

**CYCLE INDICES, SUBDEGREES AND
SUBORBITAL GRAPHS OF $\text{PSL}(2,q)$
ACTING ON THE COSETS OF SOME OF
ITS SUBGROUPS**

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DECLARATION

This thesis is my original work and has not been presented for a degree in any other university or any other award.

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DEDICATION

*To my wife, Annet; daughter, Precy; son, Pavel and mum, Anjeline:
the driving force in my academic endeavor.*

ACKNOWLEDGMENT

With affectionate regards

to

Dr. Kamuti N. I.
who instilled in me an everlasting zeal
for the culture that is
MATHEMATICS

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ABSTRACT

The action of Projective Special Linear group $PSL(2, q)$ on the cosets of its subgroups is studied. Primitive permutation representations of $PSL(2, q)$ have been previously studied by Tchuda (1986), Bon and Cohen (1989) and Kamuti (1992). In particular, the permutation representations on the cosets of $C_{\frac{q-1}{k}}$, $C_{\frac{q+1}{k}}$, P_q , A_4 and $D_{\frac{2(q-1)}{k}}$ are studied. In the case where it was previously done, we employ a different method or otherwise quote the results for completeness purpose. Thus, this thesis deals with determination of disjoint cycle structures, cycle indices, ranks and subdegrees when $PSL(2, q)$ acts on the cosets of its five subgroups mentioned above. Cycle indices are obtained using a method coined by Kamuti (1992), while ranks and subdegrees are determined using two methods, either algebraic arguments, use of table of marks, a method proposed by Ivanov et al. (1983) or both. Subdegrees of $PSL(2, q)$ acting on the cosets of its cyclic subgroup $C_{\frac{q-1}{k}}$ are shown to be $1^{(2)}, (q-1)^{(q+2)}$ when q is even and $1^{(2)}, (\frac{q-1}{2})^{(2q+4)}$ when q is odd using reduced pair group action. In this representation, the number of self-paired suborbits is determined to be $q+2$ when $p=2$, $q+3$ when $q \equiv 1 \pmod{4}$ and $q+1$ when $q \equiv -1 \pmod{4}$. The suborbit $\Delta_{(1, \infty)}$ is shown to be paired with $\Delta_{(0, 1)}$. A method for the construction of suborbital graphs corresponding to the action $PSL(2, q)$ on the cosets of $C_{\frac{q-1}{k}}$ is given. The constructed suborbital graphs that are directed are shown to be of girth three.

CHAPTER 1

INTRODUCTION

In this chapter, the subgroup structure of $PSL(2, q)$ has been briefly examined, details of which may be found in Dickson (1901, Chap. 12) or Huppert (1967, Sec. 8). It was first obtained in a paper by Wiman (1899).

The chapter is divided into seven sections. In Section 1.1, definitions of the general terms to be used throughout the study are given. In Section 1.2, we look at the subgroup structure of $PSL(2, q)$, more emphasis being put on the conjugacy classes of the subgroups $C_{\frac{q-1}{k}}$, $C_{\frac{q+1}{k}}$, P_q , A_4 and $D_{\frac{2(q-1)}{k}}$. Section 1.3 gives the problem statement and justification. The objectives of the study, both general and specific are given in Section 1.4. In Section 1.5, we present the significance of the study to group theorists. In Section 1.6, we give general preliminary results to be used in the thesis. The last section of the chapter presents the general layout of the entire thesis.

1.1 Definition of terms

Definition 1.1.1. *When a group is represented by means of permutations performed on a finite set of n distinct symbols, the integer n is called the degree of the group. The group is said to be a permutation group of degree n .*

Definition 1.1.2. For any positive integer n , define $\mu(n)$ as the sum of the primitive n^{th} roots of unity. It has values in $\{-1, 0, 1\}$ depending on the factorization of n into prime factors:

- $\mu(n) = 1$ if n is a square-free positive integer with an even number of prime factors.
- $\mu(n) = -1$ if n is a square-free positive integer with an odd number of prime factors.
- $\mu(n) = 0$ if n has a squared prime factor.

Definition 1.1.3. If a finite group G acts on a set X with n elements, each $g \in G$ corresponds to a permutation σ of X , which can be written uniquely as a product of disjoint cycles. If σ has α_1 cycles of length 1, α_2 cycles of length 2, \dots , α_n cycles of length n , we say that σ and hence g has cycle type $(\alpha_1, \alpha_2, \dots, \alpha_n)$.

Definition 1.1.4. If a finite group G acts on X , $|X| = n$ and $g \in G$ has cycle type $(\alpha_1, \alpha_2, \dots, \alpha_n)$, we define the monomial of g to be

$$\text{mon}(g) = t_1^{\alpha_1} t_2^{\alpha_2} \dots t_n^{\alpha_n}$$

Definition 1.1.5. The cycle index of the action of G on X is the polynomial (say over the rational field \mathbb{Q}) in t_1, t_2, \dots, t_n indeterminates given by

$$\begin{aligned} Z(G) &= Z_{G,X}(t_1, t_2, \dots, t_n) \\ &= \frac{1}{|G|} \sum_{g \in G} \text{mon}(g), \end{aligned}$$

where $\text{mon}(g)$ is the monomial of g . If G has conjugacy classes K_1, K_2, \dots, K_m with $g_i \in K_i$ for all i , then

$$Z(G) = \frac{1}{|G|} \sum_{i=1}^m |K_i| \text{mon}(g_i).$$

Definition 1.1.6. The cycle index of the regular representation of the cyclic group C_n is given by

$$Z(C_n) = \frac{1}{n} \sum_{d|n} \phi\left(\frac{n}{d}\right) t_{\frac{n}{d}}^d,$$

where ϕ is the Euler ϕ -function. (Krishnamurthy, 1985, p. 116)

Definition 1.1.7. A permutation group G acting on a set X is said to be transitive if for all $x, y \in X$ there exists an element $g \in G$ such that

$$gx = y.$$

Definition 1.1.8. The orbit of G containing a point $x \in X$ is the set

$$\text{Orb}_G(x) = \{gx : g \in G\}.$$

Remark 1.1.1. If G acts transitively on X , then the number of orbits of G is one.

Definition 1.1.9. The stabilizer G_x (Isotropy group) of a point x consists of all those elements $g \in G$ for which $gx = x$, i.e.

$$G_x = \{g \in G : gx = x\}.$$

G_x is a subgroup of G . It is also denoted by $\text{Stab}_G(x)$.

Definition 1.1.10. Let G be transitive on a set X and let G_x be the stabilizer in G of a point $x \in X$. The orbits $\Delta_0 = \{x\}, \Delta_1, \dots, \Delta_{r-1}$ of G_x on X are known as the suborbits of G . The rank of G in this case is r .

Definition 1.1.11. The sizes $n_i = |\Delta_i|$, ($i = 0, 1, 2, \dots, r - 1$) often called the 'lengths' of the suborbits are known as subdegrees of G .

Definition 1.1.12. Let Δ be an orbit of G_x on X . Define

$$\Delta^* = \{gx : g \in G, x \in \Delta\},$$

then Δ^* is also an orbit of G_x and is called the G_x -orbit paired with Δ . Clearly, $|\Delta| = |\Delta^*|$. If $\Delta^* = \Delta$, then Δ is said to be a self-paired orbit of G_x .

Definition 1.1.13. Let G act on a set X , then the number of fixed elements (character) π of permutation representation of G on X is defined by

$$\pi(g) = |\text{Fix}(g)| \text{ for all } g \in G.$$

Definition 1.1.14. Let (G_1, X_1) and (G_2, X_2) be finite permutation groups (i.e. G_i acts on X_i , $i = 1, 2$). To say that $(G_1, X_1) \cong (G_2, X_2)$ (permutation isomorphism) we mean that there exists a group isomorphism $\phi : G_1 \rightarrow G_2$ and a bijection $\theta : X_1 \rightarrow X_2$ so that $\theta(gx) = \phi g(\theta(x))$ for all $g \in G_1$, $x \in X_1$ or $\theta g = \phi g \theta$ for all $g \in G_1$.

Let G be a transitive permutation group acting on a set X . Then G acts on $X \times X$ by

$$g(x, y) = (gx, gy), \quad g \in G, \quad x, y \in X.$$

If $O \subseteq X \times X$ is a G -orbit, then for a fixed $x \in X$,

$$\Delta = \{y \in X : (x, y) \in O\}$$

is a G_x -orbit. On the other hand, if $\Delta \subseteq X$ is a G_x -orbit, then $O = \{(gx, gy) \mid g \in G, y \in \Delta\}$ is a G -orbit on $X \times X$. We say Δ corresponds to O . The G_x -orbits on X are called suborbits and G -orbits on $X \times X$ are called suborbitals.

The orbit containing (x, y) is denoted by $O(x, y)$. From $O(x, y)$, we can form a suborbital graph $\Gamma_{(x,y)}$; its vertices are the elements of X , and there is a directed edge from γ to δ if $(\gamma, \delta) \in O(x, y)$. Clearly, $O(y, x)$ is also a suborbital, and it is either equal or disjoint from $O(x, y)$. In the former case, $\Gamma_{(y,x)} = \Gamma_{(x,y)}$ and the graph consists of pairs of oppositely directed edges. It is convenient to replace each such pair by a single undirected edge, so that we have undirected graph which we call self-paired. In the latter case, $\Gamma_{(y,x)}$ is just $\Gamma_{(x,y)}$ with arrows reversed, and we call $\Gamma_{(x,y)}$ and $\Gamma_{(y,x)}$ paired suborbital graphs.

These ideas were first introduced by Sims (1967), and are also described in papers by Akbas and Baskan (1996), Neumann (1977), Kamuti et al. (2012), Refik and Bahar (2009), Besenk et al. (2010) and Bahadir et al. (2008). The emphasis being put on applications to finite groups.

1.2 Brief information on subgroup structure of $PSL(2, q)$

$SL(2, q)$ is the special linear group of 2×2 unimodular matrices over the finite field $GF(q)$ of prime power order q , that is $q = p^f$ for a prime p and positive integer f . Projective special linear group $PSL(2, q)$ is the quotient group

$$PSL(2, q) = \frac{SL(2, q)}{C[SL(2, q)]},$$

where $C[SL(2, q)]$ is the center of $SL(2, q)$. Equivalently, the group $PSL(2, q)$ can be viewed as consisting of linear fractional transformations of the form;

$$x \rightarrow \frac{ax + b}{cx + d},$$

with $a, b, c, d \in GF(q)$, where $ad - bc = 1$. The order of this group is given by;

$$|PSL(2, q)| = \frac{q(q^2 - 1)}{k},$$

where $k = (2, q - 1)$. Each non-identity element of $PSL(2, q)$ has at most two fixed points on the projective line $PG(1, q) = GF(q) \cup \{\infty\}$. Therefore, any element of $PSL(2, q)$ is contained in one of the following categories of subgroups.

Commutative subgroups of order q

The transformations of the form

$$t = \begin{pmatrix} 1 & \alpha \\ 0 & 1 \end{pmatrix},$$

where $\alpha \in GF(q)$ form a commutative subgroup $R = P_q$ of order q containing all transformations of $PSL(2, q)$, leaving the single element ∞ fixed. Each non-identity element of R has order p . The normalizer of R has transformations of the form

$$s = \begin{pmatrix} \alpha & \beta \\ 0 & \alpha^{-1} \end{pmatrix},$$

where $\alpha, \beta \in GF(q)$. Thus, $|N_{PSL(2,q)}(R)| = \frac{q}{k}(q-1)$ and the number of subgroups of $PSL(2, q)$ conjugate to R is $q+1$. Each of these subgroups fix some point in $PG(1, q)$.

Cyclic subgroups of order $\frac{q-1}{k}$

Suppose $\alpha \in GF(q)$ is a primitive root, the transformation of the form

$$u = \begin{pmatrix} \alpha & 0 \\ 0 & \alpha^{-1} \end{pmatrix}$$

generates a cyclic subgroup $S = C_{\frac{q-1}{k}}$ of $PSL(2, q)$ of order $\frac{q-1}{k}$, whose elements fix 0 and ∞ . Any element in $PSL(2, q)$ that normalizes u interchanges 0 and ∞ and hence has the form

$$r = \begin{pmatrix} 0 & -\beta^{-1} \\ \beta & 0 \end{pmatrix},$$

where $\beta \in GF(q)$. The transformation r and s generate a dihedral group of order $\frac{2(q-1)}{k}$. Thus the number of subgroups of $PSL(2, q)$ conjugate to S is $\frac{1}{2}q(q+1)$, all of which intersect at the identity.

Cyclic subgroups of order $\frac{q+1}{k}$

$PSL(2, q)$ has a cyclic subgroup $T = C_{\frac{q+1}{k}}$ of order $\frac{q+1}{k}$. The normalizer of this subgroup is a dihedral group of order $\frac{2(q+1)}{k}$. Thus the number of subgroups of $PSL(2, q)$ conjugate to T is $\frac{1}{2}q(q-1)$. Each pair of distinct conjugates of T intersects at the identity. Any non-identity element of T has no fixed point in $PG(1, q)$.

Theorem 1.2.1. .

a.) Let $P = \{R^g, S^g, T^g\}$, where R^g , S^g and T^g denote the conjugation of R , S^g and T^g respectively with $g \in PSL(2, q)$. Then each non-identity element of $PSL(2, q)$ is contained in exactly one of the groups in P . Thus P forms a partition of $PSL(2, q)$.

b.) Let $g \in PSL(2, q)$ and $\pi(g)$ denote the number of fixed points of g in $PG(1, q)$. We define

$$\tau_i = \{g | g \in PSL(2, q), \pi(g) = i\},$$

where $i = 0, 1, 2$. Thus

$$\tau_0 = \bigcup_{g \in PSL(2, q)} (T - I)^g,$$

$$\tau_1 = \bigcup_{g \in PSL(2, q)} (R - I)^g$$

and

$$\tau_2 = \bigcup_{g \in PSL(2, q)} (S - I)^g,$$

where I is the identity in $PSL(2, q)$.

(Huppert, 1967, Ch. 8)

An element $x \in \tau_0$ is called elliptic, $y \in \tau_1$ is parabolic and $z \in \tau_2$ is hyperbolic. The following theorem gives all the subgroups of $PSL(2, q)$.

Theorem 1.2.2. (Huppert, 1967, Sec. 8.27)

The group $PSL(2, p^f)$ contains only the following subgroups:

- (a.) elementary abelian p -groups;
- (b.) cyclic groups of order z with $z \mid \frac{p^f \pm 1}{k}$, where $k = (p^f - 1, 2)$;
- (c.) dihedral groups of order $2z$, with z as in (b.) above;
- (d.) alternating groups A_4 for $p > 2$ or $p = 2$ and $f \equiv 0(2)$;
- (e.) symmetric groups S_4 for $p^{2f} - 1 \equiv 0(16)$;
- (f.) alternating groups A_5 for $p = 5$ or $p^{2f} - 1 \equiv 0(5)$;
- (g.) semi direct products of elementary abelian groups of order p^m by cyclic groups of order t , where $t \mid \frac{p^m - 1}{k}$ and $t \mid p^f - 1$ and
- (h.) groups $PSL(2, p^m)$ for $m \mid f$, and $PGL(2, p^m)$ for $2m \mid f$. □

1.3 Problem statement and justification

Group action is an active field that has been in existence for a long time. Redfield (1927) studied some of the links between combinatorial analysis and permutation groups during which he introduced some symmetric functions he called group reduction functions. Ten years later, Pólya (1937) used the same functions although he called them cycle indices to devise a powerful general method for enumerating isomers.

Tchuda (1986) calculated subdegrees of $PSL(2, q)$ acting on the cosets of all its maximal subgroups. These subdegrees have been used by Faradžev and Ivanov (1990) to determine distance transitive representations of groups G with $PSL(2, q) \trianglelefteq G < P\Gamma L(2, q)$. Similarly, subdegrees of $PGL(2, q)$ acting on the cosets of its maximal subgroups were calculated by Kamuti (2006).

Kamuti (1992) constructed suborbital graphs corresponding to the action of $PSL(2, q)$ and $PGL(2, q)$ acting on the cosets of their maximal dihedral subgroups and also investigated their properties. In his work, he gave an alternative method of constructing the Coxeter graph which was first constructed by Coxeter (1983).

So far very little has been done on the action of $PSL(2, q)$ on the cosets of some of its subgroups (especially the non-maximal subgroups). Consequently, this research is set to investigate the following;

- cycle index formulas,
- the rank and subdegrees,
- the suborbital graphs and their theoretic properties

of $PSL(2, q)$ acting on the cosets of $C_{\frac{q-1}{k}}, C_{\frac{q+1}{k}}, P_q, A_4$ and $D_{\frac{2(q-1)}{k}}$ subgroups.

1.4 Objectives of the study

Main objective To study the action of $PSL(2, q)$ on the cosets of some of its subgroups, namely; $C_{\frac{q-1}{k}}, C_{\frac{q+1}{k}}, P_q, A_4$ and $D_{\frac{2(q-1)}{k}}$.

Specific objectives

- i.) To determine the cycle index formulas corresponding to each action;
- ii.) To determine the ranks and subdegrees corresponding to these representations;
- iii.) To construct the suborbital graphs corresponding to the action of $PSL(2, q)$ on the cosets of its cyclic subgroup $C_{\frac{q-1}{k}}$ and to study some of their theoretic properties.

1.5 Significance of the study

Representation theory is both an application of the group concept and an important concept for a deeper understanding of groups. Given a group action, representation gives further means to study the object being acted upon, yielding more information about the group. Thus group representations are an organizing principle in the theory of finite groups. In particular, this research will provide valuable information in the classification of distance-transitive graphs. The study will also be of great help to various fields. To combinatorics, calculation of cycle index formulas will enhance processes by which we organize sets so that we can interpret and apply the data they contain. To graph

theorists, graphs are used to model many types of relations and processes in physical, biological, social and information systems. Many practical problems can be represented by graphs.

1.6 Preliminary results

Theorem 1.6.1. *Let G act transitively on the set X . Let $x \in X$ and let $H = \text{stab}_G(x)$. Then the action of G on X is equivalent to action by multiplication on the set of (right) cosets of H in G . (Rose, 1978, p. 76)*

Lemma 1.6.1. *(Cauchy-Frobenius lemma)*

Let G be a group acting on a finite set X . Then the number of G -orbits on X is

$$\frac{1}{|G|} \sum_{g \in G} |\text{Fix}(g)|$$

where $\text{Fix}(g) = \{x \in X; gx = x\}$ (Krishnamurthy, 1985)

Theorem 1.6.2. *(Orbit-Stabilizer Theorem)*

Let G be a group acting on a finite set X , and let $x \in X$. Then

$$|\text{Orb}_G(x)| = |G : \text{Stab}_G(x)|.$$

(Rose, 1978)

Lemma 1.6.2. *Let g be a permutation with cycle type $(\alpha_1, \alpha_2, \dots, \alpha_n)$, then*

a) the number $\pi(g^l)$ of 1-cycles in g^l is

$$\pi(g^l) = \sum_{i|l} i\alpha_i$$

b) and

$$\alpha_l = \frac{1}{l} \sum_{i|l} \pi(g^{l/i}) \mu(i)$$

where μ is the Möbius function. (Kamuti, 1992, p. 6) For the definition of μ , see Definition 1.1.2 and also Hardy and Wright (1938, p. 234).

Theorem 1.6.3. *Two permutations in S_n are conjugate if and only if they have the same cycle type; and if $g \in S_n$ has cycle type $(\alpha_1, \alpha_2, \dots, \alpha_n)$, then the number of permutations in S_n conjugate to g is*

$$\frac{n!}{\prod_{i=1}^n \alpha_i! i^{\alpha_i}}$$

(Krishnamurthy, 1985)

Theorem 1.6.4. *Let G be a finite transitive permutation group acting on the (right) cosets of its subgroup H . If $g \in G$ and $n = [G : H]$, then*

$$\pi(g) = \frac{n|C^g \cap H|}{|C^g|}.$$

(Kamuti, 1992, p. 5)

1.7 Outline of the thesis

In Chapter 1, the subgroup structure of $PSL(2, q)$ has been briefly examined, details of which are found in Dickson (1901, Chap. 12) or Huppert (1967, Sec. 8). Known results which make the basis of the arguments to be used herein are revisited.

In Chapter 2, literature on related work is perused. The focus being on the cycle index formulas, subdegrees and suborbital graphs of permutation groups.

In Chapter 3, intersections of conjugacy class of elements $g \in G$ with certain subgroups of G , are used to determine the disjoint cycle structure of g and hence cycle index of G corresponding to some given permutation representations. In essence, the action of $PSL(2, q)$ on the cosets of $C_{\frac{q-1}{k}}$, $C_{\frac{q+1}{k}}$, P_q , A_4 and $D_{\frac{2(q-1)}{k}}$ is looked at.

In Chapter 4, using disjoint cycle structures obtained in chapter three, the subgroup structure described in chapter one and Lemma 1.6.1, the rank of $PSL(2, q)$ and subdegrees corresponding to each permutation representation are determined. This is done using two different approaches confirming the results.

In Chapter 5, borrowing a leaf from Theorem 1.6.1, it is shown that the action of $PSL(2, q)$ on the cosets of $C_{\frac{q-1}{k}}$ is equivalent to its action on ordered pairs from $PG(1, q)$. The corresponding subdegrees are determined using reduced pair group action and their properties discussed. Further, suborbital graphs corresponding to the representation of $PSL(2, q)$ on the cosets of $C_{\frac{q-1}{k}}$ are constructed and some of their theoretic properties investigated.

The conclusions and recommendations are given in Chapter 6.

CHAPTER 2

LITERATURE REVIEW

In this chapter, previous work related to the current study is looked at. The chapter is divided into three sections.

In Section 2.1, we examine the literature on cycle indices, especially about its inception and the study in the recent past. In Section 2.2, we peruse literature on ranks and subdegrees. Finally, in Section 2.3, we pay particular attention to the suborbits and suborbital graphs.

2.1 Cycle indices

Cycle index of the action of a group G on a finite set X has attracted interest before from various mathematicians as can be seen below.

Redfield (1927) studied some of the links between combinatorial analysis and permutation groups. In this study, every permutation group was associated with a type of symmetric function which was called a group reduction function. These are nowadays the so called cycle indices.

The group reduction functions were later renamed cycle indices by Pólya (1937) who used them to devise a powerful general method for enumerating the number of orbits of a group on particular configurations. The result was

greatly popularized by applying it to many counting problems, in particular to the enumeration of chemical compounds called isomers.

Kamuti (1992) computed the disjoint cycle structures of elements of $PSL(2, q)$ and $PGL(2, q)$ for any primitive permutation representation of these groups. The work further sketched general formulas for the cycle indices of these representations.

Cycle indices for the natural actions of the general linear groups and affine groups (on a vector space) and for the projective linear groups (on a projective space) over a finite field were computed by Friperfinger (1996). The author went further to demonstrate how to enumerate isometry classes of linear codes by using these cycle indices.

In 2004, Kamuti showed how the cycle index of a semi direct product group $G = M \rtimes H$ can be expressed in terms of the cycle indices of M and H by considering semi direct products called Frobenius groups. Later Kamuti (2012) expressed the cycle index of $G = M \times H$ in terms of the cycle indices of M and H where G is the direct product of its subgroups M and H .

2.2 Ranks and subdegrees

The rank and subdegrees of a permutation group G have been considered by several mathematicians before.

Higman (1964) defined the rank of a transitive permutation group as the number of orbits of the stabilizer of a point. Finite permutation groups of rank 3 were investigated. It was found that the symmetric group S_n on $X =$

$\{1, 2, \dots, n\}, n \geq 4$ acts as a rank 3 group on 2-element subsets of X , with subdegrees $1, 2(n-2)$ and $\binom{n-2}{2}$.

The great little monograph was written by Wielandt (1964) where finite permutation groups were dealt with. This work showed that a primitive group of degree $2p, p$ a prime has rank at most 3.

In 1971, Quirin studied primitive permutation groups with small suborbits. The study classified all primitive permutation groups G which have a suborbit Δ of length 4 for which $G_x^\Delta \cong A_4$ or $G_x^\Delta \cong S_4$ is faithful. The study also considered primitive permutation groups G which have a suborbit Δ of prime length $p \geq 5$ such that $|G_x^\Delta| \leq 2p$, and further reduced the classification problem to that of classifying simple groups with maximal dihedral subgroups of order $2p$.

Cameron (1972) worked on suborbits of multiply transitive permutation groups. In this paper, it was proved using combinatorial techniques that if G is a primitive permutation group on X which is not 2-transitive, and if the stabilizer G_x of a point x is 2-transitive on an orbit $F(x)$ with $|F(x)| = v > 2$; then G_x has an orbit Δ_x with $|\Delta_x| = w$, where $w > v$ and $w|v(v-1)$. Later, (Cameron, 1975) studied suborbits in transitive permutation groups. In this paper, the author established some combinatorial and algebraic relations among suborbits. Again Cameron (1983), dealt with the orbits of permutation groups on unordered sets. Construction and characterization of a 3-homogeneous but not 2-primitive permutation group H of countable degree was done in this paper, and it was shown that it has a transitive extension J which is 5-homogeneous but not 3-primitive.

Using subdegrees of primitive permutation representations of $PSL(2, q)$ calculated by Tchuda (1986), Faradžev and Ivanov (1990) showed that if $G = PSL(2, q)$ acts on the cosets of its maximal dihedral subgroup H , then the rank is at least $|G|/|H|^2$ and if $q > 100$, the rank is more than 5.

The subdegrees of primitive permutation representations of $PSL(2, q)$ and $PGL(2, q)$ were computed by Kamuti (1992). The work showed that when $PSL(2, q)$ acts on the cosets of its maximal dihedral subgroup of order $\frac{2(q-1)}{k}$, then its rank is $\frac{3}{4}(q+3)$ if $q \equiv 1 \pmod{4}$, $\frac{1}{4}(3q+7)$ if $q \equiv -1 \pmod{4}$ and $\frac{1}{2}(q+2)$ if q is even.

Dragan and Roman (1998) characterized transitive permutation groups having a non-self paired suborbit of length 2 in terms of their point stabilizers. As a consequence, elementary abelian groups were proved to be the only possible abelian point stabilizers arising from such actions.

Finite primitive permutation groups with a small suborbit were studied by Cai et al. (2004). Based on the classification result of Quirin (1971) and Wang (1992), the study produced a precise list of primitive permutation groups with a suborbit of length 4. In particular, it was showed that there exists no examples of such groups with the point stabiliser of order $2^4 3^6$, clarifying an uncertain question (since 1970s).

Kamuti (2006) computed the subdegrees of primitive representations of $PGL(2, q)$ using a method proposed by Ivanov et al. (1983) which uses marks of a permutation group. In the action of $PGL(2, q)$ on the cosets of S_4 , he found out that the rank is $\frac{1}{576}(q^3 + 189q \mp 82)$ when $q \equiv \pm 5 \pmod{12}$ and $q \equiv \pm 3 \pmod{8}$; and is $\frac{1}{576}(q^3 + 189q \mp 46)$ when $q \equiv \pm 1 \pmod{12}$ and $q \equiv \pm 3 \pmod{8}$. He also showed the subdegrees of $PGL(2, q)$ corresponding to this action to be

1,4,6,8,12 and 24. The paper further exhibited that when $PGL(2, q)$ acts on the cosets of its maximal dihedral subgroup of order $2(q-1)$, its rank is $\frac{1}{2}(q+3)$ if q is odd and $\frac{1}{2}(q+2)$ if q is even.

Ranks of the permutation representations of the simple groups $B_l(q)$, $C_l(q)$ and $D_l(q)$ on the cosets of the parabolic maximal subgroups were determined by Korableva (2008). The research determined the rank $rk(D_l)$ for $1 \leq k \leq l$ of the permutation representations of the group $D_l(q)$ for $l \geq 3$ with respect to the parabolic maximal subgroup can be computed by the following recursive method.

$$\begin{aligned} rk(D_l) &= rk(D_{l-1}) + k + 1 \text{ for } 3 \leq k \leq [l/2] + 1, \\ rk(D_l) &= rk(D_{l-1}) + l - k + 2 \text{ for } [l/2] + 1 < k < l, \\ r_1(D_l) &= r_2(D_l) = [l/2] + 1, \\ r_1(D_l) &= 3. \end{aligned}$$

Nyaga et al. (2011) generalized Higman's work to S_n acting on r -element subsets; $X^{(r)}$. The work established that if $n \geq 2$, then the subdegrees of S_n acting on $X^{(r)}$ are:

$$1, \binom{r}{1} \binom{n-r}{r-1}, \binom{r}{2} \binom{n-r}{r-2}, \dots, \binom{r}{r-1} \binom{n-r}{1}, \binom{n-r}{r}.$$

The rank and subdegrees of the symmetric group S_n acting on $X^{[r]}$, the set of all ordered r -element subsets from $X = \{1, 2, \dots, n\}$ for $n \geq 2r$

were determined by Rimmer et al. (2012). The paper showed that S_n acts transitively on $X^{[r]}$.

2.3 Suborbital graphs

In 1967, Sims introduced the idea of suborbital graphs for a finite permutation group G acting on a set X . In this work a suborbital graph Γ_i corresponding to suborbital $O_i \subseteq X \times X$ was defined as a graph in which the vertex set is X and the edge set E consists of directed edges xy such that $(x, y) \in O_i$.

Praeger et al. (1987) defined wreath product (Hamming schemes) as follows. Let Σ be a set of c points ($c \geq 2$), and let $\Omega = \Sigma^d$ for some $d \geq 2$. Two vertices A and B are joined if they differ in exactly one coordinate. The graph (called a Hamming graph) has valency $v = d(c - 1)$ and diameter d . It is G -distance transitive with $G = S_c \wr S_d$, and is primitive whenever $c \geq 3$. If $c = 2$ then the graph is both antipodal and bipartite. The antipodal quotient (that is, the graph obtained by identifying each vertex with the unique vertex at distance d from it) is a distance transitive graph of valency v and diameter $\frac{1}{2}d$. This graph is primitive if d is odd; if d is even, this graph is still bipartite, but the distance-2 graph obtained from it has two isomorphic connected components, each primitive distance transitive of valency $\binom{d}{2}$ and diameter $\frac{1}{4}d$. Alternatively, the distance-2 graph of the original graph has two isomorphic connected components, each distance transitive of valency $\binom{d}{2}$ and diameter $\frac{1}{2}d$. These are primitive if d is odd but are antipodal if d is even: taking antipodal quotients leads to primitive distance transitive

graphs of valency $\binom{d}{2}$ and diameter $\frac{1}{4}d$. Finally, if one of the above graphs has diameter 2 then its complementary graph is also distance transitive. In particular, if $d = 2$ and $c \geq 3$, the complementary graph of the Hamming graph is distance transitive of valency $(c - 1)^2$.

The suborbital graphs Γ_i corresponding to the action of $PSL(2, \mathbb{Z})$ on the rational projective line $\widehat{\mathbb{Q}} = \mathbb{Q} \cup \{\infty\}$ was constructed by Jones et al. (1991). The simplest was the Farey graph. In this graph, vertex ∞ is joined to integers, while two rational numbers $\frac{r}{s}$ and $\frac{x}{y}$ (in reduced form) are adjacent if and only if $ry - sx = \pm 1$, or equivalently if they are consecutive terms in the Farey sequence (F_m) consisting of rationals $\frac{x}{y}$ with $|y| \leq m$, arranged in an ascending order. The work found out that Γ_i is connected, contains anti-directed triangles, such as $\infty \rightarrow 1 \leftarrow 2 \rightarrow \infty$ and further conjectured that any other suborbital graph is a forest if and only if it contains no triangles.

Kamuti (1992) devised a method for constructing some of the suborbital graphs of $PSL(2, q)$ acting on the cosets of its maximal dihedral subgroup of order $q - 1$. This method gave an alternative way of constructing the Coxeter graph which was first constructed by Coxeter (1983). This is a non-Hamiltonian cubic graph on 28 vertices and 42 edges with girth 7.

Akbas and Baskan (1996) examined some properties of suborbital graphs for the normalizer \mathcal{N} of $\Gamma_0(N)$ in $PSL(2, \mathbb{R})$, where elements of this subgroup were taken to be matrices of the form

$$\Gamma_0(N) = \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \Gamma_0 \mid c \equiv 0 \pmod{N} \right\}.$$

The work showed that, if $\mathcal{N}/\Gamma_0(N)$ and the set of orbit representatives denoted by B and Ω respectively, the permutation group (B, Ω) was found to be regular and m -regular where m is an odd natural number.

A conjecture of Jones et al. (1991) that a suborbital graph for the modular group is a forest if and only if it contains no triangles was proved by Akbas (2001).

Cai et al. (2004) analyzed the orbital graphs of primitive permutation groups with a suborbit of length 3 or of length 4. They obtained a complete classification of vertex-primitive arc-transitive graphs of valency 3 and valency 4, and proved that there exists no vertex-primitive half-arc-transitive graphs of valency less than 10. Finally, they concluded by constructing vertex-primitive half-arc-transitive graphs of valency $2k$ for infinitely many integers k , with 14 as the smallest valency.

In 2007, Zai Zai (2007) constructed infinitely many primitive half-transitive graphs with automorphism groups being the symmetric groups of prime degrees, and showed that there exists at least one primitive half-transitive graph of valency $2p$ for a prime p not less than 7 and $p \neq 13$. The work managed to construct 2-arc-regular Cayley graphs.

Bahadir et al. (2008) worked on the suborbital graphs of the congruence subgroup $\Gamma_0(N)$ of the modular group Γ_0 . Taking N to be a prime p , they showed that the action of $\Gamma_0(p)$ on $\widehat{\mathbb{Q}}$ is both intransitive and imprimitive.

They further showed that the orbits of $\Gamma_0(p)$ on $\widehat{\mathbb{Q}}$ are $\begin{pmatrix} 1 \\ 1 \end{pmatrix}$ and $\begin{pmatrix} 1 \\ p \end{pmatrix}$.

Finally, the study determined the suborbital graphs of $\Gamma_0(p)$ on $\begin{pmatrix} 1 \\ p \end{pmatrix}$; each

suborbital was found to contain a pair (∞, u) for some $v \in \begin{pmatrix} 1 \\ p \end{pmatrix}$, $v = \frac{u}{p}$ and was denoted by $O_{u,p}$. The corresponding suborbital graph was denoted by $\Gamma_{u,p}$, and it was shown to contain a triangle if and only if $u^2 \mp u + 1 \equiv 0 \pmod{p}$.

The number of sides of circuits in suborbital graph for the normalizer of $\Gamma_0(m)$ in $PSL(2, \mathbb{R})$, where m is of the form $2p^2$, p a prime and $p \equiv 1 \pmod{4}$ were determined by Kader et al. (2010). The study went further to give a theoretical result which states that the prime divisors p of $2u^2 \pm 2u + 1$ are of the form $p \equiv 1 \pmod{4}$.

Guler et al. (2011) considered the action of a permutation group on a set in the spirit of the theory of permutation groups, and graphs arising from this action in hyperbolic geometric terms. In this paper, they examined some relations between elliptic elements and circuits in graph for the normalizer of $\Gamma_0(N)$ in $PSL(2, \mathbb{R})$. They conjectured that if $N = 2^\alpha 3^\beta p^2$, where $F(\infty, \frac{u}{p^2})$ is the subgraph of $G(\infty, \frac{u}{p^2})$, then $F(\infty, \frac{u}{p^2})$ has a circuit as follows.

Table 2.1: Circuit for the subgraph $F(\infty, \frac{u}{p^2})$

α	β	Circuit	Conditions
0,2,4,6	0,2	triangle	$p \equiv 1 \pmod{3}$
1,3,5,7	0,2	rectangle	$p \equiv 1 \pmod{4}$
0,2,4,6	1,3	hexagon	$p \equiv 1 \pmod{3}$

In 2011, Kader and Guler introduced the notion of conjugacy in suborbital graphs. They went ahead to show that the conjugate elliptic elements of the modular group Γ and congruence subgroups $\Gamma_0(p)$ give rise to conjugate circuits, corresponding to the related elliptic elements in the Farey graph F and in the suborbital graph $F_{u,p}$ of the action of $\Gamma_0(p)$ respectively.

The suborbital graphs for the action of the normalizer of $\Gamma_0(N)$ in $PSL(2, \mathbb{R})$, where N was of the form $2^8 p^2$, $p > 3$ and p is a prime were investigated by Besenk et al. (2012). In addition the study gave conditions to be a forest for normalizer in the suborbital graph $F(\infty, \frac{u}{2^8 p^2})$.

Kesicioglu et al. (2013) introduced a different Γ -invariant equivalence relation by using the congruence subgroup $\Gamma_1(n)$ instead of $\Gamma_0(n)$ used by Guler et al. (2011) to construct a suborbital graph $H_{u,n}$, where $\frac{u}{n} \in \widehat{\mathbb{Q}}$ and $n \geq 0$, $(u, n) = 1$. They gave a necessary and sufficient condition for the graph $H_{u,n}$ to be connected and a forest. They went further to show some relations between the lengths of circuits in $H_{u,n}$ and the elliptic elements of the group $\Gamma_1(n)$.

CHAPTER 3

CYCLE INDICES OF $G = \text{PSL}(2, \mathbb{Q})$ ACTING ON COSETS OF SOME OF ITS SUBGROUPS

In this chapter, we use intersection of conjugacy class of elements of G with various subgroups of G to determine the disjoint cycle structures and hence cycle index of G corresponding to a given permutation representation. In essence, we look at G acting on the cosets of $C_{\frac{q-1}{k}}$, $C_{\frac{q+1}{k}}$, P_q , A_4 and $D_{\frac{2(q-1)}{k}}$. The chapter is presented in two sections. In Section 3.1, we highlight well known results to be used later in the chapter. Disjoint cycle structures for the permutation representations on the cosets of the above mentioned subgroups in that order are determined in Sections 3.2. In each case, the corresponding cycle index is obtained.

3.1 Preliminary results

Suppose $g \in G$ and let g' be the induced permutation when g acts on the cosets of a subgroup H of G . To determine disjoint cycle structures of this representation, we require the following results.

Lemma 3.1.1. *If $g \in \tau_0$ or $g \in \tau_2$ and is of order greater than 2 or $g \in \tau_1$, then its centralizer in G consists of all elliptic, hyperbolic or parabolic elements respectively with the same fixed point set, together with the identity element. On the other hand, if $g \in \tau_0$ or $g \in \tau_2$ and g is of order 2, then its centralizer is the dihedral group of order $\frac{2(q+1)}{k}$ or $\frac{2(q-1)}{k}$ respectively. (Dickson, 1901, Sec.242)*

Lemma 3.1.2. *The $q - 1$ non-identity elements of the Sylow p -subgroup P_q of G are all conjugate if $p = 2$, but are separated into two sets of $\frac{q-1}{2}$ conjugate elements if $p > 2$. (Dickson, 1901, Sec.241)*

Lemma 3.1.3. *The $p - 1$ non-identity elements of a cyclic subgroup C_p of G belong half to one set of conjugacy classes and half to the other if $p > 2$ and f is odd but all belong to the same set if f is even or $p = 2$. (Dickson, 1901, Sec.241)*

From the above information on the subgroups of G , the lengths of the corresponding conjugacy classes $|C^g|$ are determined. For instance, if $g \in \tau_0$ and the order of g is greater than 2 then,

$$|C^g| = \frac{|G|}{|C_G(g)|} = q(q - 1).$$

Thus when $d > 2$ (the order of g), then:

$$|C^g| = \begin{cases} q(q - 1) & g \in \tau_0; \\ \frac{1}{2}(q + 1)(q - 1) & g \in \tau_1; \\ q(q + 1) & g \in \tau_2. \end{cases}$$

Otherwise, when $d = 2$, then:

$$|C^g| = \begin{cases} \frac{1}{2}q(q-1) & g \in \tau_0; \\ (q+1)(q-1) & g \in \tau_1; \\ \frac{1}{2}q(q+1) & g \in \tau_2. \end{cases}$$

Theorem 3.1.1. *The cycle index of G , q even, on the cosets of its subgroup H is*

$$Z(G) = \frac{1}{|G|} \left[t_1^{|G|/|H|} + (q^2 - 1)mon(x') + \frac{q(q+1)}{2} \sum_{g \in C_{q-1} \setminus I} mon(g') \right. \\ \left. + \frac{q(q-1)}{2} \sum_{g \in C_{q+1} \setminus I} mon(g') \right]$$

where $x \in \tau_1$. (Kamuti, 1992, p. 68)

Theorem 3.1.2. *The cycle index of G , q odd, on the cosets of its subgroup H is one of the following:*

a)

$$Z(G) = \frac{1}{|G|} \left[t_1^{|G|/|H|} + (q^2 - 1)mon(x') + \frac{q(q+1)}{2} \sum_{g \in C_{\frac{q-1}{2}} \setminus I} mon(g') \right. \\ \left. + \frac{q(q-1)}{2} \sum_{g \in C_{\frac{q+1}{2}} \setminus I} mon(g') \right],$$

b)

$$Z(G) = \frac{1}{|G|} \left[t_1^{|G|/|H|} + \frac{(q^2 - 1)}{2} mon(x'_1) + \frac{(q^2 - 1)}{2} mon(x'_2) \right]$$

$$\left. \begin{aligned} & + \frac{q(q+1)}{2} \sum_{g \in C_{\frac{q-1}{2}} \setminus I} \text{mon}(g') + \frac{q(q-1)}{2} \sum_{g \in C_{\frac{q+1}{2}} \setminus I} \text{mon}(g') \end{aligned} \right]$$

where $x'_1, x'_2, x' \in \tau_1$. (Kamuti, 1992, p. 69)

Remark 3.1.1. From Lemma 3.1.2 above,

- i) if elements of order p in H lie in the two conjugacy classes of parabolics in G , or if H contains no element of order p , then the $q^2 - 1$ parabolic elements have the same monomial, hence formula (a).
- ii) if elements of order p in H lie in one of the two conjugacy classes of parabolics in G , we have two different types of monomials with half of the parabolic elements sharing each, hence formula (b).

3.2 Permutation representations of G on the cosets of its subgroups

3.2.1 Permutation representations of G on the cosets of

$$H = C_{\frac{q-1}{k}}$$

3.2.1.1 Disjoint cycle structures of elements of G acting on cosets of H

Remark 3.2.1. If d is the order of $g \in G$ and no $h \in H$ exists such that its order is d , then

$$|C^g \cap H| = 0.$$

In case such an h exists, the intersection is obtained using the properties of the three non-intersecting sets that partition G described in Chapter 1. Given the order of H is $\frac{(q-1)}{k}$, then

$$[G : H] = q(q + 1).$$

From the remark above, if $g \in \tau_0$ or $g \in \tau_1$, then $C^g \cap H = \phi$, hence

$$|C^g \cap H| = 0.$$

On the other hand, since H is cyclic, if $g \in \tau_2$ and $d = 2$, $\phi(2) = 1$, then H has a single element of order 2. Thus each conjugate of H in G has one element of order 2 (conjugate to g). Therefore

$$|C^g \cap H| = 1.$$

If $g \in \tau_2$ and $d > 2$, then

$$|C^g \cap H| = 2,$$

since $|C^g| = q(q + 1)$ and $[G : N_G(H)] = \frac{q(q+1)}{2}$, so each conjugate of H must contain two elements of C^g .

From Theorem 1.6.4 and the information in the preceding section, we determine $\pi(g)$ whose values are as given in the Table 3.1 below.

Next, we calculate in details the cycle lengths of the element g' corresponding to g in this representation.

Table 3.1: Number of fixed points of g' when G acts on the cosets of $C_{\frac{q-1}{k}}$

	$ C^g $	$ C^g \cap H $	$\pi(g)$
$g \in \tau_0$	$\frac{1}{2}q(q-1); d=2$ $q(q-1); d \neq 2$	0	0
$g \in \tau_1$	$(q+1)(q-1); p=2$ $\frac{1}{2}(q+1)(q-1); p \neq 2$	0	0
$g \in \tau_2$	$\frac{1}{2}q(q+1); d=2$ $q(q+1); d \neq 2$	1 2	2 2

Note: If $o(g) = d$ (order of g) then:

$$\pi(g^d) = q(q+1).$$

I: $g \in \tau_0$

Using Lemma 1.6.2 (b) and Table 3.1, we have;

If $l < d$, then $\pi(g^l) = 0$. Thus

$$\alpha_l = 0. \tag{3.1}$$

If $l = d$, then

$$\begin{aligned} \alpha_d &= \frac{1}{d} \sum_{i|d} \pi(g^{d/i}) \mu(i) \\ &= \pi(g^d) \mu(1) \\ &= \frac{q(q+1)}{d}. \end{aligned} \tag{3.2}$$

II: $g \in \tau_1$

Similarly, if $l < d$, then $\alpha_l = 0$ and if $l = d = p$, then

$$\alpha_p = p^{f-1}(q+1). \tag{3.3}$$

III: $g \in \tau_2$

From Table 3.1, $\pi(g) = 2$, thus

Case 1; when $l = 1$, then

$$\alpha_1 = 2. \quad (3.4)$$

Case 2; when $1 < l < d$, then $\pi(g^l) = 2$ and thus

$$\begin{aligned} \alpha_l &= \frac{1}{l} \sum_{i|l} \pi(g^{l/i}) \mu(i) \\ &= \frac{2}{l} \sum_{i|l} \mu(i) \\ &= 0. \end{aligned} \quad (3.5)$$

Case 3; when $l = d$ then

$$\begin{aligned} \alpha_d &= \frac{1}{d} \sum_{i|d} \pi(g^{d/i}) \mu(i) \\ &= \frac{1}{d} \left[\sum_{1 \neq i|d} \pi(g^{d/i}) \mu(i) + \pi(g^d) \mu(1) \right] \\ &= \frac{1}{d} \left[2 \sum_{i|d} \mu(i) - 2\mu(1) + q(q+1) \right] \\ &= \frac{(q+2)(q-1)}{d}. \end{aligned} \quad (3.6)$$

Summary of the results

Table 3.2: Disjoint cycle structures of g' when G acts on the cosets of $C_{\frac{q-1}{k}}$

	τ_0	τ_1	τ_2
Cycle length of g'	d	p	1 d
Number of cycles	$\frac{q(q+1)}{d}$	$p^{f-1}(q+1)$	$2 \frac{(q+2)(q-1)}{d}$

3.2.1.2 Cycle index of G acting on the cosets of H

In view of Theorems 3.1.1 and 3.1.2 and the results in Table 3.2 above, the cycle index of G on the cosets of $C_{\frac{q-1}{k}}$ is as follows.

Theorem 3.2.1. *The cycle index of G acting on the cosets of its cyclic subgroup $C_{\frac{q-1}{k}}$ is given by:*

a) when q is even

$$Z(G) = \frac{1}{|G|} \left[t_1^{q(q+1)} + (q^2 - 1)t_2^{p^{(f-1)(q+1)}} + \frac{q(q+1)}{2} \sum_{1 \neq d|q-1} \phi(d)t_1^2 t_d^{\frac{(q+2)(q-1)}{d}} \right. \\ \left. + \frac{q(q-1)}{2} \sum_{1 \neq d|q+1} \phi(d)t_d^{\frac{q(q+1)}{d}} \right],$$

b) when q is odd

$$Z(G) = \frac{1}{|G|} \left[t_1^{q(q+1)} + (q^2 - 1)t_p^{p^{(f-1)(q+1)}} + \frac{q(q+1)}{2} \sum_{1 \neq d|\frac{q-1}{2}} \phi(d)t_1^2 t_d^{\frac{(q+2)(q-1)}{d}} \right. \\ \left. + \frac{q(q-1)}{2} \sum_{1 \neq d|\frac{q+1}{2}} \phi(d)t_d^{\frac{q(q+1)}{d}} \right].$$

Proof:

a) The identity element contributes $t_1^{|G|/|H|}$ to the sum of the monomials. All the $q^2 - 1$ parabolics lie in the same conjugacy class. Hence they have the same monomial. Thus they contribute

$$(q^2 - 1)t_2^{p^{(f-1)(q+1)}}$$

to the sum of monomials. Any hyperbolic element of G is contained in a unique cyclic group C_{q-1} whose normalizer is a dihedral group of order $2(q-1)$. Thus there are $\frac{q(q+1)}{2}$ subgroups of G conjugate to C_{q-1} . Hence the contribution by hyperbolics to the total sum of monomials is

$$\frac{q(q+1)}{2} \sum_{1 \neq d|q-1} \phi(d) t_1^2 t_d^{\frac{(q+2)(q-1)}{d}},$$

where ϕ is the Euler function. Similarly, each elliptic element of G is contained in a unique cyclic group C_{q+1} and there are in total $\frac{q(q-1)}{2}$ conjugate subgroups of C_{q+1} in G . Thus their contribution to the sum of monomials is

$$\frac{q(q-1)}{2} \sum_{1 \neq d|q+1} \phi(d) t_d^{\frac{q(q+1)}{d}}.$$

Summing up all the contributions and dividing by the order of G gives the result.

b) Similar arguments as those used above apply. □

Examples

1. $\text{PSL}(2,3)$ acting on the cosets of $C_{\frac{q-1}{k}} = C_1 = \{1\}$ gives;

$$Z(G) = \frac{1}{12} (t_1^{12} + 8t_3^4 + 3t_2^6).$$

2. $\text{PSL}(2,5)$ acting on the cosets of $C_{\frac{q-1}{k}} = C_2$ gives;

$$Z(G) = \frac{1}{60} (t_1^{30} + 24t_5^6 + 15t_1^2 t_2^{14} + 20t_3^{10}).$$

3. $\text{PSL}(2,4)$ acting on the cosets of $C_{\frac{q-1}{k}} = C_3$ gives;

$$Z(G) = \frac{1}{60} (t_1^{20} + 24t_5^4 + 15t_2^{10} + 20t_1^2t_3^6).$$

3.2.2 Permutation representations of G on the cosets of

$$H = C_{\frac{q+1}{k}}$$

3.2.2.1 Disjoint cycle structures of elements of G acting on cosets of H

When $H = C_{\frac{q+1}{k}}$, since H is again cyclic, similar arguments as used in section 3.2.1 above are used to arrive to the fact that

$$|C^g \cap H| = \begin{cases} 1; d = 2 \text{ and } g \in \tau_0 \\ 2; d \neq 2 \text{ and } g \in \tau_0 \\ 0; \text{ otherwise} \end{cases}.$$

The order of H is $\frac{q+1}{k}$, thus

$$[G : H] = q(q-1).$$

Using Remarks 3.2.1, Lemma 1.6.2 and Theorem 1.6.4, the following table is constructed.

Using Lemma 1.6.2 (b) and Table 3.3 above, we calculate in details the cycle lengths of g' corresponding to g in the representation as follows;

I: $g \in \tau_0$

From Table 3.3, $\pi(g) = 2$ and thus

Table 3.3: Number of fixed points of g' when G acts on the cosets of $C_{\frac{q+1}{k}}$

	$ C^g $	$ C^g \cap H $	$\pi(g)$
$g \in \tau_0$	$\frac{1}{2}q(q-1); d=2$	1	2
	$q(q-1); d \neq 2$	2	2
$g \in \tau_1$	$(q+1)(q-1); p=2$	0	0
	$\frac{1}{2}(q+1)(q-1); p \neq 2$	0	0
$g \in \tau_2$	$\frac{1}{2}q(q+1); d=2$	0	0
	$q(q+1); d \neq 2$	0	0

Case 1; when $l = 1$, then

$$\begin{aligned}\alpha_1 &= \pi(g)\mu(1) \\ &= 2.\end{aligned}\tag{3.7}$$

Case 2; when $1 < l < d$, then $\alpha_l = 0$.

Case 3; when $l = d$, then

$$\alpha_d = \frac{(q+1)(q-2)}{d}.\tag{3.8}$$

II: $g \in \tau_1$

If $1 \leq l < d$, then $\alpha_l = 0$ and if $l = d = p$, then $\pi(g^d) = q(q-1)$. Thus

$$\alpha_p = p^{f-1}(q-1).\tag{3.9}$$

III: $g \in \tau_2$

Similarly, if $1 \leq l < d$, then $\alpha_l = 0$ and if $l = d$, then

$$\alpha_d = \frac{q(q-1)}{d}.\tag{3.10}$$

Summary of the results

Table 3.4: Disjoint cycle structures of g' when G acts on the cosets of $C_{\frac{q+1}{k}}$

	τ_0	τ_1	τ_2
Cycle length of g'	1	d	\bar{d}
Number of cycles	2	$\frac{(q-2)(q+1)}{d}$	$p^{f-1}(q-1)$

3.2.2.2 Cycle index of G acting on the cosets of H

In view of Theorem 3.1.2, Theorem 1.6.4 and the results in Table 3.4 above, the cycle index of G on the cosets of $C_{\frac{q+1}{k}}$ is as follows.

Theorem 3.2.2. *The cycle index of G acting on the cosets of $C_{\frac{q+1}{k}}$ is given by;*

a) *when q is even*

$$Z(G) = \frac{1}{|G|} \left[t_1^{q(q-1)} + (q^2 - 1)t_p^{p^{(f-1)(q-1)}} + \frac{q(q+1)}{2} \sum_{1 \neq d | q-1} \phi(d)t_d^{\frac{q(q-1)}{d}} \right. \\ \left. + \frac{q(q-1)}{2} \sum_{1 \neq d | q+1} \phi(d)t_1^2 t_d^{\frac{(q-2)(q+1)}{d}} \right],$$

b) *when q is odd*

$$Z(G) = \frac{1}{|G|} \left[t_1^{q(q-1)} + (q^2 - 1)t_p^{p^{(f-1)(q-1)}} + \frac{q(q+1)}{2} \sum_{1 \neq d | \frac{q-1}{2}} \phi(d)t_d^{\frac{q(q-1)}{d}} \right. \\ \left. + \frac{q(q-1)}{2} \sum_{1 \neq d | \frac{q+1}{2}} \phi(d)t_1^2 t_d^{\frac{(q-2)(q+1)}{d}} \right].$$

Proof: Same arguments as used in Theorem 3.2.1 above apply. □

Examples

1. $\text{PSL}(2,3)$ acting on the cosets of $C_{\frac{q+1}{k}} = C_2$ gives;

$$Z(G) = \frac{1}{12} (t_1^6 + 8t_3^2 + 3t_1^2t_2^2).$$

2. $\text{PSL}(2,5)$ acting on the cosets of $C_{\frac{q+1}{k}} = C_3$ gives;

$$Z(G) = \frac{1}{60} (t_1^{20} + 24t_5^4 + 15t_2^{10} + 20t_1^2t_3^6).$$

3. $\text{PSL}(2,4)$ acting on the cosets of $C_{\frac{q+1}{k}} = C_5$ gives;

$$Z(G) = \frac{1}{60} (t_1^{12} + 24t_1^2t_5^2 + 15t_2^6 + 20t_3^4).$$

3.2.3 Permutation representations of G on the cosets of

$$H = P_q$$

3.2.3.1 Disjoint cycle structures of elements of G acting on cosets of H

Let $g \in G$. We consider the case only when $g \in \tau_1$ with $o(g) = d$ (where $o(g)$ denotes the order of g). Otherwise the results remain as per the Remark 3.2.1 above. H is an elementary abelian group whose elements are all of order p . Within G , $q - 1$ non-identity elements of H are all conjugate if $p = 2$, but

separate into two sets of $\frac{q-1}{2}$ conjugates if $p > 2$. (see Lemma 3.1.2) Therefore

$$|C^g \cap H| = \begin{cases} q-1; p=2 \text{ and } g \in \tau_1 \\ \frac{q-1}{2}; p \neq 2 \text{ and } g \in \tau_1 \\ 0; \text{ otherwise} \end{cases} .$$

The order of H is q , thus

$$[G : H] = \frac{(q+1)(q-1)}{k} = \begin{cases} (q-1)(q+1); p=2 \\ \frac{1}{2}(q-1)(q+1); p \neq 2 \end{cases} .$$

Using Remarks 3.2.1, Lemma 3.1.2, Lemma 1.6.2 and Theorem 1.6.4, the following table is constructed.

Table 3.5: Number of fixed points of g' when G acts on the cosets of P_q

	$ C^g $	$ C^g \cap H $	$\pi(g)$
$g \in \tau_0$	$\frac{1}{2}q(q-1); d=2$ $q(q-1); d \neq 2$	0	0
$g \in \tau_1$	$(q+1)(q-1); p=2$ $\frac{1}{2}(q+1)(q-1); p \neq 2$	$q-1$ $\frac{q-1}{2}$	$q-1$ $\frac{q-1}{2}$
$g \in \tau_2$	$\frac{1}{2}q(q+1); d=2$ $q(q+1); d \neq 2$	0	0

In view of Lemma 1.6.2 (b) and Table 3.5 above, we calculate in details the cycle lengths of the element g' corresponding to g in this representation.

I: $g \in \tau_0$

If $l < d$, then $\pi(g^l) = 0$ and hence $\alpha_l = 0$. If $l = d$ then,

$$\begin{aligned}
\alpha_d &= \frac{1}{d} \sum_{i|d} \pi(g^{d/i}) \mu(i) \\
&= \frac{1}{d} \pi(g^d) \mu(1) \\
&= \begin{cases} \frac{q^2-1}{d}; p = 2 \\ \frac{q^2-1}{2d}; p \neq 2 \end{cases}.
\end{aligned} \tag{3.11}$$

II: $g \in \tau_1$

If $l = 1$, then from Table 3.5, $\pi(g) = \frac{q-1}{k}$, hence

$$\begin{aligned}
\alpha_1 &= \pi(g) \mu(1) \\
&= \begin{cases} q-1; p = 2 \\ \frac{q-1}{2}; p \neq 2 \end{cases}.
\end{aligned} \tag{3.12}$$

If $1 < l < p$, then $\alpha_l = 0$ and if $l = p$, then

$$\begin{aligned}
\alpha_p &= \frac{1}{p} \sum_{i|p} \pi(g^{p/i}) \mu(i) \\
&= \frac{1}{p} \left[\pi(g^p) \mu(1) + \frac{q-1}{k} \sum_{i|p} \mu(i) - \frac{q-1}{k} \mu(1) \right] \\
&= \begin{cases} 2^{f-1}(q-1); p = 2 \\ \frac{p^{f-1}(q-1)}{2}; p \neq 2 \end{cases}.
\end{aligned} \tag{3.13}$$

III: $g \in \tau_2$

Since $\pi(g) = 0$ for $l < d$, then $\alpha_l = 0$ and if $l = d$, then

$$\alpha_d = \begin{cases} \frac{q^2-1}{2}; p = 2 \\ \frac{q^2-1}{2d}; p \neq 2 \end{cases}. \quad (3.14)$$

Summary of the results

Table 3.6: Disjoint cycle structures of g' when G acts on the cosets of P_q

	τ_0	τ_1	τ_2
Cycle length of g'	d	1	2
Number of cycles when $p = 2$	$\frac{q^2-1}{d}$	$q-1$	$2^{f-1}(q-1)$
Number of cycles when $p \neq 2$	$\frac{q^2-1}{2d}$	$\frac{(q-1)}{2}$	$\frac{q(q-1)}{2d}$

3.2.3.2 Cycle index of G acting on the cosets of H

In view of Theorem 3.1.2 and the results in Table 3.6 above, the cycle index of G acting on the cosets of P_q is as follows.

Theorem 3.2.3. *The cycle index of G acting on the cosets of its subgroup P_q is given by*

a) when $p = 2$

$$Z(G) = \frac{1}{|G|} \left[t_1^{q^2-1} + (q^2-1)t_1^{q-1}t_2^{2^{(f-1)(q-1)}} + \frac{q(q+1)}{2} \sum_{1 \neq d|q-1} \phi(d)t_d^{\frac{q^2-1}{d}} \right. \\ \left. + \frac{q(q-1)}{2} \sum_{1 \neq d|q+1} \phi(d)t_d^{\frac{q^2-1}{d}} \right],$$

b) when $p \neq 2$

$$Z(G) = \frac{1}{|G|} \left[t_1^{\frac{q^2-1}{2}} + (q^2-1)t_1^{\frac{q-1}{2}}t_p^{\frac{q(q-1)}{2d}} + \frac{q(q+1)}{2} \sum_{1 \neq d|\frac{q-1}{2}} \phi(d)t_d^{\frac{q^2-1}{2d}} \right]$$

$$\left. + \frac{q(q-1)}{2} \sum_{1 \neq d | \frac{q+1}{2}} \phi(d) t_d^{\frac{q^2-1}{2d}} \right].$$

Proof: Same arguments as used in Theorem 3.2.1 above apply. □

Examples

1. PSL(2,3) acting on the cosets of P_q give;

$$Z(G) = \frac{1}{12} (t_1^4 + 8t_1^1 t_3^1 + 3t_2^2).$$

2. PSL(2,4) acting on the cosets of P_q give;

$$Z(G) = \frac{1}{60} (t_1^{15} + 24t_5^3 + 15t_1^3 t_2^6 + 20t_3^5).$$

3. PSL(2,5) acting on the cosets of P_q give;

$$Z(G) = \frac{1}{60} (t_1^{12} + 24t_1^2 t_5^2 + 15t_2^6 + 20t_3^4).$$

3.2.4 Permutation representations of G on the cosets of

$$H = A_4$$

3.2.4.1 Disjoint cycle structures of elements of G acting on cosets of H

There are subgroups $H \cong A_4$ of G if and only if $p > 2$ or $p = 2$ and $f \equiv 0 \pmod{2}$ where $q = p^f$ (Dickson, 1901, Sec. 247). To discuss the disjoint structures in this case, we consider the following cases.

Case (a) $q \equiv 5 \pmod{12}$

Case (b) $q \equiv 7 \pmod{12}$

Case (c) $q \equiv 1 \pmod{12}$

Case (d) $q \equiv -1 \pmod{12}$

Case (e) q even i.e. $p = 2$

Case (f) $p = 3$ and f even

Case (g) $p = 3$ and f odd

From Jones A. Gareth on monodromy genus of $PSL(2, q)$, we obtain the following results on conjugacy class of elements of G and the size of the intersection of the conjugacy classes with the subgroups of G isomorphic to A_4 . G has a single conjugacy class C^g containing $\frac{|G|}{q-\delta}$ elements of order two, where:

$$\delta = \begin{cases} -1 & q \equiv -1 \pmod{4}; \\ 0 & q \equiv 0 \pmod{2}; \\ 1 & q \equiv 1 \pmod{4}. \end{cases}$$

Therefore,

$$|C^g| = \begin{cases} \frac{1}{2}q(q-1) & q \equiv -1 \pmod{4}; \\ q^2 - 1 & q \equiv 0 \pmod{2}; \\ \frac{1}{2}q(q+1) & q \equiv 1 \pmod{4}. \end{cases}$$

Elements of order three in G are distributed as follows. If $p = 3$ there are two conjugacy classes, each containing $\frac{|G|}{q}$ elements. If $p \neq 3$ there is a single conjugacy class containing $q(q + \varepsilon)$ elements, where:

$$\varepsilon = \begin{cases} -1 & q \equiv -1 \pmod{3}; \\ 1 & q \equiv 1 \pmod{3}. \end{cases}$$

Thus,

$$|C^g| = \begin{cases} q(q-1) & q \equiv -1 \pmod{3}; \\ q(q+1) & q \equiv 1 \pmod{3}. \end{cases}$$

These conjugacy classes intersect H as follows. Since there are three elements of order 2 in H , the single conjugacy class of elements of order 2 in G satisfy $|C^g \cap H| = 3$. The remaining eight elements of H are of order 3. If $p \neq 3$ then the single conjugacy class of elements of order 3 in G satisfy $|C^g \cap H| = 8$. If $p = 3$ and f is odd, then there are two mutually inverse conjugacy classes of elements of order 3 in G , each satisfying $|C^g \cap H| = 4$. If $p = 3$ and f is even, then the two conjugacy classes are self-inverse satisfying $|C^g \cap H| = 0$ or 8.

The order of H is 12, thus

$$[G : H] = \frac{(q+1)(q-1)}{12k} = \begin{cases} \frac{q}{12}(q^2-1) & p = 2; \\ \frac{q}{24}(q^2-1) & p \neq 2. \end{cases}$$

Table 3.7: Number of fixed points of g' when G acts on the cosets of $H \cong A_4$

$g \in \tau_i$	$ C^g $	$ C^g \cap H $	$\pi(g)$
$g \in \tau_0$ $\left\{ \begin{array}{l} \text{Cases (b), (d) and (g), } d = 2 \\ \text{Cases (a), (d), (e) and (f), } d = 3 \\ \text{Case (a) - (g), } d \neq 2, d \neq 3 \end{array} \right.$	$\frac{q}{2}(q-1)$ $q(q-1)$ $q(q-1)$	3 8 0	$\frac{1}{4}(q+1)$ $\frac{1}{3}(q+1)$ 0
$g \in \tau_1$ Cases (a) - (g)	$\frac{1}{2}(q^2-1)$	0	0
$g \in \tau_2$ $\left\{ \begin{array}{l} \text{Cases (b), (c) and (g) } d = 3 \\ \text{Cases (a), (c), (e) and (f) } d = 2 \\ \text{Cases (a) - (g), } d \neq 2, d \neq 3 \end{array} \right.$	$q(q+1)$ $\frac{q}{2}(q+1)$ $q(q+1)$	8 3 0	$\frac{1}{3}(q-1)$ $\frac{1}{4}(q-1)$ 0

Since (Kamuti, 1992, p. 28) had tackled cases (a) - (d) when A_4 is maximal, we simply quote the results after adding cases (e) - (g) when A_4 is not maximal.

Case (e) q even i.e. $p = 2$

I: $g \in \tau_0$

If $1 \leq l < d$, then from Table 3.7,

$$\pi(g) = \begin{cases} \frac{1}{4}(q-1) & \text{if } l = \frac{d}{2} \\ \frac{1}{3}(q+1) & \text{if } l = \frac{2d}{3} \text{ and } l = \frac{d}{3} \\ 0 & \text{if } l \neq \frac{d}{2}, l \neq \frac{2d}{3} \text{ and } l \neq \frac{d}{3} \end{cases}.$$

For $l \neq \frac{d}{2}, l \neq \frac{2d}{3}$ and $l \neq \frac{d}{3}$, $\pi(g^l) = 0$. Hence

$$\alpha_l = 0. \quad (3.15)$$

If $l = \frac{d}{2}$, then

$$\begin{aligned} \alpha_{\frac{d}{2}} &= \frac{2}{d}\pi(g^{d/2}) \\ &= \frac{1}{2d}(q+1). \end{aligned} \quad (3.16)$$

If $l = \frac{2d}{3}$, then

$$\begin{aligned} \alpha_{\frac{2d}{3}} &= \frac{3}{2d} [\pi(g^{2d/3}) - \pi(g^{d/3})] \\ &= 0. \end{aligned} \quad (3.17)$$

If $l = \frac{d}{3}$, then

$$\begin{aligned} \alpha_{\frac{d}{3}} &= \frac{3}{d}\pi(g^{d/3}) \\ &= \frac{1}{d}(q+1). \end{aligned} \quad (3.18)$$

Finally, if $l = d$, we have

$$\begin{aligned}\alpha_d &= \frac{1}{d} [\pi(g^d) - \pi(g^{d/2}) - \pi(g^{d/3})] \\ &= \frac{1}{24d} (q^3 - 15q - 14).\end{aligned}\tag{3.19}$$

II: $g \in \tau_1$

If $1 \leq l < d$, then from Table 3.7 above, we have

$$\pi(g) = \begin{cases} \frac{1}{3}q & \text{if } l = \frac{d}{3} \\ 0 & \text{otherwise} \end{cases}.$$

If $l = \frac{d}{3}$, then

$$\begin{aligned}\alpha_{\frac{d}{3}} &= \frac{3}{d} \pi(g^{d/3}) \\ &= \frac{1}{d} q.\end{aligned}\tag{3.20}$$

If $l \neq dd$ and $l \neq \frac{d}{3}$, then $\alpha_l = 0$. lastly, if $l = d$, we have

$$\begin{aligned}\alpha_d &= \frac{1}{d} [\pi(g^d) - \pi(g^{d/3})] \\ &= \frac{1}{12d} (q^3 - 5q).\end{aligned}\tag{3.21}$$

III: $g \in \tau_2$

If $1 \leq l < d$, then from Table 3.7,

$$\pi(g) = \begin{cases} \frac{1}{4}(q-1) & \text{if } l = \frac{d}{2} \\ \frac{1}{3}(q-1) & \text{if } l = \frac{2d}{3} \text{ and } l = \frac{d}{3} \\ \frac{1}{3}(q-1) & \text{if } l = \frac{d}{3} \\ 0 & \text{if } l \neq \frac{d}{2}, l \neq \frac{2d}{3} \text{ and } l \neq \frac{d}{3} \end{cases}.$$

For $l \neq \frac{d}{2}, l \neq \frac{2d}{3}$ and $l \neq \frac{d}{3}$, $\pi(g^l) = 0$. Hence $\alpha_l = 0$. If $l = \frac{d}{2}$, then

$$\begin{aligned} \alpha_{\frac{d}{2}} &= \frac{2}{d}\pi(g^{d/2}) \\ &= \frac{1}{2d}(q-1). \end{aligned} \tag{3.22}$$

If $l = \frac{2d}{3}$, then

$$\begin{aligned} \alpha_{\frac{2d}{3}} &= \frac{3}{2d} [\pi(g^{2d/3}) - \pi(g^{d/3})] \\ &= 0. \end{aligned} \tag{3.23}$$

If $l = \frac{d}{3}$, then

$$\begin{aligned} \alpha_{\frac{d}{3}} &= \frac{3}{d}\pi(g^{d/3}) \\ &= \frac{1}{d}(q-1). \end{aligned} \tag{3.24}$$

Finally, if $l = d$, we have

$$\begin{aligned} \alpha_d &= \frac{1}{d} [\pi(g^d) - \pi(g^{d/2}) - \pi(g^{d/3})] \\ &= \frac{1}{24d}(q^3 - 15q + 14). \end{aligned} \tag{3.25}$$

Remark 3.2.2. *Similar arguments as used in case (e) are applicable to cases (f) and (g).*

Using Lemma 1.6.2 (b) and Table 3.7 above, we calculate in details the cycle lengths of the element g' corresponding to g in this permutation representation.

From Table 3.7, Kamuti (1992) obtained the following disjoint structures.

Table 3.8: Disjoint cycle structures of g' when G acts on the cosets of $H \cong A_4$

Cycle length of g'		τ_0		
		$\frac{d}{2}$	$\frac{d}{3}$	d
Number of cycles for all cases (a) – (g)	with $\left\{ \begin{array}{l} 2 d, 3 d \\ 2 d, 3 \nmid d \\ 2 \nmid d, 3 d \\ 2 \nmid d, 3 \nmid d \end{array} \right.$	$\frac{1}{2d}(q+1)$	$\frac{1}{d}(q+1)$	$\frac{1}{24d}(q^3 - 15q - 14)$
		$\frac{1}{2d}(q+1)$	0	$\frac{1}{24d}(q^3 - 7q - 6)$
		0	$\frac{1}{d}(q+1)$	$\frac{1}{24d}(q^3 - 9q - 8)$
		0	0	$\frac{1}{24d}(q^3 - q)$

τ_1		τ_2		
$\frac{d}{3}$	d	$\frac{d}{2}$	$\frac{d}{3}$	$d = p$
0	$\frac{1}{24d}(q^3 - q)$	$\frac{1}{2d}(q-1)$	$\frac{1}{d}(q-1)$	$\frac{1}{24d}(q^3 - 15q + 14)$
0	$\frac{1}{24d}(q^3 - q)$	$\frac{1}{2d}(q-1)$	0	$\frac{1}{24d}(q^3 - 7q + 6)$
$\frac{q}{d}$	$\frac{1}{12d}(q^3 - 5q)$	0	$\frac{1}{d}(q-1)$	$\frac{1}{24d}(q^3 - 9q + 8)$
0	$\frac{1}{24d}(q^3 - q)$	0	0	$\frac{1}{24d}(q^3 - q)$

3.2.4.2 Cycle index of G acting on the cosets of H

Kamuti (1992, p. 70) gave cycle index of G acting on the cosets of A_4 with $q \equiv 1 \pmod{12}$ as follows;

$$Z(G) = \frac{1}{|G|} \left[t_1^{\frac{1}{24}(q^3-q)} + (q^2 - 1)t_p^{\frac{1}{24}(q^2-1)} + \frac{q(q+1)}{2} \sum_{\substack{2|d, q-1 \\ 3 \nmid d}} \phi(d)t_{\frac{d}{2}}^{\frac{1}{d}(q-1)}t_d^{\frac{1}{24d}(q^3-7q+6)} \right. \\ \left. + \frac{q(q+1)}{2} \sum_{\substack{3|d, q-1 \\ 2 \nmid d}} \phi(d)t_{\frac{d}{3}}^{\frac{1}{d}(q-1)}t_d^{\frac{1}{24d}(q^3-9q+8)} \right]$$

$$\begin{aligned}
& + \frac{q(q+1)}{2} \sum_{\substack{3|d| \frac{q-1}{2} \\ 2|d| \frac{q-1}{2}}} \phi(d) t_d^{\frac{1}{2}(q-1)} t_d^{\frac{1}{3}(q-1)} t_d^{\frac{1}{24d}(q^3-15q+14)} \\
& \left. + \frac{q(q+1)}{2} \sum_{\substack{2|d| \frac{q-1}{2} \\ 3|d| \frac{q-1}{2}}} \phi(d) t_d^{\frac{1}{24d}(q^3-q)} + \frac{q(q-1)}{2} \sum_{1 \neq d | \frac{q+1}{2}} \phi(d) t_d^{\frac{1}{24d}(q^3-q)} \right]
\end{aligned}$$

3.2.5 Permutation representations of G on the cosets of

$$H = D_{\frac{2(q-1)}{k}}$$

3.2.5.1 Disjoint cycle structures of elements of G acting on cosets of H

The group $G = PSL(2, q)$ contains subgroups isomorphic to dihedral group of order $\frac{2(q-1)}{k}$. Thus, the number n of cosets of H in G is

$$[G : H] = \frac{q(q+1)}{2}.$$

Lemma 3.2.1. *If $C_{\frac{q-1}{k}}$ is the maximal cyclic subgroup of H , then the $\frac{q-1}{k}$ involutions in $H \setminus C_{\frac{q-1}{k}}$ are all conjugate in H if $q \equiv -1 \pmod{4}$ or $p = 2$. If $q \equiv 1 \pmod{4}$, then involutions lie in two conjugacy classes of $\frac{q-1}{4}$ elements each. (Dickson, 1901, Sec. 246)*

Involutions in G form a single conjugacy class. Thus:

$$|C^g| = \begin{cases} \frac{q}{2}(q+1) & q \equiv 1 \pmod{4}; \\ q^2 - 1 & p = 2; \\ \frac{q}{2}(q-1) & q \equiv -1 \pmod{4}. \end{cases}$$

It follows that, if d (order of g) is 2, then:

$$|C^g \cap H| = \begin{cases} \frac{q+1}{2} & q \equiv 1 \pmod{4}; \\ q-1 & p=2; \\ \frac{q-1}{2} & q \equiv -1 \pmod{4}. \end{cases}$$

From Section 1.1 and Lemma 3.2.1 above, if $d > 2$, then:

$$|C^g \cap H| = \begin{cases} 2 & q \equiv 1 \pmod{4}; \\ 0 & \text{otherwise.} \end{cases}$$

Involutions in H lie in one of the τ_i ($i = 0, 1, 2$) giving us the following cases to consider.

- a) $p = 2$
- b) $q \equiv 1 \pmod{4}$
- c) $q \equiv -1 \pmod{4}$

In cases (b) and (c), it is worthy noting when $d = 2$ or $d \neq 2$. In consideration of these cases, we calculate the fixed points by each element $g \in G$ using Theorem 1.6.4, whose values are as displayed in the table below.

Next, we calculate in details the cycle lengths of g' corresponding to g in this representation. From Table 3.9 and Lemma 1.6.2 (b), we have;

I $g \in \tau_0$

Case a): $\pi(g^l) = 0$ for $1 \leq l < d$ and therefore

$$\alpha_l = 0. \tag{3.26}$$

Table 3.9: Number of fixed points of g' when G acts on the cosets of $D_{\frac{2(q-1)}{k}}$

	$g \in \tau_i$	$ C^g $	$ C^g \cap H $	$\pi(g)$
Case (a)	τ_0	$q(q-1)$	0	0
	τ_1	q^2-1	$q-1$	$\frac{q}{2}$
	τ_2	$q(q+1)$	2	1
Cases (b) and (c)	$\tau_0; d=2$	$\frac{q(q-1)}{2}$	$\frac{q-1}{2}$	$\frac{q+1}{2}$
	$\tau_0; d>2$	$q(q-1)$	0	0
	τ_1	$\frac{q^2-1}{2}$	0	0
	$\tau_2; d=2$	$\frac{q(q+1)}{2}$	$\frac{q+1}{2}$	$\frac{q+1}{2}$
	$\tau_2; d>2$	$q(q+1)$	2	1

If $l = d$, then

$$\begin{aligned} \alpha_d &= \frac{1}{d}\pi(g^d) \\ &= \frac{q(q+1)}{2d}. \end{aligned} \tag{3.27}$$

Case b): For $1 \leq l < d$

$$\pi(g^l) = \begin{cases} \frac{q+1}{2} & \text{if } l = \frac{d}{2} \\ 0 & \text{otherwise} \end{cases}.$$

Therefore if $1 \leq l < d$ and $l \neq \frac{d}{2}$, then $\alpha_l = 0$. If $l = \frac{d}{2}$, we have

$$\begin{aligned} \alpha_{\frac{d}{2}} &= \frac{2}{d}\pi(g^{\frac{d}{2}}) \\ &= \frac{q+1}{d}. \end{aligned} \tag{3.28}$$

If $l = d$, then

$$\begin{aligned}\alpha_d &= \frac{1}{d} \left[\pi(g^d) - \pi(g^{\frac{d}{2}}) \right] \\ &= \frac{q^2 - 1}{2d}.\end{aligned}\tag{3.29}$$

Case c): $\pi(g^l) = 0$ for $1 \leq l < d$ and therefore $\alpha_l = 0$

If $l = d$, then

$$\begin{aligned}\alpha_d &= \frac{1}{d} \pi(g^d) \\ &= \frac{q(q+1)}{2d}.\end{aligned}\tag{3.30}$$

II $g \in \tau_1$

Case a): $\pi(g^l) = 2^{f-1}$ for $1 \leq l < d$. Therefore $\alpha_l = 0$ for $1 < l < d$. If $l = 1$, then

$$\alpha_1 = 2^{f-1}.\tag{3.31}$$

If $l = d$, then

$$\begin{aligned}\alpha_d &= \frac{1}{d} \left[\pi(g^d) \mu(1) + \sum_{1 \neq i|d} \pi(g^{\frac{d}{i}}) \right] \\ &= \frac{1}{d} \left[\frac{q(q+1)}{2} + \frac{q}{2} \sum_{i|d} \mu(i) - \frac{q}{2} \mu(1) \right] \\ &= \frac{q^2}{2d}.\end{aligned}\tag{3.32}$$

Case b) and c): $\pi(g^l) = 0$ for $1 \leq l < d$. Therefore $\alpha_l = 0$. If $l = d$, then

$$\begin{aligned}\alpha_d &= \frac{1}{d}\pi(g^d) \\ &= \frac{q(q+1)}{2d}.\end{aligned}\tag{3.33}$$

III $g \in \tau_2$

Case a): $\pi(g^l) = 1$ for $1 \leq l < d$. Therefore, if $l = 1$ then

$$\alpha_l = 1.\tag{3.34}$$

and if $1 < l < d$, then $\alpha_l = 0$. If $l = d$, then

$$\begin{aligned}\alpha_d &= \frac{1}{d} \left[\pi(g^d)\mu(1) + \sum_{i|d} \mu(1) - \mu(1) \right] \\ &= \frac{(q-1)(q+2)}{2d}.\end{aligned}\tag{3.35}$$

Case b): For $1 \leq l < d$ we have

$$\pi(g^l) = \begin{cases} \frac{q-1}{2} & \text{for } l = \frac{d}{2} \\ 1 & \text{otherwise} \end{cases}.$$

Therefore, if $l = 1$ we have,

$$\alpha_l = 1.\tag{3.36}$$

If $l = \frac{d}{2}$, then

$$\alpha_{\frac{d}{2}} = \frac{q-1}{d}.\tag{3.37}$$

If $1 < l < d$ and $l \neq \frac{d}{2}$, then $\alpha_l = 0$. If $l = d$, we have

$$\begin{aligned}\alpha_d &= \frac{1}{d} \left[\pi(g^d) - \pi(g^{\frac{d}{2}}) \right] \\ &= \frac{q^2 - 1}{2d}.\end{aligned}\tag{3.38}$$

Case c): $\pi(g^l) = 1$ for $1 \leq l < d$. Therefore, if $l = 1$ we have,

$$\alpha_l = 1.\tag{3.39}$$

If $1 < l < d$, then $\alpha_l = 0$. If $l = d$, we have

$$\begin{aligned}\alpha_d &= \frac{1}{d} \left[\pi(g^d)\mu(1) + \sum_{i|d} \mu(1) - \mu(1) \right] \\ &= \frac{(q-1)(q+2)}{2d}.\end{aligned}\tag{3.40}$$

Table 3.10: Disjoint cycle structures of g' when G acts on the cosets of $D_{\frac{2(q-1)}{k}}$

	τ_0	τ_1	τ_2
Cycle length of g'	$\frac{d}{2} \quad d$	$1 \quad p$	$1 \quad \frac{d}{2} \quad d$
case (a)	$0 \quad \frac{q(q+1)}{2d}$	$2^{f-1} \quad \frac{q^2}{2d}$	$1 \quad 0 \quad \frac{(q-1)(q+2)}{2p}$
Cases (b) and (c) with $\begin{cases} 2 \mid d \\ 2 \nmid d \end{cases}$	$\frac{q+1}{2} \quad \frac{q^2-1}{4}$ $0 \quad \frac{q(q+1)}{2d}$	$0 \quad \frac{q(q+1)}{4}$ $0 \quad \frac{q(q+1)}{2p}$	$1 \quad \frac{q-1}{2} \quad \frac{q^2-1}{4}$ $1 \quad 0 \quad \frac{(q-1)q+2}{2d}$

3.2.5.2 Cycle index of G acting on the cosets of H

In view of Theorem 3.1.2 and the results above, the cycle index of G on the cosets of $D_{\frac{2(q-1)}{k}}$ is as follows.

Theorem 3.2.4. *The cycle index of G acting on the cosets of $D_{\frac{2(q-1)}{k}}$ is given by*

a) when q is even

$$Z(G) = \frac{1}{|G|} \left[t_1^{\frac{q}{2}(q-1)} + (q^2 - 1)t_1^{p(f-1)} t_2^{p^2(f-1)} + \frac{q(q+1)}{2} \sum_{1 \neq d|q-1} \phi(d) t_1 t_d^{\frac{(q+2)(q-1)}{2d}} \right. \\ \left. + \frac{q(q-1)}{2} \sum_{1 \neq d|q+1} \phi(d) t_d^{\frac{q(q+1)}{2d}} \right],$$

b) when q is odd

$$Z(G) = \frac{1}{|G|} \left[t_1^{\frac{q}{2}(q+1)} + (q^2 - 1)t_p^{\frac{q(q+1)}{2}} + \frac{q(q+1)}{2} \sum_{1 \neq d|\frac{q-1}{2}} \phi(d) t_1 t_d^{\frac{(q-1)(q+2)}{2d}} \right. \\ \left. + \frac{q(q-1)}{2} \sum_{1 \neq d|\frac{q+1}{2}} \phi(d) t_d^{\frac{q(q+1)}{2d}} \right].$$

Proof: Same arguments as used in Theorem 3.2.1 above apply. \square

Examples

1. $\text{PSL}(2,5)$ acting on the cosets of D_4 gives;

$$Z(G) = \frac{1}{60} (t_1^{15} + 15t_1 t_2^7 + 20t_3^5 + 24t_5^3).$$

2. $\text{PSL}(2,7)$ acting on the cosets of D_6 gives;

$$Z(G) = \frac{1}{168} (t_1^{28} + 56t_1 t_3^9 + 21t_2^{14} + 42t_4^7 + 48t_7^4).$$

3. $\text{PSL}(2,4)$ acting on the cosets of D_6 gives;

$$Z(G) = \frac{1}{60} (t_1^{10} + 24t_5^2 + 15t_1^2 t_2^4 + 20t_1 t_3^3).$$

CHAPTER 4

RANKS AND SUBDEGREES OF $PSL(2, Q)$

ACTING ON COSETS OF SOME OF ITS

SUBGROUPS

Using disjoint cycle structures obtained in Chapter 3, the subgroup structure described in Chapter 1 and Lemma 1.6.1, we determine the rank of $PSL(2, q)$ for each permutation representation in this chapter.

The chapter is given in two sections. In Section 4.1, we use algebraic arguments to determine the rank of $PSL(2, q)$ acting on the cosets of $C_{\frac{q-1}{k}}$ and $C_{\frac{q+1}{k}}$. Using Definition 1.1.10 and the ranks calculated in this section, we obtain the corresponding subdegrees of $PSL(2, q)$. In Section 4.2 using table of marks of the respective subgroups, we calculate the ranks of $PSL(2, q)$ and the corresponding subdegrees for all the five subgroups.

4.1 Determination of ranks and subdegrees of $G = PSL(2, q)$ acting on the cosets of some of its subgroups using algebraic arguments

4.1.1 Rank and subdegrees of G acting on the cosets of

$$H = C_{\frac{q-1}{k}}$$

Since H is cyclic, then its generator u is of order $\frac{q-1}{k}$. Therefore, from Table 3.2, the cycle structure of u' is as follows;

a) when $p = 2$, u' has 2 trivial cycles and $q + 2$ cycles of length $q - 1$. Since the rank is the sum of all suborbits, then it is

$$\begin{aligned} r &= 2 + (q + 2) \\ &= q + 4. \end{aligned}$$

b) when $p \neq 2$, u' has 2 trivial cycles and $2(q + 2)$ cycles of length $\frac{q-1}{2}$. The rank in this case is

$$r = 2(q + 3).$$

Therefore the subdegrees of G acting on the cosets of H are as given in the table below.

Table 4.1: Subdegrees of G acting on the cosets of $C_{\frac{q-1}{k}}$

	$p = 2$		$p \neq 2$	
Suborbit length	1	$q - 1$	1	$\frac{q-1}{2}$
Number of suborbits	2	$q + 2$	2	$2(q + 2)$

4.1.2 Rank and subdegrees of G acting on the cosets of

$$H = C_{\frac{q+1}{k}}$$

Let $h \in G$ be a generator of H , then the order of h is $\frac{q+1}{k}$. Therefore, from Table 3.4, the cycle structure of h' is as follows;

a) when $p = 2$, h' has 2 trivial cycles and $q - 2$ cycles of length $q + 1$. In this case the rank is

$$r = q.$$

b) when $p \neq 2$, h' has 2 trivial cycles and $2(q - 2)$ cycles of length $\frac{q+1}{2}$. The rank in this case is

$$r = 2(q - 1).$$

Hence, the subdegrees of G acting on the cosets of H are as given in the table below.

Table 4.2: Subdegrees of G acting on the cosets of $C_{\frac{q+1}{k}}$

	$p = 2$		$p \neq 2$	
Suborbit length	1	$q + 1$	1	$\frac{q+1}{2}$
Number of suborbits	2	$q - 2$	2	$2(q - 2)$

4.1.3 Rank and subdegrees of G acting on the cosets of

$$H = P_q$$

From the results given in Table 3.6, the elements of H have fixed points (cosets) as follows.

Case I: When $p = 2$

The identity fixes all $q^2 - 1$ cosets. The remaining $q - 1$ elements fix $q - 1$ cosets each. Thus from Lemma 1.6.1

$$\begin{aligned} r &= \frac{1}{q} [q^2 - 1 + (q - 1) \times (q - 1)] \\ &= 2(q - 1). \end{aligned}$$

Case II: When $p \neq 2$

$$\begin{aligned} r &= \frac{1}{q} \left[\frac{q^2 - 1}{2} + (q - 1) \times \frac{q - 1}{2} \right] \\ &= q - 1. \end{aligned}$$

4.1.4 Rank of G acting on the cosets of $H = A_4$

From 3.2.4 above, since cases (a) - (d) were computed by Kamuti (1992, p. 89), we simply quote the results and add cases (e) - (g).

a.) When $q \equiv 5 \pmod{12}$

$$r = \frac{q^3 + 81q + 46}{288}.$$

b.) When $q \equiv 7 \pmod{12}$

$$r = \frac{q^3 + 81q - 46}{288}.$$

c.) When $q \equiv 1 \pmod{12}$

$$r = \frac{q^3 + 81q - 82}{288}.$$

d.) When $q \equiv -1 \pmod{12}$

$$r = \frac{q^3 + 81q + 82}{288}.$$

e.) When $p = 2$

In this case, there are 8 elements of order 3 each fixing $\frac{2(q-1)}{3}$ cosets and 3 elements of order 2 fixing $\frac{q}{4}$ cosets each. Therefore

$$\begin{aligned} r &= \frac{1}{12} \left[\frac{q^3 - q}{12} + 8 \left(\frac{2(q-1)}{3} \right) + \frac{3q}{4} \right] \\ &= \frac{q^3 + 72q - 64}{144}. \end{aligned}$$

f.) When $p = 3$ and f even

We have 3 elements of order 2 each fixing $\frac{q-1}{4}$ cosets and 8 elements of order 3 each fixing $\frac{2q}{3}$ cosets. Since identity fixes every coset, therefore

$$\begin{aligned} r &= \frac{1}{12} \left[\frac{q^3 - q}{24} + 8 \left(\frac{2q}{3} \right) + 3 \left(\frac{q-1}{4} \right) \right] \\ &= \frac{q^3 + 145q - 18}{288}. \end{aligned}$$

g.) When $p = 3$ and f odd

There are 8 elements of order 3 each fixing $\frac{q}{3}$ cosets and 3 elements of order 2 each fixing $\frac{q+1}{4}$ cosets. Since identity fixes every coset, the rank becomes

$$\begin{aligned} r &= \frac{1}{12} \left[\frac{q^3 - q}{24} + 8 \left(\frac{q}{3} \right) + 3 \left(\frac{q+1}{4} \right) \right] \\ &= \frac{q^3 + 81q + 18}{288}. \end{aligned}$$

4.1.5 Rank and subdegrees of G acting on the cosets of

$$H = D_{\frac{2(q-1)}{k}}$$

Again, these rank and subdegrees were determined by Kamuti (1992, p. 81) which were as follows.

Case (a): When $p = 2$

$$r = \frac{q+2}{2}.$$

The subdegrees were found to be;

Table 4.3: Subdegrees of G acting on the cosets of $D_{2(q-1)}$

Suborbit length	1	$q-1$	$2(q-1)$
Number of suborbits	1	$\frac{1}{2}(q-1)$	1

Case (b): When $q \equiv -1 \pmod{4}$

$$r = \frac{3q+7}{4}. \tag{4.1}$$

Thus, the subdegrees were found to be;

Table 4.4: Subdegrees of G acting on the cosets of $D_{(q-1)}$

Suborbit length	1	$\frac{1}{2}(q-1)$	$q-1$
Number of suborbits	1	$\frac{1}{2}(q-1)$	$\frac{1}{4}(q+5)$

Case (c): When $q \equiv 1 \pmod{4}$

$$r = \frac{3(q+3)}{4}.$$

In this case, the subdegrees were found to be;

Table 4.5: Subdegrees of G acting on the cosets of $D_{(q-1)}$

Suborbit length	1	$\frac{1}{4}(q-1)$	$\frac{1}{2}(q-1)$	$q-1$
Number of suborbits	1	2	$\frac{1}{2}(q-5)$	$\frac{1}{4}(q+7)$

4.2 Determination of subdegrees of $PSL(2, q)$ acting on the cosets of some of its subgroups using table of marks

4.2.1 Background information

Rank and subdegrees of a permutation group can also be obtained using table of marks, a method proposed by Ivanov et al. (1983). Any permutation representation of a finite group G is produced when the group acts on a finite set $X = \{x_1, x_2, \dots, x_n\}$. The permutation representation P_G is the set of permutations P_g on X , each of which is associated with an element $g \in G$ so

that P_G and G are homomorphic,

$$i.e. \quad P_g P_{g'} = P_{gg'}$$

for every $g, g' \in G$.

Let H be a subgroup of index n in G . The set of all left cosets of H in G partitions G .

$$i.e. \quad G = g_1H \cup g_2H \cup \cdots \cup g_nH$$

where $g_1 = I$, the identity in G and $g_i \in G, (i = 1, 2, \dots, n)$.

Consider the set of left cosets $\{g_1H, g_2H, \dots, g_nH\}$. For any $g \in G$, the set of permutations of degree n

$$G(/H)_{(g)} = \begin{pmatrix} g_1H & g_2H & - & - & - & g_nH \\ gg_1H & gg_2H & - & - & - & gg_nH \end{pmatrix}$$

constructs a permutation representation of G , which is sometimes called the coset representation of G by H .

Theorem 4.2.1. *Suppose that the number of subgroups in a finite group G is s (where a set of conjugacy class is counted once). If we arrange the complete set of these subgroups $G_i, (i = 1, 2, \dots, s)$ in an ascending order of their orders*

$$|G_1| \leq |G_2| \leq \cdots \leq |G_s|$$

where G_1 is the identity and $G_s = G$, then the set of corresponding coset representations; $G(/G_i)(i = 1, 2, \dots, s)$ is the complete set of different transitive permutation representations of G . (Burnside, 1911, p. 236)

Theorem 4.2.2. *Any permutation representation P_G of a finite group G acting on X can be reduced into transitive coset representation with the following equation.*

$$P_G = \sum \alpha_i G(/G_i) (i = 1, 2, \dots, s)$$

where the multiplicity α_i is a non-negative integer. (Burnside, 1911, p. 238)

Definition 4.2.1. *Let P_G be a permutation representation (transitive or intransitive) of G on X . The mark of the subgroup H of G in P_G is the number of points of X fixed by every permutation of H . (Burnside, 1911, p. 236)*

In case $G(/G_i)$ is a coset representation, the mark of G_j in $G(/G_i)$ denoted by $m(G_j, G_i, G)$ is the number of cosets of G_i in G left fixed by every permutation of G_j .

Definition 4.2.2. (White, 1975)

$$m(G_j, G_i, G) = \frac{1}{|G|} \sum_{g \in G} \chi(g^{-1}G_jg \subseteq G_i)$$

where

$$\chi(\text{statement}) = \begin{cases} 1 & \text{if the statement is true;} \\ 0 & \text{otherwise.} \end{cases}$$

Definition 4.2.3. (Ivanov et al., 1983) *If $G_j \leq G_i \leq G$ and $\{G_{j_1}, G_{j_2}, \dots, G_{j_n}\}$ is a complete set of conjugacy class representatives of subgroups of G_i that are conjugate to G_j in G , then*

$$m(G_j, G_i, G) = \sum_{k=1}^n \left[N_G(G_{j_k}) : N_{G_i}(G_{j_k}) \right].$$

In particular if $n = 1$, G_j is conjugate in G_i to all subgroups G_{j_i} that are contained in G_i and conjugate to G_j in G . Therefore

$$m(G_j, G_i, G) = [N_G(G_j) : N_{G_i}(G_j)].$$

Remark 4.2.1. *These definitions are all equivalent. (Kamuti, 1992, p. 77)*

Definition 4.2.4. *The table of marks of a group G is the matrix $M(G)$, with m_{ij} -entry equal to $m(G_j, G_i, G)$, the mark of the subgroup G_j in the coset representation $G(/G_i)$.*

Table 4.6: Table of marks of a group G

	G_1	G_2	\cdots	G_s
$G(/G_1)$	m_{11}	m_{12}	\cdots	m_{1s}
$G(/G_2)$	m_{21}	m_{22}	\cdots	m_{2s}
\vdots	\vdots	\vdots		
$G(/G_s)$	m_{s1}	m_{s2}	\cdots	m_{ss}

Note:

- i) $m_{ij} = 0$ unless G_j is conjugate to a subgroup of G_i and $m_{ii} \geq 1$ for any i . Since representations are listed in order of increasing size (order), the table of marks is a lower triangular invertible matrix.
- ii) The multiplicities α_i are obtained by using the table of marks as

$$\mu_j = \sum_{i=1}^s \alpha_i M_{ij}, (j = 1, 2, \cdots, s),$$

where μ_j is the mark of G_j in P_G . If $\mu = (\mu_1, \mu_2, \cdots, \mu_s)$ is a vector with μ_j as components, the mark of G_j in the permutation representation P_G

of G on X , $\alpha = (\alpha_1, \alpha_2, \dots, \alpha_s)$ is a vector with α_i as components and $M(G)$ is the table of marks of G , then

$$\mu = \alpha M(G). \quad (4.2)$$

In this section, we determine subdegrees of $G = PSL(2, q)$ acting on the cosets of its subgroup H where H is $C_{\frac{q-1}{k}}$, $C_{\frac{q+1}{k}}$, P_q , A_4 and $D_{\frac{2(q-1)}{k}}$ using their table marks.

4.2.2 Ranks and subdegrees of $G = PSL(2, q)$ acting on the cosets of its subgroups

a.) **Rank and subdegrees of G on the cosets of $H = C_{\frac{q-1}{k}}$**

The subgroups of H are of the form C_m where $m | \frac{q-1}{k}$. Since H is abelian, each of its subgroups is normal. Suppose H has r classes of conjugacy subgroups, say

$$C_{1'} = I, C_{2'}, \dots, C_{r'} = H$$

with $i' | \frac{q-1}{k}$ and $i' \leq (i+1)'$, $(i = 1, 2, \dots, r-1)$. Then the table of marks of H is as shown in the table below.

Table 4.7: Table of marks of $H = C_{\frac{q-1}{k}}$

	$C_{1'} = I$	$C_{2'}$	\dots	$C_{(r-1)'}$	$C_{r'}$
$H(/C_{1'})$	m_{11}				
$H(/C_{2'})$	m_{21}	m_{22}			
\vdots	\vdots	\vdots			
$H(/C_{(r-1)'})$	m_{r-11}	m_{r-12}	\dots	m_{r-1r-1}	
$H(/C_{r'})$	1	1	\dots	1	1

For $C_{i'} \leq H$, the values of $m(C_{i'}, H, G) = m(C_{i'})$ are as follows:
when $i = 1$, we have:

$$\begin{aligned}
 m(C_{1'}) &= m(C_{1'}, H, G) \\
 &= [N_G(C_{1'}) : N_H(C_{1'})] \\
 &= \frac{q(q^2 - 1)}{k} \div \frac{q - 1}{k} \\
 &= q(q + 1).
 \end{aligned}$$

Since H is abelian, we have;

$$|N_H(C_{i'})| = \frac{q - 1}{k}$$

for every i .

When $i \neq 1$, since $C_{i'}$ s are cyclic of order i' such that $i' | \frac{q-1}{k}$, their normalizers in G is the dihedral group of order $\frac{2(q-1)}{k}$. (Dickson, 1901, Sec. 242) Therefore;

$$|N_G(C_{i'})| = \frac{2(q - 1)}{k}$$

for every i . Consequently

$$\begin{aligned}
 m(C_{i'}) &= \frac{2(q - 1)}{k} \div \frac{q - 1}{k} \\
 &= 2, \quad 1 < i \leq r.
 \end{aligned}$$

Hence $\mu = (q(q+1), 2, 2, \dots, 2)$ an r -tuple. Let $\alpha = (\alpha_1, \alpha_2, \dots, \alpha_r)$, then by Equation 4.2 we obtain the following system of linear equations

$$\begin{aligned}\alpha_1 m_{11} + \alpha_2 m_{21} + \dots + \alpha_{r-1} m_{r-11} + \alpha_r &= q(q+1) \\ \alpha_2 m_{22} + \dots + \alpha_{r-1} m_{r-12} + \alpha_r &= 2 \\ &\vdots \\ \alpha_{r-1} m_{r-1r-1} + \alpha_r &= 2 \\ \alpha_r &= 2.\end{aligned}$$

Since $m_{ii} \neq 0$ for all i , $\alpha_j \geq 0$, $1 \leq j \leq r$ and $\alpha_j = 0$, $2 \leq j \leq r-1$, the solution to the above system of linear equations is as given below.

$$\begin{aligned}\alpha_1 m_{11} + \alpha_r &= q(q+1) \\ \alpha_r &= 2 \\ \Rightarrow \alpha_1 &= \frac{(q-1)(q+2)}{m_{11}}.\end{aligned}$$

But

$$\begin{aligned}m_{11} &= [N_H(C_{1'}) : N_{C_{1'}}(C_{1'})] \\ &= \frac{q-1}{k}.\end{aligned}$$

Therefore

$$\begin{aligned}\alpha_1 &= (q-1)(q+2) \div \frac{q-1}{k} \\ &= k(q+2).\end{aligned}$$

for $k = (2, q - 1)$. Thus $\alpha = (k(q + 2), 0, 0, \dots, 2)$. From Theorem 4.2.2, it follows that

$$P_G = k(q + 2)G(/C_{1'}) + 2G(/H). \quad (4.3)$$

Hence, by Theorem 1.6.2, the action of G on the cosets of H yields $k(q + 2) + 2$ orbits, i.e the rank r is

$$r = k(q + 2) + 2 = \begin{cases} 2(q + 3) & \text{if } p \neq 2; \\ q + 4 & \text{if } p = 2. \end{cases}$$

The subdegrees of G in this case are $k(q + 2)$ orbits of length $\frac{q-1}{k}$ with C_1 as the stabilizer and two trivial orbits with $C_r = H$ as the stabilizer.

b.) **Rank and subdegrees of G on the cosets of $H = C_{\frac{q+1}{k}}$**

Since H is cyclic, similar arguments as used above apply. The table of marks of H is as follows.

Table 4.8: Table of marks of $H = C_{\frac{q+1}{k}}$

	$C_{1'} = I$	$C_{2'}$	\dots	$C_{(r-1)'}$	$C_{r'}$
$H(/C_{1'})$	m_{11}				
$H(/C_{2'})$	m_{21}	m_{21}			
\vdots	\vdots	\vdots			
$H(/C_{(r-1)'})$	m_{r-11}	m_{r-12}	\dots	m_{r-1r-1}	
$H(/C_{r'})$	1	1	\dots	1	1

For $C_i \leq H$, ($i = 1, 2, \dots, r$), the values of $m(C_i)$ are obtained as follows.

$$\begin{aligned}
 m(C_1) &= m(C_1, H, G) \\
 &= [N_G(C_{1'}) : N_H(C_{1'})] \\
 &= \frac{q(q^2 - 1)}{k} \div \frac{q + 1}{k} \\
 &= q(q - 1).
 \end{aligned}$$

As noted above

$$|N_H(C_{i'})| = \frac{q + 1}{k}$$

for every i . From background information on H , the normalizer of subgroup of H in G is the dihedral group of order $\frac{2(q+1)}{k}$. Therefore

$$|N_G(C_{i'})| = \frac{2(q + 1)}{k}$$

for every $i \neq 1$. Consequently, for $i \neq 1$

$$\begin{aligned}
 m(C_{i'}) &= m(C_{i'}, H, G) \\
 &= [N_G(C_{i'}) : N_H(C_{i'})] \\
 &= \frac{2(q + 1)}{k} \div \frac{q + 1}{k} \\
 &= 2.
 \end{aligned}$$

Hence $\mu = (q(q - 1), 2, 2, \dots, 2)$ an r -tuple. Let $\alpha = (\alpha_1, \alpha_2, \dots, \alpha_r)$, then by Equation 4.2, we obtain the following system of linear equations.

$$\begin{aligned}
\alpha_1 m_{11} + \alpha_2 m_{21} + \cdots + \alpha_{r-1} m_{r-11} + \alpha_r &= q(q-1) \\
\alpha_2 m_{22} + \cdots + \alpha_{r-1} m_{r-12} + \alpha_r &= 2 \\
&\vdots \\
\alpha_{r-1} m_{r-1r-1} + \alpha_r &= 2 \\
\alpha_r &= 2.
\end{aligned}$$

Since $m_{ii} \neq 0$ for all i , $\alpha_j \geq 0$, $1 \leq j \leq r$ the solution to the above system of linear equations is as follows.

$$\begin{aligned}
m_{11} &= [N_G(C_{1'}) : N_H(C_{1'})] \\
&= \frac{q+1}{k} \\
\alpha_r &= 2 \\
\alpha_i &= 0, 1 < i < r \\
\alpha_1 &= [q(q-1) - 2] \div \frac{q+1}{k} \\
&= k(q-2).
\end{aligned}$$

Thus, $\alpha = (k(q-2), 0, 0, \dots, 0, 2)$. Hence by Theorem 4.2.2,

$$P_G = k(q-2)G(/C_{1'}) + 2G(/C_{r'})$$

which by Theorem 1.6.2 implies that the action of G on the cosets of H yields $k(q-2)$ orbits of length $\frac{q+1}{k}$ with the identity $C_{1'}$ as the stabilizer and two trivial orbits with $C_{r'} = H$ as the stabilizer. The rank in this case

is

$$r = k(q - 2) + 2 = \begin{cases} 2(q - 1) & \text{if } p \neq 2; \\ q & \text{if } p = 2. \end{cases}$$

c.) **Rank and subdegrees of G on the cosets of $H = P_q$**

H is an elementary abelian group. Thus every subgroup of H is normal and is in its own conjugacy class. Let $H_1 = I, H_2, \dots, H_r = H$ be a complete set of conjugacy class representatives of subgroups of H arranged such that $1, p, p^2, \dots, p^f = q$ are their orders respectively. Then the table of marks of H is as shown below.

Table 4.9: Table of marks of $H = P_q$

	H_1	H_2	H_3	\dots	H_{r-1}	H_r
$H(/H_1)$	q					
$H(/H_2)$	p^{f-1}	p^{f-1}				
$H(/H_3)$	p^{f-2}	p^{f-2}	p^{f-2}			
\vdots	\vdots	\vdots	\vdots			
$H(/H_{r-1})$	p	p	p	\dots	p	
$H(/H_r)$	1	1	1	\dots	1	1

H is a Sylow p -subgroup of G and hence there are $q+1$ conjugate subgroups in G . Therefore;

$$\begin{aligned} |N_G(H)| &= \frac{q(q^2 - 1)}{k} \div (q + 1) \\ &= \frac{q(q - 1)}{k}. \end{aligned}$$

For $H_i \leq H$, the values of m_{ii} for all $i = 1, 2, \dots, r$ are as follows. We consider two cases.

Case I: When $i = 1$

$$\begin{aligned}
 m(H_1) &= m(H_1, H, G) \\
 &= [N_G(H_1) : N_H(H_1)] \\
 &= \frac{q^2 - 1}{k}.
 \end{aligned}$$

Case II: When $i \neq 1$

$$\begin{aligned}
 m(H_i) &= m(H_i, H, G) \\
 &= [N_G(H_i) : N_H(H_i)] \\
 &= \frac{q - 1}{k}.
 \end{aligned}$$

Hence $\mu = \left(\frac{q^2-1}{k}, \frac{q-1}{k}, \frac{q-1}{k}, \dots, \frac{q-1}{k} \right)$. Let $\alpha = (\alpha_1, \alpha_2, \dots, \alpha_r)$, then by Equation 4.1, we obtain the following system of linear equations.

$$\begin{aligned}
 \alpha_1 p^f + \alpha_2 p^{f-1} + \dots + \alpha_{r-1} p + \alpha_r &= \frac{q^2 - 1}{k} \\
 \alpha_2 p^{f-1} + \dots + \alpha_{r-1} p + \alpha_r &= \frac{q - 1}{k} \\
 &\vdots \\
 \alpha_{r-1} p + \alpha_r &= \frac{q - 1}{k} \\
 \alpha_r &= \frac{q - 1}{k}.
 \end{aligned}$$

Since $m_{ii} \neq 0$ for all i , $\alpha_j \geq 0$, $1 \leq j \leq r$. The solution to the above system will be as follows.

$$\begin{aligned}\alpha_r &= \frac{q-1}{k} \\ \alpha_i &= 0, \quad 1 < i < r \\ \alpha_1 &= \frac{q-1}{k}.\end{aligned}$$

Therefore $\alpha = (\frac{q-1}{k}, 0, \dots, 0, \frac{q-1}{k})$. Hence using Theorem 4.4.2

$$P_G = \frac{q-1}{k}G(/H_1) + \frac{q-1}{k}G(/H)$$

which by Theorem 4.1.1 translates to the action of G on the cosets of H yielding $\frac{q-1}{k}$ orbits of length q with identity as the stabilizer and $\frac{q-1}{k}$ trivial orbits with H as the stabilizer. The rank in this is

$$r = \frac{2(q-1)}{k} = \begin{cases} q-1 & \text{if } p \neq 2; \\ 2(q-1) & \text{if } p = 2. \end{cases}$$

d.) **Rank and subdegrees of G on the cosets of $H = A_4$**

Referring to Section 3.2.4, if d is the order of any element in H , then $d = 1, 2$ or 3 . Any non-identity element $g \in G$ belongs to either;

- i) τ_0 i.e. g is an element of a subgroup of G of order $\frac{q+1}{k}$ in which case $d \mid \frac{q+1}{k}$,
 - ii) τ_1 i.e. g is an element of a subgroup of G of order q in which case $d = p$
- or,

iii) τ_2 i.e. g is an element of a subgroup of G of order $\frac{q-1}{k}$ in which case

$$d \mid \frac{q-1}{k}.$$

Therefore, from Theorem 1.2.2 part (d), for any element g to belong to a subgroup H of G isomorphic to A_4 , we have the following cases to consider.

I) $q \equiv 1 \pmod{12}$

II) $q \equiv -1 \pmod{12}$

III) $q \equiv 5 \pmod{12}$

IV) $q \equiv -5 \pmod{12}$

V) q even i.e. $p = 2$

VI) $p = 3$ and f even

VII) $p = 3$ and f odd

Before we consider each case, we highlight some results we are going to use in the determination of subdegrees of G acting on the cosets of H .

Burnside (1911, p. 241), gives the following table of marks of H where H_i , $1 \leq i \leq 5$ are conjugacy class representative subgroups. In this case, the subgroup structure of H is as follows;

- An identity subgroup $I = H_1$;
- Three conjugate subgroups of order 2, $C_2 = H_2$;
- Four conjugate cyclic subgroups of order 3, $C_3 = H_3$;
- A normal subgroup of order 4 isomorphic to Klein 4-group, $V_4 = H_4$;
- $A_4 = H_5$.

Table 4.10: Table of marks of $H = A_4$

	H_1	H_2	H_3	H_4	H_5
$H(/H_1)$	12				
$H(/H_2)$	6	2			
$H(/H_3)$	4	0	1		
$H(/H_4)$	3	3	0	3	
$H(/H_5)$	1	1	1	1	1

Lemma 4.2.1. *Let C_p be a cyclic group of order p in G . Then*

$$|N_G(C_p)| = \begin{cases} \frac{q}{2}(p-1) & p \text{ odd and } f \text{ odd;} \\ q(p-1) & p \text{ odd and } f \text{ even;} \\ q & p = 2. \end{cases}$$

(Dickson, 1901, Sec. 249)

Lemma 4.2.2. *Let C_d , $(d, p) = 1$, be a cyclic group of order d in G .*

Then

$$|N_G(C_d)| = \begin{cases} q \pm 1 & p \text{ odd;} \\ 2(q \pm 1) & p = 2. \end{cases}$$

with \pm sign as $d | q \pm 1$. (Dickson, 1901, Sec. 246)

Lemma 4.2.3. *Let $d > 2$ be a divisor of $\frac{q \pm 1}{k}$ and δ be the quotient. Then*

$$N_G(D_{2d}) = \begin{cases} D_{2d} & \delta \text{ odd;} \\ D_{4d} & \delta \text{ even.} \end{cases}$$

(Dickson, 1901, Sec. 246)

Theorem 4.2.3. *Given P_q is the elementary abelian subgroup of G , then*

$$N_G(P_q) = S_{\frac{q(q-1)}{k}}$$

where $S_{\frac{q(q-1)}{k}}$ denotes a semi-direct product group (of elementary abelian group P_q and the cyclic group $C_{\frac{q-1}{k}}$) of order $\frac{q(q-1)}{k}$. (Huppert, 1967, p. 191)

Next we compute $m(H_i)$ in each cases (a) - (g) above, the number of suborbits Δ_i on which the action of G is equivalent to its action on the cosets of H_i , ($i = 1, 2, 3, 4, 5$).

I) $q \equiv 1 \pmod{12}$

Table 4.11: $m(H_i)$ when $q \equiv 1 \pmod{12}$

H_i	$ N_G(H_i) $	$ N_H(H_i) $	$m(H_i)$
I	$\frac{q}{2}(q^2 - 1)$	12	$\frac{q(q^2-1)}{24}$
C_2	$q - 1$	4	$\frac{q-1}{4}$
C_3	$q - 1$	3	$\frac{q-1}{3}$
V_4	12	12	1
A_4	12	12	1

Now, let $\check{N} = (m(I), m(C_2), m(C_3), m(V_4), m(A_4))$, $\check{Q} = (Q_1, Q_2, Q_3, Q_4, Q_5)$ and M the matrix of the table of marks given above, we have;

$$M^T \check{Q}^T = \check{N}^T$$

$$\begin{pmatrix} 12 & 6 & 4 & 3 & 1 \\ 0 & 2 & 0 & 3 & 1 \\ 0 & 0 & 1 & 0 & 1 \\ 0 & 0 & 0 & 3 & 1 \\ 0 & 0 & 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} Q_1 \\ Q_2 \\ Q_3 \\ Q_4 \\ Q_5 \end{pmatrix} = \begin{pmatrix} \frac{q}{2}(q^2 - 1) \\ \frac{q-1}{4} \\ \frac{q-1}{3} \\ 1 \\ 1 \end{pmatrix}.$$

Solving for Q in the system of linear equations that results above, we get;

$$Q = \left(\frac{q^3 - 51q + 194}{288}, \frac{q - 5}{8}, \frac{q - 4}{3}, 0, 1 \right).$$

The rank in this case is;

$$r = \frac{q^3 + 81q - 82}{288}.$$

It follows that the subdegrees of G acting on the cosets of H under these conditions are;

Table 4.12: Subdegrees of G acting on cosets of H when $q \equiv 1 \pmod{12}$

Suborbital length	12	6	4	1
Number of suborbits	$\frac{q^3 - 51q + 194}{288}$	$\frac{q - 5}{8}$	$\frac{q - 4}{3}$	1

After dealing with case (a) when $q \equiv 1 \pmod{12}$ in details, for the other cases, (b) - (g), we just give the results in Tables 4.14, 4.16, \dots , 4.24 respectively.

II) $q \equiv -1 \pmod{12}$

Table 4.13: $m(H_i)$ when $q \equiv -1 \pmod{12}$

H_i	$ N_G(H_i) $	$ N_{H_i}(F) $	$m(H_i)$
I	$\frac{q}{2}(q^2 - 1)$	12	$\frac{q(q^2 - 1)}{24}$
C_2	$q + 1$	4	$\frac{q + 1}{4}$
C_3	$q + 1$	3	$\frac{q + 1}{3}$
V_4	12	12	1
A_4	12	12	1

The solution to the system of linear equations formed with regards to the matrix of table of marks of A_4 will be

$$Q = \left(\frac{q^3 - 51q + 94}{288}, \frac{q-3}{8}, \frac{q-2}{3}, 0, 1 \right).$$

The rank in this case is

$$r = \frac{q^3 + 81q + 82}{288}.$$

The subdegrees are as follows.

Table 4.14: Subdegrees of G acting on cosets of H when $q \equiv -1 \pmod{12}$

Suborbital length	12	6	4	1
Number of suborbits	$\frac{q^3-51q+94}{288}$	$\frac{q-3}{8}$	$\frac{q-2}{3}$	1

III) $q \equiv 5 \pmod{12}$

Table 4.15: $m(H_i)$ when $q \equiv 5 \pmod{12}$

H_i	$ N_G(H_i) $	$ N_H(H_i) $	$m(H_i)$
I	$\frac{q}{2}(q^2 - 1)$	12	$\frac{q(q^2-1)}{24}$
C_2	$q - 1$	4	$\frac{q-1}{4}$
C_3	$q + 1$	3	$\frac{q+1}{3}$
V_4	12	12	1
A_4	12	12	1

This gives

$$Q = \left(\frac{q^3 - 51q + 130}{288}, \frac{q-5}{8}, \frac{q-2}{3}, 0, 1 \right).$$

The rank in this case is

$$r = \frac{q^3 + 81q + 46}{288}.$$

The subdegrees are as follows.

Table 4.16: Subdegrees of G acting on cosets of H when $q \equiv 5 \pmod{12}$

Suborbital length	12	6	4	1
Number of suborbits	$\frac{q^3-51q+130}{288}$	$\frac{q-5}{8}$	$\frac{q-2}{3}$	1

IV) $q \equiv -5 \pmod{12}$

Table 4.17: $m(H_i)$ when $q \equiv -5 \pmod{12}$

H_i	$ N_G(H_i) $	$ N_H(H_i) $	$m(H_i)$
I	$\frac{q}{2}(q^2 - 1)$	12	$\frac{q(q^2-1)}{24}$
C_2	$q + 1$	4	$\frac{q+1}{4}$
C_3	$q - 1$	3	$\frac{q-1}{3}$
V_4	12	12	1
A_4	12	12	1

This results in

$$Q = \left(\frac{q^3 - 51q + 158}{288}, \frac{q-3}{8}, \frac{q-4}{3}, 0, 1 \right).$$

The subdegrees are as follows.

Table 4.18: Subdegrees of G acting on cosets of H when $q \equiv -5 \pmod{12}$

Suborbital length	12	6	4	1
Number of suborbits	$\frac{q^3-51q+158}{288}$	$\frac{q-3}{8}$	$\frac{q-4}{3}$	1

The rank in this case is

$$r = \frac{q^3 + 81q - 46}{288}.$$

V) q even i.e. $p = 2$

From Dickson (1901, section 242),

$$|N_G(C_2)| = 2q \text{ and } |N_G(C_3)| = 2(q - 1).$$

Similarly, the normalizers of V_4 and A_4 are $N_G(V_4) = S_4$ and $N_G(A_4) = S_4$ respectively. Therefore,

Table 4.19: $m(H_i)$ when $p = 2$

H_i	$ N_G(H_i) $	$ N_H(H_i) $	$m(H_i)$
I	$q^3 - q$	12	$\frac{q(q^2-1)}{12}$
C_2	$2q$	4	$\frac{q}{2}$
C_3	$2(q - 1)$	3	$\frac{2(q-1)}{3}$
V_4	24	12	2
A_4	24	12	2

The solution to the system of linear equations resulting from this will be

$$Q = \left(\frac{q^3 - 60q + 176}{144}, \frac{q - 4}{4}, \frac{2q - 8}{3}, 0, 2 \right).$$

The subdegrees become;

Table 4.20: Subdegrees of G acting on cosets of H when $p = 2$

Suborbital length	12	6	4	1
Number of suborbits	$\frac{q^3-60q+176}{144}$	$\frac{q-4}{4}$	$\frac{2(q-4)}{3}$	2

The rank in this case is

$$r = \frac{q^3 + 72q - 64}{144}.$$

VI) $p = 3$ and f even

From Lemma 4.2.1, we obtain the following results.

Table 4.21: $m(H_i)$ when $p = 3$ and f even

H_i	$ N_G(H_i) $	$ N_{H_i}(F) $	$m(H_i)$
I	$\frac{q}{2}(q^2 - 1)$	12	$\frac{q}{24}(q^2 - 1)$
C_2	$q - 1$	4	$\frac{q-1}{4}$
C_3	$2q$	3	$\frac{2q}{3}$
V_4	12	12	1
A_4	12	12	1

This gives

$$Q = \left(\frac{q^3 - 83q + 162}{288}, \frac{q - 5}{8}, \frac{2q - 3}{3}, 0, 1 \right).$$

The subdegrees are as follows.

Table 4.22: Subdegrees of G acting on cosets of H when $p = 3$ and f even

Suborbital length	12	6	4	1
Number of suborbits	$\frac{q^3 - 83q + 162}{288}$	$\frac{q-5}{8}$	$\frac{2q-3}{3}$	1

Therefore

$$r = \frac{q^3 + 145q - 18}{288}.$$

VII) $p = 3$ and f odd

From Lemma 4.2.1, we obtain the following table.

This gives

$$Q = \left(\frac{q^3 - 51q + 126}{288}, \frac{q - 3}{8}, \frac{q - 3}{3}, 0, 1 \right).$$

The subdegrees in this case are as follows.

Table 4.23: $m(H_i)$ when $p = 3$ and f odd

H_i	$ N_G(H_i) $	$ N_{H_i}(F) $	$m(H_i)$
I	$\frac{q}{2}(q^2 - 1)$	12	$\frac{q}{24}(q^2 - 1)$
C_2	$q + 1$	4	$\frac{q+1}{4}$
C_3	q	3	$\frac{q}{3}$
V_4	12	12	1
A_4	12	12	1

Table 4.24: Subdegrees of G acting on cosets of H when $p = 3$ and f odd

Suborbital length	12	6	4	1
Number of suborbits	$\frac{q^3 - 51q + 126}{288}$	$\frac{q-3}{8}$	$\frac{q-3}{3}$	1

Therefore

$$r = \frac{q^3 + 81q + 18}{288}.$$

e.) **Rank and subdegrees of G on the cosets of $H = D_{\frac{2(q-1)}{k}}$**

The subgroups of H are either cyclic or dihedral. The structure of H depends on whether $\frac{q-1}{k}$ is odd or even. Therefore, we have the following cases to consider.

Case (I): $p = 2$

Case (II): $q \equiv -1 \pmod{4}$

Case (III): $q \equiv 1 \pmod{4}$

In Cases (I) and (II), $\frac{q-1}{k}$ is odd and hence subgroups of order two lie in one conjugacy class of length $\frac{q-1}{k}$. Thus the subgroup structure of H is the following.

- Identity denoted by I;

- A single conjugacy class of $\frac{q-1}{k}$ cyclic subgroups of order 2 denoted by C_2 ;
- Normal cyclic subgroups $C_{m_1}, C_{m_2}, \dots, C_{m_r}$ contained in $C_{\frac{q-1}{k}}$ where $m_i | \frac{q-1}{k}$ for $1 \leq i \leq r$;
- Dihedral subgroups $D_{m_1}, D_{m_2}, \dots, D_{m_r}$ where $m_i | \frac{q-1}{k}$ and $1 \leq i \leq r$;
- A normal subgroup of order $\frac{q-1}{k}$ denoted by $C_{\frac{q-1}{k}}$;
- H .

The corresponding table of marks of H with $\frac{q-1}{k}$ odd is as shown in Table 4.25.

Table 4.25: Table of marks of $D_{\frac{2(q-1)}{k}}$ with $\frac{q-1}{k}$ odd

	I	C_2	C_{m_1}	\dots	C_{m_r}	D_{m_1}	\dots	D_{m_r}	$C_{\frac{q-1}{k}}$	H
$G(/I)$	$\frac{2(q-1)}{k}$									
$G(/C_2)$	$\frac{q-1}{k}$	1								
$G(/C_{m_1})$	m_{31}	m_{32}	m_{33}							
\vdots	\vdots	\vdots	\vdots							
$G(/C_{m_r})$	m_{r+21}	m_{r+22}	m_{r+23}	\dots	m_{r+2r+2}					
$G(/D_{m_1})$	m_{r+31}	m_{r+32}	m_{r+33}	\cdot	m_{r+3r+2}	m_{r+3r+3}				
\vdots	\vdots	\vdots	\vdots		\vdots	\vdots				
$G(/D_{m_r})$	m_{2r+21}	m_{2r+22}	m_{2r+23}	\dots	$m_{2r+2r+2}$	$m_{2r+2r+3}$	\dots	$m_{2r+22r+2}$		
$G(/C_{\frac{q-1}{k}})$	m_{2r+31}	m_{2r+32}	m_{2r+33}	\dots	$m_{2r+3r+2}$	$m_{2r+3r+3}$	\dots	$m_{2r+32r+2}$	$m_{2r+32r+3}$	
$G(/H)$	1	1	1	1	1	1	1	1	1	1

Next, we compute $m(H_i) = m(H_i, H, G)$.

Case (I): $p = 2$

Using Lemma 4.2.1 - 4.2.3, we have the following.

$$\begin{aligned}
 m(I) &= [N_G(I) : N_H(I)] \\
 &= q(q^2 - 1) \div 2(q - 1) \\
 &= \frac{q(q + 1)}{2}.
 \end{aligned} \tag{4.4}$$

$$\begin{aligned}
 m(C_2) &= [N_G(C_2) : N_H(C_2)] \\
 &= q \div 2 \\
 &= \frac{q}{2}.
 \end{aligned} \tag{4.5}$$

$$\begin{aligned}
 m(C_{m_i}) &= [N_G(C_{m_i}) : N_H(C_{m_i})] \\
 &= [N_G(C_{q-1}) : N_H(C_{q-1})] \\
 &= [N_G(H) : N_H(H)] \\
 &= 2(q - 1) \div 2(q - 1) \\
 &= 1.
 \end{aligned} \tag{4.6}$$

$$\begin{aligned}
 m(D_{m_i}) &= [N_G(D_{m_i}) : N_H(D_{m_i})] \\
 &= m_i \div m_i \\
 &= 1.
 \end{aligned} \tag{4.7}$$

Table 4.26: $m(H_i)$ when $q - 1$ is odd

H_i	$ N_G(H_i) $	$ N_H(H_i) $	$m(H_i)$
I	$q(q^2 - 1)$	$2(q - 1)$	$\frac{q}{2}(q + 1)$
C_2	q	2	$\frac{q}{2}$
C_{m_i}	$2(q - 1)$	$2(q - 1)$	1
D_{m_i}	$2d$	$2d$	1
C_{q-1}	$2(q - 1)$	$2(q - 1)$	1
H	$2(q - 1)$	$2(q - 1)$	1

Let $\mu = (\frac{q(q+1)}{2}, \frac{q}{2}, 1, \dots, 1)$ be an s -tuple where $s = 2r + 4$ and $\alpha = (\alpha_1, \alpha_2, \dots, \alpha_s)$, then by Equation 4.2, we obtain the following system of linear equations.

$$\begin{aligned}
\alpha_1 m_{11} + \alpha_2 m_{21} + \alpha_3 m_{31} + \dots + \alpha_{s-1} m_{s-11} + \alpha_s &= \frac{q}{2}(q + 1) \\
\alpha_2 m_{22} + \alpha_3 m_{32} + \dots + \alpha_{s-1} m_{s-12} + \alpha_s &= \frac{q}{2} \\
\alpha_3 m_{33} + \dots + \alpha_{s-1} m_{s-13} + \alpha_s &= 1 \\
&\vdots \\
\alpha_{s-1} m_{s-1s-1} + \alpha_s &= 1 \\
\alpha_s &= 1.
\end{aligned}$$

The solution to this system is $\alpha = (1, \frac{q-2}{2}, 0, \dots, 0, 1)$. Therefore, from Theorem 4.2.2, we have

$$P_G = G(/I) + \left(\frac{q-2}{2}\right) G(/C_2) + G(/H).$$

Hence, by Theorem 1.6.2, the action of G on the cosets of H yields one orbit of length $2(q - 1)$ with the identity as the stabilizer, $\frac{q-2}{2}$ orbits of

length $q - 1$ with C_2 as the stabilizer and a trivial orbit with H as the stabilizer. The rank is

$$\begin{aligned} r &= 1 + \frac{q-2}{2} + 1 \\ &= \frac{q+2}{2}. \end{aligned}$$

Case (II): $q \equiv -1 \pmod{4}$

Table 4.27: $m(H_i)$ when $\frac{q-1}{2}$ is odd

H_i	$ N_G(H_i) $	$ N_H(H_i) $	$m(H_i)$
I	$\frac{q}{2}(q^2 - 1)$	$q - 1$	$\frac{q}{2}(q + 1)$
C_2	$q + 1$	2	$\frac{1}{2}(q + 1)$
C_{m_i}	$q - 1$	$q - 1$	1
D_{m_i}	$2d$	$2d$	1
$C_{\frac{q-1}{2}}$	$q - 1$	$q - 1$	1
H	$q - 1$	$q - 1$	1

This gives $\alpha = (\frac{q+5}{4}, \frac{q-1}{2}, 0, \dots, 0, 1)$.

$$\begin{aligned} \therefore r &= \frac{q+5}{4} + \frac{q-1}{2} + 1 \\ &= \frac{3q+7}{4}. \end{aligned}$$

Hence, by Theorem 1.6.2, the action of G on the cosets of H yields $\frac{q+5}{4}$ orbit of length $q - 1$ with the identity as the stabilizer, $\frac{q-1}{2}$ orbits of length $\frac{q-1}{2}$ with C_2 as the stabilizer and a trivial orbit with H as the stabilizer.

Case (III): $q \equiv 1 \pmod{4}$

Here $\frac{q-1}{2}$ is even. Hence the subgroups of H of order 2 lie in two conjugacy classes each of length $q - 1$. Thus its subgroups are as follows;

- Identity denoted by I ;
- A normal cyclic subgroup of order 2 denoted by $C_2(N)$;
- A conjugacy class of $\frac{q-1}{4}$ cyclic subgroups of order 2 denoted by $C_2(a)$;
- A conjugacy class of $\frac{q-1}{4}$ cyclic subgroups of order 2 denoted by $C_2(b)$;
- Normal cyclic subgroups $C_{m_1}, C_{m_2}, \dots, C_{m_r}$ contained in $C_{\frac{q-1}{2}}$ where $m_i | \frac{q-1}{2}$ and $m_i \neq 2$, $1 \leq i \leq r$;
- Dihedral subgroups $D_{m_1}, D_{m_2}, \dots, D_{m_r}$ where $m_i | \frac{q-1}{2}$ and $1 \leq i \leq r$;
- A normal subgroup of order $\frac{q-1}{2}$ denoted by $C_{\frac{q-1}{2}}$;
- H .

The corresponding table of marks of H is as shown in Table 4.28.

Table 4.28: Table of marks of D_{q-1} with $\frac{q-1}{2}$ even

	I	$C_2(N)$	$C_2(a)$	$C_2(b)$	C_{m_1}	\dots	C_{m_r}	D_{m_1}	\dots	D_{m_r}	$C_{\frac{q-1}{2}}$	D_{2z}
$H(/I)$	$q-1$											
$H(/C_2(N))$	$\frac{q-1}{2}$	$\frac{q-1}{2}$										
$H(/C_2(a))$	$\frac{q-1}{4}$	0	2									
$H(/C_2(b))$	$\frac{q-1}{4}$	0	0	2								
$H(/C_{m_1})$	m_{51}	m_{52}	m_{53}	m_{54}	m_{55}							
\vdots	\vdots	\vdots	\vdots	\vdots	\vdots							
$H(/C_{m_r})$	m_{r+41}	m_{r+42}	m_{r+43}	m_{r+44}	$m_{r+45} \dots m_{r+4r+4}$							
$H(/D_{m_1})$	m_{r+51}	m_{r+52}	m_{r+53}	m_{r+54}	$m_{r+55} \dots m_{r+5r+4}$	m_{r+5r+5}						
\vdots	\vdots	\vdots	\vdots	\vdots	\vdots	\vdots						
$H(/D_{m_r})$	m_{2r+41}	m_{2r+42}	m_{2r+43}	m_{2r+44}	$m_{2r+45} \dots m_{2r+4r+4}$	$m_{2r+4r+5} \dots m_{2r+42r+4}$						
$H(/C_{\frac{q-1}{2}})$	m_{2r+51}	m_{2r+52}	m_{2r+53}	m_{2r+54}	$m_{2r+55} \dots m_{2r+5r+4}$	$m_{2r+5r+5} \dots m_{2r+52r+4}$	$m_{2r+52r+5}$					
$H(/H)$	1	1	1	1	1	\dots	1	1	\dots	1	1	1

From Lemma 4.2.2,

$$|N_G(C_d)| = q - 1.$$

where $d \mid \frac{q-1}{2}$ and q is odd. Similarly, from Lemma 4.2.3

$$|N_G(D_{2d})| = 4d.$$

where $d \mid \frac{q-1}{4}$ and $\frac{q-1}{4}$ is even. The orders of normalizers of H_i , ($i = 1, 2, \dots, s$) where $s = 2r + 6$ in G and H , and the values of $m(H_i)$ are as shown in Table 4.29.

Table 4.29: $m(H_i)$ when $\frac{q-1}{2}$ is even

H_i	$ N_G(H_i) $	$ N_H(H_i) $	$m(H_i)$
I	$\frac{q}{2}(q^2 - 1)$	$q - 1$	$\frac{q}{2}(q + 1)$
$C_2(a)$	$q - 1$	4	$\frac{q-1}{4}$
$C_2(b)$	$q - 1$	4	$\frac{q-1}{4}$
$C_2(N)$	$q - 1$	$q - 1$	1
C_{m_i}	$q - 1$	$q - 1$	1
D_{m_i}	$4d$	$4d$	1
$C_{\frac{q-1}{2}}$	$q - 1$	$q - 1$	1
\bar{H}	$q - 1$	$q - 1$	1

This gives $\alpha = (\frac{q+7}{4}, \frac{q-5}{2}, 2, 0, \dots, 0, 1)$.

$$\begin{aligned} \therefore r &= \frac{q+7}{4} + \frac{q-5}{2} + 2 + 1 \\ &= \frac{3}{4}(q+3). \end{aligned}$$

Hence, by Theorem 1.6.2, the action of G on the cosets of H in this case yields $\frac{q+7}{4}$ orbits of length $q-1$ with the identity as the stabilizer, $\frac{q-5}{2}$ orbits of length $\frac{q-1}{2}$ with the normal cyclic subgroup of order two as the stabilizer, 2 orbits of length $\frac{q-1}{4}$ with cyclic subgroup of order two (not normal in G) as the stabilizer and a trivial orbit with H as the stabilizer.

CHAPTER 5

SUBORBITAL GRAPHS OF $PSL(2, Q)$

ACTING ON THE COSETS OF $C_{\frac{Q-1}{K}}$

After obtaining subdegrees in the previous chapter, we tackle a quite interesting task of analyzing suborbits of $PSL(2, q)$ acting on the cosets of its cyclic subgroup $C_{\frac{q-1}{k}}$, constructing and determining properties of suborbital graphs corresponding to the suborbits obtained through reduced pair group action.

The current chapter is presented in three sections. In Section 5.1, background information is given. Here, preliminary results to be used in the chapter are given and the general concept of reduced pair group action discussed. In Section 5.2, suborbits of $PSL(2, q)$ acting on the cosets of $C_{\frac{q-1}{k}}$ are determined and their properties discussed. Construction of suborbital graphs corresponding to some of the suborbits discussed in Section 5.2 is given in Section 5.3 and general theoretical properties of the constructed suborbital graphs examined.

5.1 Background information

5.1.1 Preliminary results

In this subsection, we give some results which will be used in the determination of properties of suborbits and their corresponding suborbital graphs to be described later in this chapter.

Theorem 5.1.1. *G_x has an orbit different from $\{x\}$ and paired with itself if and only if G has even order. Wielandt (1964, Sec. 16.5)*

Theorem 5.1.2. *Let G act transitively on a set X , and let $g \in G$. Suppose π is the character of the permutation of G on X , then the number n_π of self-paired suborbits is given by*

$$n_\pi = \frac{1}{|G|} \sum_{g \in G} \pi(g^2).$$

(Cameron, 1975)

Theorem 5.1.3. *Let Γ_i be the suborbital graph corresponding to the suborbital O_i . Let the suborbits $\Delta_i (i = 0, 1, \dots, r-1)$ correspond to the suborbital O_i . Then Γ_i is undirected if Δ_i is self-paired and is directed if Δ_i is not self-paired. (Sims, 1967)*

Theorem 5.1.4. *Let G be transitive on X . Then G is primitive if and only if each suborbital graph Γ_i , $i = 1, 2, \dots, r-1$ is connected. (Sims, 1967)*

5.1.2 Reduced pair group action

Let (G, X) be a finite permutation group. Denote by $X^{[2]}$ the set of all the ordered pairs from X . If g is a permutation in (G, X) with $mon(g) = t_1 t_2 \cdots t_n$,

our task is to determine $\text{mon}(g')$ where g' is the permutation induced by g on $X^{[2]}$. To do this, we have the following contributions of g to the corresponding term of $\text{mon}(g)$ to be considered.

- i.) Pair of points coming from a common cycle of g .
- ii.) Pair of points coming from distinct cycles of g .

Case i: Pair of points coming from a common cycle of g

Let θ be a cycle of length $m \geq 2$. The stabilizer H of an ordered pair from θ is isomorphic to $I \times S_{m-2} \cong S_{m-2}$. Using Theorem 1.6.4 and Lemma 1.6.2, we determine the monomial of g' as follows.

$$|C^{\theta^l} \cap H| = \begin{cases} 0 & l < m; \\ 1 & l = m. \end{cases}$$

$$\begin{aligned} \pi(\theta^l) &= \begin{cases} 0 & l < m \\ \frac{m!}{(m-2)!} \times 1 & l = m \end{cases} \\ &= \begin{cases} 0 & l < m \\ m(m-1) & l = m \end{cases} \end{aligned}$$

$$\begin{aligned} \alpha_m &= \frac{1}{m} \sum_{i|m} \pi(\theta^{m/i} \mu(i)) \\ &= \frac{1}{m} \pi(\theta^m \mu(1)) \\ &= m - 1 \end{aligned}$$

Hence, $t_m \rightarrow t_m^{m-1}$. Therefore, if there are j_m cycles of length m , then

$$t_m^{j_m} \rightarrow t_m^{j_m(m-1)} \quad (5.1)$$

Case ii: Pair of points coming from distinct cycles of g

Suppose one of the points come from a cycle θ_r of length r and the other comes from a cycle θ_s of length s . The ordered pair will be in a cycle of length $[r, s]$ (where $[r, s] = lcm(r, s)$). The total number of ordered pairs which can be formed from θ_r and θ_s is $2rs$. We have $\frac{2rs}{[r, s]} = 2(r, s)$ (where $(r, s) = gcd(r, s)$) cycles of length $[r, s]$. Therefore, if $r \neq s$

$$t_r^{j_r} t_s^{j_s} \rightarrow t_{[r, s]}^{2(r, s)j_r j_s} \quad (5.2)$$

When $r = s = m$

$$t_m^{j_m} \rightarrow t_m^{\beta}, \quad \text{where } \beta = 2m \begin{pmatrix} j_m \\ 2 \end{pmatrix} \quad (5.3)$$

Multiplication of the right hand sides of Equation 5.1, 5.2 and 5.3 over all applicable cases gives the desired result.

5.2 Suborbits of $G = PSL(2, q)$ acting on the cosets of its cyclic subgroup $H = C_{\frac{q-1}{k}}$

Theorem 5.2.1. *The action of G on the cosets of its cyclic subgroup H is equivalent to its action on the ordered pairs from $PG(1, q)$.*

Proof: Since G acts doubly transitive on $PG(1, q)$, it acts transitively on ordered pairs from $PG(1, q)$. From Theorem 1.6.1, the action of G on ordered pairs from $PG(1, q)$ is equivalent to the action of G on the cosets of a stabilizer of an ordered pair, which in this case is H . \square

H is the stabilizer of an ordered pair $(\infty, 0)$ for $\infty, 0 \in PG(1, q)$. We use the results on reduced pair group action described above to classify the H -orbits considering cases when q is even and when q is odd.

Case 1: q even

Let u be a generator of H . In the natural action of G on $PG(1, q)$, u contains 2 trivial cycles and 1 cycle of length $q - 1$,

$$i.e. \quad \langle u \rangle = \begin{pmatrix} \alpha & 0 \\ 0 & \alpha^{-1} \end{pmatrix} = (0)(\infty)(1, \alpha^2, \alpha^4, \dots, \alpha^{q-2}, \alpha, \alpha^3, \dots, \alpha^{q-3}).$$

Classifying the H -orbits in this representation, we have the following.

- i.) Points coming from the $q - 1$ non-trivial cycle contribute;

$$t_{q-1} \rightarrow t_{q-1}^{q-2}. \quad (by \text{ Equation 5.1})$$

Here we get $q - 2$ H -orbits of length $q - 1$.

- ii.) Points coming from distinct cycles of different lengths contribute;

$$t_1^2 t_{q-1} \rightarrow t_{q-1}^4. \quad (by \text{ Equation 5.2})$$

We obtain 4 H -orbits of length $q - 1$.

iii.) Points coming from distinct cycles of equal lengths contribute;

$$t_1^2 \rightarrow t_1'^2. \quad (\text{by Equation 5.3})$$

These are the two trivial H -orbits; namely $\{(\infty, 0)\}$ and $\{(0, \infty)\}$. Thus, we have 2 H -orbits of length 1.

Next, we list all the suborbits of G , i.e. the orbits of $G_{(\infty, 0)}$ when $p = 2$.

$$\left. \begin{array}{l} \Delta_{(\infty, 0)} \\ \Delta_{(0, \infty)} \end{array} \right\} \quad 2 \text{ orbits of length one}$$

$$\left. \begin{array}{l} \Delta_{(\infty, 1)} \\ \Delta_{(1, \infty)} \\ \Delta_{(0, 1)} \\ \Delta_{(1, 0)} \end{array} \right\} \quad 4 \text{ orbits of length } q - 1$$

$$\left. \begin{array}{l} \Delta_{(1, \alpha^2)} \\ \Delta_{(1, \alpha^4)} \\ \vdots \\ \Delta_{(1, \alpha^{q-2})} \\ \Delta_{(1, \alpha)} \\ \Delta_{(1, \alpha^3)} \\ \vdots \\ \Delta_{(1, \alpha^{q-3})} \end{array} \right\} \quad q - 2 \text{ orbits of length } q - 1$$

The total number of orbits is

$$2 + 4 + q - 2 = q + 4$$

Therefore, subdegrees of $PSL(2, q)$ acting on the cosets of its cyclic subgroup C_{q-1} are as given in the table below.

Table 5.1: Subdegrees of G acting on the cosets of H when q is even

Suborbit length	1	$q - 1$
Number of suborbits	2	$q + 2$

Theorem 5.2.2. *Let G act on the cosets of its cyclic subgroup $C_{\frac{q-1}{k}}$. The number n_π of self-paired suborbits when $p = 2$, is given by;*

$$n_\pi = q + 2.$$

Proof: The identity in G fixes every pair, so its contribution to n_π is $q(q + 1)$. Suppose $g \in G$ is parabolic, then when squared, it fixes all $q(q + 1)$ ordered pairs. But there are $q - 1$ such elements contained in $q + 1$ conjugate subgroups in G . Therefore, parabolics contribute

$$q(q + 1) \times (q - 1) \times (q + 1).$$

If $g \in G$ is hyperbolic, then it fixes two pairs when squared. Since there are $\frac{q(q+1)}{2}$ conjugate subgroups in G each containing $q - 1$ hyperbolic elements, their contribution is

$$2 \times (q - 1) \times \frac{q(q + 1)}{2}.$$

Elliptics fix no point, consequently, their contribution is zero. The total of all these contributions divided by the order of G gives the desired results. \square

Lemma 5.2.1. *There are only two suborbits paired with each other when G acts on the cosets of $C_{\frac{q-1}{k}}$ for $p = 2$ and these suborbits are $\Delta_{(1,\infty)}$ and $\Delta_{(0,1)}$*

Proof: Since the rank of G on the cosets of $C_{\frac{q-1}{k}}$ when $p = 2$ is $q+4$, the first part of the result follow. The element $\begin{pmatrix} 0 & 1 \\ -1 & 1 \end{pmatrix} \in G$ maps $(1, \infty) \in \Delta_{(1,\infty)}$ to $(\infty, 0)$ and $(\infty, 0)$ to $(0, 1) \in \Delta_{(0,1)}$. Hence by Definition 1.1.12, $\Delta_{(1,\infty)}$ is paired with $\Delta_{(0,1)}$ \square

Case 2: q odd

Let u be a generator of H . In the natural action of G on $PG(1, q)$, u contains 2 trivial cycles and 2 non-trivial cycles of equal length $\frac{q-1}{2}$,

$$i.e. \langle u \rangle = \begin{pmatrix} \alpha & 0 \\ 0 & \alpha^{-1} \end{pmatrix} = (0)(\infty)(1, \alpha^2, \alpha^4, \dots, \alpha^{q-2})(\alpha, \alpha^3, \dots, \alpha^{q-3})$$

The non-trivial cycles one contains residues while the other contains non-residues.

Lemma 5.2.2. *If -1 is a square mod q , then $q \equiv 1 \pmod{4}$.*

Proof: Let $x \in GF(q)$, then

$$\begin{aligned} -1 = x^2 &\Leftrightarrow x \text{ is of order } 4 \\ &\Leftrightarrow |x| \mid |GF(q)^*| \\ &\Leftrightarrow q \equiv 1 \pmod{4}. \quad \square \end{aligned}$$

Lemma 5.2.3. *1 and -1 lie in the same cycle in the generator u of $C_{\frac{q-1}{k}}$ if and only if $q \equiv 1 \pmod{4}$.*

Proof: Let α be the primitive root of $GF(q)^*$ and take u to be

$$\begin{pmatrix} \alpha & 0 \\ 0 & \alpha^{-1} \end{pmatrix} = (0)(\infty)(1, \alpha^2, \dots, \alpha^{q-3})(\alpha, \alpha^3, \dots, \alpha^{q-2}).$$

The cycle containing 1 consists of all even powers of α ; that is all non-zero squares in $GF(q)$. Hence, the results follows from Lemma 5.2.2. \square

Corollary 5.2.1. *Let $x \in GF(q)$, then α and $-\alpha$ or α^{-1} lie in the same cycle of u if and only if $q \equiv 1 \pmod{4}$.*

Next, we discuss the suborbits using reduced pair group action. Classifying the H -orbits in this representation gives the following results.

i.) Points coming from a common cycle

The only contributors in this case are the two non-trivial cycles. Therefore

$$t_{\frac{q-1}{2}}^2 \rightarrow S_{\frac{q-1}{2}}^{q-3}, \quad (\text{by Equation 5.1})$$

i.e. $q - 3$ H -orbits of length $\frac{q-1}{2}$.

ii.) Points coming from distinct cycles of different lengths contribute;

$$t_1^2 t_{\frac{q-1}{2}} \rightarrow S_{\frac{q-1}{2}}^8, \quad (\text{by Equation 5.2})$$

i.e. 8 H -orbits of length $\frac{q-1}{2}$.

iii.) Points coming from distinct cycles of equal lengths

The trivial cycles contribute 2 H -orbits each of length 1,

$$\text{i.e. } t_1^2 \rightarrow S_1^2. \quad (\text{by Equation 5.3})$$

The non-trivial cycles contribute

$$t_{\frac{q-1}{2}}^2 \rightarrow S_{\frac{q-1}{2}}^{q-1}, \quad (\text{by Equation 5.3})$$

i.e. $q - 1$ H -orbits of length $\frac{q-1}{2}$.

Next, we list all the suborbits of G i.e. the orbits of $G_{(\infty,0)}$ when $p > 2$.

$$\left. \begin{array}{l} \Delta_{(\infty,0)} \\ \Delta_{(0,\infty)} \end{array} \right\} \quad 2 \text{ orbits of length one}$$

$$\left. \begin{array}{l} \Delta_{(\infty,1)} \\ \Delta_{(1,\infty)} \\ \Delta_{(0,1)} \\ \Delta_{(1,0)} \\ \Delta_{(\infty,\alpha)} \\ \Delta_{(\alpha,\infty)} \\ \Delta_{(0,\alpha)} \\ \Delta_{(\alpha,0)} \end{array} \right\} \quad 8 \text{ orbits of length } \frac{q-1}{2}$$

$$\left. \begin{array}{l} \Delta_{(1,\alpha^2)} \\ \Delta_{(1,\alpha^4)} \\ \vdots \\ \Delta_{(1,\alpha^{q-3})} \end{array} \right\} \frac{q-3}{2} \text{ orbits of length } \frac{q-1}{2}$$

$$\left. \begin{array}{l} \Delta_{(\alpha,\alpha^3)} \\ \Delta_{(\alpha,\alpha^5)} \\ \vdots \\ \Delta_{(\alpha,\alpha^{q-2})} \end{array} \right\} \frac{q-3}{2} \text{ orbits of length } \frac{q-1}{2}$$

$$\left. \begin{array}{l} \Delta_{(1,\alpha)} \\ \Delta_{(1,\alpha^3)} \\ \vdots \\ \Delta_{(1,\alpha^{q-2})} \end{array} \right\} \frac{q-1}{2} \text{ orbits of length } \frac{q-1}{2}$$

$$\left. \begin{array}{l} \Delta_{(\alpha,1)} \\ \Delta_{(\alpha,\alpha^2)} \\ \vdots \\ \Delta_{(\alpha,\alpha^{q-3})} \end{array} \right\} \frac{q-1}{2} \text{ orbits of length } \frac{q-1}{2}$$

The total number of orbits is

$$\begin{aligned} r &= 2 + 8 + \frac{q-3}{2} + \frac{q-3}{2} + \frac{q-1}{2} + \frac{q-1}{2} \\ &= 2(q+3). \end{aligned}$$

Therefore, subdegrees of G are as follows.

Theorem 5.2.3. $G_{(\infty,0)}$ has an orbit different from $\Delta_{(\infty,0)}$ and paired with itself.

Table 5.2: Subdegrees of G acting on the cosets of H when q is odd

Suborbit length	1	$\frac{1}{2}(q-1)$
Number of suborbits	2	$2(q+2)$

Proof: From results in Table 5.1 and Table 5.2 above, $G_{(\infty,0)}$ has $q+4$ and $2(q+3)$ orbits respectively. Since $|G|$ is always even, the result follows from Theorem 5.1.1. \square

Using Theorem 5.1.2 and subgroup structure of G discussed in Chapter 1, we now determine the number of self-paired suborbits.

Theorem 5.2.4. *Let G act on the cosets of its cyclic subgroup $C_{\frac{q-1}{k}}$. The number n_π of self-paired suborbits when $p > 2$ is given by;*

a.) $n_\pi = q + 3$ when $q \equiv 1 \pmod{4}$ and

b.) $n_\pi = q + 1$ when $q \equiv -1 \pmod{4}$.

Proof: In both cases, the identity in G fixes every pair, so its contribution to n_π is $q(q+1)$.

a.) If $q \equiv 1 \pmod{4}$, parabolics have orders different from two, hence there contribution is zero. On the other hand, parabolics which are of order greater than two are $\frac{q-5}{2}$ each fixing two pairs with or without squaring. Since there are $\frac{q(q+1)}{2}$ conjugate subgroups in G containing parabolics, their contribution is

$$2 \times \frac{q-5}{2} \times \frac{q(q+1)}{2}.$$

The single hyperbolic element of order two fixes every pair when squared. Therefore given there are $\frac{q(q+1)}{2}$ conjugate subgroups of hyperbolics, their contribution is

$$1 \times q(q+1) \times \frac{q(q+1)}{2}.$$

Elliptics contribute zero. Summing all these contributions and dividing by the order of G gives the result.

- b.) If $q \equiv -1 \pmod{4}$, parabolics contribute zero since none is of order two. The single element of order two is elliptic contributing

$$1 \times q(q+1) \times \frac{q(q-1)}{2}.$$

$\frac{q-3}{2}$ hyperbolics fix two pairs each. Therefore given there are $\frac{q(q+1)}{2}$ conjugate subgroups of hyperbolics, their contribution is

$$2 \times \frac{(q-3)}{2} \times \frac{q(q+1)}{2}.$$

Dividing the sum of all these contributions by the order of G gives the required result. \square

Corollary 5.2.2. *Half of suborbits formed when G acts on the cosets of $C_{\frac{q-1}{k}}$ for $q \equiv 1 \pmod{4}$ are self-paired.*

Theorem 5.2.5. *When $q \equiv 1 \pmod{4}$, $\Delta_{(\infty,1)}$ and $\Delta_{(1,0)}$ are self-paired.*

Proof: From Lemma 5.2.3, $(1,0), (-1,0) \in \Delta_{(1,0)}$. $\begin{pmatrix} 1 & 0 \\ -1 & 1 \end{pmatrix}$ maps $(1,0)$ to $(\infty,0)$ and $(\infty,0)$ to $(-1,0)$. Hence by Definition 1.1.12, $\Delta_{(1,0)}$ is self-paired.

Similarly, using Lemma 5.2.3, $(\infty,1), (\infty,-1) \in \Delta_{(\infty,1)}$. $\begin{pmatrix} 1 & -1 \\ 0 & 1 \end{pmatrix}$ maps $(\infty,1)$ to $(\infty,0)$ and $(\infty,0)$ to $(\infty,-1)$. Thus by Definition 1.1.12, $\Delta_{(\infty,1)}$ is self-paired. \square

Theorem 5.2.6. *When $q \equiv 1 \pmod{4}$, $\Delta_{(\infty,\alpha)}$ and $\Delta_{(\alpha,0)}$ are self-paired.*

Proof: From Corollary 5.2.1, $(\infty, \alpha), (\infty, -\alpha) \in \Delta_{(\infty, \alpha)}$. $\begin{pmatrix} 1 & -\alpha \\ 0 & 1 \end{pmatrix}$ maps (∞, α) to $(\infty, 0)$ and $(\infty, 0)$ to $(\infty, -\alpha)$. Hence by Definition 1.1.12 $\Delta_{(\infty, \alpha)}$ is self-paired.

Similarly, $(\infty, 1), (\infty, -1) \in \Delta_{(\infty, 1)}$. $\begin{pmatrix} 1 & -1 \\ 0 & 1 \end{pmatrix}$ maps $(\infty, 1)$ to $(\infty, 0)$ and $(\infty, 0)$ to $(\infty, -1)$. Thus by Definition 1.1.12, $\Delta_{(\infty, 1)}$ is self-paired. \square

Theorem 5.2.7. *The orbit $\Delta_{(0, \infty)}$ is self-paired.*

Proof: The action of G on the cosets of its cyclic subgroup $C_{\frac{q-1}{k}}$ has two suborbits of length one; namely the trivial orbit $\Delta_{(\infty, 0)}$ and $\Delta_{(0, \infty)}$. Since the trivial orbit is self-paired, so is $\Delta_{(0, \infty)}$. \square

Theorem 5.2.8. *When $q \equiv 1 \pmod{4}$, $\Delta_{(1, \infty)}$ is paired with $\Delta_{(0, 1)}$.*

Proof: Taking $g = \begin{pmatrix} 0 & -1 \\ -1 & 1 \end{pmatrix}$ with Lemma 5.2.3 and Definition 1.1.12, the results follow. \square

5.3 Suborbital graphs of $G = PSL(2, q)$ acting on the cosets of its cyclic subgroup $H =$

$$C_{\frac{q-1}{k}}$$

From the preceding section, there are $k(q+2)$ suborbits of length $\frac{q-1}{k}$ and one orbit of length one to be considered since the other is trivial. Since G is doubly transitive on $PG(1, q)$, given an ordered pair (v, h) , where $v, h \in PG(1, q)$, then there exists a $g \in G$ such that $g(\infty) = v$ and $g(0) = h$.

We express g in terms of v and h as follows. Let v be represented as $\frac{v}{1}$ and h as $\frac{x}{y}$ where $x = h(v - h)^{-1}$ and $y = (v - h)^{-1}$. Then

$$g = \begin{pmatrix} v & h(v - h)^{-1} \\ 1 & (v - h)^{-1} \end{pmatrix} \in PSL(2, q). \quad (5.4)$$

If one of v and h is ∞ , then g becomes either;

$$\begin{pmatrix} 1 & h \\ 0 & 1 \end{pmatrix} \quad (5.5)$$

or

$$\begin{pmatrix} v & 1 \\ 1 & 0 \end{pmatrix}. \quad (5.6)$$

In view of Theorem 5.2.1, we construct the suborbital graphs corresponding to this action. In each case g chosen can be as in Equation 5.4, 5.5 or 5.6.

Case I: Suborbital graphs corresponding to suborbits of G formed by pairs of points of the form $(0, a)$ where a is a square and $(0, b)$ where b is not a square

Theorem 5.3.1. *$((v, h), (c, d))$ is an edge in $\Gamma_{(0, a)}$ for each of the following cases and only for these.*

- (a) $v \neq \infty, h \neq \infty, c = h$ and $d = (va + h(v - h)^{-1})(a + (v - h)^{-1})^{-1}$.
- (b) $v = \infty, c = h$ and $d = a + h$.
- (c) $h = c = \infty$ and $d = (va + 1)a^{-1}$.

Proof:

(a) If $v \neq \infty, h \neq \infty$, then by Equation 5.4,

$$\begin{pmatrix} v & h(v-h)^{-1} \\ 1 & (v-h)^{-1} \end{pmatrix} \begin{pmatrix} 0 & a \\ 1 & 1 \end{pmatrix} = \begin{pmatrix} h(v-h)^{-1} & va + h(v-h)^{-1} \\ (v-h)^{-1} & a + (v-h)^{-1} \end{pmatrix}$$

$\Rightarrow c = h$ and $d = (va + h(v-h)^{-1})(a + (v-h)^{-1})^{-1}$ as required.

(b) If $v = \infty$, by Equation 5.5, we have;

$$\begin{pmatrix} 1 & h \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 0 & a \\ 1 & 1 \end{pmatrix} = \begin{pmatrix} h & a+h \\ 1 & 1 \end{pmatrix}$$

$\Rightarrow c = h$ and $d = a + h$ as desired.

(c) If $h = \infty$, Equation 5.6 gives

$$\begin{pmatrix} v & 1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} 0 & a \\ 1 & 1 \end{pmatrix} = \begin{pmatrix} 1 & va+1 \\ 0 & 1 \end{pmatrix}$$

$\Rightarrow c = \infty$ and $d = va + 1$. □

Remark 5.3.1. *The construction for $\Gamma_{(0,b)}$ takes similar argument in which case a is substituted by b .*

Note: From now on, examples of suborbital graphs constructed are taken where $G = PSL(2, 5)$, and $u = (0)(\infty)(14)(23)$.

In view of Theorem 5.2.1 the suborbits of G acting on the cosets of C_2 become:

1. $\Delta_0 = Orb_{G_{(\infty,0)}}(\infty, 0) = \{(\infty, 0)\}$

2. $\Delta_1 = Orb_{G_{(\infty,0)}}(0, \infty) = \{(0, \infty)\}$
3. $\Delta_2 = Orb_{G_{(\infty,0)}}(0, 1) = \{(0, 1), (0, 4)\}$
4. $\Delta_3 = Orb_{G_{(\infty,0)}}(1, 0) = \{(1, 0), (4, 0)\}$
5. $\Delta_4 = Orb_{G_{(\infty,0)}}(0, 2) = \{(0, 2), (0, 3)\}$
6. $\Delta_5 = Orb_{G_{(\infty,0)}}(2, 0) = \{(2, 0), (3, 0)\}$
7. $\Delta_6 = Orb_{G_{(\infty,0)}}(\infty, 1) = \{(\infty, 1), (\infty, 4)\}$
8. $\Delta_7 = Orb_{G_{(\infty,0)}}(1, \infty) = \{(1, \infty), (4, \infty)\}$
9. $\Delta_8 = Orb_{G_{(\infty,0)}}(\infty, 2) = \{(\infty, 2), (\infty, 3)\}$
10. $\Delta_9 = Orb_{G_{(\infty,0)}}(2, \infty) = \{(2, \infty), (3, \infty)\}$
11. $\Delta_{10} = Orb_{G_{(\infty,0)}}(1, 2) = \{(1, 2), (4, 3)\}$
12. $\Delta_{11} = Orb_{G_{(\infty,0)}}(2, 1) = \{(2, 1), (3, 4)\}$
13. $\Delta_{12} = Orb_{G_{(\infty,0)}}(4, 2) = \{(4, 2), (1, 3)\}$
14. $\Delta_{13} = Orb_{G_{(\infty,0)}}(2, 4) = \{(2, 4), (3, 1)\}$
15. $\Delta_{14} = Orb_{G_{(\infty,0)}}(1, 4) = \{(1, 4), (4, 1)\}$
16. $\Delta_{15} = Orb_{G_{(\infty,0)}}(2, 3) = \{(2, 3), (3, 2)\}$

Example 1. $\Gamma_{(0,1)}$ for $PSL(2, 5)$

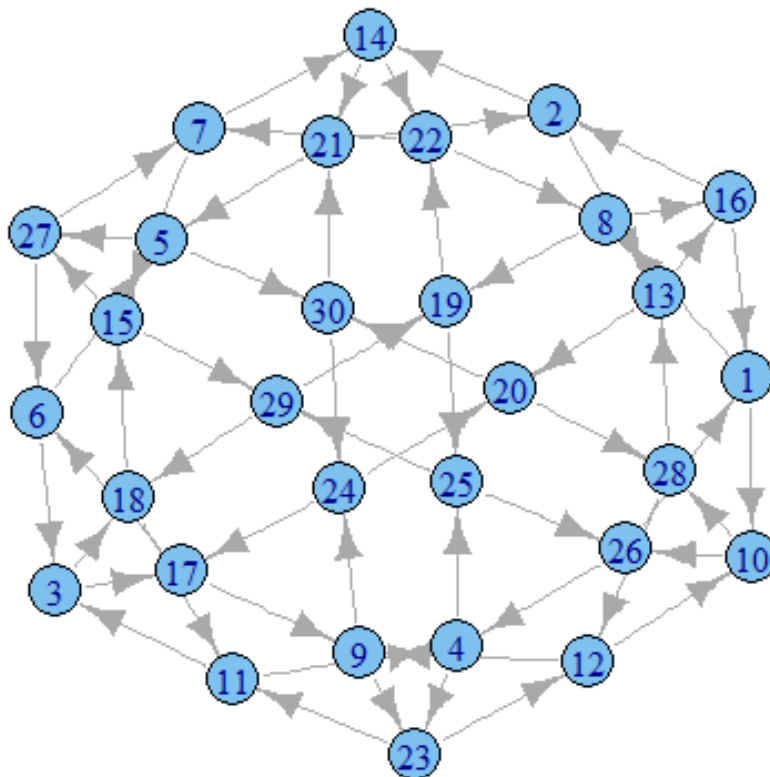


Figure 5.1: Suborbital graph $\Gamma_{(0,1)}$ corresponding to the suborbit Δ_2

Case II: Suborbital graphs corresponding to suborbits of G formed by pairs of points of the form $(a, 0)$ where a is a square and $(b, 0)$ where b is not a square

Theorem 5.3.2. $((v, h), (c, d))$ is an edge in $\Gamma_{(a,0)}$ for each of the following cases and only for these.

- (a) $v \neq \infty, h \neq \infty, d = h$ and $c = (va + h(v - h)^{-1})(a + (v - h)^{-1})^{-1}$
- (b) $v = \infty, d = h$ and $c = a + h$
- (c) $h = \infty, d = \infty$ and $c = (va + 1)a^{-1}$

Proof: Arguments similar to those in Theorem 5.3.1 are used. □

Remark 5.3.2. *The construction for $\Gamma_{(b,0)}$ is similarly done replacing b for a .*

Example 2. $\Gamma_{(1,0)}$ for $PSL(2, 5)$

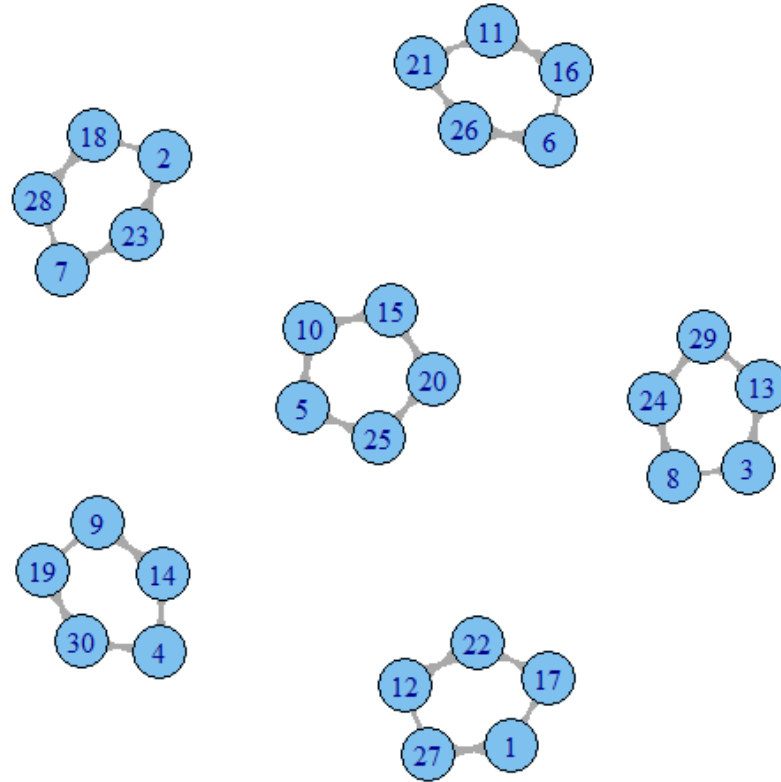


Figure 5.2: Suborbital graph $\Gamma_{(1,0)}$ corresponding to the suborbit Δ_3

Case III: Suborbital graphs corresponding to suborbits of G formed by pairs of points of the form (∞, a) where a is a square and (∞, b) where b is not a square

Theorem 5.3.3. *$((v, h), (c, d))$ is an edge in $\Gamma_{(\infty, a)}$ for each of the following cases and only for these.*

- (a) $v \neq \infty, h \neq \infty, c = v$ and $d = (va + h(v - h)^{-1})(a + (v - h)^{-1})^{-1}$.

(b) $c = v = \infty$ and $d = a + h$.

(c) $h = \infty$, $c = v$ and $c = (va + 1)a^{-1}$.

Proof: Arguments similar to those in Theorem 5.3.1 are used. □

Remark 5.3.3. *The construction for $\Gamma_{(\infty,b)}$ takes similar argument in which case a is substituted by b .*

Example 3. $\Gamma_{(\infty,1)}$ for $PSL(2,5)$

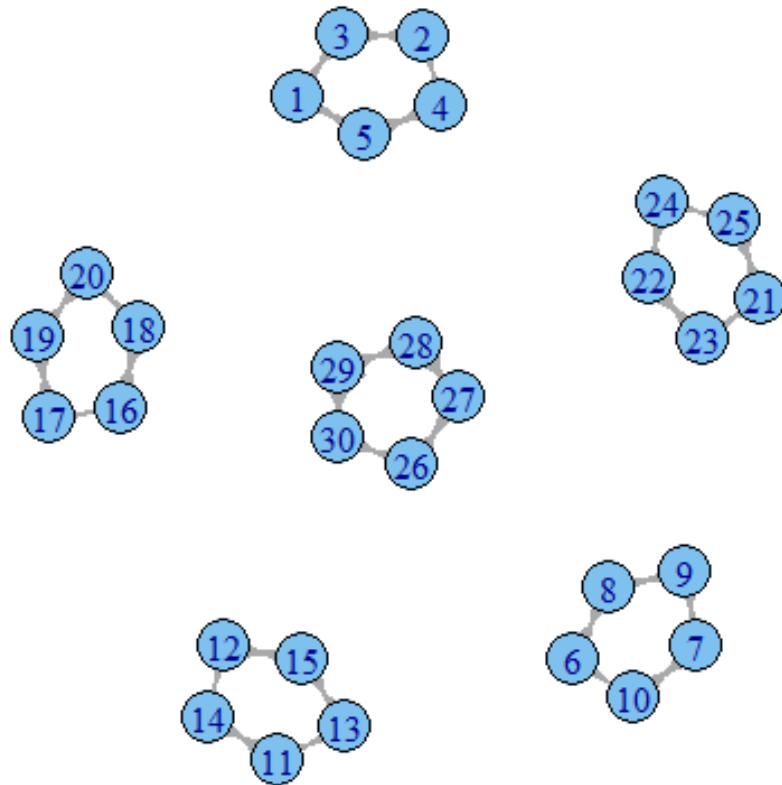


Figure 5.3: Suborbital graph $\Gamma_{(\infty,1)}$ corresponding to the suborbit Δ_6

Theorem 5.3.4. *When $q \equiv 1 \pmod{4}$, $\Gamma_{(\infty,1)}$ and $\Gamma_{(1,0)}$ are self-paired.*

Proof: This follows from Theorem 5.2.5

Case IV: Suborbital graphs corresponding to suborbits of G formed by pairs of points of the form (a, ∞) where a is a square and (b, ∞) where b is not a square

Theorem 5.3.5. $((v, h), (c, d))$ is an edge in $\Gamma_{(a, \infty)}$ for each of the following cases and only for these.

- (a) $v \neq \infty, h \neq \infty, d = v$ and $c = (va + h(v - h)^{-1})(a + (v - h)^{-1})^{-1}$.
- (b) $d = v = \infty$ and $c = a + h$.
- (c) $h = \infty, d = v$ and $c = (va + 1)a^{-1}$.

Proof: Arguments similar to those in Theorem 5.3.1 are used. □

Remark 5.3.4. The construction for $\Gamma_{(b, \infty)}$ is similarly done replacing b for a .

Example 4. $\Gamma_{(1, \infty)}$ for $PSL(2, 5)$

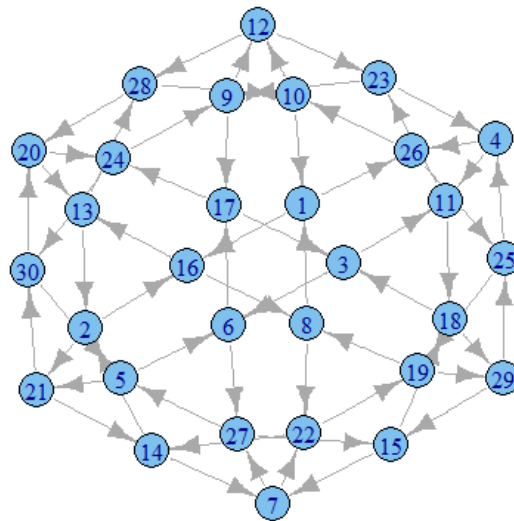


Figure 5.4: Suborbital graph $\Gamma_{(1, \infty)}$ corresponding to the suborbit Δ_7

Theorem 5.3.6. *When $q \equiv 1 \pmod{4}$, $\Gamma_{(0,\alpha)}$ is paired with $\Gamma_{(\alpha,\infty)}$.*

Proof: The result follows from Lemma 5.2.1. \square

Lemma 5.3.1. *The suborbital graphs $\Gamma_{(1,\infty)}$, $\Gamma_{(0,1)}$, $\Gamma_{(b,\infty)}$ and $\Gamma_{(0,b)}$ all have girth three.*

Proof: From definition of a suborbital graph, $((\infty, 0)(1, \infty))$ is an edge in $\Gamma_{(1,\infty)}$. Since $((\infty, 0)(0, 1))$ is an edge in $\Gamma_{(0,1)}$ and from Lemma 5.2.1, $\Delta_{(1,\infty)}$ is paired with $\Delta_{(0,1)}$, $((0, 1)(\infty, 0))$ is an edge in $\Gamma_{(1,\infty)}$. Next, we find that $\begin{pmatrix} 1 & -1 \\ 1 & 0 \end{pmatrix} \in G$ maps the edge $((\infty, 0)(1, \infty))$ to $((1, \infty)(0, 1))$. Therefore $((1, \infty)(0, 1))$ is an edge in $\Gamma_{(1,\infty)}$. We therefore have a sequence of edges

$$(\infty, 0) \longrightarrow (1, \infty) \longrightarrow (0, 1) \longrightarrow (\infty, 0)$$

in $\Gamma_{(1,\infty)}$. So $\Gamma_{(1,\infty)}$ has a directed triangle and hence it is of girth three.

Since $\Gamma_{(0,1)}$ is paired with $\Gamma_{(1,\infty)}$, and constructions of $\Gamma_{(b,\infty)}$ and $\Gamma_{(0,b)}$ are similar to those of $\Gamma_{(1,\infty)}$ and $\Gamma_{(0,1)}$ respectively, similar arguments as used above show that all are of girth three. \square

Theorem 5.3.7. *The action of $PSL(2, q)$ on the cosets of $C_{\frac{q-1}{k}}$ is not primitive.*

Proof: This follows from Theorem 5.1.4. \square

Case V: Suborbital graphs corresponding to suborbits of G formed by pairs of points of the form $(0, \infty)$

Theorem 5.3.8. *$((v, h), (c, d))$ is an edge in $\Gamma_{(0,\infty)}$ only if $c = h$ and $d = v$.*

Proof: Arguments similar to those in Theorem 5.3.1 are used. \square

Example 5. $\Gamma_{(0,\infty)}$ for $PSL(2, 5)$; this is the suborbital graph corresponding to Δ_1 .

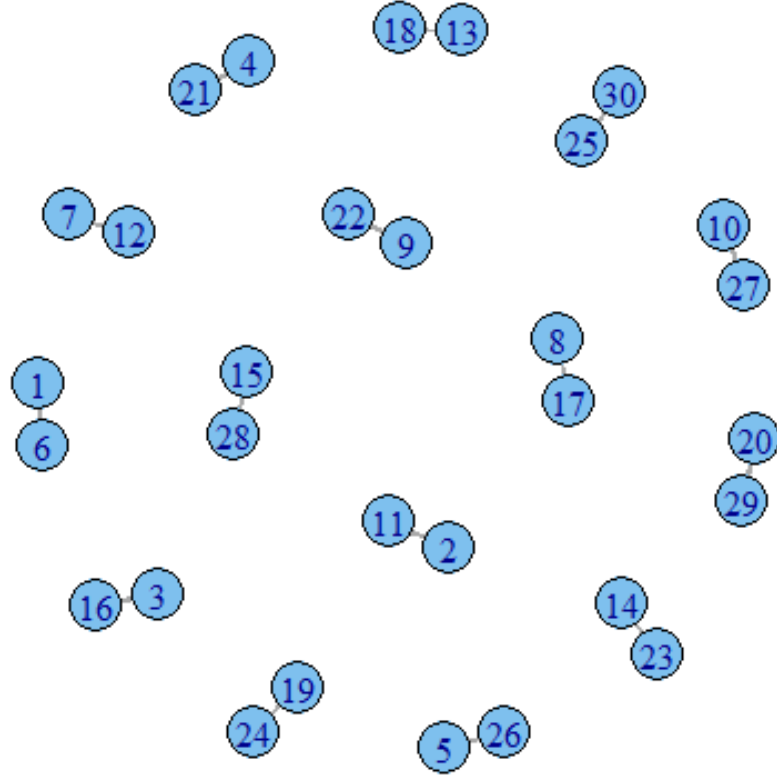


Figure 5.5: Suborbital graph $\Gamma_{(0,\infty)}$ corresponding to the suborbit Δ_1

Theorem 5.3.9. *The suborbital graph $\Gamma_{(0,\infty)}$ is regular of degree one.*

Proof: Using Theorem 5.3.8, any vertex is adjacent to only one other vertex. □

Theorem 5.3.10. *The number of connected components in the suborbital graph $\Gamma_{(0,\infty)}$ is $\binom{q+1}{2}$.*

Proof: From Theorem 5.3.9, it follows that the number of connected components is a half the number of vertices. □

Theorem 5.3.11. *The girth of the suborbital graph $\Gamma_{(0,\infty)}$ is zero.*

Proof: From Theorem 5.3.9, $\Gamma_{(0,\infty)}$ is a forest. □

Corollary 5.3.1. *The action of G on the cosets of $C_{\frac{q-1}{k}}$ is imprimitive.*

This follows from Theorem 5.3.11 and Theorem 5.1.4.

Theorem 5.3.12. *The suborbital graph $\Gamma_{(0,\infty)}$ is undirected.*

Proof: Using Theorem 5.1.3, this is a consequence of the self-paired suborbit $\Delta_{(0,\infty)}$ to which this suborbital graph corresponds. □

CHAPTER 6

CONCLUSION AND RECOMMENDATIONS

6.1 Conclusion

The study was set out to investigate the permutation representations of $PSL(2, q)$ on the cosets of its cyclic subgroups $C_{\frac{q-1}{k}}$ and $C_{\frac{q+1}{k}}$, elementary abelian subgroup P_q , alternating subgroup A_4 and dihedral subgroup $D_{\frac{2(q-1)}{k}}$. The outcomes of the research are contained in Chapters 3, 4 and 5.

In Chapter 3, we determined disjoint cycle structures corresponding to the permutation representations using Lemma 1.6.2. The disjoint cycle structures obtained were used together with Theorem 3.1.1 and 3.1.2 to come up with cycle index formulas. From the specific examples given, it can be shown that these permutation representations correspond to the cycle index of the action of A_4 and A_5 on the cosets of their relevant subgroups, which alludes to the fact that $PSL(2, 3) \cong A_4$ and $PSL(2, 4) \cong PSL(2, 4) \cong A_5$. (Huppert, 1967, Sec.6.14)

In Chapter 4, the disjoint cycle structures obtained in Chapter 3 together with Lemma 1.6.1 (Cauchy-Fröbenius lemma), were used to determine the rank and subdegrees of $PSL(2, q)$ for each permutation representation. This was done first using algebraic arguments and later in the chapter, confirmed using a method that uses table of marks of the respective subgroups. For instance,

the subdegrees of $PSL(2, q)$ acting on the cosets of its dihedral subgroup $D_{\frac{2(q-1)}{k}}$ with $\frac{q-1}{k}$ odd and $q \equiv -1 \pmod{4}$, were found to be $1^{(1)}$, $(\frac{q-1}{2})^{\frac{(q-1)}{2}}$ and $(q-1)^{\frac{(q+5)}{4}}$.

In Chapter 5, subdegrees of $PSL(2, q)$ acting on the cosets of its cyclic subgroup $C_{\frac{q-1}{k}}$ were determined using reduced pair group action. In view of Theorem 5.2.1, a construction of suborbital graphs was given using suborbits obtained from pair group action. The number n_π of self-paired suborbits was found to be

$$n_\pi = \begin{cases} q+2, & p=2 \\ q+3, & q \equiv 1 \pmod{4} \\ q+1, & q \equiv -1 \pmod{4} \end{cases} .$$

The orbit $\Delta_{(0,1)}$ was found to be paired with $\Delta_{(1,\infty)}$. Lastly, the connected suborbital graphs were all found to be of girth three.

6.2 Recommendations

In the course of this study, only five subgroups of $PSL(2, q)$ were considered. Since Tchuda (1986) and Kamuti (1992) handled maximal subgroups, more is yet to be done on none maximal subgroups. According to Theorem 1.2.2, this will be quite a substantial amount of work which can be exhaustively tackled if a programming language is put to use. Groups, Algorithms and Programming (GAP) language can be handy in this case since $PSL(2, q)$ is a finite group. This will resonate very well with the method that uses table of marks in the determination of subdegrees and hence rank.

Therefore, in this study we considered permutation representation using algebraic arguments with little use of programming. With good knowledge in GAP programming language and in graph packages such as igraph and ggplot2, the study can be more interesting and exhaustive.

We also dealt with $PSL(2, q)$ acting on the cosets of its subgroups. Similar research can be carried out on other groups $PSL(n, q)$ where $n > 2$.

We gave a construction of suborbital graphs corresponding to suborbits of the form $\Delta_{(a,b)}$ where a and b is either 0 or ∞ . The case where both a and b are not equal to 0 or ∞ is open.

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APPENDIX

VERTICES AND EDGES OF THE CONSTRUCTED SUBORBITAL GRAPHS

Suborbital graph $\Gamma_{(0,a)}$

i.) Vertices of $\Gamma_{(0,a)}$

1 – (0, 1)	7 – (1, 2)	13 – (2, 3)	19 – (3, 4)	25 – (4, ∞)
2 – (0, 2)	8 – (1, 3)	14 – (2, 4)	20 – (3, ∞)	26 – (∞ , 0)
3 – (0, 3)	9 – (1, 4)	15 – (2, ∞)	21 – (4, 0)	27 – (∞ , 1)
4 – (0, 4)	10 – (1, ∞)	16 – (3, 0)	22 – (4, 1)	28 – (∞ , 2)
5 – (0, ∞)	11 – (2, 0)	17 – (3, 1)	23 – (4, 2)	29 – (∞ , 3)
6 – (1, 0)	12 – (2, 1)	18 – (3, 2)	24 – (4, 3)	30 – (∞ , 4)

ii.) Edges of $\Gamma_{(0,a)}$

$((0, 1), (1, 3))$	$((0, 2), (2, 3))$	$((0, 3), (3, 1))$	$((\infty, 4), (4, 3))$
$((0, 4), (4, 2))$	$((0, \infty), (\infty, 1))$	$((1, 0), (0, 3))$	$((\infty, 3), (3, 2))$
$((1, 2), (2, 4))$	$((1, 3), (3, 0))$	$((1, 4), (4, 2))$	$((\infty, 2), (2, 1))$
$((1, \infty), (\infty, 0))$	$((2, 0), (0, 3))$	$((2, 1), (1, 4))$	$((\infty, 1), (1, 0))$
$((2, 3), (3, 0))$	$((2, 4), (4, 0))$	$((2, \infty), (\infty, 3))$	$((\infty, 0), (0, 4))$
$((3, 0), (0, 1))$	$((3, 1)(1, 0))$	$((3, 2)(2, 0))$	$((4, \infty), (\infty, 0))$
$((3, 4), (4, 1))$	$((3, \infty), (\infty, 4))$	$((0, 1), (1, \infty))$	$((4, \infty), (\infty, 0))$
$((0, 2), (2, 4))$	$((0, 3), (3, 2))$	$((0, 4), (4, \infty))$	$((3, 1), (1, 4))$
$((0, \infty), (\infty, 4))$	$((1, 0), (0, \infty))$	$((1, 2), (2, \infty))$	$((3, 2), (2, \infty))$
$((1, 3), (3, 4))$	$((1, 4), (4, 3))$	$((1, \infty), (\infty, 2))$	$((\infty, 4), (4, 0))$
$((2, 0), (0, 4))$	$((2, 1), (1, \infty))$	$((2, 3), (3, \infty))$	$((\infty, 3), (3, 4))$
$((2, 4), (4, 1))$	$((2, \infty)(\infty, 1))$	$((3, 0), (0, 2))$	$((\infty, 2), (2, 3))$
$((3, \infty), (\infty, 2))$	$((4, 0), (0, \infty))$	$((4, 1), (1, 3))$	$((\infty, 1), (1, 0))$
$((4, 2), (2, 1))$	$((4, 3)(3, \infty))$	$((4, \infty)(\infty, 2))$	$((\infty, 0), (0, 1))$
$((4, 1), (1, 2))$	$((4, 2)(2, 0))$	$((4, 3)(3, 1))$	$((3, 4), (4, \infty))$

iii.) Code of $\Gamma_{(0,a)}$ for R

$m = matrix(c(1, 8, 1, 10, 2, 13, 2, 14, 3, 17, 3, 18, 4, 23, 4, 25, 5, 27, 5, 30, 6, 3, 6, 5, 7,$
 $14, 7, 15, 8, 16, 8, 19, 9, 23, 9, 24, 10, 26, 10, 28, 11, 3, 11, 4, 12, 9, 12, 10, 13, 16, 13, 20,$
 $14, 21, 14, 22, 15, 27, 15, 29, 16, 1, 16, 2, 17, 6, 17, 9, 18, 11, 18, 15, 19, 22, 19, 25, 20,$
 $28, 20, 30, 21, 2, 21, 5, 22, 7, 22, 8, 23, 11, 23, 12, 24, 17, 24, 20, 25, 26, 25, 29, 26, 1, 26,$

4, 27, 6, 27, 7, 28, 12, 28, 13, 29, 18, 29, 19, 30, 21, 30, 24), $nc = 2$, $byrow = TRUE$)

Suborbital graph $\Gamma_{(a,0)}$

i.) Vertices of $\Gamma_{(a,0)}$

1 – (0, 1)	7 – (1, 2)	13 – (2, 3)	19 – (3, 4)	25 – (4, ∞)
2 – (0, 2)	8 – (1, 3)	14 – (2, 4)	20 – (3, ∞)	26 – (∞ , 0)
3 – (0, 3)	9 – (1, 4)	15 – (2, ∞)	21 – (4, 0)	27 – (∞ , 1)
4 – (0, 4)	10 – (1, ∞)	16 – (3, 0)	22 – (4, 1)	28 – (∞ , 2)
5 – (0, ∞)	11 – (2, 0)	17 – (3, 1)	23 – (4, 2)	29 – (∞ , 3)
6 – (1, 0)	12 – (2, 1)	18 – (3, 2)	24 – (4, 3)	30 – (∞ , 4)

ii.) Edges of $\Gamma_{(a,0)}$

((0, 1), (3, 1))	((0, 1), (∞ , 1))	((0, 2), (3, 2))	((0, 2), (4, 2))
((0, 3), (2, 3))	((0, 4), (2, 4))	((0, 4), (∞ , 4))	((0 ∞), (1, ∞))
((1, 0), (3, 0))	((1, 0), (∞ , 0))	((1, 2), (4, 2))	((1, 2), (∞ , 2))
((1, 3), (4, 3))	((1, 4), (2, 4))	((1, 4), (3, 4))	((1, ∞), (0, ∞))
((2, 0), (3, 0))	((2, 0), (4, 0))	((2, 1), (4, 1))	((2, 1), (∞ , 1))
((2, 2), (∞ , 3))	((2, 4), (0, 4))	((2, 4), (1, 4))	((2, ∞), (1, ∞))
((0, 3), (1, 0))	((3, 0), (2, 0))	((3, 1), (0, 1))	((3, 1), (4, 1))
((3, 2), (∞ , 2))	((3, 4), (1, 4))	((3, 4), (∞ , 4))	((3, ∞), (2, ∞))
((4, 0), (2, 0))	((4, 0), (∞ , 0))	((4, 1), (2, 1))	((4, 1), (3, 1))
((4, 2), (1, 2))	((4, 3), (1, 3))	((4, 3), (∞ , 3))	((4, ∞), (0, ∞))
((∞ , 0), (1, 0))	((∞ , 0), (4, 0))	((∞ , 1), (2, 1))	((∞ , 1), (0, 1))

$$\begin{array}{cccc}
((\infty, 2), (1, 2)) & ((\infty, 3), (4, 3)) & ((\infty, 3), (2, 3)) & ((\infty, 4), (0, 4)) \\
((0, 3), (1, 3)) & ((0, \infty), (4, \infty)) & ((1, 3), (0, 3)) & ((1, \infty), (2, \infty)) \\
((2, 3), (0, 3)) & ((2, \infty), (3, \infty)) & ((3, 2), (0, 2)) & ((3, \infty), (4, \infty)) \\
((4, 2), (0, 2)) & ((4, \infty), (3, \infty)) & ((\infty, 2), (3, 2)) & ((\infty, 4), (3, 4))
\end{array}$$

iii.) Code of $\Gamma_{(a,0)}$ for R

$m = \text{matrix}(c(1, 17, 1, 27, 2, 18, 2, 23, 3, 8, 3, 13, 4, 14, 4, 30, 5, 10, 5, 25, 6, 16, 6, 26, 7,$
 $23, 7, 28, 8, 3, 8, 24, 9, 14, 9, 19, 10, 5, 10, 15, 11, 16, 11, 21, 12, 22, 12, 27, 13, 3, 13, 29,$
 $14, 4, 14, 9, 15, 10, 15, 20, 16, 6, 16, 11, 17, 1, 17, 22, 18, 2, 18, 28, 19, 9, 19, 30, 20, 15,$
 $20, 25, 21, 11, 21, 26, 22, 12, 22, 17, 23, 2, 23, 7, 24, 8, 24, 29, 25, 5, 25, 20, 26, 6, 26,$
 $21, 27, 12, 27, 1, 28, 18, 28, 7, 29, 24, 29, 13, 30, 4, 30, 19), nc = 2, byrow = TRUE)$

Suborbital graph $\Gamma_{(\infty,a)}$

i.) Vertices of $\Gamma_{(\infty,a)}$

$$\begin{array}{ccccc}
1 - (0, 1) & 7 - (1, 2) & 13 - (2, 3) & 19 - (3, 4) & 25 - (4, \infty) \\
2 - (0, 2) & 8 - (1, 3) & 14 - (2, 4) & 20 - (3, \infty) & 26 - (\infty, 0) \\
3 - (0, 3) & 9 - (1, 4) & 15 - (2, \infty) & 21 - (4, 0) & 27 - (\infty, 1) \\
4 - (0, 4) & 10 - (1, \infty) & 16 - (3, 0) & 22 - (4, 1) & 28 - (\infty, 2) \\
5 - (0, \infty) & 11 - (2, 0) & 17 - (3, 1) & 23 - (4, 2) & 29 - (\infty, 3) \\
6 - (1, 0) & 12 - (2, 1) & 18 - (3, 2) & 24 - (4, 3) & 30 - (\infty, 4)
\end{array}$$

ii.) Edges of $\Gamma_{(\infty, a)}$

$((0, 1), (0, 3))$	$((1, 2), (1, \infty))$	$((2, 3), (2, 0))$	$((3, 4), (3, \infty))$	$((\infty, 0), (\infty, 1))$
$((0, 1), (0, \infty))$	$((1, 2), (1, 4))$	$((2, 3), (2, \infty))$	$((3, 4), (3, 1))$	$((\infty, 0), (\infty, 4))$
$((0, 2), (0, 4))$	$((1, 3), (1, 4))$	$((2, 4), (2, 0))$	$((3, \infty), (3, 4))$	$((\infty, 1), (\infty, 2))$
$((0, 2), (0, 3))$	$((1, 3), (1, 0))$	$((2, 4), (2, 1))$	$((3, \infty), (3, 2))$	$((\infty, 1), (\infty, 0))$
$((0, 3), (0, 2))$	$((1, 4), (1, 3))$	$((2, \infty), (2, 3))$	$((4, 0), (4, 2))$	$((\infty, 2), (\infty, 1))$
$((0, 3), (0, 1))$	$((1, 4), (1, 2))$	$((2, \infty), (2, 1))$	$((4, 0), (4, \infty))$	$((\infty, 2), (\infty, 3))$
$((0, 4), (0, 2))$	$((1, \infty), (1, 2))$	$((3, 0), (3, 1))$	$((4, 1), (4, 2))$	$((\infty, 3), (\infty, 2))$
$((0, 4), (0, \infty))$	$((1, \infty), (1, 0))$	$((3, 0), (3, 2))$	$((4, 1), (4, 3))$	$((\infty, 3), (\infty, 4))$
$((0, \infty), (0, 1))$	$((2, 0), (2, 4))$	$((3, 1), (3, 4))$	$((4, 2), (4, 0))$	$((\infty, 4), (\infty, 3))$
$((0, \infty), (0, 4))$	$((2, 0), (2, 3))$	$((3, 1), (3, 0))$	$((4, 2), (4, 1))$	$((\infty, 4), (\infty, 3))$
$((1, 0), (1, 3))$	$((2, 1), (2, \infty))$	$((3, 2), (3, 0))$	$((4, 3), (4, 1))$	$((4, \infty), (4, 0))$
$((1, 0), (1, \infty))$	$((2, 1), (2, 4))$	$((3, 2), (3, \infty))$	$((4, 3), (4, \infty))$	$((4, \infty), (4, 3))$

iii.) Code of $\Gamma_{(\infty, a)}$ for R

```

m = matrix(c(1, 3, 1, 5, 2, 3, 2, 4, 3, 1, 3, 2, 4, 2, 4, 5, 5, 1, 5, 4, 6, 8, 6, 10, 7, 9, 7, 10, 8,
6, 8, 9, 9, 7, 9, 8, 10, 6, 10, 7, 11, 13, 11, 14, 12, 14, 12, 15, 13, 11, 13, 15, 14, 11, 14, 12,
15, 12, 15, 13, 16, 17, 16, 18, 17, 16, 17, 19, 18, 16, 18, 20, 19, 17, 19, 20, 20, 18, 20, 19,
21, 23, 21, 25, 22, 23, 22, 24, 23, 21, 23, 22, 24, 22, 24, 25, 25, 21, 25, 24, 26, 27, 26, 30,
27, 28, 27, 26, 28, 29, 28, 27, 29, 30, 29, 28, 30, 26, 30, 29), nc = 2, byrow = TRUE)

```

Suborbital graph $\Gamma_{(a, \infty)}$

i.) Vertices of $\Gamma_{(a,\infty)}$

1 – (0, 1)	7 – (1, 2)	13 – (2, 3)	19 – (3, 4)	25 – (4, ∞)
2 – (0, 2)	8 – (1, 3)	14 – (2, 4)	20 – (3, ∞)	26 – (∞ , 0)
3 – (0, 3)	9 – (1, 4)	15 – (2, ∞)	21 – (4, 0)	27 – (∞ , 1)
4 – (0, 4)	10 – (1, ∞)	16 – (3, 0)	22 – (4, 1)	28 – (∞ , 2)
5 – (0, ∞)	11 – (2, 0)	17 – (3, 1)	23 – (4, 2)	29 – (∞ , 3)
6 – (1, 0)	12 – (2, 1)	18 – (3, 2)	24 – (4, 3)	30 – (∞ , 4)

ii.) Edges of $\Gamma_{(a,\infty)}$

((0, 1), (0, 3))	((1, 2), (∞ , 1))	((2, 3), (0, 2))	((3, 4), (∞ , 3))	((∞ , 0), (1, ∞))
((0, 1), (∞ , 0))	((1, 2), (4, 1))	((2, 3), (∞ , 2))	((3, 4), (1, 3))	((∞ , 0), (4, ∞))
((0, 2), (4, 0))	((1, 3), (4, 1))	((2, 4), (0, 2))	((3, ∞), (4, 3))	((∞ , 1), (2, ∞))
((0, 2), (3, 0))	((1, 3), (0, 1))	((2, 4), (1, 2))	((3, ∞), (2, 3))	((∞ , 1), (0, ∞))
((0, 3), (2, 0))	((1, 4), (3, 1))	((2, ∞), (3, 2))	((4, 0), (2, 4))	((∞ , 2), (1, ∞))
((0, 3), (1, 0))	((1, 4), (2, 1))	((2, ∞), (1, 2))	((4, 0), (3, 4))	((∞ , 2), (3, ∞))
((0, 4), (2, 0))	((1, ∞), (2, 1))	((3, 0), (1, 3))	((4, 1), (2, 4))	((∞ , 3), (2, ∞))
((0, 4), (∞ , 0))	((1, ∞), (0, 1))	((3, 0), (2, 3))	((4, 1), (3, 4))	((∞ , 3), (4, ∞))
((0, ∞), (1, 0))	((2, 0), (4, 2))	((3, 1), (4, 3))	((4, 2), (0, 4))	((∞ , 4), (3, ∞))
((0, ∞), (4, 0))	((2, 0), (3, 2))	((3, 1), (0, 3))	((4, 2), (1, 4))	((∞ , 4), (0, ∞))
((1, 0), (3, 1))	((2, 1), (∞ , 2))	((3, 2), (0, 3))	((4, 3), (1, 4))	((4, ∞), (0, 4))
((1, 0), (∞ , 1))	((2, 1), (4, 2))	((3, 2), (∞ , 3))	((4, 3), (∞ , 4))	((4, ∞), (2, 4))

iii.) Code of $\Gamma_{(a,\infty)}$ for R

$$m = \text{matrix}(c(1, 16, 1, 26, 2, 16, 2, 21, 3, 6, 3, 11, 4, 11, 4, 26, 5, 6, 5, 21, 6, 17, 6, 27,$$

7, 27, 7, 22, 8, 1, 8, 22, 9, 12, 9, 17, 10, 1, 10, 12, 11, 18, 11, 23, 12, 28, 12, 23, 13, 2, 13,
 28, 14, 2, 14, 7, 15, 7, 15, 18, 16, 8, 16, 13, 17, 3, 17, 24, 18, 3, 18, 29, 19, 8, 19, 29, 20, 13,
 20, 24, 21, 14, 21, 30, 22, 14, 22, 19, 23, 4, 23, 9, 24, 9, 24, 30, 25, 4, 25, 19, 26, 10, 26,
 25, 27, 5, 27, 15, 28, 10, 28, 20, 29, 15, 29, 25, 30, 5, 30, 20), $nc = 2$, $byrow = TRUE$)

Suborbital graph $\Gamma_{(0,\infty)}$

i.) Vertices of $\Gamma_{(0,\infty)}$

1 – (0, 1)	7 – (1, 2)	13 – (2, 3)	19 – (3, 4)	25 – (4, ∞)
2 – (0, 2)	8 – (1, 3)	14 – (2, 4)	20 – (3, ∞)	26 – (∞ , 0)
3 – (0, 3)	9 – (1, 4)	15 – (2, ∞)	21 – (4, 0)	27 – (∞ , 1)
4 – (0, 4)	10 – (1, ∞)	16 – (3, 0)	22 – (4, 1)	28 – (∞ , 2)
5 – (0, ∞)	11 – (2, 0)	17 – (3, 1)	23 – (4, 2)	29 – (∞ , 3)
6 – (1, 0)	12 – (2, 1)	18 – (3, 2)	24 – (4, 3)	30 – (∞ , 4)

ii.) Edges of $\Gamma_{(0,\infty)}$

$((0, 1), (1, 0))$	$((1, 2), (2, 1))$	$((2, 3), (3, 2))$	$((3, 4), (4, 3))$
$((0, 2), (2, 0))$	$((1, 3), (3, 1))$	$((2, 4), (4, 2))$	$((3, \infty), (\infty, 3))$
$((0, 3), (3, 0))$	$((1, 4), (4, 1))$	$((2, \infty), (\infty, 2))$	$((4, 0), (0, 4))$
$((0, 4), (4, 0))$	$((1, \infty), (\infty, 1))$	$((3, 0), (0, 3))$	$((4, 1), (1, 4))$
$((0, \infty), (\infty, 0))$	$((2, 0), (0, 2))$	$((3, 1), (1, 3))$	$((4, 2), (2, 4))$
$((1, 0), (0, 1))$	$((2, 1), (1, 2))$	$((3, 2), (2, 3))$	$((4, 3), (3, 4))$
$((4, \infty), (\infty, 4))$	$((\infty, 0), (0, \infty))$	$((\infty, 1), (1, \infty))$	$((\infty, 2), (2, \infty))$
$((\infty, 3), (3, \infty))$	$((\infty, 4), (4, \infty))$		

iii.) Code of $\Gamma_{(a,\infty)}$ for R

```
m = matrix(c(1, 6, 2, 11, 3, 16, 4, 21, 5, 26, 6, 1, 7, 12, 8, 17, 9, 22, 10, 27, 11, 2,  
12, 7, 13, 18, 14, 23, 15, 28, 16, 3, 17, 8, 18, 13, 19, 24, 20, 29, 21, 4, 22, 9, 23,  
14, 24, 19, 25, 30, 26, 5, 27, 10, 28, 15, 29, 20, 30, 25), nc = 2, byrow = TRUE)
```