVAL CENSORING**

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DECLARATION

This research project is my original work and has not been presented elsewhere for a degree award.

Signature.....Date.....

Peter Tumwa Situma

I confirm that the work reported in this project was carried out by the candidate under my supervision

Signature.....Date.....

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DEDICATION

I dedicate this work to my mum Roda Situma, brothers and sisters. Thank you for supporting me throughout my education. May God bless you all.

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I am sincerely thankful to God for granting me good health, knowledge and understanding throughout my education. His grace has sufficiently guided me towards the completion of this work.

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ABBREVIATIONS AND ACRONYMS

PE	Poisson Exponential
PTI	Progressive Type I
PDF	Probability Density Function
CDF	Cumulative Distribution Function
EM	Expectation-Maximization
NR	Newton-Raphson
MLEs	Maximum Likelihood Estimators
MSE	Mean Squared Error
X_i	Number of failures occurred in $(t_{i-1}, t_i]$
R_i	Surviving units in the experiment removed randomly from life test.
$p_{(i)}$	Different censoring schemes
CPU	Central Processing Unit
RMSE	Root Mean Squared Error
CI	Confidence Interval
CR	Complementary Risks

Abstract

The problem of estimating the parameters of Poisson-Exponential distribution under progressive type-I interval censoring is considered. Previously, researchers have considered maximum likelihood estimation under the progressive type-I interval censoring scheme for various distribution, but no research has considered Poisson-Exponential. Poisson-Exponential is a two-parameter lifetime distribution having an increasing hazard function. It has been applied in complementary risks problems in latent risks, that is in scenarios where maximum lifetime values are observed but information concerning factors accounting for component failure is unavailable, which can be experienced in fields such as public health. Under progressive type I interval censoring, observations are known within two consecutively pre-arranged times and items would be withdrawn at pre-scheduled time points. Progressive type-I interval censoring scheme is most suitable in those cases where the continuous examination is impossible. Based on progressive type I interval censored data, the Maximum Likelihood Estimates of Poisson-Exponential parameters are obtained via the Expectation-Maximization algorithm. The Expectation-Maximization algorithm is preferred as it has been confirmed to be a more superior tool when dealing with incomplete data sets having missing values, or models having truncated distributions. In this study, the estimates derived are compared through simulation based on bias and the mean squared error under different censoring schemes and parameter values. It is concluded that for an increasing sample size, the estimated values of the parameters tend to the true value. Among the four censoring schemes considered, the third scheme $p_{(3)}$ provides the most precise and accurate results followed by $p_{(4)}$, $p_{(1)}$ and lastly $p_{(2)}$.

CHAPTER ONE

INTRODUCTION

1.1 Background to the Study

Survival analysis pertains to analysis of data that corresponds to time from a well-defined time origin until the occurrence of endpoint or some well-defined event. Survival analysis is applicable when analyzing data from medical, actuarial science, industrial reliability, biomedical studies and demography since it often happens that data are in a way incomplete. Such an incomplete observation of the failure time is called censored data. Collett, (1994) pointed out that an individual in the study is censored when for that particular individual the endpoint of interest has not been observed, which may be due to: study termination before all individuals experience the event of interest, particular individual may have been lost to follow-up at some point, some individuals may have been sacrificed or removed from the study due to cost, time or other factor(s) as discussed by Kaplan and Meier, (1958).

In life testing and reliability experiments, a subset of the population say of n items is tested, and termination of the experiment takes place when all items have failed. In instances where the duration of the distribution of lifetimes has a thick tail, the procedure may prolong. In addition, units removal may be deliberate or unplanned in life test experiments. Such experiments yield data referred to as censored samples.

Poisson Exponential (PE) is a two-parameter lifetime distribution which was first presented by Cancho *et al.*, (2011). The distribution is widely applicable in complementary risks (CR) problems where latent risks exists. PE distribution has an increasing hazard function. Louzada-Neto *et al.*, (2011) discussed Bayes estimators and statistical properties of Poisson-Exponential. More research on PE distribution is found in Singh *et al.*, (2014), Rodrigues *et al.*, (2016), Gitahi *et al.*, (2017) and Belaghi *et al.*, (2018).

In lifetime analysis, various types of censoring schemes have widely been discussed. Type I and type II censoring schemes being the most common censoring schemes in life test experiments. The time T of termination of the experiment is pre-arranged in the ordinary type-I censoring scheme. On the other hand, the experiment continues until a pre-arranged number of failures occurs in the ordinary type-II censoring scheme. A mixture of type I and II censoring schemes is referred to as hybrid censoring. However, Type I, Type II and hybrid censoring schemes do not permit removal of experimental units at any other point other than the final termination point of the experiment. In many practical situations, there is need for the removal of test items at different time points prior to the termination of the experiment.

Progressive censoring scheme allows the removal of experimental units at different time points other than the termination point of the experiment as suggested by Cohen, (1963). Progressive type I and type II censoring schemes and their applications are extensively discussed in Balakrishnan, (2007) and Balakrishnan and Cramer, (2014). However, the two censoring schemes cannot apply for those cases where it is impossible for continuous examination of experimental units. Aggarwala, (2001) came up with a remedy to limitation from progressive type I and progressive type II censoring schemes by proposing progressive type I interval censoring scheme. Under progressive type I interval censoring, observations are known within two consecutively pre-scheduled times and items are removed at pre-scheduled time points. Although interval censoring scheme can be applied in any type of lifetime experiment, its main application is in clinical biological studies where experimental subjects are animal or human beings that are examined at discrete intervals; for instance, on monthly basis and the events under consideration are known to occur at given times between examination points as pointed out by Kokosa *et al.*, (1993), Sakamoto *et al.*, (1997) and Samuelson and Kongerud, (1994). In this study, progressive type I interval censoring scheme is considered; that is a combination of type I censoring and progressive censoring schemes. Previously, researches have been done

on various distributions basing on progressively type I interval censoring scheme. For instance, in Aggarwala, (2001) the first study on progressive type I interval censoring and the statistical inference for the exponential distribution was proposed. In addition, there was consideration of the mathematical properties in the study and concluded that progressive type I interval censoring scheme can be of use to biological experiments when experimental units are humans. Lio *et al.*, (2011) estimated Parameters of Generalized Rayleigh distribution by considering progressively type I interval-censored data. MLE, probability plot and method of moments were employed in derivation of the estimators of the distribution parameters. Based on obtained results, MLE via the Expectation-Maximization algorithm gave the most precise parameter estimates among the three methods. Recently, Singh and Tripathi, (2018) estimated the parameters of an Inverse Weibull distribution under progressively type I interval censoring in classical and Bayesian frameworks. Under the Bayesian framework the study found out that estimates improve with higher values of sample sizes while in the classical approach, estimation for the undisclosed model parameters was obtained based on proposed probability plot method. The purpose of this study will be the Maximum Likelihood Estimation of Parameters for Poisson-Exponential (PE) distribution based on Progressive Type I Interval censoring. Maximum Likelihood Estimation is employed since it is the frequently used method in parameter estimation of a distribution because of attractive properties such as asymptotic normality and consistency of the MLEs. Additionally, from literature, Maximum Likelihood Estimation method performed better than other methods such as method of moments.

1.2 Statement of the Problem

The Poisson-Exponential (PE) distribution has widely been used in application areas of complementary risks (CR) problems in latent risks, that is in scenarios where maximum lifetime values are observed but information concerning factors accounting for component failure is unavailable. However, in cases where experimental units cannot be monitored on a

continuous basis owing to insufficient time and cost, an interval censoring scheme is applied. In the interval censoring scheme, the number of failed units is recorded in pre-specified time intervals.

Different researchers for instance; Ng and Wang, (2009), Cheng *et al.*, (2010), Chen and Lio, (2010), Lio *et al.*, (2011), Teimouri and Gupta, (2012) and Kaushik *et al.*, (2017) have considered maximum Likelihood Estimation under the progressive type I (PTI) interval censoring scheme for various distributions. However, none of them has considered PE distribution.

In this work, we shall consider the Maximum Likelihood Estimation of parameters for Poisson-exponential distribution under progressive type I interval censoring.

1.3 Objectives

1.3.1 Main objective

The main aim of this work is to study the properties of the Maximum Likelihood Estimation of parameters for Poisson-Exponential distribution based on progressive type I interval censoring.

1.3.2 Specific objectives

- i. To derive the MLEs for the parameters of Poisson-Exponential distribution under progressive type I interval censoring scheme via the EM Algorithm.
- ii. To derive the variance-covariance matrix of the MLEs of the parameters of Poisson-Exponential distribution obtained in (i) above.
- iii. To assess the precision and accuracy of the MLEs of the parameters of Poisson – Exponential distribution under different censoring schemes and parameter values using simulation studies.

1.4 Significance of the study

Poisson-Exponential (PE) distribution which is applicable in CR studies when latent risks exist, that is when no information is available about factors responsible for component failure, which can be often experienced in field data, such as public health is investigated under PTI interval censoring. In this study, we provide a method of obtaining maximum likelihood estimates of the unknown parameters of PE distribution under PTI interval censoring scheme. The methodology will enable the investigator to remove some units at pre-scheduled time points when observations are known within two consecutively pre-scheduled times.

1.5 Definition of terms

Survival analysis- refers to analysis of data that corresponds to the time from a well-defined time origin until the occurrence of endpoint or some well-defined event.

Censoring-The survival time of an individual is said to be **censored** when the endpoint of interest had not been observed for that individual.

Type I Censoring – Here discontinuation of the test is at some pre-specified time T , such that beyond this time no failure will be observed.

Type II Censoring – Here the observations on study subjects cease after a predetermined number of failures have been observed.

Hybrid Censoring – A mixture of type I and type II censoring schemes.

Progressive Censoring - permits surviving experimental units to be removed at a stage before the terminal point of the experiment.

Progressive type I censoring is a type of censoring whereby experimental units are removed at a point before the terminal point of the experiment at some pre-specified time T .

Progressive type II censoring occurs when some units are removed from the experiment in the course of the experiment, that is, after the first r_1 failures some study units are removed, then after the next r_2 failures some more are removed and so on.

Interval censoring – In this type of censoring, individuals are known to have experienced failure within an interval of time, say $(t_{i-1}, t_i]$.

Progressive Type I interval censoring – under this censoring scheme, observations are only known within two consecutively pre-scheduled times and items would be allowed to withdraw at pre-scheduled time points.

The Expectation-Maximization (EM) Algorithm – Refers to an iterative step to calculate MLEs when data under observation is considered to be incomplete in some manner. Dempster *et al.*, (1977) introduced the algorithm. It involves two steps: -

- a) **The Expectation step or E-step** – In this step, the missing or censored data (or any function of missing data) are replaced with their expected values.
- b) **The Maximization step or M-step** – In this step, we maximize the expectation of the complete data likelihood obtained in the E-step to yield new parameter estimates.

Latent Risks – Refers to existing risks but not yet visible i.e. still hidden.

1.6 Outline of the project

In Chapter Two, review of literature on PE distribution is presented. Besides, the literature on progressive type I interval censoring scheme and the EM algorithm are discussed. In Chapter Three, maximum likelihood estimators of unknown parameters for PE distribution under PTI interval censoring are derived via the EM algorithm. In addition, the variance-covariance matrix of the MLEs of the PE distribution is derived.

In Chapter Four, results and discussions are given and finally, summary, conclusion and areas for further study are presented in Chapter Five.

CHAPTER TWO

LITERATURE REVIEW

2.1 Introduction

In this chapter, literature on Poisson-Exponential and related work on the distribution is reviewed. In addition, the literature on progressive type I interval censoring and the EM algorithm are discussed.

2.2 Poisson-Exponential Distribution

In reliability and life testing studies exponential distribution has proved to be a distribution with a simple, elegant and closed form of solution, Tomazella *et al.*, (2013). However, its usefulness is limited based on the fact that it has a constant hazard function. In order to overcome this drawback, different authors have come up with new lifetime distributions based on modification of Exponential distribution. Gupta and Kundu, (1999) suggested a Generalized Exponential distribution (GED) that fits the data with decreasing and increasing hazard function. Modifying an Exponential distribution to a distribution with a decreasing hazard function was done by Kus, (2007). The distribution proposed by Kus, (2007) was further modified by the inclusion of a power parameter by Barreto and Cribari, (2009).

PE is a two-parameter lifetime distribution which was first introduced by Cancho *et al.*, (2011). Its failure rate increases with time. The distribution is widely applicable in complementary risk studies. Louzada-Neto *et al.*, (2011) studied statistical properties of PE distribution and discussed Bayes estimators based on Squared Error Loss Function (SELF). Singh *et al.*, (2014) obtained the MLEs and Bayes estimators of the parameters of PE distribution under symmetric and asymmetric loss function and compared the proposed estimators in terms of their risks with

the maximum likelihood estimators. Rodrigues *et al.*, (2016) considered different estimation methods for parameters of PE distribution. Methods considered included Maximum likelihood estimator, Moments estimator, Maximum product spacing and Anderson-Darling Estimator (ADE). From simulation results, the Anderson-Darling estimator was the most precise and accurate in terms of bias and MSE. Gitahi *et al.*, (2017) obtained the MLEs of the parameters of PE distribution based on progressively type II censoring via the Expectation-Maximization Algorithm. It was found that by increasing the sample size the estimates of the parameters approached the true values, the variances of the estimators decreased and the width of the CI became narrower. Recently, Belaghi *et al.*, (2018) considered prediction and estimation of lifetime data following PE distribution considered under Type II censoring. Performance under informative prior of the Expectation-Maximization algorithm against Newton-Raphson was compared. Their results showed that average estimated values obtained via NR were generally higher as compared to those under the EM algorithm. In addition, performance under informative prior was better than that under non-informative prior in terms of the average estimates and the MSE values.

Cohen, (1963) introduced progressive censoring, which permits the removal of experimental units at a point other than the final termination point of the experiment. Comprehensive work on progressive censoring schemes inter alia application, mathematical results and data analysis techniques have been discussed by Balakrishnan and Aggarwala, (2000). The most common classifications of progressive censoring schemes include progressive type II censoring, progressive type II hybrid censoring and progressive type I interval censoring.

2.3 Progressive Type I Interval Censoring Scheme

In progressive type I (PTI) interval censoring surviving experimental units are removed at a pre-specified time interval at points prior to the termination of the test. This is because

observations are known within two succeeding pre-determined times and removal of items would be at pre-specified time points.

Aggarwala, (2001) proposed progressive type I interval censoring and provided the statistical inference for the exponential distribution. Also, in the study mathematical properties were considered and concluded that when experimental units are humans, the PTI interval censoring scheme can be of use to biological experiments. Their research considered four Progressive type I censoring schemes, labelled $p_{(1)}$, $p_{(2)}$, $p_{(3)}$ and $p_{(4)}$, whereby scheme $p_{(3)}$ produced the most precise estimates followed by $p_{(4)}$, $p_{(1)}$ and then $p_{(2)}$ as depicted from the estimated curves. Ng and Wang, (2009) dealt with the estimation of Weibull distribution parameters basing on the PTI interval-censored sample. Different methods of estimation were considered, including; method of moments, mid-point method, maximum likelihood estimation and Weibull plot estimation. Their results revealed that the mid-point method had the highest bias while maximum likelihood estimation and Weibull plot had similar and smaller biases. In addition, from their simulation, it was observed that censoring scheme $p_{(3)}$ gave the most precise parameter estimates followed by $p_{(4)}$, $p_{(1)}$ and $p_{(2)}$ in terms of MSE.

Cheng *et al.*, (2010) introduced a new Algorithm for Maximum Likelihood Estimation under PTI interval-censored data. Exponential and Weibull distributions were considered. They formulated an algorithm for finding the Maximum Likelihood Estimator of the hazard function for exponential distribution in scenarios of unequal interval sizes. Based on their simulation results, the new algorithm had smaller biases and MSEs compared with the traditional EM algorithm. Chen and Lio, (2010) estimated parameters of Generalized exponential distribution under PTI interval censoring. Parameter estimates were obtained by applying maximum likelihood method, probability plot and method of moments. The behaviour of proposed estimates was investigated through a simulation study. Amongst the three methods, the maximum likelihood estimator via the EM algorithm gave the most precise parameter estimates

in terms of MSE followed by the method of moments and finally probability plot. Results were based on four censoring schemes first described in Aggarwala, (2001) whereby scheme $p_{(3)}$ outperformed the rest since it yields the most precise results as evident in the MSE and bias values, followed by $p_{(4)}$, $p_{(1)}$ and finally $p_{(2)}$. Lio *et al.*, (2011) considered estimation of Parameters of Generalized Rayleigh distribution based on progressively type I interval-censored data. Derivation of the estimators of the distribution parameters was done by applying probability plot, MLE and method of moments. Based on these results, MLE via the Expectation-Maximization algorithm gave the most precise parameter estimates among the three methods.

Lin and Lio, (2012) estimated the parameters of Weibull and Generalized Exponential distributions under PTI interval censoring by the Bayesian method. The proposed model selection procedure confirmed that the Generalized Exponential model is a better model than Weibull in estimating mix proportion in a supervised mixture process. Teimouri and Gupta, (2012) considered a Maximum Likelihood Estimator via the Expectation-Maximization algorithm, mid-point estimator and moment-based estimator for Gompertz Makeham distribution under PTI interval censoring and found Maximum Likelihood Estimator via Expectation-Maximization algorithm to be the best among the three by comparing their RMSE values.

Kaushik *et al.*, (2017) considered the Maximum Likelihood and Bayesian estimations of Generalized Exponential distribution parameters under the PTI interval censoring scheme with Random Removals. The results showed that bias and MSEs decrease as n increases in all the considered cases. Also, MSE of the MLE was more than that of the corresponding Bayes estimate in all cases. Recently, Singh and Tripathi, (2018) estimated Inverse Weibull distribution parameters under PTI interval censoring in classical and Bayesian frameworks. In classical approach, estimation for the undisclosed model parameters was obtained based on the

proposed probability plot method while under the Bayesian framework the study found out that the estimates improved with higher values of sample sizes. Teimouri, (2018) corrected the MLE via the EM algorithm obtained by Chen and Lio, (2010). The research pointed out that the EM algorithm proposed by Chen and Lio, (2010) was incorrect since in the research expectations were taken from the complete data log-likelihood function after differentiating with respect to the parameters which is not usual in the EM algorithm framework. In the paper by Teimouri, (2018), in the E-step, expectations were taken from the complete data log-likelihood before differentiating with respect to the unknown parameters. The simulation results obtained from the proposed EM by Teimouri, (2018) outperformed the one obtained by Chen and Lio, (2010) in terms of bias and MSE.

2.4 Expectation-Maximization (EM) Algorithm

EM algorithm introduced by Dempster *et al.*, (1977) has extensively been applied in incomplete data problems since it has been confirmed to be a superior tool when dealing with incomplete data sets or models having truncated distributions. The algorithm is applied in estimating the unknown parameters by the method of maximum likelihood for survival distributions. In a study by Little and Rubin, (1983), it was shown that estimates obtained using the EM algorithm converge to maximum likelihood estimates.

Several authors have applied the EM algorithm for progressive type I interval censoring scheme. They include; Ng and Wang, (2009), Chen and Lio, (2010), Lio *et al.*, (2011), Teimouri and Gupta, (2012), Singh and Tripathi, (2018) and Teimouri, (2018). The EM algorithm is an iterative procedure. Each iteration has two steps (E-step and M-step).

From the above discussions, it is clearly evident that no research has been done on the Maximum Likelihood Estimation of parameters for Poisson-exponential distribution based on the PTI interval censoring scheme. Hence, the main aim of the study.

2.5 Chapter Summary

The above literature review starts by introduction of PE distribution. The distribution is widely preferred due to its property of an increasing hazard function and its application in complementary risk scenario. Previous studies on the distribution are well articulated. In addition, existing researches on PTI interval censoring are discussed in details, that is; the type of distribution, methods applied and the results obtained. Lastly a discussion on EM algorithm is presented and its merits.

CHAPTER THREE

MAXIMUM LIKELIHOOD ESTIMATION OF PARAMETERS FOR POISSON-EXPONENTIAL DISTRIBUTION UNDER PROGRESSIVE TYPE I INTERVAL CENSORING

3.1 Introduction

This chapter presents a discussion on Poisson-Exponential distribution. Its probability density function, cumulative distribution function, survival function and hazard function are given. Additionally, moments, progressive type I interval censoring scheme and maximum likelihood method are presented. Lastly, MLEs for the parameters of PE distribution via EM Algorithm under the PTI censoring scheme and the variance-covariance matrix of the MLEs of the PE distribution are derived.

3.2 Poisson-Exponential Distribution

Poisson-Exponential distribution is derived as follows; let W be a random variable representing the number of complementary risks corresponding to the occurrence of an event of interest under an assumption that W has a zero truncated Poisson distribution with probability mass function given by

$$P(W = w) = \frac{e^{-\theta} \theta^w}{w!(1 - e^{-\theta})}, w = 1, 2, \dots, \theta > 0 \quad (3.1)$$

Let T_j for $j = 1, 2, \dots$ be time to occurrence due to j^{th} complementary risk of an event of interest. Furthermore, given $W = w$, the random variables T_j ($j = 1, 2, \dots, m$) are assumed to be independent and identically distributed with a common distribution function with PDF given by

$$f(t; \lambda) = \lambda e^{-\lambda t}, t > 0, \lambda > 0 \quad (3.2)$$

In latent CR scenario, the number of causes W and the lifetime T_j associated with a particular cause are not observable that is; latent variables, but only the maximum lifetime X among all causes is usually observed, Louzada-Neto *et al.*, (2011). Therefore, the component lifetime is defined as

$$X = \max(T_1, T_2, \dots, T_W) \quad (3.3)$$

The maximum lifetime X follows Poisson-Exponential distribution with scale parameter λ and shape parameter θ having the PDF and CDF respectively given by

$$f(x; \theta, \lambda) = \frac{\theta \lambda e^{-\lambda x - \theta e^{-\lambda x}}}{1 - e^{-\theta}}, x > 0, \theta > 0, \lambda > 0 \quad (3.4)$$

$$F(x; \theta, \lambda) = 1 - \frac{1 - e^{-\theta e^{-\lambda x}}}{1 - e^{-\theta}}, x > 0, \theta > 0, \lambda > 0 \quad (3.5)$$

The conditional distribution of X in (3.3) given $W = w$ is thus given by

$$f(x / W = w; \lambda) = w \lambda (1 - e^{-\lambda x})^{w-1} e^{-\lambda x}, t > 0, \lambda > 0, w = 1, 2, \dots \quad (3.6)$$

This is proved below: -

Using equations (3.1) and (3.6) PDF of X is given by

$$f(x; \theta, \lambda) = \sum_{w=1}^{\infty} w \lambda (1 - e^{-\lambda x})^{w-1} e^{-\lambda x} \times \frac{e^{-\theta} \theta^w}{w! (1 - e^{-\theta})}$$

$$f(x; \theta, \lambda) = \frac{\theta \lambda e^{-\theta - \lambda x}}{1 - e^{-\theta}} \sum_{w=1}^{\infty} \frac{[\theta (1 - e^{-\lambda x})]^{w-1}}{(w-1)!}$$

$$f(x; \theta, \lambda) = \frac{\theta \lambda e^{-\theta - \lambda x}}{1 - e^{-\theta}} \times e^{\theta - \theta e^{-\lambda x}}$$

$f(x; \theta, \lambda) = \frac{\theta \lambda e^{-\lambda x - \theta e^{-\lambda x}}}{1 - e^{-\theta}}$, hence the proof

PE distribution approaches an exponential distribution with parameter λ as θ tends to zero.

Its PDF decreases if $0 < \theta < 1$ and unimodal for $\theta \geq 1$. The mode of the distribution is $\frac{\lambda}{e}$ which

is obtained at $x = \log\left(\frac{\theta}{\lambda}\right)$. Louzada-Neto *et al.*, (2011), noted that parameters λ and θ can be directly interpreted in terms of CR. That is; λ represents lifetime failure rate while θ denotes the mean number of CR.

The survival and hazard functions of PE distribution are given respectively by

$$S(x; \theta, \lambda) = \frac{1 - e^{-\theta e^{-\lambda x}}}{1 - e^{-\theta}}, x > 0, \theta > 0, \lambda > 0 \quad (3.7)$$

and

$$h(x; \theta, \lambda) = \frac{\theta \lambda e^{-\lambda x - \theta e^{-\lambda x}}}{1 - e^{-\theta e^{-\lambda x}}}, x > 0, \theta > 0, \lambda > 0 \quad (3.8)$$

The hazard function is increasing for all $\theta, \lambda > 0$ as depicted in Louzada-Neto *et al.*, (2011).

3.3 Moments

The r^{th} moment is generally expressed as $\mu'_r = E(X^r)$. According to Louzada-Neto *et al.*, (2011), the r^{th} moment of the PE distribution is obtained by considering the generalized hypergeometric function defined by

$$F_{u,v}(\underset{\sim}{a}, \underset{\sim}{b}, \theta) = \sum_{j=0}^{\infty} \frac{\theta^j \prod_{i=1}^u \Gamma(a_i + j) \Gamma(a_i)^{-1}}{\Gamma(j+1) \prod_{i=1}^v \Gamma(b_i + j) \Gamma(b_i)^{-1}} \quad (3.9)$$

Where $\underline{a} = [a_1, \dots, a_u]$ and $\underline{b} = [b_1, \dots, b_v]$, u and v being the number of terms of \underline{a} and \underline{b} respectively.

Let X be a random variable distributed as PE with parameters $\theta > 0, \lambda > 0$ whose PDF is given by equation (3.4); then its r^{th} moment is

$$\mu_r' = E[X^r] = \frac{\theta r!}{\lambda^r (1 - e^{-\theta})} F_{r+1, r+1}([1, \dots, 1], [2, \dots, 2], -\theta) \quad (3.10)$$

Mean and variance of PE distribution are obtained by considering equation (3.10) as follows:

For the mean, we take $r = 1$

$$\text{Therefore; } E(X) = \frac{\theta}{\lambda(1 - e^{-\theta})} F_{2,2}([1, 1], [2, 2], -\theta) \quad (3.11)$$

The variance is generally given by

$$\text{var}(X) = E(X^2) - [E(X)]^2 \quad (3.12)$$

First, we compute $E(X^2)$ by considering $r = 2$, for which we get

$$E(X^2) = \frac{2\theta}{\lambda^2(1 - e^{-\theta})} F_{3,3}([1, 1, 1], [2, 2, 2], -\theta) \quad (3.13)$$

Substituting (3.11) and (3.13) in (3.12) and simplifying, we obtain the variance as

$$\text{Var}(X) = \frac{\theta}{\lambda^2(1 - e^{-\theta})} \left[2F_{3,3}([1, 1, 1], [2, 2, 2], -\theta) - \frac{\theta}{1 - e^{-\theta}} F_{2,2}([1, 1], [2, 2], -\theta)^2 \right] \quad (3.14)$$

3.4 Progressive type I interval censoring

On a lifetime experiment, let n units be placed on a life testing simultaneously at time $t_0 = 0$ and under inspection at m pre-arranged times $t_1 < t_2 < \dots < t_m$. Let t_m be the planned time for

test termination when all surviving items are removed. The number of failures, X_i , and the number of randomly removed surviving items from the test, R_i , is recorded in the interval $(t_{i-1}, t_i]$ at time t_i , ($i = 1, 2, \dots, m$). In the interval $(t_{i-1}, t_i]$, the number of surviving items, Y_i , is a random variable and $R_i \leq Y_i$. At time t_i , R_i could be determined through pre-determined percentages of surviving remaining items. Given pre-specified percentages p_1, p_2, \dots, p_{m-1} and $p_m = 1$ for removal at time $t_1 < t_2 < \dots < t_m$ respectively, $R_i = p_i Y_i$ at each inspection time t_i . Hence, the PTI interval-censored sample is represented by $\{X_i, R_i, t_i\}$ and the size of the sample

$$n = \sum_{i=1}^m (X_i + R_i).$$

3.5 Maximum Likelihood Estimation

Maximum Likelihood Estimation, first introduced by Fisher, (1912) is the most commonly used method in estimating parameters of a distribution because of attractive properties such as consistency and asymptotic normality of the MLEs.

Let $\underline{X} = (x_1, x_2, \dots, x_n)^T$ be a random vector with PDF $f(x_1, x_2, \dots, x_n; \Theta)$ where $\Theta = (\lambda, \theta) \in \Omega$. Further, let the joint density of \underline{X} be given by

$$f(x_1, x_2, \dots, x_n; \Theta) = f(x_1; \Theta) f(x_2; \Theta), \dots, f(x_n; \Theta)$$

$$= \prod_{i=1}^n f(x_i; \Theta) = L(x_i; \Theta)$$

$L(x_i; \Theta)$ is the likelihood function considered as a function of an unknown parameter, Θ . The

Maximum likelihood estimate of Θ is the value of Θ that maximizes $L(x_i; \Theta)$; that is,

$\hat{\Theta} = \arg \max_{\Theta} L(x_i; \Theta)$. Instead of maximizing the likelihood function, it is often easier to

maximize the log-likelihood, basing on the fact that the logarithm is monotonically increasing function and therefore the maximum values of the likelihood function and log-likelihood coincide.

Log-likelihood function is defined as $l = \ln L(x_i; \Theta)$.

Maximizing l with respect to Θ yields Maximum Likelihood Estimator; which is obtained by solving the first-order partial derivative equations below

$$\left. \frac{\partial \ln L(x_i; \Theta)}{\partial \Theta} \right|_{\Theta = \hat{\Theta}} = 0$$

3.6 Parameter Estimation

Given a PTI interval-censored sample, $\{X_i, R_i, t_i\}$ for $i = 1, 2, \dots, m$, of size n , from a continuous lifetime distribution with (CDF), $F(t; \Theta)$, where parameter vector $\Theta = (\theta, \lambda)$, the likelihood function in general will be derived as follows; Aggarwala, (2001).

$$\begin{aligned} L(\Theta) &\propto [F(t_1; \Theta)]^{X_1} [1 - F(t_1; \Theta)]^{R_1} \times [F(t_2; \Theta) - F(t_1; \Theta)]^{X_2} [1 - F(t_2; \Theta)]^{R_2} \\ &\times \dots \times [F(t_m; \Theta) - F(t_{m-1}; \Theta)]^{X_m} [1 - F(t_m; \Theta)]^{R_m} \\ L(\Theta) &\propto \prod_{i=1}^m [F(t_i; \Theta) - F(t_{i-1}; \Theta)]^{X_i} [1 - F(t_i; \Theta)]^{R_i} \end{aligned} \quad (3.15)$$

Since for $t_0 = 0$, $F(t_1; \Theta) - F(t_0; \Theta) = F(t_1; \Theta)$

The likelihood function (3.15) reduces to the likelihood function for the conventional type I censored case if $R_1 = R_2 = \dots = R_{m-1} = 0$

By substituting equation (3.5) in equation (3.15) and simplifying, we obtain

$$L(\Theta) \propto \prod_{i=1}^m \left[\frac{e^{-\theta e^{-\lambda t_i}} - e^{-\theta e^{-\lambda t_{i-1}}}}{1 - e^{-\theta}} \right]^{X_i} \left[\frac{1 - e^{-\theta e^{-\lambda t_i}}}{1 - e^{-\theta}} \right]^{R_i} \quad (3.16)$$

Obtaining the log-likelihood of equation (3.16) above and simplifying, yields,

$$l(\Theta) = \sum_{i=1}^m X_i \log \left[e^{-\theta e^{-\lambda t_i}} - e^{-\theta e^{-\lambda t_{i-1}}} \right] + \sum_{i=1}^m R_i \log \left[1 - e^{-\theta e^{-\lambda t_i}} \right] - \left(\sum_{i=1}^m X_i + \sum_{i=1}^m R_i \right) \log \left[1 - e^{-\theta} \right] \quad (3.17)$$

Differentiating equation (3.17) partially with respect to θ and λ and equating to zero, yields the following normal equations

$$\frac{\partial l}{\partial \theta} = \frac{\sum_{i=1}^m X_i e^{-\lambda t_{i-1}} e^{-\theta e^{-\lambda t_{i-1}}} - \sum_{i=1}^m X_i e^{-\lambda t_i} e^{-\theta e^{-\lambda t_i}}}{e^{-\theta e^{-\lambda t_i}} - e^{-\theta e^{-\lambda t_{i-1}}}} + \frac{\sum_{i=1}^m R_i e^{-\lambda t_i} e^{-\theta e^{-\lambda t_i}}}{1 - e^{-\theta e^{-\lambda t_i}}} - \frac{\left(\sum_{i=1}^m X_i + \sum_{i=1}^m R_i \right) e^{-\theta}}{1 - e^{-\theta}} = 0 \quad (3.18)$$

$$\frac{\partial l}{\partial \lambda} = \frac{\theta \sum_{i=1}^m X_i t_i e^{-\lambda t_i} e^{-\theta e^{-\lambda t_i}} - \theta \sum_{i=1}^m X_i t_{i-1} e^{-\lambda t_{i-1}} e^{-\theta e^{-\lambda t_{i-1}}}}{e^{-\theta e^{-\lambda t_i}} - e^{-\theta e^{-\lambda t_{i-1}}}} - \frac{\theta \sum_{i=1}^m R_i t_i e^{-\lambda t_i} e^{-\theta e^{-\lambda t_i}}}{1 - e^{-\theta e^{-\lambda t_i}}} = 0 \quad (3.19)$$

Clearly, normal equations (3.18) and (3.19) cannot yield solutions to θ and λ in a closed-form.

We therefore apply the EM Algorithm to obtain the solution of normal equations.

3.7 The EM Algorithm

The EM Algorithm was introduced by Dempster *et al.*, (1977) to deal with any incomplete data problem. Progressive type I interval censoring can be considered as an incomplete data problem and therefore EM Algorithm is used as the most suitable method in obtaining the MLEs of the unknown parameters.

Let X_i represent the number of failed units observed in the time interval $(t_{i-1}, t_i]$ and R_i represent withdrawn units from the experiment at a specified time t_i where $i = 1, 2, \dots, m$.

Additionally, we assume T_{ij} and Z_{ij} to represent lifetimes of observed failures for $j = 1, 2, \dots, X_i$

and lifetimes of missing units (censored units) for $j = 1, 2, \dots, R_i$ respectively from an experiment at time t_i where $i = 1, 2, \dots, m$. Therefore, under the PTI interval censoring scheme, the complete sample is denoted by, $W = (T_{ij}, Z_{ij})$. From the definition of the PTI interval censoring scheme, the likelihood function of the complete sample is given by

$$L(\Theta) \propto \prod_{i=1}^m \left[\prod_{j=1}^{X_i} f(T_{ij}; \Theta) \prod_{j=1}^{R_i} f(Z_{ij}; \Theta) \right] \quad (3.20)$$

Introducing logs on (3.20) we get

$$l(\Theta) = \ln L(\Theta) = \sum_{i=1}^m \left[\sum_{j=1}^{X_i} \ln f(T_{ij}; \Theta) + \sum_{j=1}^{R_i} \ln f(Z_{ij}; \Theta) \right]$$

$$l(\Theta) = \sum_{i=1}^m \sum_{j=1}^{X_i} \ln f(T_{ij}; \Theta) + \sum_{i=1}^m \sum_{j=1}^{R_i} \ln f(Z_{ij}; \Theta)$$

substituting for $f(T_{ij}; \Theta)$ and $f(Z_{ij}; \Theta)$ with (3.4) yields

$$l(\Theta) = \sum_{i=1}^m \sum_{j=1}^{X_i} \ln \left[\frac{\theta \lambda e^{-\lambda T_{ij} - \theta e^{-\lambda T_{ij}}}}{1 - e^{-\theta}} \right] + \sum_{i=1}^m \sum_{j=1}^{R_i} \ln \left[\frac{\theta \lambda e^{-\lambda Z_{ij} - \theta e^{-\lambda Z_{ij}}}}{1 - e^{-\theta}} \right] \quad (3.21)$$

Simplification of equation (3.21) yields

$$\begin{aligned} l(\Theta) = & n \ln(\theta \lambda) - \lambda \sum_{i=1}^m \sum_{j=1}^{X_i} (T_{ij}) - \theta \sum_{i=1}^m \sum_{j=1}^{X_i} (e^{-\lambda T_{ij}}) - \lambda \sum_{i=1}^m \sum_{j=1}^{R_i} (Z_{ij}) \\ & - \theta \sum_{i=1}^m \sum_{j=1}^{R_i} (e^{-\lambda Z_{ij}}) - n \ln(1 - e^{-\theta}) \end{aligned} \quad (3.22)$$

Where $n = \sum_{i=1}^m (X_i + R_i)$

The lifetimes of the X_i failures in the i^{th} interval $(t_{i-1}; t_i]$ are independent and follow a doubly truncated PE distribution from left and right at t_{i-1} and t_i respectively and the lifetimes of the R_i censored units in the i^{th} interval $(t_{i-1}; t_i]$ are independent and follow a truncated PE distribution from left at t_i for $i = 1, 2, \dots, m$.

The E-step requires the construction of pseudo-log-likelihood function. This is achieved by computing the conditional expectations and then replacing them in the log-likelihood function in (3.22). For our case the required conditional expectations are

$$E_{1i} = E\left[T_{ij} / T_{ij} \in (t_{i-1}, t_i]; \Theta\right]$$

$$E_{2i} = E\left[e^{-\lambda T_{ij}} / T_{ij} \in (t_{i-1}, t_i]; \Theta\right]$$

$$E_{3i} = E\left[Z_{ij} / Z_{ij} \in (t_i, \infty); \Theta\right]$$

$$E_{4i} = E\left[e^{-\lambda Z_{ij}} / Z_{ij} \in (t_i, \infty); \Theta\right]$$

Conditional expectations E_{1i} , E_{2i} , E_{3i} and E_{4i} are computed by application of general form in (3.23) in conjunction with some integration techniques. The required expected values of a doubly truncated distribution at p and q from left and right respectively with $0 < p < q < \infty$ for EM algorithm is given by

$$E_{\theta, \lambda} [y | y \in (p, q)] = \frac{\int_p^q y f(y; \theta, \lambda) dy}{F(q; \theta, \lambda) - F(p; \theta, \lambda)} \quad (3.23)$$

For E_{1i} ,

$$E_{1i} = E\left[T_{ij} / T_{ij} \in (t_{i-1}, t_i]; \Theta\right]$$

$$E_{1i} = \frac{1}{F(t_i; \theta, \lambda) - F(t_{i-1}; \theta, \lambda)} \int_{t_{i-1}}^{t_i} x f(x) dx$$

$$\text{But } \frac{1}{F(t_i; \theta, \lambda) - F(t_{i-1}; \theta, \lambda)} = \frac{1 - e^{-\theta}}{e^{-\theta e^{-\lambda t_i}} - e^{-\theta e^{-\lambda t_{i-1}}}}$$

$$\text{Therefore } E_{li} = \frac{1 - e^{-\theta}}{e^{-\theta e^{-\lambda t_i}} - e^{-\theta e^{-\lambda t_{i-1}}}} \int_{t_{i-1}}^{t_i} \frac{\theta \lambda x e^{-\lambda x - \theta e^{-\lambda x}}}{1 - e^{-\theta}} dx$$

$$= \frac{\theta \lambda}{e^{-\theta e^{-\lambda t_i}} - e^{-\theta e^{-\lambda t_{i-1}}}} \int_{t_{i-1}}^{t_i} x e^{-\lambda x - \theta e^{-\lambda x}} dx \quad (3.24)$$

$$\text{We first consider } \int_{t_{i-1}}^{t_i} x e^{-\lambda x - \theta e^{-\lambda x}} dx$$

$$\text{Let } m = e^{-\lambda x} \Rightarrow \ln m = -\lambda x \Rightarrow x = \frac{\ln m}{-\lambda} \Rightarrow dx = \frac{dm}{-\lambda m}$$

Substituting in the integral above and simplifying we get

$$\frac{1}{\lambda^2} \int \ln m e^{-\theta m} dm$$

Applying integration by parts $\int u dv = uv - \int v du$

$$\text{Let } u = \ln m \Rightarrow du = \frac{dm}{m} \text{ and } dv = e^{-\theta m} dm \Rightarrow v = \frac{e^{-\theta m}}{-\theta}$$

$$\begin{aligned} \frac{1}{\lambda^2} \int \ln m e^{-\theta m} dm &= \frac{1}{\lambda^2} \left\{ \frac{\ln m e^{-\theta m}}{-\theta} + \frac{1}{\theta} \int \frac{e^{-\theta m}}{m} dm \right\} \\ &= \frac{1}{\lambda^2} \left\{ \frac{\ln m e^{-\theta m}}{-\theta} + \frac{1}{\theta} \int \frac{1}{m} \left[1 - \theta m + \frac{\theta^2 m^2}{2!} - \frac{\theta^3 m^3}{3!} + \dots \right] dm \right\} \\ &= \frac{1}{\lambda^2} \left\{ \frac{\ln m e^{-\theta m}}{-\theta} + \frac{1}{\theta} \int \left[\frac{1}{m} - \theta + \frac{\theta^2 m}{2!} - \frac{\theta^3 m^2}{3!} + \dots \right] dm \right\} \\ &= \frac{1}{\lambda^2} \left\{ \frac{\ln m e^{-\theta m}}{-\theta} + \frac{1}{\theta} \left[\ln m - \theta m + \frac{\theta^2 m^2}{2 \times 2!} - \frac{\theta^3 m^3}{3 \times 3!} + \dots \right] \right\} \\ &= \frac{1}{\lambda^2 \theta} \left\{ -\ln m e^{-\theta m} + \ln m + \sum_{n=1}^{\infty} \frac{(-\theta m)^n}{n \times n!} \right\} \\ &= \frac{1}{\lambda^2 \theta} \left\{ (1 - e^{-\theta m}) \ln m + \sum_{n=1}^{\infty} \frac{(-\theta m)^n}{n \times n!} \right\} \quad (3.25) \end{aligned}$$

But $m = e^{-\lambda x}$

$$\text{Therefore } \int_{t_{i-1}}^{t_i} x e^{-\lambda x - \theta e^{-\lambda x}} dx = \frac{1}{\lambda^2 \theta} \left[\left(1 - e^{-\theta e^{-\lambda x}} \right) \ln e^{-\lambda x} + \sum_{n=1}^{\infty} \frac{(-\theta e^{-\lambda x})^n}{n \times n!} \right]_{t_{i-1}}^{t_i} \quad (3.26)$$

$$= \frac{1}{\lambda^2 \theta} \left[\lambda t_{i-1} \left(1 - e^{-\theta e^{-\lambda t_{i-1}}} \right) - \lambda t_i \left(1 - e^{-\theta e^{-\lambda t_i}} \right) + \sum_{n=1}^{\infty} \frac{(-\theta e^{-\lambda t_i})^n}{n \times n!} - \sum_{n=1}^{\infty} \frac{(-\theta e^{-\lambda t_{i-1}})^n}{n \times n!} \right]$$

Substituting the expression for $\int_{t_{i-1}}^{t_i} x e^{-\lambda x - \theta e^{-\lambda x}} dx$ in equation (3.24) and simplifying we obtain

$$E_{1i} = \frac{1}{\lambda \left(e^{-\theta e^{-\lambda t_i}} - e^{-\theta e^{-\lambda t_{i-1}}} \right)} \left[\lambda t_{i-1} \left(1 - e^{-\theta e^{-\lambda t_{i-1}}} \right) - \lambda t_i \left(1 - e^{-\theta e^{-\lambda t_i}} \right) + \sum_{n=1}^{\infty} \frac{(-\theta e^{-\lambda t_i})^n}{n \times n!} - \sum_{n=1}^{\infty} \frac{(-\theta e^{-\lambda t_{i-1}})^n}{n \times n!} \right] \quad (3.27)$$

For E_{2i} ,

$$E_{2i} = E \left[e^{-\lambda T_{ij}} / T_{ij} \in (t_{i-1}, t_i]; \Theta \right]$$

$$E_{2i} = \frac{1}{F(t_i; \theta, \lambda) - F(t_{i-1}; \theta, \lambda)} \int_{t_{i-1}}^{t_i} e^{-\lambda x} f(x) dx$$

$$\text{But } \frac{1}{F(t_i; \theta, \lambda) - F(t_{i-1}; \theta, \lambda)} = \frac{1 - e^{-\theta}}{e^{-\theta e^{-\lambda t_i}} - e^{-\theta e^{-\lambda t_{i-1}}}}$$

$$\text{Therefore } E_{2i} = \frac{1 - e^{-\theta}}{e^{-\theta e^{-\lambda t_i}} - e^{-\theta e^{-\lambda t_{i-1}}}} \int_{t_{i-1}}^{t_i} \frac{\theta \lambda e^{-2\lambda x - \theta e^{-\lambda x}}}{1 - e^{-\theta}} dx$$

$$= \frac{\theta \lambda}{e^{-\theta e^{-\lambda t_i}} - e^{-\theta e^{-\lambda t_{i-1}}}} \int_{t_{i-1}}^{t_i} e^{-2\lambda x - \theta e^{-\lambda x}} dx \quad (3.28)$$

$$\text{We first consider } \int_{t_{i-1}}^{t_i} e^{-2\lambda x - \theta e^{-\lambda x}} dx$$

$$\text{Let } m = e^{-\lambda x} \Rightarrow \ln m = -\lambda x \Rightarrow x = \frac{\ln m}{-\lambda} \Rightarrow dx = \frac{dm}{-\lambda m}$$

Substituting in the integral above and simplifying we get

$$-\frac{1}{\lambda} \int m e^{-\theta m} dm$$

Applying integration by parts $\int u dv = uv - \int v du$

Let $u = m \Rightarrow du = dm$ and $dv = e^{-\theta m} dm \Rightarrow v = \frac{e^{-\theta m}}{-\theta}$

$$\begin{aligned} -\frac{1}{\lambda} \int m e^{-\theta m} dm &= -\frac{1}{\lambda} \left[\frac{m e^{-\theta m}}{-\theta} + \int \frac{e^{-\theta m}}{\theta} dm \right] \\ &= -\frac{1}{\lambda} \left[\frac{m e^{-\theta m}}{-\theta} - \frac{e^{-\theta m}}{\theta^2} \right] = \frac{1}{\theta \lambda} \left[m e^{-\theta m} + \frac{e^{-\theta m}}{\theta} \right] \end{aligned}$$

But $m = e^{-\lambda x}$

$$\text{Therefore } \int_{t_{i-1}}^{t_i} e^{-2\lambda x - \theta e^{-\lambda x}} dx = \frac{1}{\theta \lambda} \left[e^{-\lambda x - \theta e^{-\lambda x}} + \frac{e^{-\theta e^{-\lambda x}}}{\theta} \right]_{t_{i-1}}^{t_i} \quad (3.29)$$

$$\int_{t_{i-1}}^{t_i} e^{-2\lambda x - \theta e^{-\lambda x}} dx = \frac{1}{\theta \lambda} \left[e^{-\lambda t_i - \theta e^{-\lambda t_i}} - e^{-\lambda t_{i-1} - \theta e^{-\lambda t_{i-1}}} + \frac{e^{-\theta e^{-\lambda t_i}} - e^{-\theta e^{-\lambda t_{i-1}}}}{\theta} \right]$$

Substituting the expression for $\int_{t_{i-1}}^{t_i} e^{-2\lambda x - \theta e^{-\lambda x}} dx$ in equation (3.28) and simplifying we obtain

$$\begin{aligned} E_{2i} &= \frac{1}{e^{-\theta e^{-\lambda t_i}} - e^{-\theta e^{-\lambda t_{i-1}}}} \left[\frac{\theta e^{-\lambda t_i - \theta e^{-\lambda t_i}} - \theta e^{-\lambda t_{i-1} - \theta e^{-\lambda t_{i-1}}} + e^{-\theta e^{-\lambda t_i}} - e^{-\theta e^{-\lambda t_{i-1}}}}{\theta} \right] \\ E_{2i} &= \frac{\theta e^{-\lambda t_i - \theta e^{-\lambda t_i}} - \theta e^{-\lambda t_{i-1} - \theta e^{-\lambda t_{i-1}}} + e^{-\theta e^{-\lambda t_i}} - e^{-\theta e^{-\lambda t_{i-1}}}}{\theta (e^{-\theta e^{-\lambda t_i}} - e^{-\theta e^{-\lambda t_{i-1}}})} \quad (3.30) \end{aligned}$$

For E_{3i} ,

$$E_{3i} = E \left[Z_{ij} / Z_{ij} \in (t_i, \infty); \Theta \right]$$

$$E_{3i} = \frac{1}{1 - F(t_i; \theta, \lambda)} \int_{t_i}^{\infty} x f(x) dx$$

$$E_{3i} = \frac{\theta \lambda (1 - e^{-\theta})^{-1}}{1 - F(t_i; \theta, \lambda)} \int_{t_i}^{\infty} x e^{-\lambda x - \theta e^{-\lambda x}} dx = \frac{\theta \lambda}{1 - e^{-\theta e^{-\lambda t_i}}} \int_{t_i}^{\infty} x e^{-\lambda x - \theta e^{-\lambda x}} dx \quad (3.31)$$

From (3.24)
$$\int x e^{-\lambda x - \theta e^{-\lambda x}} dx = \frac{1}{\lambda^2 \theta} \left[(1 - e^{-\theta e^{-\lambda x}}) \ln e^{-\lambda x} + \sum_{n=1}^{\infty} \frac{(-\theta e^{-\lambda x})^n}{n \times n!} \right]$$

Therefore
$$\int_{t_i}^{\infty} x e^{-\lambda x - \theta e^{-\lambda x}} dx = \frac{1}{\lambda^2 \theta} \left[(1 - e^{-\theta e^{-\lambda x}}) \ln e^{-\lambda x} + \sum_{n=1}^{\infty} \frac{(-\theta e^{-\lambda x})^n}{n \times n!} \right] \Bigg|_{t_i}^{\infty}$$

Substituting limits and simplification yields

$$\int_{t_i}^{\infty} x e^{-\lambda x - \theta e^{-\lambda x}} dx = \frac{1}{\lambda^2 \theta} \left[\lambda t_i (1 - e^{-\theta e^{-\lambda t_i}}) - \sum_{n=1}^{\infty} \frac{(-\theta e^{-\lambda t_i})^n}{n \times n!} \right]$$

Substituting the expression for $\int_{t_i}^{\infty} x e^{-\lambda x - \theta e^{-\lambda x}} dx$ in equation (3.31) gives

$$E_{3i} = \frac{\theta \lambda}{1 - e^{-\theta e^{-\lambda t_i}}} \times \frac{1}{\lambda^2 \theta} \left[\lambda t_i (1 - e^{-\theta e^{-\lambda t_i}}) - \sum_{n=1}^{\infty} \frac{(-\theta e^{-\lambda t_i})^n}{n \times n!} \right]$$

$$E_{3i} = t_i - \frac{1}{\lambda (1 - e^{-\theta e^{-\lambda t_i}})} \sum_{n=1}^{\infty} \frac{(-\theta e^{-\lambda t_i})^n}{n \times n!} \quad (3.32)$$

For E_{4i} ,

$$E_{4i} = E \left[e^{-\lambda Z_{ij}} / Z_{ij} \in (t_i, \infty); \Theta \right]$$

$$E_{4i} = \frac{1}{1 - F(t_i; \theta, \lambda)} \int_{t_i}^{\infty} e^{-\lambda x} f(x) dx \quad \text{But } 1 - F(t_i; \theta, \lambda) = \frac{1 - e^{-\theta e^{-\lambda t_i}}}{1 - e^{-\theta}}$$

$$E_{4i} = \frac{\theta \lambda (1 - e^{-\theta})^{-1}}{1 - F(t_i; \theta, \lambda)} \int_{t_i}^{\infty} e^{-2\lambda x - \theta e^{-\lambda x}} dx = \frac{\theta \lambda}{1 - e^{-\theta e^{-\lambda t_i}}} \int_{t_i}^{\infty} e^{-2\lambda x - \theta e^{-\lambda x}} dx \quad (3.33)$$

From (3.29)
$$\int e^{-2\lambda x - \theta e^{-\lambda x}} dx = \frac{1}{\theta \lambda} \left[e^{-\lambda x - \theta e^{-\lambda x}} + \frac{e^{-\theta e^{-\lambda x}}}{\theta} \right]$$

Therefore
$$\int_{t_i}^{\infty} e^{-2\lambda x - \theta e^{-\lambda x}} dx = \frac{1}{\theta \lambda} \left[e^{-\lambda x - \theta e^{-\lambda x}} + \frac{e^{-\theta e^{-\lambda x}}}{\theta} \right] \Bigg|_{t_i}^{\infty}$$

$$\int_{t_i}^{\infty} e^{-2\lambda x - \theta e^{-\lambda x}} dx = \frac{1}{\theta\lambda} \left[\frac{1 - \theta e^{-\lambda t_i - \theta e^{-\lambda t_i}} - e^{-\theta e^{-\lambda t_i}}}{\theta} \right]$$

Substituting the expression for $\int_{t_i}^{\infty} e^{-2\lambda x - \theta e^{-\lambda x}} dx$ in equation (3.33) and simplifying we obtain

$$E_{4i} = \frac{1 - \theta e^{-\lambda t_i - \theta e^{-\lambda t_i}} - e^{-\theta e^{-\lambda t_i}}}{\theta(1 - e^{-\theta e^{-\lambda t_i}})} \quad (3.34)$$

Replacing the conditional expectations E_{1i} , E_{2i} , E_{3i} and E_{4i} in equation (3.22) completes the E-step. We get the pseudo-log-likelihood function as

$$\begin{aligned} l(\Theta) = & n \ln(\theta\lambda) - \lambda \sum_{i=1}^m X_i E_{1i} - \theta \sum_{i=1}^m X_i E_{2i} - \lambda \sum_{i=1}^m R_i E_{3i} \\ & - \theta \sum_{i=1}^m R_i E_{4i} - n \ln(1 - e^{-\theta}) \end{aligned} \quad (3.35)$$

After substituting the conditional expectations, partial derivatives of (3.35) with respect to parameters are derived in order to maximize the pseudo-log-likelihood function as follows:-

$$\frac{\partial l(\Theta)}{\partial \lambda} = \frac{n}{\lambda} - \sum_{i=1}^m X_i E_{1i} - \sum_{i=1}^m R_i E_{3i} \quad (3.36)$$

and

$$\frac{\partial l(\Theta)}{\partial \theta} = \frac{n}{\theta} - \frac{n}{e^{\theta} - 1} - \sum_{i=1}^m X_i E_{2i} - \sum_{i=1}^m R_i E_{4i} \quad (3.37)$$

Suppose that $(\theta^{(k)}, \lambda^{(k)})$ are estimates of (θ, λ) at the k^{th} step. Then, by equating (3.36) and (3.37) to zero, we obtain

$$\frac{n}{\lambda^{(k)}} = \sum_{i=1}^m X_i E_{1i}(\theta; \lambda) + \sum_{i=1}^m R_i E_{3i}(\theta; \lambda) \quad (3.38)$$

and

$$\frac{n}{\theta^{(k)}} = \frac{n}{e^\theta - 1} + \sum_{i=1}^m X_i E_{2i}(\theta; \lambda) + \sum_{i=1}^m R_i E_{4i}(\theta; \lambda) \quad (3.39)$$

The M-step requires solutions to equations (3.38) and (3.39) to obtain the next values $\lambda^{(k+1)}$ and $\theta^{(k+1)}$ of λ and θ given respectively by

$$\lambda^{(k+1)} = \frac{n}{\sum_{i=1}^m X_i E_{1i}(\theta^{(k)}; \lambda^{(k)}) + \sum_{i=1}^m R_i E_{3i}(\theta^{(k)}; \lambda^{(k)})} \quad (3.40)$$

and

$$\theta^{(k+1)} = \frac{n}{\frac{n}{e^{\theta^{(k)}} - 1} + \sum_{i=1}^m X_i E_{2i}(\theta^{(k)}; \lambda^{(k)}) + \sum_{i=1}^m R_i E_{4i}(\theta^{(k)}; \lambda^{(k)})} \quad (3.41)$$

Checking convergence, if the convergence occurs at $(\theta^{(k)}; \lambda^{(k)})$ then the current $\theta^{(k)}$ and $\lambda^{(k)}$ are the approximated MLEs of θ and λ via EM algorithm. Otherwise set $k = k + 1$ and go back to equations (3.40) and (3.41). The process is repeated until convergence occurs. At convergence $|\theta^{(k+1)} - \theta^{(k)}| < \epsilon$ and $|\lambda^{(k+1)} - \lambda^{(k)}| < \epsilon$ for some given $\epsilon > 0$, say $\epsilon = 0.0001$.

3.8 Variance-covariance Matrix

In most applications, a variance-covariance matrix is used to calculate the standard errors of estimators. Applying large sample approximation, $\hat{\Theta} = (\hat{\theta}, \hat{\lambda})$ has a bivariate normal distribution with mean Θ and variance-covariance matrix given by the inverse of the Fisher information matrix.

The likelihood function for the PE distribution with parameter vector $\Theta = (\theta, \lambda)$ on the basis of the observed sample $\underline{t} = t_1, t_2, \dots, t_n$ of size n is given by

$$L(\Theta) = e^{n \log(\theta\lambda) - \lambda \sum_{i=1}^n t_i - \theta \sum_{i=1}^n e^{-\lambda t_i} - n \log(1 - e^{-\theta})} \quad (3.42)$$

Introducing logarithms on (3.42) yields

$$\ln L(\Theta) = n \log(\theta\lambda) - \lambda \sum_{i=1}^n t_i - \theta \sum_{i=1}^n e^{-\lambda t_i} - n \log(1 - e^{-\theta}) \quad (3.43)$$

The variance-covariance matrix is derived from the log-likelihood function (3.43). The inverse of the Fisher information matrix, Cox and Hinkley, (1979) obtained from the variance-covariance matrix is given as below,

$$\begin{bmatrix} \text{var}(\hat{\theta}) & \text{cov}(\hat{\theta}, \hat{\lambda}) \\ \text{cov}(\hat{\theta}, \hat{\lambda}) & \text{var}(\hat{\lambda}) \end{bmatrix} = \begin{bmatrix} -E \left[\frac{\partial^2 \ln L(\Theta)}{\partial \theta^2} \right] & -E \left[\frac{\partial^2 \ln L(\Theta)}{\partial \theta \partial \lambda} \right] \\ -E \left[\frac{\partial^2 \ln L(\Theta)}{\partial \theta \partial \lambda} \right] & -E \left[\frac{\partial^2 \ln L(\Theta)}{\partial \lambda^2} \right] \end{bmatrix}^{-1} \quad (3.44)$$

Where: -

$$\frac{\partial^2 \ln L}{\partial \theta^2} = -\frac{n}{\theta^2} + \frac{ne^{-\theta}}{(1 - e^{-\theta})^2}$$

$$\frac{\partial^2 \ln L}{\partial \theta \partial \lambda} = \sum_{i=1}^n t_i e^{-\lambda t_i}$$

$$\frac{\partial^2 \ln L}{\partial \lambda^2} = -\frac{n}{\lambda^2} - \theta \sum_{i=1}^n t_i^2 e^{-\lambda t_i}$$

Expected values of $te^{-\lambda t}$ and $t^2e^{-\lambda t}$ are given by

$$E(te^{-\lambda t}) = \frac{n\theta}{4\lambda(1 - e^{-\theta})} F_{2,2}([2, 2], [3, 3], -\theta)$$

$$E(t^2 e^{-\lambda t}) = \frac{n\theta}{4\lambda^2(1-e^{-\theta})} F_{3,3}([2, 2, 2], [3, 3, 3], -\theta)$$

Therefore, the elements of the Fisher information matrix for complete information, $I_{comp}(\Theta)$ are

$$\left. \begin{aligned} -E\left[\frac{\partial^2 \ln L(\Theta)}{\partial \theta^2}\right] &= \frac{n}{\theta^2} - \frac{ne^{-\theta}}{(1-e^{-\theta})^2} \\ -E\left[\frac{\partial^2 \ln L(\Theta)}{\partial \theta \partial \lambda}\right] &= -E\left[\frac{\partial^2 \ln L(\Theta)}{\partial \lambda \partial \theta}\right] = -\frac{n\theta}{4\lambda(1-e^{-\theta})} F_{2,2}([2, 2], [3, 3], -\theta) \\ -E\left[\frac{\partial^2 \ln L(\Theta)}{\partial \lambda^2}\right] &= \frac{n}{\lambda^2} + \frac{n\theta^2}{4\lambda^2(1-e^{-\theta})} F_{3,3}([2, 2, 2], [3, 3, 3], -\theta) \end{aligned} \right\} \quad (3.45)$$

In this project, the Fisher information matrix of observed elements is derived using the idea of missing information principle developed by Louis, (1982). The idea suggested that $I_{obs}(\Theta) = I_{comp}(\Theta) - I_{mis}(\Theta)$ where $I_{obs}(\Theta)$, $I_{comp}(\Theta)$ and $I_{mis}(\Theta)$ represents the observed, complete and missing information respectively.

Fisher information matrix at the i^{th} failure for a single observation is given by

$$I_{mis}^i(\Theta) = -E\left[\frac{\partial^2 \ln f_{z/t}(z_{ij} / z_{ij} > t_i; \Theta)}{\partial \Theta^2}\right]$$

Given $T_i = t_i$, the conditional distribution of z_{ij} given t_i follows a truncated PE distribution truncated from left at t_i . Which is expressed as

$$f_{z/t}(z_{ij} / z_{ij} > t_i; \Theta) = \frac{f_t(z_{ij})}{1 - F_T(t_i)}, z_{ij} > t_i \quad (3.46)$$

On substituting (3.4) and (3.5) in (3.46) and introducing logarithms on both sides yields

$$\ln f_{z/t} (z_{ij} / z_{ij} > t_i; \Theta) = \ln \theta + \ln \lambda - \lambda z_i - \theta e^{-\lambda z_i} - \ln(1 - e^{-\theta e^{-\lambda t_i}}) \quad (3.47)$$

2nd Partial derivatives of (3.47) with respect to θ and λ gives

$$\frac{\partial^2 \ln f_{z/t} (z_{ij} / z_{ij} > t_i; \Theta)}{\partial \theta^2} = -\frac{1}{\theta^2} + \frac{e^{-2\lambda t_i - \theta e^{-\lambda t_i}}}{(1 - e^{-\theta e^{-\lambda t_i}})^2}$$

$$\frac{\partial^2 \ln f_{z/t} (z_{ij} / z_{ij} > t_i; \Theta)}{\partial \theta \partial \lambda} = z_i e^{-\lambda z_i} + \frac{t_i (e^{-\lambda t_i - \theta e^{-\lambda t_i}} - \theta e^{-2\lambda t_i - \theta e^{-\lambda t_i}} - e^{-\lambda t_i - 2\theta e^{-\lambda t_i}})}{(1 - e^{-\theta e^{-\lambda t_i}})^2}$$

$$\frac{\partial^2 \ln f_{z/t} (z_{ij} / z_{ij} > t_i; \Theta)}{\partial \lambda^2} = -\frac{1}{\lambda^2} - \theta z_i^2 e^{-\lambda z_i} + \frac{\theta t_i^2 (-e^{-\lambda t_i - \theta e^{-\lambda t_i}} + \theta e^{-2\lambda t_i - \theta e^{-\lambda t_i}} + e^{-\lambda t_i - 2\theta e^{-\lambda t_i}})}{(1 - e^{-\theta e^{-\lambda t_i}})^2}$$

Expected values of $z e^{-\lambda z}$ and $z^2 e^{-\lambda z}$ are

$$E(z e^{-\lambda z}) = \frac{\theta}{4\lambda(1 - e^{-\theta})} F_{2,2}([2, 2], [3, 3], -\theta)$$

$$E(z^2 e^{-\lambda z}) = \frac{\theta}{4\lambda^2(1 - e^{-\theta})} F_{3,3}([2, 2, 2], [3, 3, 3], -\theta)$$

Hence, elements of the Fisher information matrix for missing information, $I_{mis}(\Theta)$ are

$$\left. \begin{aligned} -E \left[\frac{\partial^2 \ln f_{z/t} (z_{ij} / z_{ij} > t_i; \Theta)}{\partial \theta^2} \right] &= \frac{1}{\theta^2} + \frac{e^{-2\lambda t_i - \theta e^{-\lambda t_i}}}{(1 - e^{-\theta e^{-\lambda t_i}})^2} = A_{11} \\ -E \left[\frac{\partial^2 \ln f_{z/t} (z_{ij} / z_{ij} > t_i; \Theta)}{\partial \theta \partial \lambda} \right] &= -\frac{\theta}{4\lambda(1 - e^{-\theta})} F_{2,2}([2, 2], [3, 3], -\theta) - \frac{t_i (e^{-\lambda t_i - \theta e^{-\lambda t_i}} - \theta e^{-2\lambda t_i - \theta e^{-\lambda t_i}} - e^{-\lambda t_i - 2\theta e^{-\lambda t_i}})}{(1 - e^{-\theta e^{-\lambda t_i}})^2} = A_{12} \\ -E \left[\frac{\partial^2 \ln f_{z/t} (z_{ij} / z_{ij} > t_i; \Theta)}{\partial \lambda^2} \right] &= \\ &= \frac{1}{\lambda^2} + \frac{\theta^2}{4\lambda^2(1 - e^{-\theta})} F_{3,3}([2, 2, 2], [3, 3, 3], -\theta) - \frac{\theta t_i^2 (-e^{-\lambda t_i - \theta e^{-\lambda t_i}} + \theta e^{-2\lambda t_i - \theta e^{-\lambda t_i}} + e^{-\lambda t_i - 2\theta e^{-\lambda t_i}})}{(1 - e^{-\theta e^{-\lambda t_i}})^2} = A_{22} \end{aligned} \right\} \quad (3.48)$$

Therefore the variance-covariance matrix is finally obtained by substituting the elements for $I_{comp}(\Theta)$ from equation (3.45) and the elements for $I_{mis}(\Theta)$ from equation (3.48) in the equation (3.49) and simplifying.

$$I_{obs}^{-1}(\Theta) = [I_{comp}(\Theta) - I_{mis}(\Theta)]^{-1} \quad (3.49)$$

The variance-covariance matrix (3.49) can be applied in the construction of asymptotic confidence interval for the unknown parameters.

3.9 Chapter Summary

In this chapter, probability density function, cumulative distribution function and moments for PE distribution are given. PTI interval censoring is also discussed. Derivation of Maximum Likelihood Estimates of parameters for PE distribution is presented via EM algorithm under PTI interval censoring. Lastly, variance-covariance matrix of the MLEs is derived.

CHAPTER FOUR

RESULTS AND DISCUSSION

4.1 Introduction

In this chapter, a simulation study is conducted to investigate the behaviour of the proposed MLEs of Poisson-Exponential distribution parameters under progressive type I interval censoring scheme via EM algorithm on simulated data. In particular, we assess the precision and accuracy of the MLEs obtained under different censoring schemes and parameter values.

4.2 Simulation study

The progressive type I interval censored samples are generated from PE distribution by considering four different censoring schemes similar to those considered by Aggarwala, (2001). Data is simulated by employing an algorithm proposed by Aggarwala, (2001) to generate number of failures X_1, X_2, \dots, X_m in each interval $(t_{l-1}, t_l]$, for $l = 1, 2, \dots, m$ from an initial sample of size n put on a life test at time $t_0 = 0$.

Progressively type I interval-censored data $\{X_l, R_l, t_l\}$ for $l = 1, 2, \dots, m$ where, X_l is the number of failures, R_l is the number of randomly removed surviving units and t_l is the l^{th} inspection time from PE distribution with CDF (3.5) is generated as follows:-

Let $X_0 = 0, R_0 = 0$ and for $l = 1, 2, \dots, m, \Theta \in (\theta; \lambda)$.

$$X_l | X_{l-1}, \dots, X_0, R_{l-1}, \dots, R_0 \sim rbinom \left[n - \sum_{k=1}^{l-1} (X_k + R_k), \frac{F(t_l; \Theta) - F(t_{l-1}; \Theta)}{1 - \sum_{k=1}^{l-1} [F(t_k; \Theta) - F(t_{k-1}; \Theta)]} \right]$$

$$\begin{aligned}
&= rbinom \left[n - \sum_{k=1}^{l-1} (X_k + R_k), \frac{F(t_l; \Theta) - F(t_{l-1}; \Theta)}{1 - F(t_{l-1}; \Theta)} \right] \\
&= rbinom \left[n - \sum_{k=1}^{l-1} (X_k + R_k), \frac{e^{-\theta e^{-\lambda t_l}} - e^{-\theta e^{-\lambda t_{l-1}}}}{1 - e^{-\theta e^{-\lambda t_{l-1}}}} \right] \\
R_l &= floor \left\{ p_l \times \left[n - \sum_{k=1}^{l-1} (X_k + R_k) - X_l \right] \right\}
\end{aligned}$$

$floor()$ returns the greatest integer equal to or less than the argument and p_l is the pre-specified percentage of withdrawing.

Clearly, if $p_1 = p_2 = \dots = p_{m-1} = 0$, then $R_1 = R_2 = \dots = R_{m-1} = 0$ since $R_l = p_l Y_l$ where Y_l is the number of surviving units and hence $X_1, X_2, \dots, X_m, X_{m+1} = R_m$ becomes a simulated sample from the conventional type I interval censoring. The above algorithm is an improvement from the one suggested by Kemp and Kemp, (1987) for multinomial distribution involving the generation of m binomial random variables with the pseudo-code as follows:-

1. Set $l=0$ and let $xsum = rsum = 0$
2. $l=l+1$

- Generate X_l as a binomial random variable with parameters

$$n - xsum - rsum \text{ and } \frac{e^{-\theta e^{-\lambda t_l}} - e^{-\theta e^{-\lambda t_{l-1}}}}{1 - e^{-\theta e^{-\lambda t_{l-1}}}}.$$

- Calculate $R_l^{obs} = floor \left\{ p_l \times \left[n - \sum_{k=1}^{l-1} (X_k + R_k) - X_l \right] \right\}$ or

$$R_l^{obs} = \min(n - xsum - rsum - X_l; R_l)$$
 which will be dependent on

implemented censoring scheme by percentage p_l or R_l .

3. Set $xsum = xsum + X_l$ and $rsum = rsum + R_l^{obs}$.
4. If $l < m$, go to step 2; otherwise, stop.

A simulation study was conducted in R language; a software package that was designed by Ihaka and Gentleman, (1996). To compare how the proposed estimator performs in the estimation of parameters of PE distribution under PTI interval censoring, we used four different progressive interval censoring schemes similar to those considered by Aggarwala, (2001).

Which are: -

$$\text{scheme 1, } p_{(1)} = (0.25, 0.25, 0.25, 0.25, 0.5, 0.5, 0.5, 0.5, 1)$$

$$\text{scheme 2, } p_{(2)} = (0.5, 0.5, 0.5, 0.5, 0.25, 0.25, 0.25, 0.25, 1)$$

$$\text{scheme 3, } p_{(3)} = (0, 0, 0, 0, 0, 0, 0, 0, 1)$$

$$\text{scheme 4, } p_{(4)} = (0.25, 0, 0, 0, 0, 0, 0, 0, 1)$$

The schemes are chosen to specify the percentage of surviving units to be withdrawn at the 9 censoring and monitoring points. Whereby in $p_{(1)}$ for the first four intervals the removal is lighter as compared to the last four intervals, while in $p_{(2)}$ the reverse scenario of $p_{(1)}$ is applied. In $p_{(3)}$ no removal is done prior to termination which is a case similar to conventional type I interval censoring. Lastly, in $p_{(4)}$ removal is conducted at the left-most and right-most ends.

We considered the parameter values and sample sizes respectively as

$$(\theta, \lambda) \in \{(0.02, 0.2), (0.02, 0.4), (0.03, 0.2), (0.03, 0.4)\} \text{ and } n=20 \text{ (small), } 200 \text{ (large)}$$

Then we generate progressive type I interval-censored data for PE distribution.

Once the samples have been generated, we obtain the parameter estimates via the EM algorithm by repeating the process $N=100$ times for different sample sizes $n=20$ and $n=200$ respectively. In this work, convergence is assumed to occur when the absolute difference between successive estimates is less than 0.0001.

Suppose $\hat{\Theta}_i$ is the MLE of Θ for the i^{th} replication of the simulated EM algorithm, then from the simulation runs the absolute value of bias and mean squared error (MSE) of $\hat{\Theta}$ are respectively computed by applying the formulae below:-

$$(i) \quad Bias(\hat{\Theta}) = \frac{1}{100} \sum_{i=1}^{100} \left| \Theta - \hat{\Theta}_i \right|, \text{ where } \Theta = (\theta, \lambda)$$

$$(ii) \quad MSE(\hat{\Theta}) = \frac{1}{100} \sum_{i=1}^{100} \left(\Theta - \hat{\Theta}_i \right)^2$$

4.3 Results

Table 4.1: Bias and MSE of $\hat{\theta}$ and $\hat{\lambda}$ under different censoring schemes when $\theta = 0.02$ and $\lambda = 0.2$

Sample size	Censoring scheme	Estimated values		Bias ($\hat{\theta}$)	Bias ($\hat{\lambda}$)	MSE ($\hat{\theta}$)	MSE ($\hat{\lambda}$)
		$\hat{\theta}$	$\hat{\lambda}$				
n=20	Scheme 1	0.02017589	0.1010482	0.0001759	0.0989518	3.094E-08	0.00979
	Scheme 2	0.02022346	0.0710573	0.0002235	0.1289427	4.993E-08	0.01663
	Scheme 3	0.02010807	0.1428157	0.0001081	0.0571843	1.168E-08	0.00327
	Scheme 4	0.02014845	0.14102	0.0001484	0.05898	2.204E-08	0.00348
n=200	Scheme 1	0.02015016	0.1226911	0.0001502	0.0773089	2.255E-08	0.00598
	Scheme 2	0.02022257	0.0774668	0.0002226	0.1225332	4.954E-08	0.01501
	Scheme 3	0.02006497	0.1770925	0.0000650	0.0229075	4.221E-09	0.00052
	Scheme 4	0.02012954	0.1453538	0.0001295	0.0546462	1.678E-08	0.00299

Table 4.2: Bias and MSE of $\hat{\theta}$ and $\hat{\lambda}$ under different censoring schemes when $\theta = 0.02$ and $\lambda = 0.4$

Sample size	Censoring scheme	Estimated values		Bias ($\hat{\theta}$)	Bias ($\hat{\lambda}$)	MSE ($\hat{\theta}$)	MSE ($\hat{\lambda}$)
		$\hat{\theta}$	$\hat{\lambda}$				
n=20	Scheme 1	0.02025872	0.2181204	0.0002587	0.1818796	6.694E-08	0.03308
	Scheme 2	0.02031863	0.1479301	0.0003186	0.2520699	1.015E-07	0.06354
	Scheme 3	0.02013722	0.2795695	0.0001372	0.1204305	1.883E-08	0.0145
	Scheme 4	0.02021796	0.2531296	0.0002180	0.1468704	4.751E-08	0.02157
n=200	Scheme 1	0.02021691	0.2446643	0.0002169	0.1553357	4.705E-08	0.02413
	Scheme 2	0.02027654	0.1842143	0.0002765	0.2157857	7.647E-08	0.04656
	Scheme 3	0.0201086	0.3091464	0.0001086	0.0908536	1.179E-08	0.00825
	Scheme 4	0.02017807	0.2812259	0.0001781	0.1187741	3.171E-08	0.01411

Table 4.3: Bias and MSE of $\hat{\theta}$ and $\hat{\lambda}$ under different censoring schemes when $\theta = 0.03$ and $\lambda = 0.2$

Sample size	Censoring scheme	Estimated values		Bias ($\hat{\theta}$)	Bias ($\hat{\lambda}$)	MSE ($\hat{\theta}$)	MSE ($\hat{\lambda}$)
		$\hat{\theta}$	$\hat{\lambda}$				
n=20	Scheme 1	0.03038365	0.1005903	0.0003837	0.0994097	1.472E-07	0.00988
	Scheme 2	0.03048287	0.07007791	0.0004829	0.12992209	2.332E-07	0.01688
	Scheme 3	0.03023907	0.1426577	0.0002391	0.0573423	5.715E-08	0.00329
	Scheme 4	0.03032899	0.1409852	0.0003290	0.0590148	1.082E-07	0.00348
n=200	Scheme 1	0.03035381	0.116287	0.0003538	0.083713	1.252E-07	0.00701
	Scheme 2	0.03048001	0.07636214	0.0004800	0.12363786	2.304E-07	0.01529
	Scheme 3	0.03014612	0.1766993	0.0001461	0.0233007	2.135E-08	0.00054
	Scheme 4	0.03028618	0.145192	0.0002862	0.054808	8.190E-08	0.003

Table 4.4: Bias and MSE of $\hat{\theta}$ and $\hat{\lambda}$ under different censoring schemes when $\theta = 0.03$ and $\lambda = 0.4$

Sample size	Censoring scheme	Estimated values		Bias ($\hat{\theta}$)	Bias ($\hat{\lambda}$)	MSE ($\hat{\theta}$)	MSE ($\hat{\lambda}$)
		$\hat{\theta}$	$\hat{\lambda}$				
n=20	Scheme 1	0.03061982	0.2177433	0.0006198	0.1822567	3.842E-07	0.03322
	Scheme 2	0.03070269	0.1469932	0.0007027	0.2530068	4.938E-07	0.06401
	Scheme 3	0.0303752	0.2795617	0.0003752	0.1204383	1.408E-07	0.01451
	Scheme 4	0.03048842	0.2529547	0.0004884	0.1470453	2.386E-07	0.02162
n=200	Scheme 1	0.03048228	0.24428	0.0004823	0.15572	2.326E-07	0.02425
	Scheme 2	0.03061033	0.1798701	0.0006103	0.2201299	3.725E-07	0.04846
	Scheme 3	0.03024274	0.3091344	0.0002427	0.0908656	5.892E-08	0.00826
	Scheme 4	0.0303978	0.2810305	0.0003978	0.1189695	1.582E-07	0.01415

From **Tables 4.1 to 4.4**, it is observed that:

- (i) The MSE and bias of the estimates decrease as the sample size increase from n=20 to n=200 for each censoring scheme. These imply that the estimates become more precise and accurate as the sample size increase.
- (ii) Among the four censoring schemes, the third scheme $p_{(3)}$ provides the most precise and accurate results as seen from the MSE and bias values, followed by scheme $p_{(4)}$,

$p_{(1)}$ and lastly $p_{(2)}$. Similar performance among the four schemes is observed when the sample size is increased from $n=20$ to $n=200$.

The results of the performance comparisons among these four censoring schemes are similar to the results obtained in Ng and Wang, (2009), Chen and Lio, (2010) and Lio *et al.*, (2011).

These phenomena are expected since the third censoring scheme $p_{(3)}$, could have the largest number of failure items observed before the termination of the life-testing, followed by $p_{(4)}$, $p_{(1)}$ and lastly $p_{(2)}$. Intuitively, these are also consistent with the statistical theory that the larger the “sample size” is the more accurate the parameter estimate is.

As for the performance among the assumed true parameters, it is observed that when the values are varied from $(\theta = 0.02 \text{ and } \lambda = 0.2)$ to $(\theta = 0.03 \text{ and } \lambda = 0.4)$ the estimators become less accurate and less precise as depicted from the bias and MSE values from **Table 4.1** and **Table 4.4**.

4.4 Chapter summary

The chapter entails a simulation study conducted to investigate the behaviour of the proposed MLEs of PE distribution parameters under PTI interval censoring via EM algorithm on simulated data. Bias and MSE of parameters are obtained under different censoring schemes and parameter values.

CHAPTER FIVE

SUMMARY, CONCLUSION AND SUGGESTED AREAS FOR FURTHER STUDY

5.1 Introduction

This chapter presents the summary, conclusion and areas for further study.

5.2 Summary

The main objective of this study was to obtain the Maximum Likelihood Estimation of parameters for Poisson-Exponential distribution based on progressive type I interval censoring.

The EM algorithm was employed to compute Maximum Likelihood estimates.

A simulation study was conducted to compare the accuracy and precision of this method in estimating the parameters for Poisson-Exponential distribution. The assessment was done through the bias and MSE under four different censoring schemes and various parameter values. Bias and MSE of estimates decreased as the sample size increased. As for the schemes, the results from the third scheme were most precise and accurate followed by fourth, first and lastly second scheme.

5.3 Conclusion

In the research, the problem of the Maximum Likelihood Estimation of parameters for Poisson-Exponential distribution based on progressive type I interval censored samples generated using the algorithm of Aggarwala, (2001) via the EM algorithm is addressed.

A comparison of the MLEs obtained is made by simulation under four different censoring schemes and various parameter values. The results have shown that:-

- (i) For an increasing sample size, the estimated values of the parameters become closer to the true values.

- (ii) Among the censoring schemes considered, the third scheme $p_{(3)}$ provides the most accurate and precise results then schemes $p_{(4)}$, $p_{(1)}$ and lastly scheme $p_{(2)}$.

5.4 Areas for further study

In this research, Maximum Likelihood is the only method of Estimation of parameters of PE distribution considered under PTI interval censoring. In the course of research, areas for further study have emerged, for instance, it was noticed that estimation could as well be done through other methods such as mid-point approximation method, method of moments, estimation based on probability plot amongst other methods and a comparison made based on results from all the methods to ascertain the method that gives the most precise and accurate parameter estimates.

The derived variance-covariance matrix in future studies can be used in the construction of confidence interval for the unknown parameters.

In addition, besides the above classical methods of analysis, Bayesian inference under progressive type I interval censoring for PE distribution can be an interesting area of study.

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APPENDIX
R-codes

```

#remove variables in memory
rm(list=ls())
ls()
library(SuppDists)
#-----
#  random sample from Poisson Exponential distribution
#  n: sample size; theta, lambda: parameters
#-----
# random number for PED
set.seed(1400)
#model parameters
n<-200
p<-c(0.25,0,0,0,0,0,0,0,1)
m<-length(p)
T<-c(0,1,2,3,4,5,6,7,8,9)
theta<-0.03
lambda<-0.4
rPed=function(n,theta,lambda){
  U=runif(n,min=0,max=1)
  return (log(log(1-(1-U)*(1-exp(-theta)))/(-theta)))/(-lambda))
}
# probability density function for PED with theta and lambda as parameters
dPed=function(x,theta,lambda)(theta*lambda*exp((-lambda*x)-theta*exp(-lambda*x)))/(1-
exp(-theta))
# Cumulative distribution function for PED with theta and lambda as parameters
pPed=function(x,theta,lambda)(1-(((1-exp((-theta)*exp(-lambda*x)))/(1-exp(-theta))))))
# hazard function for PED with theta and lambda as parameters
hPed=function(x,theta,lambda)(dPed(x,theta,lambda))/(1-pPed(x,theta,lambda))
# 2 function calls for the EM

```

```
y1=function(x,theta,lambda)(x*(theta*lambda*exp((-lambda*x))-theta*exp(-lambda*x))/(1-
exp(-theta)))
```

```
y2=function(x,theta,lambda)((((exp(-lambda*x))*(theta*lambda*exp((-lambda*x))-theta*exp(-
lambda*x)))/(1-exp(-theta))))
```

```
#-----
```

```
# Progressive Type I sample
```

```
# n is the number of items on test at t =0
```

```
# T is pre-specified schedules T[1]=0 and length(T)= length(P)+1
```

```
# P is percentage of withdraw at pre-specified schedule after initial time
```

```
# theta, lambda are parameters
```

```
#-----
```

```
PTIS=function(theta,lambda,p,T,m,n){
```

```
  xsum=0
```

```
  rsum=0
```

```
  m=length(p)
```

```
  x=R=pp=c()
```

```
  pp[1]=1-((1-exp(-theta)*exp(-lambda*T[1]))/(1-exp(-theta)))
```

```
  x[1]=rbinom(1,n,pp[1])
```

```
  R[1]=floor(p[1]*(n-x[1]))
```

```
  xsum[1]=x[1]
```

```
  rsum[1]=R[1]
```

```
  for (i in 2:m){
```

```
    pp[i]=(exp(-theta*exp(-lambda*T[i]))-exp(-theta*exp(-lambda*T[i-1])))/(1-exp(-
theta*exp(-lambda*T[i-1])))
```

```
    x[i]=rbinom(1,n-xsum[i-1]-rsum[i-1],pp[i])
```

```
    R[i]=floor(p[i]*(n-xsum[i-1]-rsum[i-1]-x[i]))
```

```
    xsum[i]=xsum[i-1]+x[i]
```

```
    rsum[i]=rsum[i-1]+R[i]
```

```
  }
```

```
  return(cbind(x,R))
```

```
}
```

PTIS(theta,lambda,p,T,m,n)

```
#####
```

```
# EM algorithm for theta and lambda
```

```
#####
```

```
theta<-0.03
```

```
lambda<-0.4
```

```
T<-c(0,1,2,3,4,5,6,7,8,9)
```

```
x<-c(0,56,39,16,12,9,7,6,2)
```

```
R<-c(50,0,0,0,0,0,0,0,3)
```

```
EM=function(x,R,T,theta,lambda){
```

```
  m=length(R)
```

```
  n=sum(x)+sum(R)
```

```
  th1=theta
```

```
  lam1=lambda
```

```
  E1=E2=E3=E4=numeric(m)
```

```
  Cont=TRUE
```

```
  while(Cont){
```

```
    the=th1
```

```
    lam=lam1
```

```
    mm=m+1
```

```
    #E-step
```

```
    for(i in 2:mm){
```

```
      d1=pPed(T[i],the,lam)-pPed(T[i-1],the,lam)
```

```
      d2=1.0-pPed(T[i],the,lam)
```

```
      # print(c(d1,d2))
```

```
      # print(c(the,lam,T[i-1],T[i]))
```

```
      E1[i-1]=integrate(y1,lower=T[i-1],upper=T[i],theta=the,lambda=lam)$value/d1
```

```
      E2[i-1]=integrate(y2,lower=T[i-1],upper=T[i],theta=the,lambda=lam)$value/d1
```

```

E3[i-1]=integrate(y1,lower=T[i],upper=Inf,theta=the,lambda=lam)$value/d2
E4[i-1]=integrate(y2,lower=T[i], upper=Inf,theta=the,lambda=lam)$value/d2
print(c(E1[i-1],E2[i-1],E3[i-1],E4[i-1]))
}
# end of E-step

#M step
lam1=n/(sum(x*E1)+sum(R*E3))
th1=n/((n/(exp(th1)-1))+sum(x*E2)+sum(R*E4))
# add a check since the search could diverge, so force to start over again
#   if (abs(th1-the) > 10^-4 || abs(lam1-lam) >10^-4)
#   cat(th1,lam1)
#   cat(the,lam)
#Convergence checking
if((abs(th1-the)<10^-4) && (abs(lam1-lam)<10^-4)) Cont= FALSE
} # end of while-loop
return(cbind(th1,lam1))
} # end of EM module
EM(x,R,T,theta,lambda)
th1<- 0.0303978
lam1<-0.2810305
MSE1<-(theta-th1)^2
MSE2<-(lambda-lam1)^2
Bias1<-abs(theta-th1)
Bias2<-abs(lambda-lam1)
MSE1
MSE2
Bias1
Bias2

```