

**INVESTIGATING SEDENION EXTENSION LOOPS AND
FRAMES OF GENERAL HYPERCOMPLEX NUMBERS**

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DECLARATION

Declaration by the candidate

This thesis is my original work and has not been presented for a degree in any other University or any other award.

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DEDICATION

I dedicate this work to my wonderful children Mukami and Mutoru, *the wind beneath my wings*.

May this work inspire you in your academic endeavours.

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LIST OF SYMBOLS

\mathbb{Z}	-	Set of integers
\mathbb{R}	-	Set of real numbers
\mathbb{C}	-	Set of complex numbers
\mathbb{H}	-	Quaternions
\mathbb{K}	-	Octonions
\mathbb{S}	-	Sedenions
\bar{y}	-	the conjugate of y
$L \times S^0$	-	Sedenion loop extension of the loop L
\oplus	-	Nim Addition

Number Theory
Number Complex

ABSTRACT

A sequence of algebras over the field of real numbers can be constructed, each with twice the dimension of the previous one. The algebra constructed by doubling complex numbers is the 2^2 -dimensional quaternions. Next we have the 2^3 -dimensional octonions, constructed by doubling the quaternions. The algebra constructed by doubling octonions is the 2^4 -dimensional sedenions. The oldest method of constructing these algebras is the Cayley-Dickson formula. Since they extend the complex numbers, they are called hypercomplex numbers in general for dimensions greater than 2^4 . Our main emphasis is on the general 2^n -ons. Split extensions of 2^n -ons are studied from the point of view of Loop Theory. Multiplication of basis elements of complex, quaternion, octonion and sedenion split extensions using the Jonathan Smith doubling formula is done. It is shown that Nim addition gives a way of determining the subscripts for the products of the basis elements for the split extensions. This result is extended to split extensions of general 2^n -ons. Subloops of sedenions are also investigated, and it is shown that they do not necessarily reduce to either subloops of octonions or sedenion split extensions. This is done by showing the existence of two-sided subloops of sedenions that are neither subloops of octonions or sedenion split extensions. It is well known that when L , the multiplicative subloop of octonions is abelian, the sedenion split extension $L \times S^0$ formed is a group. The structure of $L \times S^0$ when L is non-abelian is studied. In particular, it is shown that $L \times S^0$ fails to satisfy group properties. When the satisfaction of standard Loop theoretical properties is investigated, $L \times S^0$ is seen to satisfy the Jordan identity and flexible properties. It however fails to satisfy the left alternative, right alternative and anti-automorphism properties. Finally, multiplication of the basis elements of complex numbers, quaternions, octonions and sedenions is done using the Jonathan Smith doubling formula. In each case, it is shown that the formula gives a way of constructing Hadamard matrices, as did the Smith-Conway formula. This result is generalized for general 2^n -ons.

CHAPTER ONE

INTRODUCTION

This chapter covers background information, statement of the problem, hypotheses, objectives and significance of the study. It also includes the scope of the study, definitions and preliminary results which are used extensively in the research. A section on characterization of algebra properties is also included.

1.1 Background Information

A complex number $a + bi$ can be written as an ordered pair (a, b) of real numbers a and b with multiplication defined by

$$(a, b)(c, d) = (ac - db, da + bc) \quad (1)$$

In Mathematics, a sequence of algebras over the field of real numbers can be constructed, each with twice the dimension of the previous one. The oldest method of constructing these algebras is the Cayley-Dickson formula and the algebras thus produced are known as Cayley-Dickson algebras (Bremner and Hentzel, 2001). Since the algebras extend the complex numbers they are called hypercomplex numbers (Kantor and Solodovnikov, 1989) in general for dimensions greater than 2^4 .

The algebra constructed by doubling complex numbers is the 2^2 - dimensional quaternions. The doubling is done by forming ordered pairs (a, b) of complex numbers a and b , where multiplication is defined by:

$$(a,b)(c,d) = (ac - \bar{d}b, da + b\bar{c}) \quad (2)$$

Next we have the 2^3 -dimensional octonions, constructed by forming ordered pairs (a,b) of quaternions a and b , where multiplication is defined exactly as for the quaternions. The algebra immediately following the octonions is the 2^4 -dimensional sedenions.

In general, the Cayley-Dickson construction is a process by which a 2^n -dimensional hypercomplex number system is constructed from pairs of 2^{n-1} -dimensional hypercomplex numbers, where n is a positive integer. This is done by defining a multiplication rule for the 2^n -dimensional hypercomplex numbers.

But this, of course, raises the question: Why isn't there an infinite sequence of division algebras, each one obtained from the preceding one by the Cayley-Dickson construction? The answer is that each time we apply the construction, our algebra gets a bit worse. First the complex numbers lose the self-conjugacy and ordering property, the quaternions lose commutativity, octonions lose associativity and finally the sedenions lose the division algebra property.

Several scholars have come up with different doubling formulas with an interest of getting nicer properties in the sedenions and higher 2^n -ons. Some of the formulas are included in Section 2.4.

1.2 Definitions of terms

Below are some definitions of the terms that will be used extensively in this work.

1.2.1: A *groupoid* $\langle G, \cdot \rangle$ is a non-empty set which is closed under the binary operation \cdot , that is, for all $a, b \in G$, there exists a unique $c \in G$ such that $a \cdot b = c$.

1.2.2: A *quasigroup* is a groupoid G such that in the equation $x \cdot y = z$, knowledge of any two of the elements defines the third one uniquely.

A quasigroup can also be defined as a set Q with a binary operation \cdot , left division \backslash and right division $/$ such that the identities

$$x \backslash (x \cdot y) = y \quad x \cdot (x \backslash y) = y \quad (3)$$

$$(y \cdot x) / x = y \quad (y / x) \cdot x = y \quad (4)$$

are satisfied. In this aspect, a left quasigroup is a set Q satisfying (equation 3) while a right quasigroup satisfies (equation 4).

1.2.3: A *loop* L is a quasigroup Q with a two-sided identity e satisfying

$$e \cdot x = x \cdot e = x \quad \forall x \in Q.$$

1.2.4:

(a) A *left loop* has a left identity satisfying $x \backslash x = x \backslash (x \cdot e) = e \quad \forall x \in Q$.

(b) A *right loop* has a right identity satisfying $x / x = (e \cdot x) / x = e \quad \forall x \in Q$.

1.2.5: A loop L has

(a) *left inverse property* if $\forall x \in L, \exists x' \in L$ such that $x'(x \cdot y) = y, \forall y \in L$.

(b) *right inverse property* if $\forall x \in L, \exists x' \in L$ such that $(y \cdot x)x' = y, \forall y \in L$.

(c) *inverse property* if it has both left and right inverse properties and

$$x \cdot x^r = x^l \cdot x = e, \forall x \in L.$$

1.2.6: A *subloop* is a non-empty subset of a loop which is itself a loop.

1.2.7: A loop L is called a *Moufang loop* if the Moufang identities

$$xy \cdot zx = (x \cdot yz)x$$

$$x(y \cdot xz) = (xy \cdot x)z$$

$x(y \cdot zy) = (xy \cdot z)y$ are satisfied for every $x, y, z \in L$.

1.2.8: An algebra A is said to be *power associative* if the exponential rule

$$x^m x^n = x^{m+n} \text{ for all } x \in A, \text{ and for all } m \geq 1, n \geq 1 \text{ always holds.}$$

1.2.9: A *Bol loop* is a loop satisfying the left or right 'Bol laws'

$$\text{Left Bol Law: } (x \cdot yx)z = x(y \cdot xz).$$

$$\text{Right Bol Law: } z(xy \cdot x) = (zx \cdot y)x.$$

1.2.10: A *division algebra* is an algebra in which, given b , and given $a \neq 0$

(a) there exists z so that $a \cdot z = b$, and

(b) there exists q so that $q \cdot a = b$.

1.2.11: Let L be a multiplicative subloop of the non-zero octonions. Then its

sedenion extension $L \times S^0$ is the disjoint union $L \cup Lf$ within the sedenions.

Elements of this union are encoded as pairs (a, ε) with $\varepsilon \in S^0 = \{1, -1\}$, by $a \mapsto (a, 1)$, $af \mapsto (a, -1)$. See (equation 9).

1.3 Preliminary Results

Theorem 1.3.1: Kivunge (2004). Let L be a multiplicative subloop of the octonions. Then the following statements are equivalent:

- (i) $L \times S^0$ is a group
- (ii) $L \times S^0$ has the left inverse property
- (iii) $L \times S^0$ has the right inverse property
- (iv) L is commutative
- (v) L is abelian

Lemma 1.3.2: Kivunge (2004). Each commutative multiplicative subloop of the octonions is associative.

Corollary 1.3.3: Kivunge (2004). A commutative sedenion extension loop is associative.

Theorem 1.3.4: (Moufang theorem) Hsu (2000). Let x, y and z be elements in a Moufang loop L . Then the subloop generated by x, y and z is associative if and only if x, y and z associate.

1.4 Hypotheses

1.4.1: The structure of the sedenion extension formed when the octonion subloop from which it is formed is **abelian** is stronger than the one formed when the octonion subloop is **non-abelian**, in terms of loop properties.

1.4.2: The multiplication of split extensions for 2^n -ons can be viewed in terms of Nim addition when Jonathan Smith formula is used.

1.4.3: Subloops of sedenions do not necessarily reduce to subloops of octonions or sedenion split extensions.

1.4.4: The frame multiplication of 2^n -ons using the Jonathan Smith formula gives a way of constructing Hadamard matrices.

1.5 Objectives

1.5.1 General Objective

The objective of this research is to study complex numbers, quaternions, octonions, sedenions and the general 2^n -ons. Sedenion split extensions will be investigated from the point of view of Loop Theory.

1.5.2 Specific Objectives

- (i) To investigate the structure of the sedenion extension formed when the octonion subloop from which it is formed is non-abelian.
- (ii) To determine whether or not the multiplication of split extensions for 2^n -ons can be viewed in terms of Nim-addition.
- (iii) To determine whether subloops of sedenions reduce to subloops of octonions or sedenion split extensions
- (iv) To find out if the Jonathan Smith formula gives a way of constructing Hadamard matrices, as did the Smith-Conway formula.

1.6 Significance of the Study

The importance of orthogonal matrices in modern Discrete Mathematics and its applications is well known for many applied problems. Hadamard matrices were originally investigated in this category of matrices in the 18th century. The original investigations were connected with the Linear Algebra problem of maximizing determinants. It was later realized that Hadamard matrices were very useful in problems relating to signal processing, machine learning, and information theory. There is an inter-relation between Hadamard matrices and combinatorial configurations such as block designs, latin squares, error-correcting codes and finite geometry (Martin and Sloane, 1979).

With the wide application of Hadamard matrices, Mathematicians have devoted themselves to finding new methods of constructing them. In this research the method used to construct the matrices is the Jonathan Smith formula.

1.7 Scope of the Study

This study is on the multiplication of the general hypercomplex numbers, which are referred to as 2^n -ons. There are many formulae for doing this, but in this work the Jonathan Smith formula is used.

Sedenion split extensions are also studied, specifically when they are formed from subloops of octonions. The structure taken by the split extensions when the

octonion subloop from which they are formed is non-abelian instead of abelian is be investigated.

1.8 Statement of the Problem

The problem being studied is divided into four parts. Each part seeks to answer a specific question.

(a) Suppose the multiplicative subloop of the octonions L in the following theorem is non- **abelian**, what effect does that have on the statements in the theorem?

Theorem: Kivunge (2004). Let L be a multiplicative subloop of the octonions. Then, the following statements are equivalent:

- (i) $L \times S^0$ is a group
- (ii) $L \times S^0$ has the left inverse property
- (iii) $L \times S^0$ has the right inverse property
- (iv) L is commutative
- (v) L is **abelian**

(b) The multiplication of the 2^n -ons can be viewed in terms of Nim addition when the Smith-Conway formula is used (Kivunge 2004). Can the multiplication of split extensions for 2^n -ons using the Jonathan Smith formula also be viewed in terms of Nim addition?

(c) Does there exist two-sided subloops of sedenions that are neither subloops of octonions or sedenion split extensions, given the following:?

- (i) Each commutative multiplicative subloop of the octonions is associative (Lemma 1.3.2).
- (ii) A commutative sedenion extension loop is associative (Corollary 1.3.3).
- (d) When the frame multiplication of 2^n -ons was done using the Smith-Conway formula, it gave a way of constructing Hadamard matrices. Now, does the Jonathan Smith formula give a way of constructing the same?

1.9 Characterization of algebra properties

The following table gives some commonly used definitions characterizing properties in loops. We consider a loop L with identity, $x, y \in L$ and scalars α, β .

Property	Characterization
<i>Anti-automorphism</i>	$\overline{(xy)} = \bar{y}\bar{x}$
<i>Power distributive</i>	$(\alpha x^k + \beta x^m)y = \alpha x^k y + \beta x^m y$
<i>Right alternative</i>	$x \cdot y^2 = xy \cdot y$
<i>Left alternative</i>	$x \cdot xy = x^2 \cdot y$
<i>Left distributivity</i>	$x(y + z) = xy + xz$
<i>Right distributivity</i>	$(x + y)z = xz + yz$
<i>Flexible</i>	$xy \cdot x = x \cdot yx$
<i>Jordan</i>	$x^2 y \cdot x = x^2 \cdot yx, xy \cdot x^2 = x(y \cdot x^2)$

Table 1.1 Characterization of algebra properties

1.10 Outline of the thesis

This section contains the outline of the contents of this work. Chapter one contains the background information, with definitions and preliminary results that have

been used extensively. The statement of the problem, hypotheses and significance of the study are also included. The Chapter also contains the objectives of the study, and also a section on Characterization of algebra properties.

Chapter two concerns itself with Literature Review. It includes sections on Discovery of Quaternions, Formation of Octonions and Sedenions, Doubling Formulas and Sedenion Split extensions.

In Chapter three, the multiplication in $L \times S^0$ with L being non-abelian is performed. The properties of the split extension so formed are investigated and in particular the satisfaction of standard Loop Theoretical properties.

Chapter four deals with multiplication of basis elements of complex, quaternion, octonion and sedenion split extensions using the Jonathan Smith doubling formula. The relationship between this multiplication and Nim addition is investigated, and the result is extended to split extensions of general 2^n -ons.

In Chapter five, subloops of sedenions are investigated to determine whether or not they reduce to either subloops of octonions or sedenion split extensions. This is done by showing the existence of two-sided subloops of sedenions that are neither subloops of octonions or sedenion split extensions.

Chapter six is concerned with multiplication of the basis elements of complex numbers, quaternions, octonions and sedenions using the Jonathan Smith doubling formula. The aim is to determine whether or not the formula gives a way of constructing Hadamard matrices, as did the Smith- Conway formula. The result is extended to general 2^n - ons .

The conclusion and recommendations are contained in Chapter seven, which is followed by References

CHAPTER TWO

LITERATURE REVIEW

2.1 Discovery of Quaternions

In 1831, Gauss was convinced that there was no 'hypercomplex' number system outside the complex numbers in which the basic properties of the complex numbers persisted (Ebbinghaus et al.,1991).

In 1843, William Hamilton (Ebbinghaus et al.,1991), also known for his contribution in Mechanics, was trying to generalize complex numbers to a system of hypercomplex numbers which would be viewed as ordered n -tuples of more than two numbers. He came up with the multiplication rule for the new number system with four ordered real numbers, and he called them quaternions. It was not the discovery of quaternions which was the great achievement, but rather the great freedom to construct hypercomplex numbers.

2.2 Formation of Octonions

In December 1843, Graves discovered the 8-dimensional octonions (octaves). He communicated his results to Hamilton in a letter dated 4th January 1844, but they were not published till 1848. Octonions were rediscovered by Arthur Cayley and have since been called Cayley numbers. He outlined how to construct a theory for octonions (Cayley, 1845).

Compared to the quaternions, however, octonions have languished into such obscurity. One reason is that they lacked a tireless defender such as Hamilton, but the real reason for this is that they lacked any clear application to Geometry and Physics (Baez, 2002).

2.3 Formation of Sedenions

Arthur Cayley also came up with the Cayley-Dickson construction process (equation 2) from which the 16-dimensional sedenions and the general 2^n -ons can be constructed.

2.4 Doubling Formulas

Zassenhaus and Eichhorn (1966) found a rational 16-square identity based on defining the determinant of a 2×2 matrix whose entries are octonions. The multiplication formula is

$$(a, b)(c, d) = (a\bar{c} - b\bar{d}, \quad ad + a \cdot c \cdot a^{-1}b) \quad (5)$$

Conway and Smith (2003) gave a modification of the Cayley-Dickson process defined by

$$(a, b)(c, d) = \begin{cases} (ac, \bar{a}d) & \text{if } b = 0 \\ (ac - \bar{b}d, \quad \overline{b\bar{c} + b\bar{a} \cdot b^{-1}d}) & \text{if } b \neq 0 \end{cases} \quad (6)$$

They have also come up with a new and more recent approach based on matrix multiplications, given by

$$(a,b)(c,d) = \begin{cases} (ac, \bar{a}d) & \text{if } b = 0 \\ (ac - b\bar{d}, \overline{\overline{\overline{\overline{b\bar{c} + \bar{b}\bar{a}b^{-1}\bar{d}}}}}}) & \text{if } b \neq 0 \end{cases} \quad (7)$$

The multiplication is power associative, right linear, power distributive and has the anti- automorphism property.

Smith (1995) introduced the formula

$$(a,b)(c,d) = \begin{cases} (ac, da) & \text{if } b = 0 \\ (ab.cb^{-1} - b\bar{d}, b\bar{c} + db^{-1} \cdot ab) & \text{if } b \neq 0 \end{cases} \quad (8)$$

The non-zero sedenions form a left loop under multiplication (Smith and Romanowska, 1999) that includes a two-sided identity.

2.5 Sedenion Split Extensions

The multiplication in equation (8) gives rise to the two-sided loops namely *Sedenion Extensions Loops* denoted by $L \times S^0$, which are niners (a,b) with $b \in S^0 = \{1, -1\}$ and L a subloop of non-zero octonions.

The study of sedenions from the point of view of Loop Theory has been done (Kivunge and Smith 2004). They obtained two-sided loops and also gave conditions for the satisfaction of standard loop-theoretical identities within these loops.

Kivunge (2004) did a study on hypercomplex numbers, with a major emphasis on 16-dimensional sedenions. He introduced sedenion split extensions and gave

equivalent conditions for the split extensions to be groups. (Kivunge and Smith (2004). He was also able to show that the sign matrix of the frame multiplication in the 2^n -ons under the Smith-Conway or Cayley-Dickson process is a Hadamard matrix. It is this work that this research intends to extend, mainly using the Jonathan Smith formula (equation 8).

CHAPTER THREE

SPLIT EXTENSIONS OF OCTONIONS AND GENERAL 2^N -ONS

3.1 Introduction

Let L be a multiplicative subloop of the octonions. When L is an abelian group, the split extension $L \times S^0$ so formed is a group. When a non-abelian group L is used, the properties of the split extension so formed are investigated. In particular, multiplication in $L \times S^0$ with L being non-abelian is performed. Of interest is the effect this has on the five statements in the theorem below:

Theorem 3.1.1: Kivunge (2004) Let L be a multiplicative subloop of the octonions. Then the following statements are equivalent.

- (i) $L \times S^0$ is a group
- (ii) $L \times S^0$ has the left inverse property
- (iii) $L \times S^0$ has the right inverse property
- (iv) L is commutative
- (v) L is abelian

3.2 Octonion Split Extension

As noted earlier, quaternion multiplication is non-commutative. In this section we investigate the structure $H \times S^0$ where H is the quaternion group and S^0 is the zero-dimensional unit sphere or two-element multiplicative group $\{\pm 1\}$ as in

Theorem 3.11 above. Since H is a multiplicative group, it is a subloop of the octonions and it will therefore be used in next section to represent L .

3.2.1 Multiplication of $H \times S^0$ elements

The multiplication of elements of $L \times S^0$ in general is given by the following equations:

$$\begin{aligned}
 \text{i). } & (x,1)(y,1) = (xy,1) \\
 \text{ii). } & (x,1)(y,-1) = (yx,-1) \\
 \text{iii). } & (x,-1)(y,1) = (x\bar{y},-1) \\
 \text{iv). } & (x,-1)(y,-1) = (-x\bar{y},1)
 \end{aligned} \tag{9}$$

We will use this formula to perform the multiplication in $H \times S^0$.

For the quaternion group H , let elements of $H \times S^0$ be given by:

$$\begin{aligned}
 \alpha_0 &= (1,1), \alpha_1 = (i,1), \alpha_2 = (j,1), \alpha_3 = (k,1), \alpha_4 = (-1,-1), \alpha_5 = (-i,-1), \\
 \alpha_6 &= (-j,-1), \alpha_7 = (-k,-1), \alpha_8 = (-1,1), \alpha_9 = (-i,1), \alpha_{10} = (-j,1), \\
 \alpha_{11} &= (-k,1), \alpha_{12} = (1,-1), \alpha_{13} = (i,-1), \alpha_{14} = (j,-1), \alpha_{15} = (k,-1)
 \end{aligned}$$

Example 3.2.2:

$$\alpha_1 \cdot \alpha_1 = (i,1)(i,1) = (i^2,1) = (-1,1) = \alpha_8$$

Example 3.2.3:

$$\alpha_3 \cdot \alpha_7 = (k,1)(-k,-1) = (-k^2,-1) = (1,-1) = \alpha_{12}$$

Example 3.2.4:

$$\alpha_{13} \cdot \alpha_{11} = (i, -1)(-k, 1) = (-i\bar{k}, -1) = (-j, -1) = \alpha_6$$

Example 3.2.5:

$$\alpha_6 \cdot \alpha_{15} = (-j, -1)(k, -1) = (j\bar{k}, 1) = (-i, 1) = \alpha_9$$

The multiplication table for $H \times S^0$ is given below:

	α_0	α_1	α_2	α_3	α_4	α_5	α_6	α_7	α_8	α_9	α_{10}	α_{11}	α_{12}	α_{13}	α_{14}	α_{15}
α_0	α_0	α_1	α_2	α_3	α_4	α_5	α_6	α_7	α_8	α_9	α_{10}	α_{11}	α_{12}	α_{13}	α_{14}	α_{15}
α_1	α_1	α_8	α_3	α_{10}	α_5	α_{12}	α_{15}	α_6	α_9	α_0	α_{11}	α_2	α_{13}	α_4	α_7	α_{14}
α_2	α_2	α_{11}	α_8	α_1	α_6	α_7	α_{12}	α_{13}	α_{10}	α_3	α_0	α_9	α_{14}	α_{15}	α_4	α_5
α_3	α_3	α_2	α_9	α_8	α_7	α_{14}	α_5	α_{12}	α_{11}	α_{10}	α_1	α_0	α_{15}	α_6	α_{13}	α_4
α_4	α_4	α_{13}	α_{14}	α_{15}	α_8	α_1	α_2	α_3	α_{12}	α_5	α_6	α_7	α_0	α_9	α_{10}	α_{11}
α_5	α_5	α_4	α_{15}	α_6	α_9	α_8	α_3	α_{10}	α_{13}	α_{12}	α_7	α_{14}	α_1	α_0	α_{11}	α_2
α_6	α_6	α_7	α_4	α_{13}	α_{10}	α_{11}	α_8	α_1	α_{14}	α_{15}	α_{12}	α_5	α_2	α_3	α_0	α_9
α_7	α_7	α_{14}	α_5	α_4	α_{11}	α_2	α_9	α_8	α_{15}	α_6	α_{13}	α_{12}	α_3	α_{10}	α_1	α_0
α_8	α_8	α_9	α_{10}	α_{11}	α_{12}	α_{13}	α_{14}	α_{15}	α_0	α_1	α_2	α_3	α_4	α_5	α_6	α_7
α_9	α_9	α_0	α_{11}	α_2	α_{13}	α_4	α_7	α_{14}	α_1	α_8	α_3	α_{10}	α_5	α_{12}	α_{15}	α_6
α_{10}	α_{10}	α_3	α_0	α_9	α_{14}	α_{15}	α_4	α_5	α_2	α_{11}	α_8	α_1	α_6	α_7	α_{12}	α_{13}
α_{11}	α_{11}	α_{10}	α_1	α_0	α_{15}	α_6	α_{13}	α_4	α_3	α_2	α_9	α_8	α_7	α_{14}	α_5	α_{12}
α_{12}	α_{12}	α_5	α_6	α_7	α_0	α_9	α_{10}	α_{11}	α_4	α_{13}	α_{14}	α_{15}	α_8	α_1	α_2	α_3
α_{13}	α_{13}	α_{12}	α_7	α_{14}	α_1	α_0	α_{11}	α_2	α_5	α_4	α_{15}	α_6	α_9	α_8	α_3	α_{10}
α_{14}	α_{14}	α_{15}	α_{12}	α_5	α_2	α_3	α_0	α_9	α_6	α_7	α_4	α_{13}	α_{10}	α_{11}	α_8	α_1
α_{15}	α_{15}	α_6	α_{13}	α_{12}	α_3	α_{10}	α_1	α_0	α_7	α_{14}	α_5	α_4	α_{11}	α_2	α_9	α_8

Table 3.1 Multiplication of $H \times S^0$ elements

Observations:

1. Since each element appears only once in every row and every column, it follows that any two elements describe a third one uniquely i.e. $H \times S^0$ is a quasi-group.
2. α_0 is the two-sided identity for the extension $H \times S^0$ since $a_0 a_n = a_n a_0 = a_n$, $n = 0, 1, 2, \dots, 15$.
3. $H \times S^0$ does not satisfy the left inverse property, i.e. $x'(xy) \neq y$
 $\forall x, y \in H \times S^0$.

Example 3.2.6:

Let $x = \alpha_6$, $y = \alpha_7$. From the table, $(\alpha_6)^{-1} = \alpha_{14}$.

Now, $(\alpha_6)'(\alpha_6 \alpha_7) = (\alpha_6)'(\alpha_1) = \alpha_{14} \alpha_1 = \alpha_{15} \neq \alpha_7$.

4. $H \times S^0$ does not satisfy the right inverse property, i.e. $(xy)y' \neq x$
 $\forall x, y \in H \times S^0$.

Example 3.2.7:

$(\alpha_6 \alpha_7)(\alpha_7)^r = \alpha_1 (\alpha_7)^r = \alpha_1 \alpha_{15} = \alpha_{14} \neq \alpha_6$.

5. $H \times S^0$ is not associative since $(\alpha_i \alpha_j) \alpha_k \neq \alpha_i (\alpha_j \alpha_k) \quad \forall i, j, k$.

Example 3.2.8:

$\alpha_8 (\alpha_6 \alpha_7) = \alpha_{13} \neq \alpha_9 = (\alpha_8 \alpha_6) \alpha_7$.

Thus, $H \times S^0$ is not a group.

The following is given for the purpose of offering a clear alternative proof to Theorem 3.1.1 given by Kivunge (2004).

Theorem 3.2.5: Let L be a multiplicative subloop of octonions. Then $L \times S^0$ is a group iff L is abelian.

Proof:

Let L be abelian. We prove that $L \times S^0$ is a group.

1. *Closure:*

$$(x, \pm 1)(y, \pm 1) = (z, \pm 1) \in L \times S^0 \quad \forall x, y, z \in L.$$

Thus $L \times S^0$ is closed under multiplication.

2. *Identity:*

There exists a two-sided identity $e = (1, 1) \in L \times S^0$.

$$(x, \pm 1) \cdot e = e \cdot (x, \pm 1) = (x, \pm 1) \quad \forall x \in L.$$

$$(x, 1)(1, 1) = (x \cdot 1, 1) = (x, 1)$$

$$(1, 1)(x, 1) = (1 \cdot x, 1) = (x, 1)$$

$$(x, -1)(1, 1) = (x \cdot \bar{1}, -1) = (x, -1)$$

$$(1, 1)(x, -1) = (x \cdot 1, -1) = (x, -1).$$

3. *Inverse:*

For each $(x, \pm 1) \in L \times S^0$, there exists $(x, \pm 1)^{-1}$ such that

$$(x, \pm 1)^{-1}(x, \pm 1) = (x, \pm 1)(x, \pm 1)^{-1} = (1, 1). \text{ There are two cases:}$$

(i) If $x^{-1} = y \in L$

$$(x, 1)(x^{-1}, 1) = (xx^{-1}, 1) = (1, 1)$$

$$(x^{-1}, 1)(x, 1) = (x^{-1}x, 1) = (1, 1)$$

$$\text{So } (x, 1)^{-1} = (x^{-1}, 1)$$

$$(ii) \text{ Let } (x, -1)^{-1} = (y, -1), \text{ then } (x, -1)(x, -1)^{-1} = (1, 1)$$

$$(x, -1)(y, -1) = (-x\bar{y}, 1) = (1, 1) \text{ and } -x\bar{y} = 1$$

$$\text{But } \frac{x\bar{x}}{|x|^2} = 1 \Rightarrow -x\bar{y} = \frac{x\bar{x}}{|x|^2} \Rightarrow -\bar{y} = \frac{\bar{x}}{|x|^2} \Rightarrow \bar{y} = -\frac{\bar{x}}{|x|^2} \Rightarrow y = \frac{-x}{|x|^2}$$

$$\text{Therefore } (x, -1)^{-1} = \left(\frac{-x}{|x|^2}, -1 \right)$$

4. Associativity:

There are eight cases:

$$(i) \quad [(x, 1)(y, 1)](z, 1) = (xy, 1)(z, 1) = (xyz, 1)$$

$$(x, 1)[(y, 1)(z, 1)] = (x, 1)(yz, 1) = (xyz, 1)$$

$$(ii) \quad [(x, 1)(y, 1)](z, -1) = (xy, 1)(z, -1) = (zxy, -1)$$

$$(x, 1)[(y, 1)(z, -1)] = (x, 1)(zy, -1) = (zyx, -1)$$

Equality holds iff $zxy = zyx$ and so $xy = yx$.

$$(iii) \quad [(x, 1)(y, -1)](z, 1) = (yx, -1)(z, 1) = (yx\bar{z}, -1)$$

$$(x, 1)[(y, -1)(z, 1)] = (x, 1)(y\bar{z}, -1) = (y\bar{z}x, -1)$$

Equality holds iff $yx\bar{z} = y\bar{z}x$ and thus $x\bar{z} = \bar{z}x$.

$$(iv) \quad (x, 1)[(y, -1)(z, -1)] = (x, 1)(-y\bar{z}, 1) = (-xy\bar{z}, 1)$$

$$[(x,1)(y,-1)](z,-1) = (yx,-1)(z,-1) = (-yx\bar{z},1)$$

Equality holds iff $-xy\bar{z} = -yx\bar{z}$ and thus $xy = yx$.

$$(v) \quad (x,-1)[(y,1)(z,1)] = (x,-1)(yz,1) = (x\bar{y}\bar{z},-1) = (x\bar{z}\bar{y},-1)$$

$$[(x,-1)(y,1)](z,1) = (x\bar{y},-1)(z,1) = (x\bar{y}\bar{z},-1)$$

Equality holds iff $x\bar{z}\bar{y} = x\bar{y}\bar{z}$, thus $\bar{z}\bar{y} = \bar{y}\bar{z}$ and $yz = zy$.

$$(vi) \quad (x,-1)[(y,1)(z,-1)] = (x,-1)(zy,-1) = (-x\bar{z}\bar{y},1) = (-x\bar{y}\bar{z},1)$$

$$[(x,-1)(y,1)](z,-1) = (x\bar{y},-1)(z,-1) = (-x\bar{y}\bar{z},1)$$

Here, $x\bar{y}\bar{z} = x\bar{y}\bar{z}$ which is true.

$$(vii) \quad (x,-1)[(y,-1)(z,1)] = (x,-1)(y\bar{z},-1) = (-x\bar{y}\bar{z},1) = (-xz\bar{y},1)$$

$$[(x,-1)(y,-1)](z,1) = (-x\bar{y},1)(z,1) = (-x\bar{y}\bar{z},1)$$

Equality holds iff $-xz\bar{y} = -x\bar{y}\bar{z}$, thus $\bar{z}\bar{y} = \bar{y}\bar{z}$ and $zy = yz$.

$$(viii) \quad [(x,-1)(y,-1)](z,-1) = (-x\bar{y},1)(z,-1) = (-zx\bar{y},-1)$$

$$(x,-1)[(y,-1)(z,-1)] = (x,-1)(-y\bar{z},1) = (-x\bar{y}\bar{z},-1) = (-xz\bar{y},-1)$$

Equality holds iff $-zx\bar{y} = -xz\bar{y}$, thus $zx = xz$.

Thus $L \times S^0$ is a group.

Conversely, let $L \times S^0$ be a group. Then multiplication in $L \times S^0$ is associative. As shown in the eight cases above, L is abelian.

Remark 3.2.6

Group properties of closure, two-sided identity and two-sided inverse will hold regardless of whether L is abelian or not. In other words, the commutativity of L does not have a direct consequence on the other three group properties.

The question to ask then is, what happens to $L \times S^0$ when L is not abelian. In other words, what properties does $L \times S^0$ satisfy as L varies? We now investigate the satisfaction of loop properties by $L \times S^0$.

3.3 Loop properties of $L \times S^0$

Lemma 3.3.1: *Let $L = \{\pm 1, \pm i, \pm j, \pm k\}$ be the quaternion group. Then $L \times S^0$ satisfies the Jordan Identity and is flexible, but does not satisfy the anti-automorphism, right alternative and left alternative properties.*

Proof:

(i) **Jordan Identity,** $x^2 y \cdot x = x^2 \cdot yx$, $xyx^2 = x(y \cdot x^2)$

There are four cases, one trivial and three non-trivial. Let $x, y \in L$.

In the first case, the S^0 -components in $L \times S^0$ are both positive.

Case 1(trivial):

$$(a) [(x,1)^2 (y,1)](x,1) = [(xx,1)(y,1)](x,1) = (xxy,1)(x,1) = (xxyx,1)$$

$$(x,1)^2 [(y,1)](x,1) = (xx,1)(yx,1) = (xxyx,1)$$

Therefore $[(x,1)^2 (y,1)](x,1) = (x,1)^2 [(y,1)(x,1)] \forall x, y \in L$

$$(b) [(x,1)(y,1)](x,1)^2 = [(xy,1)(xx,1)] = (xyxx,1)$$

$$(x,1)[(y,1)(x,1)^2] = (x,1)[(y,1)(xx,1)] = (x,1)(yxx,1) = (xyxx,1)$$

$$\text{Therefore } [(x,1)(y,1)](x,1)^2 = (x,1)[(y,1)(x,1)^2] \quad \forall x, y \in L$$

In the following three cases, one or both of the S^0 -component(s) in $L \times S^0$ is/are negative.

Case 2:

(a)

$$[(x,-1)^2(y,1)](x,-1) = [(-x\bar{x},1)(y,1)](x,-1) = (-x\bar{x}y,1)(x,-1) = (-xx\bar{x}y,-1)$$

$$= (-x|x|^2 y,-1) = (-xy,-1) \text{ since } |x|^2 = 1 \quad \forall x \in L$$

$$(x,-1)^2[(y,1)(x,-1)] = (-x\bar{x},1)(xy,-1) = (-xyx\bar{x},-1) = (-xy,-1)$$

$$\text{Therefore } [(x,-1)^2(y,1)](x,-1) = (x,-1)^2[(y,1)(x,-1)] \quad \forall x, y \in L.$$

(b)

$$[(x,-1)(y,1)](x,-1)^2 = (x\bar{y},-1)(-x\bar{x},1) = (-x\bar{y}\bar{x}x,-1) = (-x\bar{y}|x|^2,-1) = (-x\bar{y},-1)$$

$$\text{since } |x|^2 = 1 \quad \forall x \in L$$

$$(x,-1)[(y,1)(x,-1)^2] = (x,-1)[(y,1)(-x\bar{x},1)] = (x,-1)(-yx\bar{x},1) = (-x\overline{yx\bar{x}},-1)$$

$$= (-x\bar{x}\bar{y},-1) = (-x|x|^2\bar{y},-1) = (-x\bar{y},-1) \text{ since } |x|^2 = 1 \quad \forall x \in L$$

$$\text{Therefore } [(x,-1)(y,1)](x,-1)^2 = (x,-1)[(y,1)(x,-1)^2] \quad \forall x, y \in L.$$

Case 3:

$$(a) [(x,1)^2(y,-1)](x,1)=[(xx,1)(y,-1)](x,1)=(yxx,-1)(x,1)=(yxx\bar{x},-1)$$

$$=(yx|x|^2,-1)=(yx,-1) \text{ since } |x|^2=1 \forall x \in L$$

$$(x,1)^2[(y,-1)(x,1)]=(xx,1)(y\bar{x},-1)=(y\bar{x}xx,-1)=(y|x|^2x,-1)=(yx,-1)$$

$$\text{since } |x|^2=1 \forall x \in L$$

Therefore $[(x,1)^2(y,-1)](x,1)=(x,1)^2[(y,-1)(x,1)] \forall x, y \in L$.

(b)

$$[(x,1)(y,-1)](x,1)^2=(yx,-1)(xx,1)=(yx\bar{x}\bar{x},-1)=(yx\bar{x}\bar{x},-1)=(yx|x|^2,-1)=(y\bar{x},-1)$$

$$(x,1)[(y,-1)(x,1)^2]=(x,1)[(y,-1)(xx,1)]=(x,1)(y\bar{x}\bar{x},-1)=(y\bar{x}\bar{x}x,-1)=(y\bar{x}|x|^2,-1)$$

$$=(y\bar{x},-1)$$

Therefore $[(x,1)(y,-1)](x,1)^2=(x,1)[(y,-1)(x,1)^2] \forall x, y \in L$.

Case 4:

$$(a) [(x,-1)^2(y,-1)](x,-1)=[(-x\bar{x},1)(y,-1)](x,-1)=(-yx\bar{x},-1)(x,-1)=(yx\bar{x}\bar{x},1)$$

$$=(y|x|^2\bar{x},1)=(y\bar{x},1)$$

$$(x,-1)^2[(y,-1)(x,-1)]=(-x\bar{x},1)(-y\bar{x},1)=(x\bar{x}y\bar{x},1)=(|x|^2y\bar{x},1)=(y\bar{x},1)$$

Therefore, $[(x,-1)^2(y,-1)](x,-1)=(x,-1)^2[(y,-1)(x,-1)] \forall x, y \in L$.

$$(b) [(x,-1)(y,-1)](x,-1)^2=(-x\bar{y},1)(-x\bar{x},1)=(x\bar{y}x\bar{x},1)=(x\bar{y}|x|^2,1)=(x\bar{y},1)$$

$$(x,-1)[(y,-1)](x,-1)^2=(x,-1)[(y,-1)(-x\bar{x},1)]=(x,-1)(-yx\bar{x},-1)=(xx\bar{x}\bar{y},1)$$

$$=(x|x|^2\bar{y},1)=(x\bar{y},1)$$

Therefore, $[(x,-1)(y,-1)](x,-1)^2=(x,-1)[(y,-1)(x,-1)^2] \forall x, y \in L$.

The Jordan property holds in the extension $L \times S^0$.

(ii) Flexible Property, $xy \cdot x = x \cdot yx$

There are four cases, one trivial and three non-trivial. Let $x, y \in L$.

Case 1 (trivial):

$$[(x,1)(y,1)](x,1) = (xy,1)(x,1) = (xyx,1)$$

$$(x,1)[(y,1)(x,1)] = (x,1)(yx,1) = (xyx,1)$$

Therefore, $[(x,1)(y,1)](x,1) = (x,1)[(y,1)(x,1)] \forall x, y \in L$.

Case 2:

$$[(x,-1)(y,1)](x,-1) = (x\bar{y},-1)(x,-1) = (-x\bar{y}\bar{x},1)$$

$$(x,-1)[(y,1)(x,-1)] = (x,-1)(xy,-1) = (-x\bar{xy},1) = (-x\bar{y}\bar{x},1)$$

Therefore, $[(x,-1)(y,1)](x,-1) = (x,-1)[(y,1)(x,-1)] \forall x, y \in L$.

Case 3:

$$[(x,1)(y,-1)](x,1) = (yx,-1)(x,1) = (yx\bar{x},-1) = (y|x|^2,-1) = (y,-1)$$

$$(x,1)[(y,-1)(x,1)] = (x,1)(y\bar{x},-1) = (y\bar{xx},-1) = (y|x|^2,-1) = (y,-1)$$

Therefore, $[(x,1)(y,-1)](x,1) = (x,1)[(y,-1)(x,1)] \forall x, y \in L$.

Case 4:

$$[(x,-1)(y,-1)](x,-1) = (-x\bar{y},1)(x,-1) = (-xx\bar{y},-1)$$

$$(x,-1)[(y,-1)(x,-1)] = (x,-1)(-y\bar{x},1) = (-xx\bar{y},-1)$$

Therefore, $[(x,-1)(y,-1)](x,-1) = (x,-1)[(y,-1)(x,-1)] \forall x, y \in L$.

The extension $L \times S^0$ is flexible.

(iii) Anti-automorphism Property, $\overline{(xy)} = \bar{y}\bar{x}$

There is one trivial and three non-trivial cases. Let $x, y \in L$.

Case 1(trivial):

$$\overline{(x, 1)(y, 1)} = \overline{(xy, 1)} = (\bar{y}\bar{x}, 1)$$

$$\overline{(y, 1)(x, 1)} = \overline{(yx, 1)} = (\bar{x}\bar{y}, 1)$$

Therefore $\overline{(x, 1)(y, 1)} = \overline{(y, 1)(x, 1)} \quad \forall x, y \in L$.

Case 2:

$$\overline{(x, 1)(y, -1)} = \overline{(yx, -1)} = (\bar{x}\bar{y}, -1)$$

$$\overline{(y, -1)(x, 1)} = \overline{(yx, -1)} = (\bar{y}\bar{x}, -1)$$

Therefore, $\overline{(x, 1)(y, -1)} \neq \overline{(y, -1)(x, 1)}$ for some $x, y \in L$.

Case 3:

$$\overline{(x, -1)(y, 1)} = \overline{(x\bar{y}, -1)} = (y\bar{x}, -1)$$

$$\overline{(y, 1)(x, -1)} = \overline{(y\bar{x}, -1)} = (\bar{x}\bar{y}, -1)$$

Therefore, $\overline{(x, -1)(y, 1)} \neq \overline{(y, 1)(x, -1)}$ for some $x, y \in L$.

Case 4:

$$\overline{(x, -1)(y, -1)} = \overline{(-x\bar{y}, 1)} = (-y\bar{x}, 1)$$

$$\overline{(y, -1)(x, -1)} = \overline{(y\bar{x}, 1)} = (-\bar{y}x, 1)$$

Therefore, $\overline{(x, -1)(y, -1)} \neq \overline{(y, -1)(x, -1)}$ for some $x, y \in L$.

The anti-automorphism property does not hold in the extension $L \times S^0$.

(iv) **Right alternative Property, $x \cdot y^2 = xy \cdot y$**

There is one trivial case and three non-trivial cases. Let $x, y \in L$.

Case 1 (trivial):

$$(x, 1)[(y, 1)(y, 1)] = (x, 1)(y^2, 1) = (xy^2, 1)$$

$$[(x, 1)(y, 1)](y, 1) = (xy, 1)(y, 1) = (xy^2, 1)$$

Therefore $(x, 1)[(y, 1)(y, 1)] = [(x, 1)(y, 1)](y, 1) \forall x, y \in L$.

Case 2:

$$(x, -1)[(y, 1)(y, 1)] = (x, -1)(y^2, 1) = (x\bar{y}^2, -1)$$

$$[(x, -1)(y, 1)](y, 1) = (x\bar{y}, -1)(y, 1) = (x\bar{y}^2, -1)$$

Therefore, $(x, -1)[(y, 1)(y, 1)] = [(x, -1)(y, 1)](y, 1) \forall x, y \in L$.

Case 3:

$$(x, 1)[(y, -1)(y, -1)] = (x, 1)(-y\bar{y}, 1) = (-xy\bar{y}, 1) = (-x|y|^2, 1) = (-x, 1)$$

$$[(x, 1)(y, -1)](y, -1) = (yx, -1)(y, -1) = (-yx\bar{y}, 1)$$

Therefore, $(x, 1)[(y, -1)(y, -1)] \neq [(x, 1)(y, -1)](y, -1)$ for some $x, y \in L$.

Case 4:

$$(x, -1)[(y, -1)(y, -1)] = (x, -1)(-y\bar{y}, 1) = (-xy\bar{y}, -1) = (-x|y|^2, -1) = (-x, -1)$$

$$[(x, -1)(y, -1)](y, -1) = (-x\bar{y}, 1)(y, -1) = (-yx\bar{y}, -1)$$

Therefore, $(x, -1)[(y, -1)(y, -1)] \neq [(x, -1)(y, -1)](y, -1)$ for some $x, y \in L$.

The extension $L \times S^0$ does not satisfy the right alternative property.

(v) **Left alternative Property, $x^2 \cdot y = x \cdot xy$**

There is one trivial case and three non-trivial cases. Let $x, y \in L$.

Case 1 (trivial):

$$[(x,1)(x,1)](y,1) = (x^2,1)(y,1) = (x^2y,1)$$

$$(x,1)[(x,1)(y,1)] = (x,1)(xy,1) = (x^2y,1)$$

Therefore, $[(x,1)(x,1)](y,1) = (x,1)[(x,1)(y,1)] \forall x, y \in L$.

Case 2:

$$[(x,1)(x,1)](y,-1) = (x^2,1)(y,-1) = (yx^2,-1)$$

$$(x,1)[(x,1)(y,-1)] = (x,1)(yx,-1) = (yx^2,-1)$$

Therefore, $[(x,1)(x,1)](y,-1) = (x,1)[(x,1)(y,-1)] \forall x, y \in L$.

Case 3:

$$[(x,-1)(x,-1)](y,1) = (-x\bar{x},1)(y,1) = (-x\bar{x}y,1) = (-|x|^2y,1) = (-y,1)$$

$$(x,-1)[(x,-1)(y,1)] = (x,-1)(x\bar{y},-1) = (-x\bar{y},1)$$

Therefore, $[(x,-1)(x,-1)](y,1) \neq (x,-1)[(x,-1)(y,1)]$ for some $x, y \in L$.

Case 4:

$$[(x,-1)(x,-1)](y,-1) = (-x\bar{x},1)(y,-1) = (-yx\bar{x},-1) = (-y|x|^2,-1) = (-y,-1)$$

$$(x,-1)[(x,-1)(y,-1)] = (x,-1)(-x\bar{y},1) = (-x\bar{y},-1)$$

Therefore, $[(x,-1)(x,-1)](y,-1) \neq (x,-1)[(x,-1)(y,-1)]$ for some $x, y \in L$.

The alternative property does not hold in the extension $L \times S^0$.

Conclusion: The Octonion split extension satisfies the Jordan and flexible properties, but does not satisfy the anti-automorphism, left and right alternative properties. If a property does not hold for the Octonion split extension constructed using H , then it does not hold in the general 2^{n+1} – dimensional split extensions of 2^n – ons constructed using 2^n – dimensional non-abelian subloops. We conclude that the anti-automorphism, left and right alternative properties do not hold for split extensions of general 2^n – ons .

CHAPTER FOUR

NIM ADDITION AND SPLIT EXTENSION BASIS FOR 2^N -ONS

4.1 Introduction

In this chapter we start by introducing the concept of Nim addition and its basic properties. We also include a few examples on how the addition is carried out.

We then perform the multiplication of basis elements of complex, quaternion, octonion and sedenion split extensions using equation (9). In each case we show how the multiplication is related to Nim addition. We conclude with a theorem that shows how the multiplication of split extensions for 2^n -ons can be viewed in terms of Nim addition.

4.2 Nim addition

Consider the non-negative integers $\mathbb{Z}^+ = \{0, 1, 2, 3, \dots\}$. Nim addition and Nim multiplication give a way of defining addition and multiplication in \mathbb{Z}^+ to make it a field of characteristic 2. In this section we only consider Nim addition and use the symbol \oplus for the operation.

The binary operation of Nim addition is governed by the following rules:

1. The Nim-sum of a number of distinct powers of 2 is the ordinary sum.

$$\text{Thus } 32 \oplus 16 \oplus 4 \oplus 1 = 53$$

2. The Nim-sum of two equal numbers is 0.

4.2.1 Properties of Nim addition

(i) $\alpha \oplus \beta = \alpha \oplus \theta$ iff $\beta = \theta$

(ii) $\alpha \oplus \beta = \beta \oplus \alpha$

(iii) $(\alpha \oplus \beta) \oplus \theta = \alpha \oplus (\beta \oplus \theta)$

(iv) $\alpha \oplus \beta = 0$, iff $\alpha = \beta$

Proposition 4.2.2: Suppose $K = \{0, 1, 2, \dots, 2^{m-1}\}$ and $\alpha, \beta \in K$, then $\alpha \oplus \beta \in K$.

Proof: By Induction

It is true for $K = \{0, 1, 2, 3\}$ as shown in Table 4.2.

Suppose it is true for $m-1$.

By Induction hypothesis, $\forall \alpha, \beta \in K_{m-1} = \{0, 1, 2, \dots, 2^{m-1} - 1\}$, $\alpha \oplus \beta \in K_{m-1}$.

Here, $\alpha = a_{m-2}2^{m-2} + a_{m-3}2^{m-3} + \dots + a_12^1 + a_02^0$,

$$\beta = b_{m-2}2^{m-2} + b_{m-3}2^{m-3} + \dots + b_12^1 + b_02^0,$$

where $a_i, b_i \in \{0, 1\}$.

We show that it is true for m .

Let $\alpha = a_{m-1}2^{m-1} + a_{m-2}2^{m-2} + \dots + a_12^1 + a_02^0$,

$$\beta = b_{m-1}2^{m-1} + b_{m-2}2^{m-2} + \dots + b_12^1 + b_02^0,$$

where $a_i, b_i \in \{0, 1\}$. There are four cases to be considered.

Case I: If $a_{m-1} = b_{m-1} = 0$,

$\alpha, \beta \in K_{m-1}$ and $\alpha \oplus \beta \in K_{m-1}$ by induction hypothesis.

Case 2: If $a_{m-1} = b_{m-1} = 1$,

$$\begin{aligned} & a_{m-1}2^{m-1} \oplus b_{m-1}2^{m-1} \\ &= 2^{m-1} \oplus 2^{m-1} = 0 \end{aligned}$$

Therefore, $\alpha \oplus \beta \in K_{m-1}$ by induction hypothesis.

Case 3: If $a_{m-1} = 0$ and $b_{m-1} = 1$ or $a_{m-1} = 1$ and $b_{m-1} = 0$,

$$\begin{aligned} \alpha \oplus \beta &= 2^{m-1} \oplus [(a_{m-2} + b_{m-2})2^{m-2} + \dots + (a_1 + b_1)2^1 + (a_0 + b_0)2^0] \\ &= 2^{m-1} + c_{m-2}2^{m-2} + \dots + c_12^1 + c_02^0 \in K_m. \text{ This completes the proof.} \end{aligned}$$

Corollary 4.2.3: If $m \in K$ and $r \notin K$, $m \oplus r \notin K$

Example 4.2.4

1. $41 \oplus 10 = (32 + 8 + 1) \oplus (8 + 2) = 32 \oplus 2 \oplus 1 = 35$
2. $27 \oplus 15 = (16 + 8 + 2 + 1) \oplus (8 + 4 + 2 + 1) = (16 \oplus 4) = 20$

Nim addition can be done by first writing the numbers in binary form and then “*adding without carrying over*”. This kind of addition is used in the game of Nim.

We now use it to describe basic properties of multiplication of split extensions for 2^n -ons.

4.3 Split Extension basis for 2^n -ons and Nim addition

In this section we obtain the multiplication tables for the basis elements of complex, quaternion and octonion split extensions. In each case we obtain the corresponding Nim addition table and compare the two tables. We also study the

relationship between Nim addition and the multiplication of sedenion split extensions. We generalize this result for split extensions for 2^n -ons.

4.3.1 Complex split extensions

Let $a_0 = (1, 1), a_1 = (i, 1), a_2 = (1, -1), a_3 = (i, -1)$ be basis elements in $\mathbb{C} \times S^0$. The multiplication of the elements gives rise to the following table.

\cdot	a_0	a_1	a_2	a_3
a_0	a_0	a_1	a_2	a_3
a_1	a_1	$-a_0$	a_3	$-a_2$
a_2	a_2	$-a_3$	$-a_0$	a_1
a_3	a_3	a_2	$-a_1$	$-a_0$

Table 4.1 Multiplication of Complex split extensions

The following table gives the Nim addition for the elements from 0 to 3.

\oplus	0	1	2	3
0	0	1	2	3
1	1	0	3	2
2	2	3	0	1
3	3	2	1	0

Table 4.2 Nim addition for the elements 0 to 3

This table is similar to Table 4.1. The subscripts of the elements in Table 4.1 make up Table 4.2.

Observations:

1. $a_k a_m = \pm a_{k \oplus m}, \forall k, m = 0, 1, 2, 3.$

2. $a_k a_m = -a_m a_k \quad \forall k, m = 1, 2, 3.$

3. $a_m^2 = -a_0 \quad \forall m = 1, 2, 3.$

4.3.2 Quaternion split extensions

Let $b_0 = (a_0, 1), b_1 = (a_1, 1), b_2 = (a_2, 1), b_3 = (a_3, 1), b_4 = (a_0, -1), b_5 = (a_1, -1),$

$b_6 = (a_2, -1), b_7 = (a_3, -1)$ basis elements in $H \times S^0$. The multiplication of the elements gives rise to the following table.

\cdot	b_0	b_1	b_2	b_3	b_4	b_5	b_6	b_7
b_0	b_0	b_1	b_2	b_3	b_4	b_5	b_6	b_7
b_1	b_1	$-b_0$	b_3	$-b_2$	b_5	$-b_4$	$-b_7$	b_6
b_2	b_2	$-b_3$	$-b_0$	b_1	b_6	b_7	$-b_4$	$-b_5$
b_3	b_3	b_2	$-b_1$	$-b_0$	b_7	$-b_6$	b_5	$-b_4$
b_4	b_4	$-b_5$	$-b_6$	$-b_7$	$-b_0$	b_1	b_2	b_3
b_5	b_5	b_4	$-b_7$	b_6	$-b_1$	$-b_0$	b_3	$-b_2$
b_6	b_6	b_7	b_4	$-b_5$	$-b_2$	$-b_3$	$-b_0$	b_1
b_7	b_7	$-b_6$	b_5	b_4	$-b_3$	b_2	$-b_1$	$-b_0$

Table 4.3 Multiplication of Quaternion split extensions

The following table gives the Nim addition for the elements from 0 to 7.

	0	1	2	3	4	5	6	7
0	0	1	2	3	4	5	6	7
1	1	0	3	2	5	4	7	6
2	2	3	0	1	6	7	4	5
3	3	2	1	0	7	6	5	4
4	4	5	6	7	0	1	2	3
5	5	4	7	6	1	0	3	2
6	6	7	4	5	2	3	0	1
7	7	6	5	4	3	2	1	0

Table 4.4 Nim addition for the elements 0 to 7

This table is similar to Table 4.3. The subscripts of the elements in Table 4.3 make up Table 4.4.

Observations:

$$1. b_k b_m = \pm b_{k \oplus m} \quad \forall k, m = 0, 1, \dots, 7$$

$$2. b_k b_m = -b_m b_k \quad \forall k, m = 1, 2, \dots, 7$$

$$3. b_m^2 = -b_0 \quad \forall m = 1, 2, \dots, 7$$

4.3.3 Octonion split extensions

Let $c_0 = (b_0, 1), c_1 = (b_1, 1), c_2 = (b_2, 1), c_3 = (b_3, 1), c_4 = (b_4, 1), c_5 = (b_5, 1), c_6 = (b_6, 1),$

$c_7 = (b_7, 1), c_8 = (b_0, -1), c_9 = (b_1, -1), c_{10} = (b_2, -1), c_{11} = (b_3, -1), c_{12} = (b_4, -1)$

$c_{13} = (b_5, -1), c_{14} = (b_6, -1), c_{15} = (b_7, -1)$ be basis elements in $K \times S^0$. The

multiplication of the elements gives rise to the following table.

	c_0	c_1	c_2	c_3	c_4	c_5	c_6	c_7	c_8	c_9	c_{10}	c_{11}	c_{12}	c_{13}	c_{14}	c_{15}
c_0	c_0	c_1	c_2	c_3	c_4	c_5	c_6	c_7	c_8	c_9	c_{10}	c_{11}	c_{12}	c_{13}	c_{14}	c_{15}
c_1	c_1	$-c_0$	c_3	$-c_2$	c_5	$-c_4$	$-c_7$	c_6	c_9	$-c_8$	$-c_{11}$	c_{10}	$-c_{13}$	c_{12}	c_{15}	$-c_{14}$
c_2	c_2	$-c_3$	c_0	c_1	c_6	c_7	$-c_4$	$-c_5$	c_{10}	c_{11}	c_8	$-c_9$	$-c_{14}$	$-c_{15}$	c_{12}	c_{13}
c_3	c_3	c_2	$-c_1$	$-c_0$	c_7	$-c_6$	c_5	$-c_4$	c_{11}	$-c_{10}$	c_9	$-c_8$	$-c_{15}$	c_{14}	$-c_{13}$	c_{12}
c_4	c_4	$-c_5$	$-c_6$	$-c_7$	$-c_0$	c_1	c_2	c_3	c_{12}	c_{13}	c_{14}	c_{15}	$-c_8$	$-c_9$	$-c_{10}$	$-c_{11}$
c_5	c_5	c_4	$-c_7$	c_6	$-c_1$	$-c_0$	c_3	$-c_2$	c_{13}	$-c_{12}$	c_{15}	$-c_{14}$	c_9	$-c_8$	$-c_{11}$	c_{10}
c_6	c_6	c_7	c_4	$-c_5$	$-c_2$	$-c_3$	$-c_0$	c_1	c_{14}	$-c_{15}$	$-c_{12}$	c_{13}	c_{10}	c_{11}	$-c_8$	$-c_9$
c_7	c_7	$-c_6$	c_5	c_4	$-c_3$	c_2	$-c_1$	$-c_0$	c_{15}	$-c_{14}$	$-c_{13}$	$-c_{12}$	c_{11}	$-c_{10}$	$-c_9$	$-c_8$
c_8	c_8	$-c_9$	$-c_{10}$	$-c_{11}$	$-c_{12}$	$-c_{13}$	$-c_{14}$	$-c_{15}$	c_0	c_1	c_2	c_3	c_4	c_5	c_6	c_7

Table 4.5 Multiplication of Octonion Split Extensions

Table 4.5 Cont-

	c_0	c_1	c_2	c_3	c_4	c_5	c_6	c_7	c_8	c_9	c_{10}	c_{11}	c_{12}	c_{13}	c_{14}	c_{15}
c_9	c_9	c_8	$-c_{11}$	c_{10}	$-c_{13}$	c_{12}	c_{15}	c_{14}	$-c_1$	$-c_0$	c_3	$-c_2$	c_5	$-c_4$	c_7	c_6
c_{10}	c_{10}	c_{11}	$-c_8$	$-c_9$	c_{14}	$-c_{15}$	c_{12}	c_{13}	$-c_2$	$-c_3$	c_0	c_1	c_6	c_7	$-c_4$	$-c_5$
c_{11}	c_{11}	$-c_{10}$	c_9	c_8	$-c_{15}$	c_{14}	$-c_{13}$	$-c_{12}$	c_3	c_2	$-c_1$	$-c_0$	c_7	$-c_6$	c_5	$-c_4$
c_{12}	c_{12}	c_{13}	c_{14}	c_{15}	c_8	$-c_9$	$-c_{10}$	$-c_{11}$	$-c_4$	$-c_5$	$-c_6$	$-c_7$	$-c_0$	c_1	c_2	c_3
c_{13}	c_{13}	$-c_{12}$	c_{15}	$-c_{14}$	c_9	c_8	$-c_{11}$	c_{10}	$-c_5$	c_4	$-c_7$	$-c_6$	$-c_1$	$-c_0$	c_3	$-c_2$
c_{14}	c_{14}	$-c_{15}$	$-c_{12}$	c_{13}	c_{10}	c_{11}	c_8	c_9	$-c_6$	$-c_7$	c_4	$-c_5$	$-c_2$	$-c_3$	$-c_0$	c_1
c_{15}	c_{15}	c_{14}	$-c_{13}$	$-c_{12}$	c_{11}	$-c_{10}$	c_9	c_8	$-c_7$	$-c_6$	c_5	c_4	$-c_3$	c_2	$-c_1$	$-c_0$

The following table gives the Nim addition for the elements from 0 to 15.

\oplus	0	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15
0	0	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15
1	1	0	3	2	5	4	7	6	9	8	11	10	13	12	15	14
2	2	3	0	1	6	7	4	5	10	11	8	9	14	15	12	13
3	3	2	1	0	7	6	5	4	11	10	9	8	15	14	13	12
4	4	5	6	7	0	1	2	3	12	13	14	15	8	9	10	11
5	5	4	7	6	1	0	3	2	13	12	15	14	9	8	11	10
6	6	7	4	5	2	3	0	1	14	15	12	13	10	11	8	9
7	7	6	5	4	3	2	1	0	15	14	13	12	11	10	9	8
8	8	9	10	11	12	13	14	15	0	1	2	3	4	5	6	7
9	9	8	11	10	13	12	15	14	1	0	3	2	5	4	7	6
10	10	11	8	9	14	15	12	13	2	3	0	1	6	7	4	5
11	11	10	9	8	15	14	13	12	3	2	1	0	7	6	5	4
12	12	13	14	15	8	9	10	11	4	5	6	7	0	1	2	3
13	13	12	15	14	9	8	11	10	5	4	7	6	1	0	3	2
14	14	15	12	13	10	11	8	9	6	7	4	5	2	3	0	1
15	15	14	13	12	11	10	9	8	7	6	5	4	3	2	1	0

Table 4.6 Nim addition for the elements 0 to 15

The table is similar to Table 4.5. The subscripts of the elements in Table 4.5 make up Table 4.6.

Observations:

1. From Tables 4.5 and 4.6, the following observations can be made:

There are 4 cases:

Case 1: $k \leq 7, m \leq 7$

$$c_k c_m = (b_k, 1)(b_m, 1) = (\pm b_{k \oplus m}, 1) = \pm c_{k \oplus m}$$

Case 2: $k \leq 7, m \geq 8$

$$\begin{aligned} c_k c_m &= (b_k, 1)(b_{m-8}, -1) \\ &= (b_{m-8} b_k, -1) \\ &= (\pm b_{(m-8) \oplus k}, -1) \\ &= \pm c_{m \oplus k} \end{aligned}$$

Case 3: $k \geq 8, m \leq 7$

$$\begin{aligned} c_k c_m &= (b_{k-8}, -1)(b_m, 1) \\ &= (b_{k-8} \overline{b_m}, -1) \\ &= (-b_{k-8} b_m, -1) \\ &= -(b_{k-8} b_m, -1) \\ &= -(\pm b_{(k-8) \oplus m}, -1) \\ &= \pm c_{m \oplus k} \end{aligned}$$

Case 4: $k \geq 8, m \geq 8$

$$\begin{aligned} c_k c_m &= (b_{k-8}, -1)(b_{m-8}, -1) \\ &= (-b_{k-8} \overline{b_{m-8}}, 1) \\ &= [(-b_{k-8})(-b_{m-8}), 1] \\ &= (b_{k-8} b_{m-8}, 1) \\ &= (\pm b_{(k-8) \oplus (m-8)}, 1) \\ &= \pm c_{m \oplus k} \end{aligned}$$

In general,

$$1. c_k c_m = \pm c_{k \oplus m} \quad \forall k, m = 0, 1, \dots, 15$$

$$2. c_k c_m = -c_m c_k \quad \forall k, m = 1, 2, \dots, 15$$

$$3. c_m^2 = -c_0 \quad \forall m = 1, 2, \dots, 15$$

4.3.4 Sedenion split extensions

Let $d_0 = (c_0, 1)$, $d_1 = (c_1, 1)$, $d_2 = (c_2, 1)$, $d_3 = (c_3, 1)$, $d_4 = (c_4, 1)$, $d_5 = (c_5, 1)$,

$d_6 = (c_6, 1)$, $d_7 = (c_7, 1)$, $d_8 = (c_8, 1)$, $d_9 = (c_9, 1)$, $d_{10} = (c_{10}, 1)$, $d_{11} = (c_{11}, 1)$, $d_{12} = (c_{12}, 1)$

, $d_{13} = (c_{13}, 1)$, $d_{14} = (c_{14}, 1)$, $d_{15} = (c_{15}, 1)$, $d_{16} = (c_0, -1)$, $d_{17} = (c_1, -1)$, $d_{18} = (c_2, -1)$,

$d_{19} = (c_3, -1)$, $d_{20} = (c_4, -1)$, $d_{21} = (c_5, -1)$, $d_{22} = (c_6, -1)$, $d_{23} = (c_7, -1)$,

$d_{24} = (c_8, -1)$, $d_{25} = (c_9, -1)$, $d_{26} = (c_{10}, -1)$, $d_{27} = (c_{11}, -1)$, $d_{28} = (c_{12}, -1)$,

$d_{29} = (c_{13}, -1)$, $d_{30} = (c_{14}, -1)$, $d_{31} = (c_{15}, -1)$ be basis elements of sedenion split

extensions.

Using the results of Section 4.3.3, the multiplication of these elements can be summarized in the following 4 cases:

Case 1: $k \leq 15, m \leq 15$

$$\begin{aligned} d_k d_m &= (c_k, 1)(c_m, 1) \\ &= (\pm c_{m \oplus k}, 1) \\ &= \pm (c_{m \oplus k}, 1) \\ &= \pm d_{m \oplus k} \end{aligned}$$

Case 2: $k \leq 15, m \geq 16$

$$\begin{aligned} d_k d_m &= (c_k, 1)(c_{m-16}, -1) \\ &= (c_{m-16} c_k, -1) \\ &= (\pm c_{(m-16) \oplus k}, -1) \\ &= \pm d_{m \oplus k} \end{aligned}$$

Case 3: $k \geq 16, m \leq 15$

$$\begin{aligned} d_k d_m &= (c_{k-16}, -1)(c_m, 1) \\ &= (c_{k-16} \overline{c_m}, -1) \\ &= (-c_{k-16} c_m, -1) \\ &= -(c_{k-16} c_m, -1) \\ &= -(\pm c_{(k-16) \oplus m}, -1) \\ &= \pm d_{m \oplus k} \end{aligned}$$

Case 4: $k \geq 16, m \geq 16$

$$\begin{aligned} d_k d_m &= (c_{k-16}, -1)(c_{m-16}, -1) \\ &= (-c_{k-16} \overline{c_{m-16}}, 1) \\ &= [(-c_{k-16})(-c_{m-16}), 1] \\ &= (c_{k-16} c_{m-16}, 1) \\ &= (\pm c_{(k-16) \oplus (m-16)}, 1) \\ &= \pm d_{k \oplus m} \end{aligned}$$

Observations:

1. $d_k d_m = \pm d_{k \oplus m} \quad \forall k, m = 0, 1, \dots, 31$
2. $d_k d_m = -d_m d_k \quad \forall k, m = 1, 2, \dots, 31$
3. $d_m^2 = -d_0 \quad \forall m = 1, 2, \dots, 31$

In general, we have the following main Theorem:

Theorem 4.3.5

Consider the split extension basis elements of dimension 2^n , $n = 1, 2, \dots$ given by

$$f_0, f_1, f_2, \dots, f_{2^n-1}, \text{ where } f_i = \begin{cases} (g_i, 1), & i = 0, 1, 2, \dots, 2^{n-1} - 1 \\ (g_{i-2^{n-1}}, -1) & i = 2^{n-1}, 2^{n-1} + 1, \dots, 2^n - 1 \end{cases} \quad (10)$$

Then $f_k f_m = \pm f_{k \oplus m}$ for $k, m = 0, 1, 2, \dots, 2^n - 1$

Proof by induction:

The multiplication is true for $n = 1, 2, 3, 4, 5$ as shown in Sections 4.31 to 4.34 above.

Let the split extension basis elements be given by equation (10) above. By induction hypothesis, suppose it's true for $n-1$ with basis elements $g_0, g_1, g_2, \dots, g_{2^{n-1}-1}$. We prove that it's true for n with basis elements

$$f_0, f_1, f_2, \dots, f_{2^n-1}.$$

If it is true for $n-1$ with basis elements $g_0, g_1, g_2, \dots, g_{2^{n-1}-1}$, then $g_k g_m = \pm g_{k \oplus m}$

$$k, m = 0, 1, \dots, 2^{n-1} - 1.$$

There are 4 cases:

Case 1: $k \leq 2^{n-1} - 1, m \leq 2^{n-1} - 1$

$$\begin{aligned} f_k f_m &= (g_k, 1)(g_m, 1) \\ &= (g_k g_m, 1) \\ &= (\pm g_{m \oplus k}, 1) \\ &= \pm (g_{m \oplus k}, 1) \\ &= \pm f_{k \oplus m} \end{aligned}$$

Case 2: $k \leq 2^{n-1} - 1$, $m \geq 2^{n-1}$

$$\begin{aligned}
 f_k f_m &= (g_k, 1)(g_{m-2^{n-1}}, -1) \\
 &= (g_{m-2^{n-1}} g_k, -1) \\
 &= (\pm g_{(m-2^{n-1}) \oplus k}, -1) \\
 &= \pm f_{m \oplus k}
 \end{aligned}$$

Case 3: $k \geq 2^{n-1}$, $m \leq 2^{n-1} - 1$

$$\begin{aligned}
 f_k f_m &= (g_{k-2^{n-1}}, -1)(g_m, 1) \\
 &= (g_{k-2^{n-1}} \overline{g_m}, -1) \\
 &= (-g_{k-2^{n-1}} g_m, -1) \\
 &= -(g_{k-2^{n-1}} g_m, -1) \\
 &= -(\pm g_{(k-2^{n-1}) \oplus m}, -1) \\
 &= \pm f_{m \oplus k}
 \end{aligned}$$

Case 4: $k \geq 2^{n-1}$, $m \geq 2^{n-1}$

$$\begin{aligned}
 f_k f_m &= (g_{k-2^{n-1}}, -1)(g_{m-2^{n-1}}, -1) \\
 &= (-g_{k-2^{n-1}} \overline{g_{m-2^{n-1}}}, 1) \\
 &= [(-g_{k-2^{n-1}})(-g_{m-2^{n-1}}), 1] \\
 &= (g_{k-2^{n-1}} g_{m-2^{n-1}}, 1) \\
 &= (\pm g_{(k-2^{n-1}) \oplus (m-2^{n-1})}, 1) \\
 &= \pm f_{k \oplus m}
 \end{aligned}$$

Therefore, $f_k f_m = \pm f_{k \oplus m}$ for $k, m = 0, 1, \dots, 2^n - 1$.

CHAPTER FIVE

SEDENION EXTENSION LOOPS

5.1 Introduction

In this chapter we show that subloops of sedenions do not necessarily reduce to either subloops of octonions or sedenion split extensions. We show the existence of two-sided subloops of sedenions that are neither subloops of octonions nor sedenion split extensions, given that each commutative multiplicative subloop of the octonions is associative (**Lemma 1.3.2**) and that a commutative sedenion extension loop is associative (**Corollary 1.3.3**). We do this by taking elements outside the octonions and sedenion split extensions and showing that they form a subloop of sedenions.

5.2 An example of a subloop of sedenions that is not a subloop of Octonions or a Split Extension

A loop is called a *proper* Moufang loop if it is Moufang but not associative. The non-zero octonions form a proper Moufang loop since they satisfy the Moufang identities:

$$xy \cdot zx = (x \cdot yz)x$$

$$x(y \cdot xz) = (xy \cdot x)z$$

$$x(x \cdot zy) = (x \cdot yz)y.$$

Any one of the three identities implies the other two (Albert, 1963). In a Moufang loop, if three elements associate, they associate in any order as guaranteed by the crucial Moufang theorem (Theorem 1.3.4).

Sedenion split extensions, which are two-sided loops, are obtained within the version obtained by Jonathan Smith (1995). Although the split extensions do not directly yield loops in the way that Moufang loops are obtained from octonions, the left loop of non-zero sedenions contains new two-sided subloops defined as sedenion split extensions. These loops are constructed abstractly as extensions from the subloops of the octonions.

5.3 Construction of a Subloop of Sedenions that is not a Subloop of Octonions or a Sedenion Split Extension

In this section we perform the multiplication of arbitrary elements that are from neither subloops of octonions nor sedenion split extensions using the Cayley-Dickson doubling formula (equation 2).

Let $e_0, e_2, e_3, \dots, e_7$ be basis elements of octonions.

Let $S = \mathbb{Z} (e_0, e_2, e_3, \dots, e_7) = \{n_0e_0 + n_1e_1 + n_2e_2 + n_3e_3 + \dots + n_7e_7 : n_i \in \mathbb{Z}\}$ and

$S^2 = \{s_1 + s_2f, s_1, s_2 \in S\}$ S^2 is not contained in octonions since it contains non-octonion elements. It is not also a sedenion split extension, whose elements are encoded as pairs

(a, ε) with $\varepsilon \in S^0 = \{1, -1\}$, by $a \mapsto (a, 1)$, $af \mapsto (a, -1)$.

We show that the multiplication in S^2 gives rise to a subloop of sedenions.

Lemma 5.3.1

Let $e_0, e_1, e_2, e_3, \dots, e_7$ be basis elements of octonions.

Let $S = \mathbb{Z}(e_0, e_1, e_2, e_3, \dots, e_7) = \{n_0e_0 + n_1e_1 + n_2e_2 + n_3e_3 + \dots + n_7e_7 : n_i \in \mathbb{Z}\}$.

Let $p = n_0e_0 + n_1e_1 + n_2e_2 + n_3e_3 + \dots + n_7e_7 \in S$ and

$$q = m_0e_0 + m_1e_1 + m_2e_2 + m_3e_3 + \dots + m_7e_7 \in S$$

Then $pq, \bar{p}q, p\bar{q} \in S$.

Proof:

Using Table 6.3 for the multiplication of $e_0, e_1, e_2, e_3, \dots, e_7$ we have:

$$\begin{aligned} pq &= (n_0e_0 + n_1e_1 + n_2e_2 + n_3e_3 + \dots + n_7e_7)(m_0e_0 + m_1e_1 + m_2e_2 + m_3e_3 + \dots + m_7e_7) \\ &= (n_0m_0 - n_1m_1 - n_2m_2 - n_3m_3 - n_4m_4 - n_5m_5 - n_6m_6 - n_7m_7) \\ &\quad + e_1(n_0m_1 + n_1m_0 + n_2m_3 + n_3m_2 + n_4m_5 - n_5m_4 - n_6m_7 + n_7m_6) \\ &\quad + e_2((n_0m_2 - n_1m_3 + n_2m_0 + n_3m_1 + n_4m_6 + n_5m_7 - n_6m_4 - n_7m_5)) \\ &\quad + e_3(n_0m_3 + n_1m_2 - n_2m_1 + n_3m_0 + n_4m_7 - n_5m_6 + n_6m_5 - n_7m_4) \\ &\quad + e_4(n_0m_4 + n_1m_5 - n_2m_6 - n_3m_7 + n_4m_0 + n_5m_1 + n_6m_2 + n_7m_3) \\ &\quad + e_5(n_0m_5 + n_1m_4 - n_2m_7 + n_3m_6 - n_4m_1 + n_5m_0 - n_6m_3 + n_7m_2) \\ &\quad + e_6(n_0m_6 + n_1m_7 + n_2m_4 - n_3m_5 - n_4m_2 + n_5m_3 + n_6m_0 - n_7m_1) \\ &\quad + e_7(n_0m_7 - n_1m_6 + n_2m_5 + n_3m_4 - n_4m_3 - n_5m_2 + n_6m_1 + n_7m_0) \in S \end{aligned}$$

$$\begin{aligned} \bar{p}q &= (n_0e_0 - n_1e_1 - n_2e_2 - n_3e_3 - n_4e_4 - n_5e_5 - n_6e_6 - n_7e_7)(m_0e_0 + m_1e_1 + m_2e_2 + m_3e_3 + \dots + m_7e_7) \\ &= (n_0m_0 + n_1m_1 + n_2m_2 + n_3m_3 + n_4m_4 + n_5m_5 + n_6m_6 + n_7m_7) \\ &\quad + e_1(n_0m_1 - n_1m_0 - n_2m_3 + n_3m_2 - n_4m_5 + n_5m_4 + n_6m_7 - n_7m_6) \\ &\quad + e_2(n_0m_2 + n_1m_3 - n_2m_0 - n_3m_1 - n_4m_6 - n_5m_7 + n_6m_4 + n_7m_5) \\ &\quad + e_3(n_0m_3 - n_1m_2 + n_2m_1 - n_3m_0 - n_4m_7 + n_5m_6 - n_6m_5 + n_7m_4) \\ &\quad + e_4(n_0m_4 + n_1m_5 + n_2m_6 + n_3m_7 - n_4m_0 - n_5m_1 - n_6m_2 - n_7m_3) \\ &\quad + e_5(n_0m_5 - n_1m_4 + n_2m_7 - n_3m_6 + n_4m_1 - n_5m_0 + n_6m_3 - n_7m_2) \\ &\quad + e_6(-n_0m_6 - n_1m_7 - n_2m_4 + n_3m_5 + n_4m_2 - n_5m_3 + n_6m_0 + n_7m_1) \\ &\quad + e_7(n_0m_7 + n_1m_6 - n_2m_5 - n_3m_4 + n_4m_3 + n_5m_2 - n_6m_1 - n_7m_0) \in S \end{aligned}$$

$$\begin{aligned}
p\bar{q} &= (n_0e_0 + n_1e_1 + n_2e_2 + n_3e_3 + \dots + n_7e_7) (m_0e_0 - m_1e_1 - m_2e_2 - m_3e_3 - m_4e_4 - m_5e_5 - m_6e_6 - m_7e_7) \\
&= (n_0m_0 - n_1m_1 - n_2m_2 - n_3m_3 - n_4m_4 - n_5m_5 - n_6m_6 - n_7e_7) \\
&+ e_1(-n_0m_1 + n_1m_0 - n_2m_3 + n_3m_2 - n_4m_5 + n_5m_4 + n_6m_7 + n_7m_6) \\
&+ e_2(-n_0m_2 + n_1m_3 + n_2m_0 - n_3m_1 - n_4m_6 - n_5m_7 + n_6m_4 + n_7m_5) \\
&+ e_3(-n_0m_3 - n_1m_2 + n_2m_1 + n_3m_0 - n_4m_7 + n_5m_6 - n_6m_5 + n_7m_4) \\
&+ e_4(-n_0m_4 + n_1m_5 + n_2m_6 + n_3m_7 + n_4m_0 - n_5m_1 - n_6m_2 - n_7m_3) \\
&+ e_5(-n_0m_5 - n_1m_4 + n_2m_7 - n_3m_6 + n_4m_1 + n_5m_0 + n_6m_3 - n_7m_2) \\
&+ e_6(-n_0m_6 - n_1m_7 - n_2m_4 + n_3m_5 + n_4m_2 - n_5m_3 + n_6m_0 + n_7m_1) \\
&+ e_7(-n_0m_7 + n_1m_6 - n_2m_5 - n_3m_4 + n_4m_3 + n_5m_2 - n_6m_1 - n_7m_0) \in S
\end{aligned}$$

Therefore $pq, \bar{p}q, p\bar{q} \in S$.

Lemma 5.3.2:

Let $e_0, e_2, e_3, \dots, e_7$ be basis elements of octonions.

Let $S = \mathbb{Z} (e_0, e_2, e_3, \dots, e_7) = \{n_0e_0 + n_1e_1 + n_2e_2 + n_3e_3 + \dots + n_7e_7 : n_i \in \mathbb{Z}\}$

$S^2 = \{s_1 + s_2f, s_1, s_2 \in S\}$ is a subloop of sedenions not contained in octonions

and not a sedenion split extension.

Proof:

1. Closure

Let $(a, b), (c, d) \in S^2$. Then $(ab)(c, d) = (ac - \bar{d}b, da + b\bar{c})$. From Lemma 5.3.1,

$ac, \bar{d}b, da$ and $b\bar{c}$ are in S . So S^2 is closed.

2. Identity

There exists a two-sided identity $1 + 0f = (1, 0) \in S^2$ since

$$(a, b)(1, 0) = (a, b) \text{ and } (1, 0)(c, d) = (c, d)$$

From the results of Table 6.4, in the equation $x \cdot y = z$, knowledge of any two of the elements uniquely defines the third one and thus S^2 is a quasi-group.

Therefore $S^2 = \{s_1 + s_2 f, s_1, s_2 \in S\}$ is a subloop of sedenions not contained in octonions and not a sedenion split extension.

Remark 5.3.3:

Even though Lemma 5.3.2 establishes the existence of a subloop of sedenions that is neither a split extension nor a subloop of octonions, the subloop is not finite unless it is either a split extension or a subloop of octonions.

Even when one starts with only two elements, the elements generated build up very fast as illustrated by the following example.

Example 5.3.4:

Let $\alpha_1 = i + 1f$, $\alpha_2 = i + 0f$

Using Cayley Dickson's doubling formula,

$$\alpha_1 \alpha_1 = (i + 1f)(i + 1f) = (i, 1)(i, 1) = (-2, 0) = \alpha_3, \text{ say}$$

$$\alpha_1 \alpha_2 = (i + 1f)(i + 0f) = (i, 1)(i, 0) = (-1, -i) = \alpha_4,$$

$$\alpha_1 \alpha_3 = (i + 1f)(-2 + 0f) = (i, 1)(-2, 0) = (-2i, -2) = \alpha_5,$$

$$\alpha_1 \alpha_4 = (i + 1f)(-1 - if) = (i, 1)(-1, -i) = (-2i, 0) = \alpha_6,$$

$$\alpha_1 \alpha_5 = (i + 1f)(-2i - 2f) = (i, 1)(-2i, -2) = (4, 0) = \alpha_7,$$

$$\alpha_1 \alpha_6 = (i + 1f)(-2i + 0f) = (i, 1)(-2i, 0) = (2, -2i) = \alpha_8,$$

$$\alpha_2 \alpha_1 = (i + 0f) (i + 1f) = (i, 0)(i, 1) = (-1, i) = \alpha_9,$$

$$\alpha_2 \alpha_2 = (i + 0f) (i + 0f) = (i, 0)(i, 0) = (-1, 0) = \alpha_{10},$$

$$\alpha_2 \alpha_3 = (i + 0f) (-2 + 0f) = (i, 0)(-2, 0) = (-2i, 0) = \alpha_6,$$

$$\alpha_2 \alpha_4 = (i + 0f) (-1 - if) = (i, 0)(-1, -i) = (-i, -1) = \alpha_{11},$$

$$\alpha_2 \alpha_5 = (i + 0f) (-2i - 2f) = (i, 0)(-2i, -2) = (2, -2i) = \alpha_8,$$

$$\alpha_2 \alpha_6 = (i + 0f) (-2i + 0f) = (i, 0)(-2i, 0) = (-2, 0) = \alpha_3$$

$$\alpha_3 \alpha_1 = (-2 + 0f) (i + 1f) = (-2, 0)(i, 1) = (-2i, -2) = \alpha_5$$

$$\alpha_3 \alpha_2 = (-2 + 0f) (i + 0f) = (-2, 0)(i, 0) = (-2i, 0) = \alpha_6$$

$$\alpha_3 \alpha_3 = (-2 + 0f) (-2 + 0f) = (-2, 0)(-2, 0) = (4, 0) = \alpha_7$$

$$\alpha_3 \alpha_4 = (-2 + 0f) (-1 - if) = (-2, 0)(-1, -i) = (2, 2i) = \overline{\alpha_8}$$

$$\alpha_3 \alpha_5 = (-2 + 0f) (-2i - 2f) = (-2, 0)(-2i, -2) = (4i, 4) = \alpha_{12}$$

$$\alpha_3 \alpha_6 = (-2 + 0f) (-2i + 0f) = (-2, 0)(-2i, 0) = (4i, 0) = \alpha_{13}$$

$$\alpha_4 \alpha_1 = (-1 - if) (i + 1f) = (-1, -i) (i, 1) = (0, -2) = \alpha_{14}$$

$$\alpha_4 \alpha_2 = (-1 - if) (i + 0f) = (-1, -i) (i, 0) = (-i, -1) = \alpha_{11}$$

$$\alpha_4 \alpha_3 = (-1 - if) (-2 + 0f) = (-1, -i) (-2, 0) = (2, 2i) = \overline{\alpha_8}$$

$$\alpha_4 \alpha_4 = (-1 - if) (-1 - if) = (-1, -i) (-1, -i) = (0, 2i) = \alpha_{15}$$

$$\alpha_4 \alpha_5 = (-1 - if) (-2i - 2f) = (-1, -i) (-2i, -2) = (0, 4) = \alpha_{16}$$

$$\alpha_4 \alpha_6 = (-1 - if) (2i + 0f) = (-1, -i) (2i, 0) = (2i, 2) = -\alpha_5$$

$$\alpha_5 \alpha_1 = (-2i - 2f)(i + 1f) = (-2i, -2)(i, 1) = (4, 0) = \alpha_7$$

$$\alpha_5 \alpha_2 = (-2i - 2f)(i + 0f) = (-2i, -2)(i, 0) = (2, 2i) = \overline{\alpha_8}$$

$$\alpha_5 \alpha_3 = (-2i - 2f)(-1 + 0f) = (-2i, -2)(-2, 0) = (4i, 4) = \alpha_{12}$$

$$\alpha_5 \alpha_4 = (-2i - 2f)(-1 - if) = (-2i, -2)(-1, -i) = (4i, 0) = \alpha_{13}$$

$$\alpha_5 \alpha_5 = (-2i - 2f)(-2i - 2f) = (-2i, -2)(-2i, -2) = (-8, 0) = \alpha_{17}$$

$$\alpha_5 \alpha_6 = (-2i - 2f)(-2i + 0f) = (-2i, -2)(-2i, 0) = (-4, -4i) = \alpha_{18}$$

$$\alpha_6 \alpha_1 = (-2i + 0f)(1 + 1f) = (-2i, 0)(i, 1) = (2, -2i) = \alpha_8$$

$$\alpha_6 \alpha_2 = (-2i + 0f)(i + 0f) = (-2i, 0)(i, 0) = (2, 0) = -\alpha_3$$

$$\alpha_6 \alpha_3 = (-2i + 0f)(-2 + 0f) = (-2i, 0)(-2, 0) = (4i, 0) = \alpha_{13}$$

$$\alpha_6 \alpha_4 = (-2i + 0f)(-1 - if) = (-2i, 0)(-1, -i) = (2i, -2) = \alpha_{19}$$

$$\alpha_6 \alpha_5 = (-2i + 0f)(-2i - 2f) = (-2i, 0)(-2i, -2) = (-4, 4i) = \overline{\alpha_{18}}$$

$$\alpha_6 \alpha_6 = (-2i + 0f)(-2i + 0f) = (-2i, 0)(-2i, 0) = (-4, 0) = -\alpha_7$$

We therefore conclude that there exists subloops of sedenions that are neither subloops of octonions or sedenion split extensions.

CHAPTER SIX

CONSTRUCTION OF HADAMARD MATRICES USING JONATHAN SMITH FORMULA

6.1 Introduction

The frame multiplication of 2^n -ons has been done using the Cayley-Dickson and Smith-Conway doubling formulae (Kivunge 2004). These formulae gave a way of constructing Hadamard matrices. In this chapter the multiplication is done using the Jonathan Smith formula (equation 8). We show that the formula also gives a way of constructing Hadamard matrices.

6.2 Hadamard Matrices

A Hadamard matrix is a square matrix whose entries are either +1 or -1 and whose rows are mutually orthogonal. This means that every two different rows in a Hadamard matrix represent two perpendicular vectors. Such matrices are used in error correcting codes such as the Reed-Muller code.

Hadamard matrices also appear in balanced repeated replication (BRR), used by statisticians to estimate the variance of a parameter. They are named after the French Mathematician Jacques Hadamard.

Examples of Hadamard matrices were actually first constructed by James Joseph Sylvester. He constructed matrices of order 2^k for every non-negative integer k . Mathematicians have devoted themselves to finding new methods of constructing

these matrices. In this research the method used is the Jonathan Smith formula (equation 8).

Definition 6.2.1: A *Hadamard matrix* of order n is an $n \times n$ matrix H with entries ± 1 such that $HH^T = nI_n$, where H^T is the transpose of H and I_n is the identity matrix of order n .

The following are some of the fundamental properties of Hadamard matrices.

6.2.2 Properties of Hadamard matrices Kivunge (2004).

1. Any two columns of H are orthogonal of weight n .
2. If some rows or columns of H are permuted, the resulting matrix is Hadamard.
3. If some rows or columns are multiplied by -1 , the resulting matrix is still Hadamard.
4. If H is Hadamard, then H^T is also Hadamard.

6.3 Construction of Hadamard matrices

In this section the structures resulting from the multiplication of general 2^n -ons using the Jonathan Smith formula are investigated. We show that the sign matrix of the frame multiplication is Hadamard.

When restricted to the frame of the 2^n -ons, equation (8) reduces to:

$$(a,0) (c,0) = (ac, 0)$$

$$(a,0) (0,d) = (0, da)$$

$$(0, b) (c, 0) = (0, b\bar{c})$$

$$(0, b) (0, d) = (-bd, 0), \text{ for } a, b, c, d \in Q \text{ where } Q \text{ is the } 2^n\text{-on in consideration.}$$

The conjugation is given by $\overline{(a, b)} = (\bar{a}, -b)$

Definition 6.3.1: Let $A = \{\pm a_1, \pm a_2, \dots, \pm a_n\}$ be a set of basis elements for the 2^n -ons with $a_i \cdot a_j \in \{\pm a_k\}$. The *sign matrix* associated with A is the $n \times n$ matrix S

$$\text{with } S_{ij} = \begin{cases} 1, & \text{if } a_i \cdot a_j = a_k \\ -1, & \text{if } a_i \cdot a_j = -a_k \end{cases}$$

6.3.2 Frame Multiplication of Complex numbers

Let $e_0 = 1, e_1 = i$ be the basis elements of \mathbb{C} . The frame for the complex numbers

is $\pm\{e_0 = 1, e_1 = i\}$. The multiplication table is given by:

\cdot	e_0	e_1
e_0	e_0	e_1
e_1	e_1	$-e_0$

Table 6.1 Multiplication of Complex basis elements

The sign matrix for this multiplication is the Hadamard matrix $S_C = \begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix}$ and

$$S_C S_C^T = 2I$$

6.3.3 Frame Multiplication of Quaternions

The frame for the quaternions is $\pm\{e_0 = 1, e_1 = i, e_2 = j, e_3 = k\}$. The multiplication table is given by:

\cdot	e_0	e_1	e_2	e_3
e_0	e_0	e_1	e_2	e_3
e_1	e_1	$-e_0$	e_3	$-e_2$
e_2	e_2	$-e_3$	$-e_0$	e_1
e_3	e_3	e_2	$-e_1$	$-e_0$

Table 6.2 Multiplication of Quaternion basis elements

The sign matrix for this multiplication is the Hadamard matrix

$$S_H = \begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & -1 & 1 & -1 \\ 1 & -1 & -1 & 1 \\ 1 & 1 & -1 & -1 \end{bmatrix} \text{ and } S_H S_H^T = 4I_4$$

6.3.4 Frame Multiplication of Octonions

The frame for the octonions is

$$\pm\{e_0 = (1, 0), e_1 = (i, 0), e_2 = (j, 0), e_3 = (k, 0), e_4 = (0, 1), e_5 = (0, i), e_6 = (0, j), e_7 = (0, k)\}$$

Multiplication is as shown in the following table

\cdot	e_0	e_1	e_2	e_3	e_4	e_5	e_6	e_7
e_0	e_0	e_1	e_2	e_3	e_4	e_5	e_6	e_7
e_1	e_1	$-e_0$	e_3	$-e_2$	e_5	$-e_4$	$-e_7$	e_6
e_2	e_2	$-e_3$	$-e_0$	e_1	e_6	e_7	$-e_4$	$-e_5$
e_3	e_3	e_2	$-e_1$	$-e_0$	e_7	$-e_6$	e_5	$-e_4$
e_4	e_4	$-e_5$	$-e_6$	$-e_7$	$-e_0$	e_1	e_2	e_3
e_5	e_5	e_4	$-e_7$	e_6	$-e_1$	$-e_0$	e_3	e_2
e_6	e_6	e_7	e_4	$-e_5$	$-e_2$	e_3	$-e_0$	$-e_1$
e_7	e_7	$-e_6$	e_5	e_4	$-e_3$	$-e_2$	e_1	$-e_0$

Table 6.3 Multiplication of Octonion basis elements

The sign matrix for this multiplication is given by

$$S_K = \begin{bmatrix} 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 \\ 1 & -1 & 1 & -1 & 1 & -1 & -1 & 1 \\ 1 & -1 & -1 & 1 & 1 & 1 & -1 & -1 \\ 1 & 1 & -1 & -1 & 1 & -1 & 1 & -1 \\ 1 & -1 & -1 & -1 & -1 & 1 & 1 & 1 \\ 1 & 1 & -1 & 1 & -1 & -1 & -1 & 1 \\ 1 & 1 & 1 & -1 & -1 & 1 & -1 & -1 \\ 1 & -1 & 1 & 1 & -1 & -1 & 1 & -1 \end{bmatrix} \quad \text{and} \quad S_K S_K^T = 8I_8 = 2^3 I_8$$

6.3.5 Frame Multiplication of Sedenions

The frame for the sedenions is

$$\pm\{e_0=(e_0,0), e_1=(e_1,0), e_2=(e_2,0), e_3=(e_3,0), e_4=(e_4,0), e_5=(e_5,0), e_6=(e_6,0), e_7=(e_7,0), e_8=(0,e_0), e_9=(0,e_1), e_{10}=(0,e_2), e_{11}=(0,e_3), e_{12}=(0,e_4), e_{13}=(0,e_5), e_{14}=(0,e_6), e_{15}=(0,e_7)\}$$

Multiplication is as shown in the following table

\cdot	e_0	e_1	e_2	e_3	e_4	e_5	e_6	e_7	e_8	e_9	e_{10}	e_{11}	e_{12}	e_{13}	e_{14}	e_{15}
e_0	e_0	e_1	e_2	e_3	e_4	e_5	e_6	e_7	e_8	e_9	e_{10}	e_{11}	e_{12}	e_{13}	e_{14}	e_{15}
e_1	e_1	$-e_0$	e_3	$-e_2$	e_5	$-e_4$	$-e_7$	e_6	e_9	$-e_8$	$-e_{11}$	e_{10}	$-e_{13}$	e_{12}	e_{15}	e_{14}
e_2	e_2	$-e_3$	$-e_0$	e_1	e_6	e_7	$-e_4$	$-e_5$	e_{10}	e_{11}	$-e_8$	$-e_9$	$-e_{14}$	$-e_{15}$	e_{12}	e_{13}
e_3	e_3	e_2	$-e_1$	$-e_0$	e_7	$-e_6$	e_5	$-e_4$	e_{11}	$-e_{10}$	e_9	$-e_8$	$-e_{15}$	e_{14}	$-e_{13}$	e_{12}
e_4	e_4	$-e_5$	$-e_6$	$-e_7$	$-e_0$	e_1	e_2	e_3	e_{12}	e_{13}	e_{14}	e_{15}	$-e_8$	$-e_9$	$-e_{10}$	$-e_1$
e_5	e_5	e_4	$-e_7$	e_6	$-e_1$	$-e_0$	$-e_3$	e_2	e_{13}	$-e_{12}$	e_{15}	$-e_{14}$	e_9	$-e_8$	e_{11}	$-e_{10}$
e_6	e_6	e_7	e_4	$-e_5$	$-e_2$	e_3	$-e_0$	$-e_1$	e_{14}	$-e_{15}$	$-e_{12}$	e_{13}	e_{10}	$-e_{11}$	$-e_8$	e_9
e_7	e_7	$-e_6$	e_5	e_4	$-e_3$	$-e_2$	e_1	$-e_0$	e_{15}	e_{14}	$-e_{13}$	$-e_{12}$	e_{11}	e_{10}	$-e_9$	$-e_8$
e_8	e_8	$-e_9$	$-e_{10}$	$-e_{11}$	$-e_{12}$	$-e_{13}$	$-e_{14}$	$-e_{15}$	$-e_0$	e_1	e_2	e_3	e_4	e_5	e_6	e_7
e_9	e_9	e_8	$-e_{11}$	e_{10}	$-e_{13}$	e_{12}	e_{15}	$-e_{14}$	$-e_1$	$-e_0$	$-e_3$	e_2	$-e_5$	e_4	e_7	$-e_6$
e_{10}	e_{10}	e_{11}	e_8	$-e_9$	$-e_{14}$	$-e_{15}$	e_{12}	e_{13}	$-e_2$	e_3	$-e_0$	$-e_1$	$-e_6$	$-e_7$	e_4	e_5
e_{11}	e_{11}	$-e_{10}$	e_9	e_8	$-e_{15}$	e_{14}	$-e_{13}$	e_{12}	$-e_3$	$-e_2$	e_1	$-e_0$	$-e_7$	e_6	$-e_5$	e_4
e_{12}	e_{12}	e_{13}	e_{14}	e_{15}	e_8	$-e_9$	$-e_{10}$	$-e_{11}$	$-e_4$	e_5	e_6	e_7	$-e_0$	$-e_1$	$-e_2$	$-e_3$
e_{13}	e_{13}	$-e_{12}$	e_{15}	$-e_{14}$	e_9	e_8	e_{11}	e_{10}	$-e_5$	e_4	e_7	$-e_6$	e_1	$-e_0$	e_3	$-e_2$
e_{14}	e_{14}	$-e_{15}$	$-e_{12}$	e_{13}	e_{10}	$-e_{11}$	e_8	e_9	$-e_6$	$-e_7$	$-e_4$	e_5	e_2	e_3	$-e_0$	e_1
e_{15}	e_{15}	e_{14}	$-e_{13}$	$-e_{12}$	e_{11}	e_{10}	$-e_9$	e_8	$-e_7$	e_6	$-e_5$	e_4	e_3	e_2	$-e_1$	$-e_0$

Table 6.4 Multiplication of Sedenion basis elements

The sign matrix for this multiplication is given by

$$S_s = \begin{bmatrix} 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 \\ 1 & -1 & 1 & -1 & 1 & -1 & -1 & 1 & 1 & -1 & -1 & 1 & -1 & 1 & 1 & -1 \\ 1 & -1 & -1 & 1 & 1 & 1 & -1 & -1 & 1 & 1 & -1 & -1 & -1 & -1 & 1 & 1 \\ 1 & 1 & -1 & -1 & 1 & -1 & 1 & -1 & 1 & -1 & 1 & -1 & -1 & 1 & -1 & 1 \\ 1 & -1 & -1 & -1 & -1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & -1 & -1 & -1 & -1 \\ 1 & 1 & -1 & 1 & -1 & -1 & -1 & 1 & 1 & -1 & 1 & -1 & 1 & -1 & 1 & -1 \\ 1 & 1 & 1 & -1 & -1 & 1 & -1 & -1 & 1 & -1 & -1 & 1 & 1 & -1 & -1 & 1 \\ 1 & -1 & 1 & 1 & -1 & -1 & 1 & -1 & 1 & 1 & -1 & -1 & 1 & 1 & -1 & -1 \\ 1 & -1 & -1 & -1 & -1 & -1 & -1 & -1 & -1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 \\ 1 & 1 & -1 & 1 & -1 & 1 & 1 & -1 & -1 & -1 & -1 & 1 & -1 & 1 & 1 & -1 \\ 1 & 1 & 1 & -1 & -1 & -1 & 1 & 1 & 1 & 1 & -1 & -1 & -1 & -1 & 1 & 1 \\ 1 & -1 & 1 & 1 & -1 & 1 & -1 & 1 & -1 & -1 & 1 & -1 & -1 & 1 & -1 & 1 \\ 1 & 1 & 1 & 1 & 1 & -1 & -1 & -1 & -1 & 1 & 1 & 1 & -1 & -1 & -1 & -1 \\ 1 & -1 & 1 & -1 & 1 & 1 & 1 & -1 & -1 & -1 & 1 & -1 & 1 & -1 & 1 & -1 \\ 1 & -1 & -1 & 1 & 1 & -1 & 1 & 1 & -1 & -1 & -1 & 1 & 1 & -1 & -1 & 1 \\ 1 & 1 & -1 & -1 & 1 & 1 & -1 & 1 & -1 & 1 & -1 & -1 & 1 & 1 & -1 & -1 \end{bmatrix}$$

$$\text{and } S_s S_s^T = 16I_{16} = 2^4 I_{16}$$

We therefore conclude that the frame multiplication of complex numbers, quaternions, octonions and sedenions using the Jonathan Smith formula (equation 8) gives a way of constructing Hadamard matrices, as did the Cayley-Dickson and Smith-Conway formulae (Kivunge 2004).

It is motivating to ask if the sign matrix of the frame multiplication for the general 2^n -ons is a Hadamard matrix. The following theorem shows that this is indeed the case.

Theorem 6.3.6: The sign matrix for the frame of the 2^n – ons under the Jonathan Smith multiplication is a $2^n \times 2^n$ Hadamard matrix.

Proof: From the preceding discussion, the sign matrix is Hadamard for $n \leq 4$.

This sets up the basis for an induction proof.

Let M, N be the sign matrices for the basis elements in the 2^n – ons and 2^{n+1} – ons respectively under Jonathan Smith multiplication. Let

$\{e_0 = 1, e_1, \dots, e_{2^n-1}\}$ be the basis elements of the 2^n – ons, then

$\{(e_0, 0), (e_1, 0), \dots, (e_{2^n-1}, 0), (0, e_0), (0, e_1), \dots, (0, e_{2^n-1})\}$ are the basis elements of the 2^{n+1} – ons.

We assume M is a $2^n \times 2^n$ Hadamard matrix and show that N is a $2^{n+1} \times 2^{n+1}$ Hadamard matrix. The matrix N is of the form

$$N = \left[\begin{array}{c|c} P & Q \\ \hline R & S \end{array} \right],$$

where P, Q, R and S are $2^n \times 2^n$ matrices specified as follows:

For $i, j > 0$

$$(e_{i-1}, 0)(e_{j-1}, 0) = (e_{i-1}e_{j-1}, 0) \Rightarrow P_{ij} = M_{ij}, \quad P = M$$

$$(e_{i-1}, 0)(0, e_{j-1}) = (0, e_{j-1}e_{i-1}) \Rightarrow Q_{ij} = M_{ji}, \quad Q = M^T$$

$$(0, e_{i-1})(e_{j-1}, 0) = (0, e_{i-1}\overline{e_{j-1}}) \Rightarrow R_{i1} = M_{i1} = 1, \quad R_{ij} = -M_{ij} \quad j \neq 1$$

$$(0, e_{i-1})(0, e_{j-1}) = (-e_{i-1}\overline{e_{j-1}}, 0) \Rightarrow S_{i1} = -M_{i1} = -1, \quad S_{ij} = -M_{ij} \quad i \neq 1$$

Thus,

$$T = \left[\begin{array}{ccc|ccc} 1 & 1 \cdots & 1 & 1 & 1 \cdots & 1 \\ \vdots & M_{ij} & & \vdots & M_{ji} & \\ \mathbf{1} & & & \mathbf{1} & & \\ \hline 1 & -1 \cdots & -1 & -1 & 1 \cdots & 1 \\ \vdots & -M_{ij} & & \vdots & M_{ji} & \\ 1 & & & -1 & & \end{array} \right]$$

We perform a column permutation $C_1 \leftrightarrow C_{2^{n+1}}$, i.e. $C_{(e_0,0)} \leftrightarrow C_{(0,e_0)}$, to get the equivalent matrix

$$H = \left[\begin{array}{ccc|ccc} 1 & 1 \cdots & 1 & 1 & 1 \cdots & 1 \\ \vdots & M_{ij} & & \vdots & M_{ji} & \\ \mathbf{1} & & & \mathbf{1} & & \\ \hline -1 & -1 \cdots & -1 & 1 & 1 \cdots & 1 \\ \vdots & -M_{ij} & & \vdots & M_{ji} & \\ -1 & & & 1 & & \end{array} \right] = \left[\begin{array}{c|c} M & M^T \\ \hline -M & M^T \end{array} \right]$$

Now,

$$\begin{aligned} HH^T &= \left[\begin{array}{c|c} M & M^T \\ \hline -M & M^T \end{array} \right] \left[\begin{array}{c|c} M^T & -M^T \\ \hline M & M \end{array} \right] = \left[\begin{array}{c|c} MM^T + M^T M & -MM^T + M^T M \\ \hline -MM^T + M^T M & MM^T + M^T M \end{array} \right] \\ &= \left[\begin{array}{cc} 2^{n+1} I & 0 \\ 0 & 2^{n+1} I \end{array} \right] = 2^{n+1} I_{2^{n+1}} \end{aligned}$$

H is Hadamard equivalent to N , and the proof is complete by Properties 6.2.2.

CHAPTER SEVEN

CONCLUSION AND RECOMMENDATIONS

7.1 Conclusion

In this research we studied sedenion extension loops from the point of view of Loop Theory. We performed the multiplication of 2^n -ons and their split extensions using the Jonathan Smith formula.

We established that if L is not abelian, $L \times S^0$ fails to satisfy group properties but satisfies loop properties like the Jordan Identity and flexible properties. It however fails to satisfy the anti-auto morphism, left and right alternative properties.

When multiplication of basis elements of 2^n -ons was carried out using the Jonathan Smith formula, we found that Nim addition gave a way of determining the subscripts for the products of the basis elements for the split extension. This was done for $n = 1, 2, 3$ and 4 and then generalized for general 2^n -ons.

We also found that there exists subloops of sedenions that are not sedenion extension loops or subloops of octonions.

Finally, we established that the multiplication of 2^n -ons using the Jonathan Smith formula gave a way of constructing Hadamard matrices.

7.2 Recommendations for further research

1. In this research we established the existence of subloops of sedenions that are neither sedenion extensions nor subloops of octonions. The problem of finding out if other subloops of a different 'structure' exist remains open.
2. The relationship between the multiplication of general 2^n -ons using the Jonathan Smith formula and Nim addition was established. Would the use of a different doubling formulae lead to the same relationship with Nim-addition?
3. Nim addition gave a way of determining the subscripts for the products of the basis elements for the split extension. The problem of determining the sign of the subscripts (positive or negative) remains open.
4. The frame multiplication of sedenion split extensions using the Jonathan Smith gave a way of constructing Hadamard Matrices, as did the Smith-Conway and Cayley-Dickson formulas previously. It would be of interest to find out if all other doubling formulas would do the same.

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