# THE ACTION OF SYMMETRY GROUPS OF PLATONIC SOLIDS ON THEIR RESPECTIVE VERTICES

WAWERU R. BRIAN (B.Sc.)
Reg. No. I56/37108/2017

A Research Project Submitted in Partial Fulfillment of the Requirements for the Award of the Degree of Master of Science (Pure Mathematics) in the School of Pure and Applied Sciences of
Kenyatta University

December 2020

# Declaration

This research project is my original work and has not been presented for a degree award in any other university or for any other award.

**WAWERU R. BRIAN** (B.Sc.)

Sign:__________________________ Date:__________________

## Supervisor

**Dr. Bunyasi F. M.**

Sign:__________________________ Date:__________________

Department of Mathematics
Kenyatta University

# Dedication

This work is dedicated to the community of researchers and math enthusiasts. It is my hope that this work will be used in improving understanding on the fascinating platonic solids.

*"Mathematics is the language in which God has written the universe" - Galileo Galilei*

# Acknowledgments

I sincerely acknowledge my friends and family for their unrelenting support during the course of my studies. I am also greatly indebted to my supervisor, Dr.Magero, who tirelessly supported and guided me through this research. His insights, encouragement and availability for consultations have enabled me to not only complete my research, but also acquire valuable intellectual and research skills which I will strive to maintain in the future. I would also like to extend my gratitude to the Chairperson of the Mathematics Department, together with other lecturers in the department who have rightly imparted knowledge on me.

Above all, I thank God for giving me strength, endurance, encouragement and provision during the entire period of my studies.

# NOTATIONS

| | |
|---|---|
| $A_n$ | Alternating group of $\frac{n!}{2}$ elements |
| $\cong$ | Congruency or isomorphic to |
| $*$ | Group operation |
| $e$ | The identity element |
| $\in$ | Member of |
| $G$ | A group set of elements |
| $\emptyset$ | An empty set |
| $\Delta_i, \delta_i, \Lambda_i, \lambda_i$ | Set of elements |
| $\mathbb{R}$ | Set of real numbers |
| $\varphi$ | Group action of a group |
| $S_n$ | Symmetry group of $n!$ elements |
| $S(O)$ | Full symmetry group |
| $S_d(O)$ | Direct symmetry group |
| $X$ | Set of elements |
| $\|X\|$ | The cardinality of a set $X$ |
| $\mathbb{Z}_n$ | Set of addition modulo $n$ elements |
| $\mathbb{G}_r$ | Rotational group of symmetry |
| $\mathbb{G}_f$ | Reflection-al group of symmetry |
| $\mathbb{G}$ | Full symmetry group of a solid |
| $[\ v_a, v_b, ..., v_c\ ]$ | Face contoured by vertices $v_a, v_b$ to $v_c$ of a solid |
| $v_a \longleftrightarrow v_b$ | An axis from vertices $v_a$ and $v_b$ |
| $v_i \boxtimes v_j \boxtimes v_k \boxtimes v_l$ | A plane through the center of a solid and vertices $v_i, v_j, v_k$ and $v_l$ |

# Contents

# List of Tables

# List of Figures

# Abstract

Platonic solids are 3-dimensional regular, convex polyhedrons. Each of the faces are equidistant and equiangular to each other in any of the solids. They derive their name from the ancient Greek philosopher, $Plato$ who wrote about them in his dialogue, the $Timaeus$ as reported by Cornford (2014). The solids features have fascinated mathematicians for decades including the renown geometer, $Euclid$: In his $Book\ XIII$ of the $Elements$, as rewrote by Heath et al. (1956), he successfully determined the exact number of solids that qualify to be Platonic Solids; tetrahedron, cube, octahedron, dodecahedron and icosahedron. In group theory, the symmetry group of an object is the group of all transformations under which the object remains unchanged, endowed with the group operation of composition. Due to their inherent symmetry of these solids many mathematicians have attempted to derive their symmetry groups. For instant, Foster (1990) who successfully enumerated the symmetry groups of the dodecahedron and recently Morandi (2004) attempted to compute these symmetric groups of the solids using a computer program called $Maple$. Although such contributions are noteworthy, a few attempts have been made to explore other features such as the symmetry groups of the platonic solids. Thus, this project investigates the properties of the group action of the symmetry groups of these platonic solids acting on their respective vertices. We embark on constructing the symmetry groups of each of the solids then employ the orbit-stabilizer and other theorems to determine the ranks and sub-degrees of each solid. The action of $\mathbb{G}$ on $V$ shows that tetrahedron has a rank of 2, the octahedron has a rank of 3, dodecahedron has a rank of 6 while the cube and icosahedron have a rank of 4.

# Chapter 1

# INTRODUCTION


Figure 1.1: Platonic solids

## 1.1 Background information

A Platonic Solid is a 3-D convex polyhedra which is made up of faces of the same regular polygon and the same number of polygons meet at each vertex (corner) as shown above on figure 1.1. Each of the solid is named by the number of faces it possesses. These solids have been subject of fascination to mathematicians across the decades due to their regular and symmetrical properties. Such fascination of these solids led both $Plato$ and $Kepler$ to use them in their theories of the cosmos. Their aesthetic properties had attracted renaissance artists and craftsmen such as Leonardo da Vinci drawings with the Hexahedron(cube). This was captured from his sketches and outlined by Williams and Xavier (2011).They are also associated with ideas in many areas of modern mathematics from algebra of group theory to the study of geometric singularities.

The ancient Greek philosopher, $Plato$ wrote about them in his dialogue $Timaeus$ (360 B.C.). This is recounted by Cornford (2014) in which the author associated the platonic solids with the 5 classical elements; air, water, fire, earth and aether. He considered what makes the elements different from each other. The Table below **1.1** shows the distinctive properties of each of the platonic solids.

Table 1.1: Distinctive properties of platonic solids

| | Faces | Vertices | Edges | Edges per Face | Dual |
|---|---|---|---|---|---|
| Tetrahedron | 4 | 4 | 6 | 3 | Tetrahedron |
| Cube | 6 | 8 | 12 | 4 | Octahedron |
| Octahedron | 8 | 6 | 12 | 3 | Cube |
| Dodecahedron | 12 | 20 | 30 | 5 | Icosahedron |
| Icosahedron | 20 | 12 | 30 | 3 | Dodecahedron |

## 1.2 Definitions

**Definition 1.2.1.** *A dual of a platonic solid is a similar solid formed by forming vertices above the centres of each face and then connecting them with the adjacent vertices.*

**Definition 1.2.2.** *Permutations of a set $X$ can be defined as a bijection from the set $X$ onto itself. All permutations of a set with $n$ elements form a symmetric group, denoted $S_n$, where the group operation is composition of functions.*

**Definition 1.2.3.** *Permutation of a finite set is either even or odd depending on whether it can be expressed as the product of an even or odd number of transpositions.*

**Definition 1.2.4.** *A subgroup of $S_n$ consisting of all even permutations of $S_n$ is called the Alternating group of degree $n$ and it is denoted by $A_n$ and has an order of $|A_n| = \frac{n!}{2}$.*

**Definition 1.2.5.** *(Lim, 2008) Let $X$ be an object in $\mathbb{R}^3$. The symmetry axes $l$ of the object $X$ are lines about which there exists $\theta \in (0, 2\pi)$ such that the object $X$ can be brought, by rotating through an angle $\theta$, to a new orientation $X_\theta$, which appears to be identical to $X$. The symmetry planes $\wp$ of $X$ are imaginary mirrors in which the object $X$ can be reflected while appearing unchanged.*

**Definition 1.2.6.** *Any rotation of a solid $Y$ about $\theta$ that leaves the solid invariant and all the vertices back to their original positions composes the identity element, $g_1 = I$. e.g. $\theta = 360°$.*

**Definition 1.2.7.** *If $G$ is a group and $X$ is a set, then a (left) group action $\varphi$ of $G$ on $X$ is a function*

$$\varphi\colon G \times X \to X\colon (g, x) \mapsto \varphi(g, x)$$

*that satisfies the following two axioms (where we denote $\varphi(g, x)$ as $gx$):*

- *$ex = x$ for all $x$ in $X$. (Here, $e$ denotes the identity element of the group $G$.)*
- *$(gh)x = g(hx)$ for all $g, h \in G$ and all $x \in X$.*

**Definition 1.2.8.** *Let $G$ act on the set $X$ and let $x \in X$. Then an orbit of the set $X$ in $G$ is the set*

$$Orb_G(x) = \{gx \mid g \in G\}$$

**Definition 1.2.9.** *Let $G$ act on a set $X$ and let $x \in X$. The stabilizer of $x$ is the set given by*

$$Stab_G(x) = \{g \in G | gx = x\}$$

*This is a subgroup of $G$ denoted by $G_x$. $G_x$ is also referred to as the isotropy group.*

**Definition 1.2.10.** *(Kamuti, 1992) If an action of group has only $1$ orbit, then such an action is said to be* transitive. *In other words, $G$ is said to act transitively on $X$ if for every pair $x, y$ in $X$ there exist a $g \in G$ such that $gx = y$.*

**Definition 1.2.11.** *(Rose, 1978) Let $G$ be a group and $g \in G$. Let $X$ be a set with $x \in X$. Then the action of $G$ on $X$ is said to be* faithful *if $gx = x, \forall g \in G$ implies $g = e$ the identity element in $G$.*

**Definition 1.2.12.** *Let a group $G$ act transitively on a non-empty set $X$. The orbits of the isotropy $G_x$ of a point $x \in X$ is called* suborbits *of $G$ on $X$. The number, $R(G)$ of these sub-orbits is known as the* rank *of $G$ on $X$ and the lengths of the suborbits are called the* sub-degrees *of $G$ on $X$.*

**Definition 1.2.13.** *Let an action of a group $G$ on a set $X$ be transitive. Then the action is said to be regular if the order of the isotropy group is $1$ and it is the identity element i.e. if $Stab_G(x)$ = the identity in $G$.*

**Definition 1.2.14.** *Suppose that the action of a group $G$ on a non-empty finite set $X$ is transitive. Then a subset $\sigma$ of $X$ is referred to as a block of the action if, for each $g \in G$, either $g\sigma = \sigma$ or $g\sigma \cap \sigma = \emptyset$. In particular, $\emptyset, X$ and all $1$ -element subsets of $X$ are obvious blocks and they are referred to as trivial blocks. If the action of $G$ on the set $X$ has only trivial blocks then action is primitive and imprimitive otherwise.*

## 1.3 Preliminary results

**Theorem 1.3.1.** ***(Orbit-Stabilizer theorem)*** *(Rose, 1978) Let $G$ act on a set $X$ and let $x \in X$ then*

$$|Orb_G(x)| = |G : Stab_G(x)|$$

**Theorem 1.3.2.** *(Wielandt et al., 1964) Let $x \in X$, $|X| > 1$. A transitive group on $X$ is primitive if and only $G_x$ is a maximal group of $G$.*

**Theorem 1.3.3.** ***(Cauchy-Frobenius theorem)*** *Let $G$ act on a non-empty set. Then the number of orbits of $X$ is equal to*

$$\frac{1}{|G|}\sum_{g \in G} |fix(g)|$$

*where $fix(g)$ is the number of elements of $X$ fixed by $g \in G$ given by;*

$$fix(g) = \{x \in G \colon gx = x\} \, (Fraleigh, \; 2003)$$

## 1.4 Statement of the problem

The Platonic Solids have an innate regular symmetry. This symmetry has sparred ideas and relational links between disciplines across the decades from Geometry, arts and even astronomy. In lieu of this development, abstract algebra in group theory hasn't been left out. Most notably, Foster (1990) employing known definitions derived the symmetry groups of the dodecahedron. Despite efforts put forward by previous authors, not a lot has been done to explore the properties of group action of the symmetry groups on the respective vertices.

## 1.5 Objectives

**General objectives**

To investigate the properties of the symmetry groups of platonic solids acting on their respective vertices.

**Specific objectives**

1. To determine transitivity of the action of the symmetry groups on the vertices.
2. To investigate regularness of the action.
3. To establish primitivity of the action.
4. To derive the subdegrees and calculate the rank of the action.

## 1.6 Significance of study

In group theory, the symmetry group of a geometric object is the group of all transformations under which the object is invariant, endowed with the group operation of composition. In Mathematics, the Platonic Solids are interesting objects that, due to their natural symmetry, we can derive their symmetry groups from their symmetric features.

Through this research, we anticipate to determine new concepts which are valuable knowledge to researchers in the field of combinatorics. The knowledge of the action of symmetry groups on platonic solids and their behaviour have several real-world applications as well as in other fields of study. For instance, in minimizing the energy of point particles, it is obtained that particles cluster around the vertices of a polyhedron Atiyah and Sutcliffe (2003). There are several other applications of the knowledge of symmetries of platonic solids which can be applied in solving geometrical problems associated with point particles. The research will help further exploration of properties of regular polyhedra.

# Chapter 2

# LITERATURE REVIEW

In three-dimensional space, $\mathbb{R}^3$, a Platonic solid is a regular, convex polyhedron. It is constructed by congruent regular polygonal faces with the same number of faces meeting at each vertex.

The first known proof that of these solids was supplied by $Theaetetus$, a friend of $Plato$ as noted by Lim (2008). He gave a mathematical description that they were exactly five regular polyhedra in existence. Later Heath et al. (1956), recounts the work of $Euclid$ in $Book\ XIII$ of the $Elements$ where the author wrote a complete mathematical description of the solids, described how they could be constructed. The author proposed why there can be no more than five Platonic solids in total. He investigated their geometrical properties and argued that there are no further convex regular polyhedra.

Coxeter (1973) introduced $2$-dimensional and $3$-dimensional description of polyhedra. The author described a combinatorial definition of ”regularity” and applies it to show that there are no other convex regular polyhedra apart from the five Platonic solids. Furthermore, the author employs the concept of ”regularity” to extended to non-convex shapes such as star polyhedra.

A renown author, (Wenninger, 1974) wrote on the convex uniform polyhedra involving the Platonic and Archimedean Solids. Later the author gave a presentation of the complex set of uniform duals of uniform polyhedral shapes starting with the simplest convex solids, the tetrahedron.

Foster (1990) offered a proof of a well-known theorem that the group of symmetry $G$ of a dodecahedron is isomorphic to $A_5$ and that the dual of a dodecahedron was the icosahedron. Another author, (Cromwell, 1997) in briefly recounts the work laid previously by historical figures such as $Plato$ in his written dialogue, the $Timaeus$ and $Kepler$ who both attempted to relate the solids with the extraterrestrial planets known at that time. Cromwell (1997) also proved the properties of regular, convex polyhedrons and went ahead and described the link between these solids and the Archimedean solids as the families of convex polyhedra with regular faces. In addition, the author recounted the work of $Euclid$ (360 B.C.) on how the regular polygons can be constructed with the euclidean tools and proceeded to show a much vivid way of constructing models of the solids.

In $2004$, it was shown that symmetry groups of platonic solids are isomorphic to common permutation groups like $S_4, A_5$ and their direct products with $\mathbb{Z}_2$ using the computer program $Maple$ by Morandi (2004). Later, (Schwartz, 2007) attempted to label the vertices of the solids in order to discuss the permutation symmetry groups of Platonic Solids. An exposition of the Platonic Solids where the symmetry groups of the $5$ Platonic Solids was given by Lim (2008). Moreover, the researcher went ahead and proved the $Theaetetus'$ proposition of the precise number of

Platonic Solids. The same researcher described the general technique of determining the groups of symmetry of all $5$ Platonic solids explicitly with the application of *Dual Spaces* and *Central Inversion*. Furthermore the researcher added proofs to $Theaetetus$ and $Euclid$ propositions that there were exactly $5$ platonic solids with which the dual platonic solids make up the same symmetry group. Winger (2012) reduced the complex symmetries of the Platonic solids to a study of simple permutation groups. These authors derived similar propositions that the direct $S_d(O)$ and full symmetry $S(O)$ groups of the:-

1. Tetrahedron is isomorphic to $A_4$ and $S_4$, respectively.

2. Cube, and consequently Octahedron, are $S_4$ and $S_4 \times \mathbb{Z}_2$ respectively.

3. Dodecahedron, and correspondingly Icosahedron, are isomorphic to $A_5$ and $A_5 \times \mathbb{Z}_2$, respectively.

Another author, (Huisinga, 2012) discussed the use of Polya's counting theory as a spectacular tool that allowed one to count the number of distinct items given a certain characteristics and ultimately suggested that the Polya's theorem can be used to calculate the number of graphs up to isomorphism with a fixed number of vertices.

Silas (2011) in a seminar article attempted to discuss and describe the rotational symmetry of the platonic solids and the maps from the groups of symmetries $S_d(O)$ to the group of permutations $S(O)$ which the researcher showed that they were injective. Additionally the researcher found the order of the symmetry groups using the Orbit-Stabilizer theorem and deduced the images of the maps in addition to having showed that the group of symmetry of a dodecahedron was simple.

In 2004 (Cheung, 2014) described the symmetry groups of the platonic solids while exposing the order of both the rotational and reflection symmetry groups. The author states that the symmetry group, $S(O)$ of any of the Platonic Solid is isomorphic to $S_d(O) \times \mathbb{Z}_2$ excluding the tetrahedron. The researcher went along to show that the symmetry groups of dual platonic solids were the same.

Homans (2016) made significant strides having showed the automorphism groups of the platonic solids skeleton graphs and then showed that for any two solids which are duals of each other, they have an identical automorphism group. The researcher went ahead and indicated the relation of the group with graphs and then derived their order.

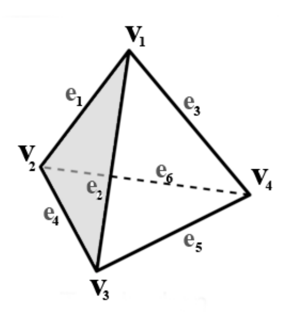
These work and related articles can comprehensively form a solid foundation on which we can sufficiently meet our objectives. These authors have derived the symmetry groups of the platonic solids. With these Symmetry groups we can postulate the group action of the said symmetry groups on their respective vertices for each platonic solid. Hence, we can explore whether the action is transitive, primitive, regular, faithful and then derive the ranks along with the sub-degrees of these platonic solids amongst other properties.

# Chapter 3

# ACTION OF $\mathbb{G}$ ON THE VERTICES OF A TETRAHEDRON

## 3.1 Symmetry Group of a Tetrahedron

In this chapter we shall begin by deriving the rotational, $\mathbb{G}_r$ and reflectional, $\mathbb{G}_f$ permutation groups of a tetrahedron. From these permutation groups we shall compose the symmetry group $\mathbb{G}$ of a tetrahedron from which we shall defined our group action of $\mathbb{G}$ on the set of vertices $V$ and consequently the properties of the action. Let $V = \{v_1, v_2, v_3, v_4\}$ be the set of vertices of a tetrahedron.




| | |
|---|---|
| $f_1$ | [$v_1$, $v_2$, $v_3$] |
| $f_2$ | [$v_1$, $v_2$, $v_4$] |
| $f_3$ | [$v_1$, $v_3$, $v_4$] |
| $f_4$ | [$v_2$, $v_3$, $v_4$] |

Figure 3.1: Faces, edges and vertices of a tetrahedron

### 3.1.1 Rotational Group of Symmetry of a Tetrahedron

Suppose we let the solid be situated with its center at the origin in $\mathbb{R}^3$. The group of rotational symmetries in $\mathbb{R}^3$ leaving the tetrahedron invariant will be denoted by $\mathbb{G}_r$. Its elements are

- the identity, $g_1 = I$,
- two rotations through $120°$ and $240°$ about each of the four axes joining vertices with centres of opposite faces,
- one rotation through angle $180°$ about each of $3$ axes joining the midpoints of opposites edges. We note this group isomorphic to $\mathbb{A}_4$.

Using the denotations for faces and edges as described in figure **3.1** we can illustrate the above elements using figure **4.2** belowThese are as illustrated in the figure **3.2** below.

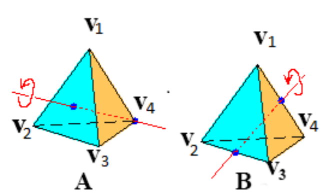



Figure 3.2: Rotational axes of a tetrahedron

**Remark 3.1.1.** *A: Rotation of an axis through the centre of* $1$ *face to the vertex not contoured by the vertices of the face whose centre the axis goes through. Turning the axis about* $120°$ *from the line that centre to the vertex on the same face and* $240°$ *from the same line to next vertex. Since a tetrahedron has* $4$ *faces, then we have* $4$ *such axes.*

Table 3.1: Rotations of axis $\boldsymbol{A}$ through $v_i$ and the centres of opposite faces

| Axis | 120° | 240° |
|---|---|---|
| $v_4 \longleftrightarrow f_1$ | $g_2 = (1, 2, 3)$ | $g_4 = (1, 3, 2)$ |
| $v_3 \longleftrightarrow f_2$ | $g_3 = (1, 2, 4)$ | $g_7 = (1, 4, 2)$ |
| $v_2 \longleftrightarrow f_3$ | $g_5 = (1, 3, 4)$ | $g_6 = (1, 4, 3)$ |
| $v_1 \longleftrightarrow f_4$ | $g_8 = (2, 3, 4)$ | $g_9 = (2, 4, 3)$ |

**Remark 3.1.2.** ***B****: An axis through the midpoint of one edges to the midpoint of the midpoint of the diagonally opposite edge about* $180°$*. There being* $3$ *such diagonally opposite edges we have* $3$ *unique axis that leave the solid invariant.*

Table 3.2: Rotation of axis $\boldsymbol{B}$ passing through the midpoints of opposites edges

| Axis | 180° |
|---|---|
| $e_1 \longleftrightarrow e_5$ | $g_{10} = (1, 2)(3, 4)$ |
| $e_2 \longleftrightarrow e_6$ | $g_{11} = (1, 3)(2, 4)$ |
| $e_3 \longleftrightarrow e_4$ | $g_{12} = (1, 4)(2, 3)$ |

**Remark 3.1.3.**

- *All permutations about* $360°$ *make up the identity,* $g_1 = I$.
- *All resultant rotational permutations of a tetrahedron are* $12$ *including the identity. i.e.* $|\mathbb{G}_r| = 12$.
- *The identity* $I$ *and the other permutations derived above makes up* $\mathbb{G}_r$.

### 3.1.2 Reflectional Group of Symmetry of a Tetrahedron

A tetrahedron has $4$ vertices, $6$ edges and $4$ faces. Suppose we let the solid be situated with its center at the origin in $\mathbb{R}^3$. There are $6$ unique reflections of a tetrahedron all derived by placing a reflection mirror between the solid from the vertex on top while it rest on one face. These reflections can be illustrated by the figure **3.3** below

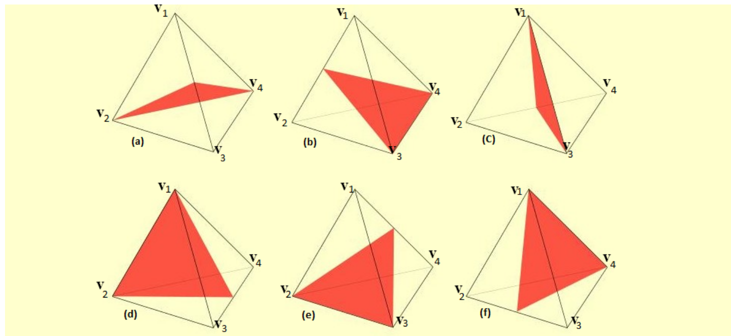



Figure 3.3: Planes of reflection of a tetrahedron

Table 3.3: Reflectional symmetries of a Tetrahedron

| | $\wp$ | permutation | | $\wp$ | permutation |
|---|---|---|---|---|---|
| 1. | **b** | $g_{13} = (1, 2)$ | 2. | **a** | $g_{14} = (1, 3)$ |
| 3. | **e** | $g_{15} = (1, 4)$ | 4. | **f** | $g_{16} = (2, 3)$ |
| 5. | **c** | $g_{17} = (2, 4)$ | 6. | **d** | $g_{18} = (3, 4)$ |

**Remarks 1.** *The reflectional group of symmetry of a tetrahedron is of order* 12*. From the work of Lim (2008), we note that the missing permutation are derived from multi-reflections.*

### 3.1.3 Full Symmetry Group of a Tetrahedron

The full symmetry group of a tetrahedron $\mathbb{G}$ is composed of the rotational $\mathbb{G}_r$, and reflection symmetries $\mathbb{G}_f$, of a tetrahedron. i.e. $|\mathbb{G}| = 24$

## 3.2 The Action of $\mathbb{G}$ on the Vertices of a Tetrahedron

**Theorem 3.2.1.** *Let $V = \{v_i\}$ with $i = 1, 2, ..., n$ be the set of vertices of a platonic solid where $v_i$ represents each vertex of the solid. Let $\mathbb{G}$ be the group of symmetries of the platonic solid. Then we define the action of $\mathbb{G}$ on $V$ by*

$$g_k(v_i) = v_j$$

*where $g_k \in \mathbb{G}$, $k = 1, 2, ..., m$ and $v_i, v_j \in V$ where $j, k = 1, 2, ..., n$*
*and $m$ represent the last element in $\mathbb{G}$ while $n$ represent the last element in $V$.*

**Theorem 3.2.2.** *The action of $\mathbb{G}$ on the set of vertices $V$ of a tetrahedron is well defined.*

*Proof.*

i. From Remark **3.1.3** and Theorem **3.2.1** with the symmetry group $\mathbb{G}$ described above, $g_1 = I \in \mathbb{G}$ and hence $g_1$ fixes all the vertices of a tetrahedron.

ii. For any $2$ elements $g_i, g_j \in \mathbb{G}$ , the action $(g_i g_j)v_k = g_i(g_j v_k)$

□

**Example 1.**

*i. Taking $g_1 = I$ and $v_4$, then $g_1(v_4) = v_4$*

*ii. Taking $g_2 = (1, 2, 3), g_3 = (1, 2, 4) \in \mathbb{G}$, we have;*

$$\begin{aligned}
\{g_2 g_3\}v_4 &= \{(1, 2, 3)(1, 2, 4)\}v_4 \\
&= (1, 3)(2, 4)v_4 \\
&= v_2
\end{aligned}$$

$$\text{On the other hand,} \qquad g_2(g_3v_4) = g_2\{(1,2,4)v_4\}$$
$$= (1,2,3)v_1$$
$$= v_2$$

**Conclusion 1.** *Hence the action is well-defined.*

**Lemma 3.2.1.** *Let $v_i \in V$, then the orbit of $v_i$ is of length 4.*

*Proof.* By Definition **1.2.8**, we get,

$$\begin{aligned} Orb_{\mathbb{G}}(v_i) &= V, \text{ and since } |V| = 4 \\ i.e. \ |Orb_{\mathbb{G}}(v_i)| &= 4. \Rightarrow \text{Hence the orbits are of length } 4 \end{aligned}$$

□

**Example 2.** *Take $v_1 \in V$ then its corresponding orbit in $\mathbb{G}$ is given by*

$$\begin{aligned} Orb_{\mathbb{G}}(v_1) &= \{v_1, v_2, v_3, v_4\} = V \\ i.e. \ |Orb_{\mathbb{G}}(v_1)| &= 4. \end{aligned}$$

**Lemma 3.2.2.** *The order of a stabilizer of $v_i \in V$ in $\mathbb{G}$ is 6.*

*Proof.* Let $X$ be the tetrahedron. The non-identity element of $\mathbb{G}$ fixing the point $v_i \in V$ is $g_j$ i.e. rotations through the point. Any rotational axis going through a vertex and the center of gravity of $X$, must go through the center of the opposite face which is an equilateral. Thus, the angle of rotation that leaves $X$ invariant is $\{\frac{2\pi}{3}\}$. Consequently, the order of $g_j$ hence the stabilizer is 3. Additionally, the reflections from any of the vertex generates 3 reflections that leaves $X$ invariant and hence the order of stabilizer is 6. □

**Example 3.** *Take $v_1 \in V$ then using Definition **1.2.9** we obtain the following result:-*

$$\begin{aligned} Stab_{\mathbb{G}}(v_1) &= \{I,\ (2,3),\ (2,4),\ (3,4),\ (2,3,4),\ (2,4,3)\} \\ i.e.\ |Stab_{\mathbb{G}}(v_1)| &= 6. \end{aligned}$$

**Theorem 3.2.3.** *The action of $\mathbb{G}$ on the set of vertices $V$ is transitive.*

*Proof.* By Theorem **1.3.1** and Lemma **3.2.2**

$$\begin{aligned} |Orb_{\mathbb{G}}(v_i)| &= |\mathbb{G} : Stab_{\mathbb{G}}(v_i)| \\ &= \frac{24}{6} = 4 \\ &= |V| \end{aligned}$$

Thus by Definition **1.2.10** the action is transitive. □

**Theorem 3.2.4.** *The action of $\mathbb{G}$ on the set of vertices $V$ is Faithful.*

*Proof.* We observe that only $g_1(v_i) = v_i$, $g_1 = I$ fixes any of the vertices while the rest of the elements of $\mathbb{G}$ maps the vertices to different points. Then by Definition **1.2.11** above the action of $\mathbb{G}$ on $V$ is indeed faithful. □

**Theorem 3.2.5.** *The action of $\mathbb{G}$ on the set of vertices $V$ is not regular.*

*Proof.* By Definition **1.2.10** and Definition **1.2.13** the action of $\mathbb{G}$ on $V$ is transitive but not regular since it is not semi-regular. $i.e.\ |Stab_{\mathbb{G}}(v_i)| \neq \mathbf{1}$ □

**Theorem 3.2.6.** *The action of $\mathbb{G}$ on the set of vertices $V$ is not primitive.*

*Proof.* For any $v_i,\ i = 1, 2, 3, 4$, the isotropy subgroup $\mathbb{G}_{v_i}$ is of order $6$. Then,

$$\begin{aligned} |Orb_{\mathbb{G}}(v_i)| &= |\mathbb{G} : \mathbb{G}_{v_i}| = \frac{24}{6} \\ &= 4, not\ Prime \end{aligned}$$

Thus, $\mathbb{G}_{v_i}$ is not maximal and hence the action is not primitive. □

**Example 4.** *Let* $\mathbb{G}$ *be as described above and* $V = \{v_1, v_2, v_3, v_4\}$*. Then with*

$$\begin{aligned} \mathbb{G}_{v_2} &= \{I, (1,3), (1,4), (3,4), (1,3,4), (1,4,3)\},\ and \\ |\mathbb{G} : \mathbb{G}_{v_2}| &= \frac{24}{6}, \\ &= 4, not\ Prime \end{aligned}$$

*Thus,* $\mathbb{G}_{v_2}$ *is not maximal and hence the action is not primitive.*

**Theorem 3.2.7.** *The action of* $\mathbb{G}$ *on* $V$ *has sub-degrees* $1^{(1)}$ *and* $1^{(3)}$*. Thus* $R(\mathbb{G})$ *is* 2.

*Proof.* By Lemma **3.2.2**, the order of the isotropy subgroup $\mathbb{G}_{v_i}$ on $V$ is $6$. The action of $\mathbb{G}_{v_i}$ has 2 orbits namely;

- the trivial, $\lambda_i = \{v_i\}$, and
- the non-trivia, $\Lambda_i = \{V \setminus v_i\}$

where $\lambda_i$ denotes the element of $V$ being acted upon by $\mathbb{G}_{v_i}$ and $\Lambda_i$ denotes the set $\{V \setminus v_i\}$ with $v_i$ being the element of $V$ being acted upon. We observe that $|\Delta_i| = 1$ and $|\Lambda_i| = 3$.

For instance, consider $\mathbb{G}_{v_1}$. The $|\mathbb{G}_{v_1}| = 6$. The orbits of $\mathbb{G}_{v_1}$ are $\Delta_1 = v_1$ and $\delta_1 = \{v_2, v_3, v_4\} = \{V \setminus v_1\}$.

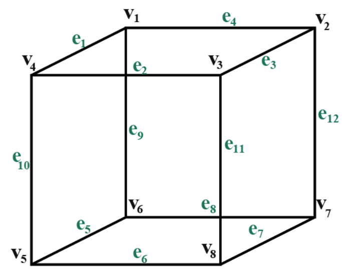
Now, with the results above and Definition **1.2.12** then we can conclude that the sub-degrees are $1^{(1)}$ and $1^{(3)}$ and the rank, $R(\mathbb{G})$ is 2. □

# Chapter 4

# ACTION OF $\mathbb{G}$ ON THE VERTICES OF A CUBE AND AN OCTAHEDRON

## 4.1 Symmetry Group of a Cube

The dual of a cube is an octahedron. In this chapter we shall explore the properties of the action of the symmetry groups $\mathbb{G}$, on the vertices of each of this duals. We shall start by showing how to derive the rotational, $G_r$ and the reflectional groups $\mathbb{G}_f$, of each of the duals a combination of which we form the full symmetry group $\mathbb{G}$ for each. In Section **4.1** we have the introduction to the chapter, in Section **4.2** and Section **4.3** we have the symmetry group and the properties of the action of $\mathbb{G}$ on the vertices of a cube and Section **4.4** and Section **4.5** we have the symmetry group and properties of the action of $\mathbb{G}$ on the vertices of an octahedron.




| | |
|---|---|
| $f_1$ | [$v_1$, $v_2$, $v_3$, $v_4$] |
| $f_2$ | [$v_3$, $v_4$, $v_5$, $v_8$] |
| $f_3$ | [$v_2$, $v_3$, $v_8$, $v_7$] |
| $f_4$ | [$v_1$, $v_2$, $v_7$, $v_6$] |
| $f_5$ | [$v_1$, $v_4$, $v_5$, $v_6$] |
| $f_6$ | [$v_5$, $v_6$, $v_7$, $v_8$] |

Figure 4.1: Face, edges and vertices of a cube

### 4.1.1 Rotational Group of Symmetry of a Cube

Suppose we let the cube be situated with its center at the origin in $\mathbb{R}^3$. The group of rotational symmetries in $\mathbb{R}^3$ leaving the cube invariant will be denoted by $\mathbb{G}_r$. Its elements are:-

***i.*** The identity, $g_1 = I$
***ii.*** Rotations of an axis through the centres of directly opposite faces about $90°, 180°$ and $270°$
***iii.*** Rotations of an axis through the midpoints of the 2 diagonally opposite edges about $180°$
***iv.*** Rotations of an axis through 2 diagonally opposite vertices about $120°$ and $240°$

Using the denotations for faces and edges as described in figure **4.1** we can illustrate the above elements using figure **4.2** below

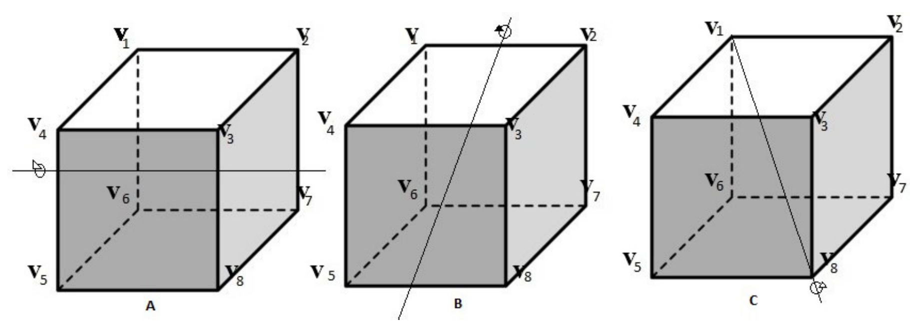



Figure 4.2: Rotational axes of a cube

**Note 1.** *The rotations **ii, iii** and **iv** are represented by **A**,**B** and **C** respectively.*

Table 4.1: Rotations about axis ***A*** through the centers of directly opposite faces

| Direct opposite faces | 90° | 180° | 270° |
|---|---|---|---|
| $f_1 \longleftrightarrow f_6$ | $g_2 = (1,2,3,4)(5,6,7,8)$ | $g_3 = (1,3)(2,4)(5,7)(6,8)$ | $g_4 = (1,4,3,2)(5,8,7,6)$ |
| $f_5 \longleftrightarrow f_3$ | $g_5 = (1,4,5,6)(2,7,8,3)$ | $g_6 = (1,5)(2,8)(3,7)(4,6)$ | $g_7 = (1,6,5,4)(2,3,8,7)$ |
| $f_4 \longleftrightarrow f_2$ | $g_8 = (1,2,7,6)(3,8,5,4)$ | $g_9 = (1,7)(2,6)(3,5)(4,8)$ | $g_{10} = (1,6,7,2)(3,4,5,8)$ |

Table 4.2: Rotations of the axis ***B*** through the midpoints of diagonally opposite edges

| Directly opposite edges | 180° |
|---|---|
| $e_4 \longleftrightarrow e_6$ | $g_{11} = (1,2)(3,6)(4,7)(5,8)$ |
| $e_1 \longleftrightarrow e_7$ | $g_{12} = (1,4)(2,5)(3,6)(7,8)$ |
| $e_9 \longleftrightarrow e_{11}$ | $g_{13} = (1,6)(2,5)(3,8)(4,7)$ |
| $e_3 \longleftrightarrow e_5$ | $g_{14} = (1,8)(2,3)(4,7)(5,6)$ |
| $e_{12} \longleftrightarrow e_{10}$ | $g_{15} = (1,8)(2,7)(3,6)(4,5)$ |
| $e_2 \longleftrightarrow e_8$ | $g_{16} = (1,8)(2,5)(3,4)(6,7)$ |

Table 4.3: Rotations of the axis ***C*** through the diagonally opposite vertices

| Diagonally opposite vertices | 120° | 240° |
|---|---|---|
| $v_1 \longleftrightarrow v_8$ | $g_{17} = (2,6,4)(3,7,5)$ | $g_{18} = (2,4,6)(3,5,7)$ |
| $v_2 \longleftrightarrow v_5$ | $g_{19} = (1,3,7)(4,8,6)$ | $g_{20} = (1,7,3)(4,6,8)$ |
| $v_3 \longleftrightarrow v_6$ | $g_{21} = (1,5,7)(2,4,8)$ | $g_{22} = (1,7,5)(2,8,4)$ |
| $v_4 \longleftrightarrow v_7$ | $g_{23} = (1,5,3)(2,6,8)$ | $g_{24} = (1,3,5)(2,8,6)$ |

**Remarks 2.** *The rotational group of symmetry of a cube,* $\mathbb{G}_r$ *is composed of all the preceding elements* $g_i, i = 1, 2, ..., 24$. *i.e* $|\mathbb{G}_r| = 24$

### 4.1.2 Reflectional Group of Symmetry of a Cube

There are $9$ unique reflections planes, $\wp$ for a cube, this can be illustrated by figure **4.1.2** below and their corresponding permutations displayed on table **4.4**.

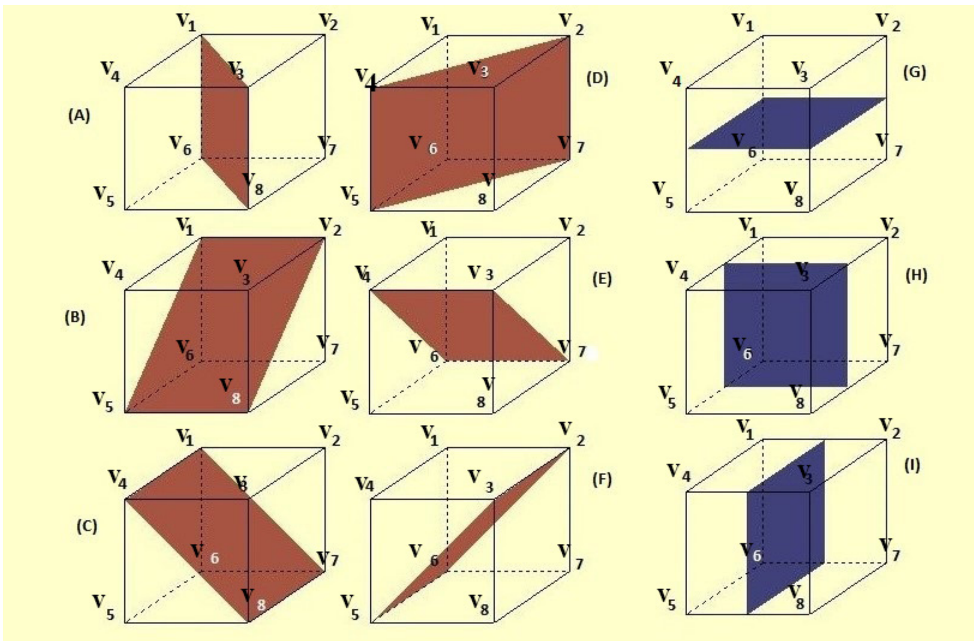



Figure 4.3: Planes of reflection of a cube

Table 4.4: Reflection-al permutations of the planes of a Cube

| $\wp$ | Cycle | $\wp$ | Cycle | $\wp$ | Cycle |
|---|---|---|---|---|---|
| **(A)** | $g_{28} = (2,4)(5,7)$ | **(D)** | $g_{25} = (1,3)(6,8)$ | **(G)** | $g_{33} = (1,6)(2,7)(3,8)(4,5)$ |
| **(B)** | $g_{30} = (3,7)(4,6)$ | **(E)** | $g_{26} = (1,5)(2,8)$ | **(H)** | $g_{32} = (1,4)(2,3)(5,6)(7,8)$ |
| **(C)** | $g_{29} = (2,6)(3,5)$ | **(F)** | $g_{27} = (1,7)(4,8)$ | **(I)** | $g_{31} = (1,2)(3,4)(5,8)(6,7)$ |

**Remarks 3.**

i. *The reflectional group of symmetry of a cube, $\mathbb{G}_f$ is composed of elements* $g_i, i = 25, ....33$

ii. *The* $|\mathbb{G}_f| = 24$*, the remaining permutations are derived from multi-reflections as explained by Lim (2008).*

### 4.1.3 Full symmetry group of a Cube, $\mathbb{G}$

The full group of symmetry of a cube denoted by $\mathbb{G}$ is composed of all the elements of the rotational symmetries $\mathbb{G}_r$, and the reflectional symmetries $\mathbb{G}_f$. i.e $|\mathbb{G}| = 48$.

## 4.2 The Action of $\mathbb{G}$ on the Vertices of a Cube

**Theorem 4.2.1.** *Let the set of vertices of a cube be define as* $V = \{v_1, v_2, ..., v_8\}$*, then the action of* $\mathbb{G}$ *on the set* $V$ *is well defined.*

*Proof.* From Theorem **3.2.1** we observe that through the action, $g_1 \in \mathbb{G}$ fixes all the elements of $V$ and any other 2 elements $g_j, g_k \in \mathbb{G}$ will in turn permute a vertex of the cube accordingly hence the action is well-defined. □

**Example 5.**

i. *Taking* $g_1 = I, \in \mathbb{G}$ *and* $v_2 \in V$*, then* $g_1(v_2) = v_2$*,*

ii. *Taking* $g_{12} = \{(1,4)(2,5)(3,6)(7,8)\}, g_5 = \{(1,4,5,6)(2,7,8,3)\} \in \mathbb{G}$*, we have;*

$$\begin{aligned}
(g_{12} * g_5)v_2 &= \{(1,4)(2,5)(3,6)(7,8) * (1,4,5,6)(2,7,8,3)\}v_2 \\
&= \{(2,8,6,4)(3,5)\} * v_2 \\
&= v_8
\end{aligned}$$

$$\begin{aligned} \textit{On the other hand,} \qquad g_{12}\{g_5 * v_2\} &= g_{12}\{(1,4,5,6)(2,7,8,3) * v_2\} \\ &= \{(1,4)(2,5)(3,6)(7,8)\}v_7 \\ &= v_8. \end{aligned}$$

**Lemma 4.2.1.** *Let* $v_i \in V$ *where* $V$ *is the set of vertices of a cube. Then the orbit of* $v_i$ *is the length* $8$*.*

*Proof.* By Definition **1.2.8**, we get,

$$\begin{aligned} Orb_{\mathbb{G}}(v_i) &= V, and\ since\ |V| = 8 \\ i.e.\ |Orb_{\mathbb{G}}(v_i)| &= 8. \Rightarrow Hence\ the\ orbits\ are\ of\ length\ 8. \end{aligned}$$

□

**Example 6.** *Take* $v_3 \in V$ *then its corresponding orbit in* $\mathbb{G}$ *is given by*

$$\begin{aligned} Orb_{\mathbb{G}}(v_3) &= \{v_1, v_2, v_3, v_4, v_5, v_6, v_7, v_8\} = V \\ |Orb_{\mathbb{G}}(v_3)| &= 8. \end{aligned}$$

**Lemma 4.2.2.** *The order of the isotropy group,* $\mathbb{G}_{v_i}$ *in* $\mathbb{G}$ *is* $6$*.*

*Proof.* Let $X$ be the cube. All rotational axes going through any $2$ vertices when $X$ is standing upright on either of those vertices such that those rotations leave $X$ and its vertices invariant are $3$ in number. Additionally, all reflection of $X$ from $2$ diagonally opposite edges which will leave $X$ and its vertices invariant are also $3$ in number. Altogether representing order of the stabilizers of $X$, which is $6$. □

**Example 7.** *Taking $v_2 \in V$ and Definition **1.2.9** we obtain*

$$\begin{aligned}
Stab_{\mathbb{G}}(v_2) &= \{g_1, g_{19}, g_{20}, g_{25}, g_{27}, g_{30}\} \\
Stab_{\mathbb{G}}(v_2) &= \left\{ \begin{array}{l} g_1 = I, \\ g_{25} = (1,3)(6,8), g_{27} = (1,7)(4,8), g_{30} = (3,7)(4,6), \\ g_{19} = (1,3,7)(4,8,6), g_{30} = (1,7,3)(4,6,8) \end{array} \right\} \\
|Stab_{\mathbb{G}}(v_2)| &= 6.
\end{aligned}$$

**Theorem 4.2.2.** *The action of $\mathbb{G}$ on the set of vertices, $V$ of a cube is transitive.*

*Proof.* From lemma **4.2.1** all the orbits of the vertices are equal to $V$ i.e.

$$Orb_{\mathbb{G}}(v_i) = V, \; \forall\, i = 1, 2, ..., 8$$

then by Definition **1.2.10** the action is transitive. □

**Theorem 4.2.3.** *The action of $\mathbb{G}$ on the set of vertices $V$ of a cube is Faithful.*

*Proof.* From Lemma **4.2.2** we observe that only $g_1 = I \in \mathbb{G}$ fixes any of the vertices while the rest of the elements of $\mathbb{G}$ maps the vertices to different points. Then by Definition **1.2.11** above, the action of $\mathbb{G}$ on $V$ is indeed faithful. □

**Theorem 4.2.4.** *The action of $\mathbb{G}$ on the set of vertices $V$ of a cube is not regular.*

*Proof.* From Definition **1.2.13** and Lemma **4.2.2** the action of $\mathbb{G}$ on $V$ is transitive but not regular since it is not semi-regular. i.e. $|Stab_{\mathbb{G}}(v_i)| \neq 1$. □

**Theorem 4.2.5.** *The action of $\mathbb{G}$ on the set of vertices $V$ of a cube is not primitive.*

*Proof.* From lemma **4.2.2** the isotropy subgroup $\mathbb{G}_{v_i} \forall\, v_i,\ i = 1, 2, ..., 8$, is of order 6. Then,

$$|Orb_{\mathbb{G}}(v_i)| = |\mathbb{G} : \mathbb{G}_{v_i}| \;=\; \frac{48}{6}$$
$$= 8, not\ Prime$$

Thus, $\mathbb{G}_{v_i}$ is not maximal and hence the action is not primitive. □

**Theorem 4.2.6.** *The action of* $\mathbb{G}$ *on the set of vertices* $V$ *of a cube has sub-degrees are* $2^{(1)}$ *and* $2^{(3)}$. *Thus the rank,* $R(\mathbb{G})$ *is* 4.

*Proof.* Let $X$ be the cube, and $v_i \longleftrightarrow v_{i^o}$ be an axis through $v_i$ to the vertex diagonally opposite it, $v_{i^o}$ while $X$ is held upright on either of this vertices.

By Lemma **4.2.2**, the order of the isotropy subgroup $\mathbb{G}_{v_i}$ on $V$ is 6. The action of $\mathbb{G}_{v_i}$ has 4 orbits namely;

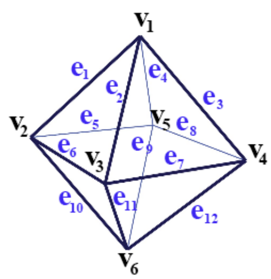
- 1 trivial element $\delta_i = v_i$,
- 1 trivial element $\Delta_i = v_{i^o}$,
- 1 non-trivial element $\Lambda_i = \{v_{i+2}, v_{i+4}, v_{i+6}\}$, and
- 1 non-trivial element $\lambda_i = \{v_{i^o+2}, v_{i^o+4}, v_{i^o+6}\}$

where $\delta_i, \Delta_i$ denotes the element $v_i$ of $V$ being acted upon by $\mathbb{G}_{v_i}$ and the vertex diagonally opposite, $v_{i^o}$ respectively. $\Lambda_i = \{v_j \in V\}$ is such that $j =$ all vertices with an odd subscript while $\lambda_i = \{v_k \in V\}$ is such that $j =$ all vertices with an even subscript and $j, k \neq i, i^o$. We observe that $|\Delta_i| = 1 = |\delta_i|$ and $|\lambda_i| = 3 = |\Lambda_i|$.

Consider $\mathbb{G}_{v_1}$. The $|\mathbb{G}_{v_1}| = 6$. The orbits of $\mathbb{G}_{v_1}$ are $\delta_1 = v_1, \Delta_1 = v_8, \Lambda_1 = \{v_3, v_5, v_7\}$ and $\lambda_1 = \{v_2, v_4, v_6\}$.

Now, with the results above and Definition **1.2.12** then we can conclude that the sub-degrees are $2^{(1)}$ and $2^{(3)}$ and the rank $R(G)$ is $4$. □

## 4.3 Symmetry Group of an Octahedron




| | | | |
|---|---|---|---|
| $f_1$ | $[v_1, v_2, v_3]$ | $f_5$ | $[v_6, v_2, v_3]$ |
| $f_2$ | $[v_1, v_3, v_4]$ | $f_6$ | $[v_6, v_3, v_4]$ |
| $f_3$ | $[v_1, v_4, v_5]$ | $f_7$ | $[v_6, v_4, v_5]$ |
| $f_4$ | $[v_1, v_5, v_2]$ | $f_8$ | $[v_6, v_5, v_2]$ |

Figure 4.4: Face, edges and vertices of an octahedron

### 4.3.1 Rotational Group of Symmetry of an Octahedron

Suppose we let the solid be situated with its center at the origin in $\mathbb{R}^3$. The group of rotational symmetries in $\mathbb{R}^3$ leaving the octahedron invariant will be denoted by $\mathbb{G}_r$. Its elements are

- The identity, $g_1 = I$,
- 4 rotations of axes through the center of directly opposite faces about $120°$ and $240°$,
- 3 rotations of axes from the midpoints of directly opposite edges about $180°$.

Using the denotations for faces and edges as described in figure **4.4** we can illustrate the above elements using figure **4.5** below.

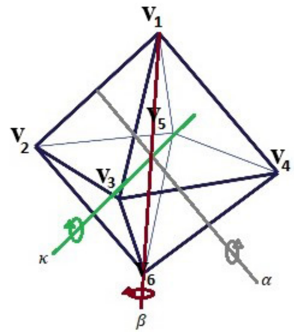



Figure 4.5: Rotational axis of an octahedron

Table 4.5: Rotations of an axis, $\kappa$, about the centres of all directly opposite faces

| Directly opposite faces | 120° | 240° |
|---|---|---|
| $f_1 \longleftrightarrow f_7$ | $g_2 = (1,2,3)(4,5,6)$ | $g_3 = (1,3,2)(4,6,5)$ |
| $f_2 \longleftrightarrow f_8$ | $g_4 = (1,3,4)(2,6,5)$ | $g_5 = (1,4,3)(2,5,6)$ |
| $f_3 \longleftrightarrow f_5$ | $g_6 = (1,4,5)(2,3,6)$ | $g_7 = (1,5,4)(2,6,3)$ |
| $f_4 \longleftrightarrow f_6$ | $g_8 = (1,5,2)(3,4,6)$ | $g_9 = (1,2,5)(3,6,4)$ |

Table 4.6: Rotation of axis, $\alpha$, from the midpoints of directly opposite edges

| | Axis | Permutation about 180° |
|---|---|---|
| 1. | $e_1 \longleftrightarrow e_{12}$ | $g_{10} = (1,2)(3,5)(4,6)$ |
| 2. | $e_3 \longleftrightarrow e_{10}$ | $g_{11} = (1,4)(2,6)(3,5)$ |
| 3. | $e_2 \longleftrightarrow e_9$ | $g_{12} = (1,3)(2,4)(5,6)$ |
| 4. | $e_4 \longleftrightarrow e_{11}$ | $g_{13} = (1,5)(2,4)(3,6)$ |
| 5. | $e_5 \longleftrightarrow e_7$ | $g_{14} = (1,6)(2,5)(3,4)$ |
| 6. | $e_6 \longleftrightarrow e_8$ | $g_{15} = (1,6)(2,3)(4,5)$ |

Table 4.7: Rotation of the axis, $\beta$, crossing through directly opposite vertices

| | Axis | 90° | 180° | 270° |
|---|---|---|---|---|
| 1. | $v_1 \longleftrightarrow v_6$ | $g_{16} = (2,3,4,5)$ | $g_{17} = (2,4)(3,5)$ | $g_{18} = (2,5,4,3)$ |
| 2. | $v_2 \longleftrightarrow v_4$ | $g_{19} = (1,5,6,3)$ | $g_{20} = (1,6)(3,5)$ | $g_{21} = (1,3,6,5)$ |
| 3. | $v_3 \longleftrightarrow v_5$ | $g_{22} = (1,2,6,4)$ | $g_{23} = (1,6)(2,4)$ | $g_{24} = (1,4,6,2)$ |

**Remarks 4.** *The rotational group of symmetry of an octahedron, $\mathbb{G}_r$ is composed of all the preceding elements $g_i, i = 1,2,...,24$. i.e $|\mathbb{G}_r| = 24$*

### 4.3.2 Reflectional Group of Symmetry of an Octahedron

There are $2$ unique planes of reflections of an octahedron. Let the group of reflectional symmetries be denoted by $\mathbb{G}_f$, then its elements are

- Three Plane of reflection through the center of the octahedron and the adjoining $4$ vertices on the same plane
- Six Planes of reflections thorough the center of the solid and the midpoints of $2$ directly opposite edges on the same plane

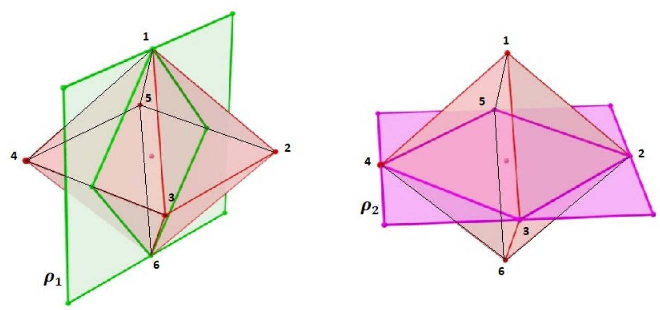



Figure 4.6: Planes of reflection of an octahedron

Table 4.8: Planes of reflections for $\rho_1$

| | Plane | Permutation |
|---|---|---|
| 1. | $v_3 \boxtimes e_1 \boxtimes v_5 \boxtimes e_{12}$ | $g_{25} = (1, 2)(4, 6)$ |
| 2. | $v_2 \boxtimes e_2 \boxtimes v_4 \boxtimes e_9$ | $g_{26} = (1, 3)(5, 6)$ |
| 3. | $v_3 \boxtimes e_3 \boxtimes v_5 \boxtimes e_{10}$ | $g_{27} = (1, 4)(2, 6)$ |
| 4. | $v_2 \boxtimes e_4 \boxtimes v_4 \boxtimes e_{11}$ | $g_{28} = (1, 5)(3, 6)$ |
| 5. | $v_1 \boxtimes e_5 \boxtimes v_6 \boxtimes e_7$ | $g_{29} = (2, 5)(3, 4)$ |
| 6. | $v_1 \boxtimes e_6 \boxtimes v_6 \boxtimes e_8$ | $g_{30} = (2, 3)(4, 5)$ |

Table 4.9: Planes of reflections for $\rho_2$

| | Plane | Permutation |
|---|---|---|
| 1. | $v_2 \boxtimes v_3 \boxtimes v_4 \boxtimes v_5$ | $g_{31} = (1, 6)$ |
| 2. | $v_1 \boxtimes v_3 \boxtimes v_6 \boxtimes v_5$ | $g_{32} = (2, 4)$ |
| 3. | $v_1 \boxtimes v_2 \boxtimes v_6 \boxtimes v_4$ | $g_{33} = (3, 5)$ |

**Remarks 5.**

i. *The reflection-al group of symmetry of an octahedron,* $\mathbb{G}_f$ *is composed of the following elements* $g_i, i = 25, ....33$

ii. *The* $|\mathbb{G}_f| = 24$*, the remaining permutations are derived from multi-reflections as explained by Lim (2008).*

### 4.3.3 Full Symmetrical Group of an Octahedron, $\mathbb{G}$

The full group of symmetry of an octahedron be denoted by $\mathbb{G}$. It is composed of all the elements of $\mathbb{G}_r$ and $\mathbb{G}_f$. Then, $|\mathbb{G}| = 48$.

## 4.4 The Action of $\mathbb{G}$ on the Vertices of an Octahedron

**Theorem 4.4.1.** *Let the set of vertices of an octahedron be defined as* $V = \{v_1, v_2, v_3, v_4, v_5, v_6\}$*, then action of* $\mathbb{G}$ *on* $V$ *is well defined.*

*Proof.* From Theorem **3.2.1** we observe that through the action $g_1 \in \mathbb{G}$ fixes all the elements of $V$. Moreover any 2 elements of $\mathbb{G}$ will in turn permute any vertex accordingly. Thus the action is well-defined. □

**Example 8.**

i. *Taking* $g_1 = I, \in \mathbb{G}$ *and* $v_5 \in V$ *of an octahedron, then* $g_1(v_5) = v_5$, *ii. Taking* $g_{17} = (2,4)(3,5), g_{32} = (2,4) \in \mathbb{G}$*, we have;*

$$\begin{aligned} (g_{17} * g_{32})v_5 &= \{(2,4)(3,5) * (2,4)\}v_5 \\ &= (3,5) * v_5 \\ &= v_3 \\ \text{On the other hand,} \qquad g_{17}(g_{32} * v_5) &= g_{17}\{(2,4) * v_5\} \\ &= (2,4)(3,5) * v_5 \\ &= v_3 \end{aligned}$$

**Lemma 4.4.1.** *Let* $v_i \in V$ *where* $V$ *is the set of vertices of a octahedron. Then the orbit of* $v_i$ *is the length* $6$*.*

*Proof.* By Definition **1.2.8**, we get,

$$\begin{aligned} Orb_{\mathbb{G}}(v_i) &= V, and\ since\ |V| = 6 \\ i.e.\ |Orb_{\mathbb{G}}(v_i)| &= 6.\ Hence\ the\ orbits\ are\ of\ length\ 6. \end{aligned}$$

□

**Example 9.** *Take* $v_5 \in V$*, then its corresponding orbit in* $\mathbb{G}$ *is given by*

$$\begin{aligned} Orb_{\mathbb{G}}(v_5) &= \{v_1, v_2, v_3, v_4, v_5, v_6\} = V \\ |Orb_{\mathbb{G}}(v_5)| &= 6. \end{aligned}$$

**Lemma 4.4.2.** *The order of a isotropy group* $\mathbb{G}_{v_i}$ *in* $\mathbb{G}$ *is* $8$*.*

*Proof.* Let $X$ be the octahedron. The non-identity element of $\mathbb{G}$ fixing the point $v_i \in V$ is $g_j$ i.e. rotations through the point. Any rotational axis going through a vertex and the center of gravity of $X$, must go through the vertex on the opposite side. Thus, the angle of rotation that leaves $X$ invariant is $\{\frac{2\pi}{4}\}$. Consequently, the order of $g_j$ is 4. Additionally, the number of reflections that leave $X$ invariant are also 4 and hence the order of the stabilizer is 8. $\square$

**Example 10.** *Take $v_3 \in V$ of the set of vertices of an octahedron then using Definition **1.2.9** we obtain the following result:-*

$$\begin{aligned} Stab_{\mathbb{G}}(v_3) &= \{g_1, g_{22}, g_{23}, g_{24}, g_{25}, g_{27}, g_{31}, g_{32}\} \\ Stab_{\mathbb{G}}(v_3) &= \left\{ \begin{array}{l} g_1 = I, \\ g_{22} = (1,2,6,4), g_{23} = (1,6)(2,4), g_{24} = (1,4,6,2), \\ g_{25} = (1,2)(4,6), g_{27} = (1,4)(2,6), g_{31} = (1,6), g_{32} = (2,4). \end{array} \right\} \\ i.e.\ |Stab_{\mathbb{G}}(v_3)| &= 8 \end{aligned}$$

**Theorem 4.4.2.** *The action of $\mathbb{G}$ on the set of vertices $V$ is transitive.*

*Proof.* From Definition **1.2.10** and Lemma **4.4.1**, we observe that

$$Orb_{\mathbb{G}}(v_i) = V,\ \forall\, i = 1, 2, ..., 6.$$

Thus the action is transitive. $\square$

**Theorem 4.4.3.** *The action of $\mathbb{G}$ on the set of vertices $V$ of an octahedron is Faithful.*

*Proof.* We observe that only $g_1(v_i) = v_i, g_1 = I$ fixes any of the vertices while the rest of the elements of $\mathbb{G}$ maps the vertices to different points. Then by Definition **1.2.11** above the action of $\mathbb{G}$ on $V$ is indeed faithful. $\square$

**Theorem 4.4.4.** *The action of $\mathbb{G}$ on the set of vertices $V$ of an octahedron is not regular.*

*Proof.* By Definition **1.2.10** and Definition **1.2.13** the action of $\mathbb{G}$ on $V$ is transitive but not regular since it is not semi-regular. i.e. $|Stab_{\mathbb{G}(v_i)}| \neq 1$. $\square$

**Theorem 4.4.5.** *The action of* $\mathbb{G}$ *on the set of vertices* $V$ *of an octahedron is not primitive.*

*Proof.* For Lemma **4.4.2** any vertex $v_i, i = 1, 2, ..., 6$ the isotropy subgroup $\mathbb{G}_{vi}$ is of order $8$.

$$\begin{aligned} |Orb_{\mathbb{G}}(v_i)| = |\mathbb{G} : \mathbb{G}_{v_i}| &= \frac{48}{8} \\ &= 6, not\ Prime \end{aligned}$$

Thus, $\mathbb{G}_{v_i}$ is not maximal and hence the action is not primitive. □

**Theorem 4.4.6.** *The action of* $\mathbb{G}$ *on the set of vertices* $V$ *of an octahedron has subdegrees* $2^{(1)}$ *and* $1^{(4)}$. *Thus the rank is* $3$.

*Proof.* Let $X$ be the octahedron, and $v_i \longleftrightarrow v_j$ be an axis through $v_i$ to the vertex directly opposite it, $v_j$ while $X$ is held upright on either of this vertices.

By Lemma **4.4.2**, the order of the isotropy subgroup $\mathbb{G}_{v_i}$ on $V$ is $8$. The action of $\mathbb{G}_{v_i}$ has $3$ orbits namely;

- the trivial, $\delta_i = v_i$,
- the trivial $\Delta_i = v_j$,
- the non-trivial $\Lambda_i = V \setminus \{v_i, v_j\}$.

Where $\delta_i$ denotes the element of $V$ being acted upon by $\mathbb{G}_{v_i}$ and $\Delta_i$ denotes the vertex directly opposite $v_j$ being acted upon by $\mathbb{G}_{v_i}$ while $\Lambda_i$ denotes the set $V \setminus \{v_i, v_j\}$ with $v_i$ being the element of $V$ being acted upon. We observe that $|\delta_i| = 1 = |\Delta_i|$ and $|\Lambda_i| = 4$.

Consider, $\mathbb{G}_{v_2}$, then $|\mathbb{G}_{v_2}| = 8$. The orbits of $\mathbb{G}_{v_2}$ are $\delta_2 = v_2, \Delta_2 = v_4$ and $\Lambda_2 = \{v_1, v_3, v_5, v_6\} = V \setminus \{v_2, v_4\}$

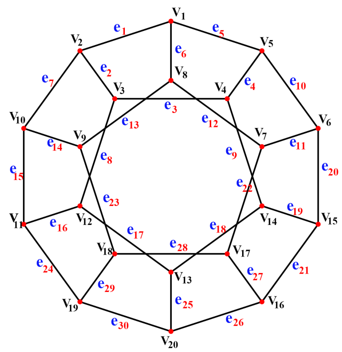
Now, with the results above and Definition **1.2.12** then we can conclude that the sub-degrees are $2^{(1)}$ and $1^{(4)}$. Hence the rank $R(G)$ is $3$. □

# Chapter 5

# ACTION OF $\mathbb{G}$ ON THE VERTICES OF A DODECAHEDRON AND AN ICOSAHEDRON

## 5.1 Symmetry Group of a Dodecahedral

As shown earlier the dual of a dodecahedron is the icosahedron. In this chapter we shall discuss the action of the symmetry group $\mathbb{G}$ on the vertices of each of this duals. We shall start by showing how to derive the rotational, $G_r$ and the reflectional groups $\mathbb{G}_f$, of each of the duals a combination of which we form the full symmetry group $\mathbb{G}$ for each dual. In Section **5.1** we have the introduction to this chapter, in Section **5.2** and Section **5.3** we have the symmetry group and the properties of the action of $\mathbb{G}$ on the vertices of a dodecahedron and Section **5.4** and Section **5.5** we have the symmetry group and properties of the action of $\mathbb{G}$ on the vertices of an icosahedron.




| | |
|---|---|
| $f_1$ | [$V_1$, $V_2$, $V_3$, $V_4$, $V_5$] |
| $f_2$ | [$V_1$, $V_2$, $V_{10}$, $V_9$, $V_8$] |
| $f_3$ | [$V_2$, $V_3$, $V_{12}$, $V_{11}$, $V_{10}$] |
| $f_4$ | [$V_3$, $V_4$, $V_{14}$, $V_{13}$, $V_{12}$] |
| $f_5$ | [$V_4$, $V_5$, $V_6$, $V_{15}$, $V_{14}$] |
| $f_6$ | [$V_1$, $V_5$, $V_6$, $V_7$, $V_8$] |
| $f_7$ | [$V_7$, $V_8$, $V_9$, $V_{18}$, $V_{17}$] |
| $f_8$ | [$V_9$, $V_{10}$, $V_{11}$, $V_{19}$, $V_{18}$] |
| $f_9$ | [$V_{11}$, $V_{12}$, $V_{13}$, $V_{20}$, $V_{19}$] |
| $f_{10}$ | [$V_{13}$, $V_{14}$, $V_{15}$, $V_{16}$, $V_{20}$] |
| $f_{11}$ | [$V_6$, $V_7$, $V_{17}$, $V_{16}$, $V_{15}$] |
| $f_{12}$ | [$V_{16}$, $V_{17}$, $V_{18}$, $V_{19}$, $V_{20}$] |

Figure 5.1: Face, edges and vertices of an dodecahedron

### 5.1.1 Rotational Group Symmetry of a Dodecahedron

The rotational group of symmetry, $\mathbb{G}_r$ is isomorphic to $\mathbb{A}_5$. The rotational symmetries are:-

- The identity, $g_1 = I$
- Four rotations of an axis through the centres of the pairs of directly opposite faces about $72°, 144°, 216°$ and $288°$
- One rotations of an axis through the midpoints of the $15$ pairs of diagonally opposite edges about $180°$
- Two rotations of an axis through the $10$ pairs of directly opposite vertices about $120°$ and $240°$

Using the denotations of faces and edges as described in figure **5.1** we can illustrate the above rotational elements using figure **5.2** below

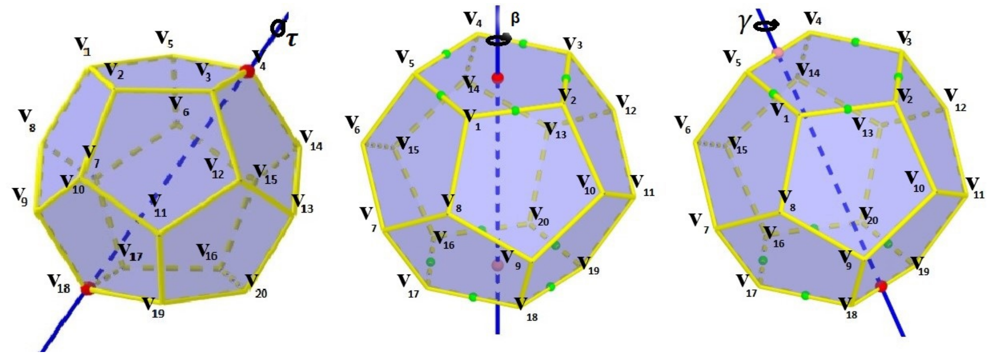



Figure 5.2: Rotational axes of a dodecahedron

Table 5.1: Rotations of axis $\beta$ through centres of $f_1$ and $f_{12}$

| $\theta$ | Permutation |
|---|---|
| $72°$ | $g_2 = (1,2,3,4,5)(6,8,10,12,14)(7,9,11,13,15)(16,17,18,19,20)$ |
| $144°$ | $g_3 = (1,3,5,2,4)(6,10,14,8,12)(7,11,15,9,13)(16,18,20,17,19)$ |
| $216°$ | $g_4 = (1,4,2,5,3)(6,12,8,14,10)(7,13,9,15,11)(16,19,17,20,18)$ |
| $288°$ | $g_5 = (1,5,4,3,2)(6,14,12,10,8)(7,15,13,11,9)(16,20,19,18,17)$ |

Table 5.2: Rotations of axis $\beta$ through centres of $f_3$ and $f_{11}$

| $\theta$ | Permutation |
|---|---|
| $72°$ | $g_6 = (1,4,13,19,9)(2,3,12,11,10)(5,14,20,18,8)(6,15,16,17,7)$ |
| $144°$ | $g_7 = (1,13,9,4,19)(2,12,10,3,11)(5,20,8,14,18)(6,16,7,15,17)$ |
| $216°$ | $g_8 = (1,19,4,9,13)(2,11,3,10,12)(5,18,14,8,20)(6,17,15,7,16)$ |
| $288°$ | $g_9 = (1,9,19,13,4)(2,10,11,12,3)(5,8,18,20,14)(6,7,17,16,15)$ |

Table 5.3: Rotations of axis $\beta$ through centers of $f_2$ and $f_{10}$

| $\theta$ | Permutation |
|---|---|
| $72°$ | $g_{10} = (1, 2, 10, 9, 8)(3, 11, 18, 7, 5)(4, 12, 19, 17, 6)(13, 20, 16, 15, 14)$ |
| $144°$ | $g_{11} = (1, 10, 8, 2, 9)(3, 18, 5, 11, 7)(4, 19, 6, 12, 17)(13, 16, 14, 20, 15)$ |
| $216°$ | $g_{12} = (1, 9, 2, 8, 10)(3, 7, 11, 5, 18)(4, 17, 12, 6, 19)(13, 15, 20, 14, 16)$ |
| $288°$ | $g_{13} = (1, 8, 9, 10, 2)(3, 5, 7, 18, 11)(4, 6, 17, 19, 12)(13, 14, 15, 16, 20)$ |

Table 5.4: Rotations of axis $\beta$ through centers of $f_6$ and $f_9$

| $\theta$ | Permutation |
|---|---|
| $72°$ | $g_{14} = (1, 8, 7, 6, 5)(2, 9, 17, 15, 4)(3, 10, 18, 16, 14)(11, 19, 20, 13, 12)$ |
| $144°$ | $g_{15} = (1, 7, 5, 8, 6)(2, 17, 4, 9, 15)(3, 18, 14, 10, 16)(11, 20, 12, 19, 13)$ |
| $216°$ | $g_{16} = (1, 6, 8, 5, 7)(2, 15, 9, 4, 17)(3, 16, 10, 14, 18)(11, 13, 19, 12, 20)$ |
| $288°$ | $g_{17} = (1, 5, 6, 7, 8)(2, 4, 15, 17, 9)(3, 14, 16, 18, 10)(11, 12, 13, 20, 19)$ |

Table 5.5: Rotations of axis $\beta$ through centers of $f_5$ and $f_8$

| $\theta$ | Permutation |
|---|---|
| $72°$ | $g_{18} = (1, 7, 16, 13, 3)(2, 8, 17, 20, 12)(4, 5, 6, 15, 14)(9, 18, 19, 11, 10)$ |
| $144°$ | $g_{19} = (1, 16, 3, 7, 13)(2, 17, 12, 8, 20)(4, 6, 14, 5, 15)(9, 19, 10, 18, 11)$ |
| $216°$ | $g_{20} = (1, 13, 7, 3, 16)(2, 20, 8, 12, 17)(4, 15, 5, 14, 6)(9, 11, 18, 10, 19)$ |
| $288°$ | $g_{21} = (1, 3, 13, 16, 7)(2, 12, 20, 17, 8)(4, 14, 15, 6, 5)(9, 10, 11, 19, 18)$ |

Table 5.6: Rotations of axis $\beta$ through centers of $f_4$ and $f_7$

| $\theta$ | Permutation |
|---|---|
| 72° | $g_{22} = (1, 6, 16, 19, 10)(2, 5, 15, 20, 11)(3, 4, 14, 13, 12)(7, 17, 18, 9, 8)$ |
| 144° | $g_{23} = (1, 16, 10, 6, 19)(2, 15, 11, 5, 20)(3, 14, 12, 4, 13)(7, 18, 8, 17, 9)$ |
| 216° | $g_{24} = (1, 19, 6, 10, 16)(2, 20, 5, 11, 15)(3, 13, 4, 12, 14)(7, 9, 17, 8, 18)$ |
| 288° | $g_{25} = (1, 10, 19, 16, 6)(2, 11, 20, 15, 5)(3, 12, 13, 14, 4)(7, 8, 9, 18, 17)$ |

Table 5.7: Rotations of the $\gamma$ axis through the midpoints of diagonally opposite edges

| Pairs of Opp. Edges | Permutation |
|---|---|
| $e_1 \longleftrightarrow e_{26}$ | $g_{26} = (1, 2)(3, 8)(4, 9)(5, 10)(6, 11)(7, 12)(13, 17)(14, 18)(15, 19)(16, 20)$ |
| $e_2 \longleftrightarrow e_{27}$ | $g_{27} = (1, 12)(2, 3)(4, 10)(5, 11)(6, 19)(7, 20)(8, 13)(9, 14)(15, 18)(16, 17)$ |
| $e_3 \longleftrightarrow e_{28}$ | $g_{28} = (1, 13)(2, 14)(3, 4)(5, 12)(6, 11)(7, 19)(8, 20)(9, 16)(10, 15)(17, 18)$ |
| $e_4 \longleftrightarrow e_{29}$ | $g_{29} = (1, 14)(2, 15)(3, 6)(4, 5)(7, 12)(8, 13)(9, 20)(10, 16)(11, 17)(18, 19)$ |
| $e_5 \longleftrightarrow e_{30}$ | $g_{30} = (1, 5)(2, 6)(3, 7)(4, 8)(9, 14)(10, 15)(11, 16)(12, 17)(13, 18)(19, 20)$ |
| $e_6 \longleftrightarrow e_{25}$ | $g_{31} = (1, 8)(2, 7)(3, 17)(4, 18)(5, 9)(6, 10)(11, 15)(12, 16)(13, 20)(14, 19)$ |
| $e_7 \longleftrightarrow e_{21}$ | $g_{32} = (1, 11)(2, 10)(3, 9)(4, 18)(5, 19)(6, 20)(7, 13)(8, 12)(14, 17)(15, 16)$ |
| $e_8 \longleftrightarrow e_{22}$ | $g_{33} = (1, 20)(2, 13)(3, 12)(4, 11)(5, 19)(6, 18)(7, 17)(8, 16)(9, 15)(10, 14)$ |
| $e_9 \longleftrightarrow e_{23}$ | $g_{34} = (1, 20)(2, 16)(3, 15)(4, 14)(5, 13)(6, 12)(7, 11)(8, 19)(9, 18)(10, 17)$ |
| $e_{10} \longleftrightarrow e_{24}$ | $g_{35} = (1, 15)(2, 16)(3, 17)(4, 7)(5, 6)(8, 14)(9, 13)(10, 20)(11, 19)(12, 18)$ |
| $e_{19} \longleftrightarrow e_{14}$ | $g_{36} = (1, 19)(2, 18)(3, 17)(4, 16)(5, 20)(6, 13)(7, 12)(8, 11)(9, 10)(14, 15)$ |
| $e_{20} \longleftrightarrow e_{15}$ | $g_{37} = (1, 20)(2, 19)(3, 18)(4, 17)(5, 16)(6, 15)(7, 14)(8, 13)(9, 12)(10, 11)$ |
| $e_{21} \longleftrightarrow e_{16}$ | $g_{38} = (1, 16)(2, 20)(3, 19)(4, 18)(5, 17)(6, 7)(8, 15)(9, 14)(10, 13)(11, 12)$ |
| $e_{12} \longleftrightarrow e_{17}$ | $g_{39} = (1, 17)(2, 16)(3, 20)(4, 19)(5, 18)(6, 9)(7, 8)(10, 15)(11, 14)(12, 13)$ |
| $e_{18} \longleftrightarrow e_{13}$ | $g_{40} = (1, 18)(2, 17)(3, 16)(4, 20)(5, 19)(6, 11)(7, 10)(8, 9)(12, 15)(13, 14)$ |

Table 5.8: Rotations of axis $\tau$ about centers of pairs of directly opposite vertices

| | Axis | $\theta$ | Permutations |
|---|---|---|---|
| 1. | $v_1 \longleftrightarrow v_{20}$ | $120^\circ$ | $g_{41} = (2,8,5)(3,9,6)(4,10,7)(11,17,14)(12,18,15)(13,19,16)$ |
| | | $240^\circ$ | $g_{42} = (2,5,8)(3,6,9)(4,7,10)(11,14,17)(12,15,18)(13,16,19)$ |
| 2. | $v_2 \longleftrightarrow v_{16}$ | $120^\circ$ | $g_{43} = (1,3,10)(4,11,8)(5,12,9)(6,13,18)(7,14,19)(15,20,17)$ |
| | | $240^\circ$ | $g_{44} = (1,10,3)(4,8,11)(5,9,12)(6,18,13)(7,19,14)(15,17,20)$ |
| 3. | $v_3 \longleftrightarrow v_{17}$ | $120^\circ$ | $g_{45} = (1,14,11)(2,4,12)(5,13,10)(6,20,9)(7,16,18)(8,15,19)$ |
| | | $240^\circ$ | $g_{46} = (1,11,14)(2,12,4)(5,10,13)(6,9,20)(7,18,16)(8,19,15)$ |
| 4. | $v_4 \longleftrightarrow v_{18}$ | $120^\circ$ | $g_{47} = (1,15,12)(2,6,13)(3,5,14)(7,20,10)(8,16,11)(9,17,19)$ |
| | | $240^\circ$ | $g_{48} = (1,12,15)(2,13,6)(3,14,5)(7,10,20)(8,11,16)(9,19,17)$ |
| 5. | $v_5 \longleftrightarrow v_{19}$ | $120^\circ$ | $g_{49} = (1,6,4)(2,7,14)(3,8,15)(9,16,12)(10,17,13)(11,18,20)$ |
| | | $240^\circ$ | $g_{50} = (1,4,6)(2,14,7)(3,15,8)(9,12,16)(10,13,17)(11,20,18)$ |
| 6. | $v_6 \longleftrightarrow v_{11}$ | $120^\circ$ | $g_{51} = (1,17,14)(2,18,13)(3,9,20)(4,8,16)(5,7,15)(10,19,12)$ |
| | | $240^\circ$ | $g_{52} = (1,14,17)(2,13,18)(3,20,9)(4,16,8)(5,15,7)(10,12,19)$ |
| 7. | $v_7 \longleftrightarrow v_{12}$ | $120^\circ$ | $g_{53} = (1,18,15)(2,19,14)(3,11,13)(4,10,20)(5,9,16)(6,8,17)$ |
| | | $240^\circ$ | $g_{54} = (1,15,18)(2,14,19)(3,13,11)(4,20,10)(5,16,9)(6,17,8)$ |
| 8. | $v_8 \longleftrightarrow v_{13}$ | $120^\circ$ | $g_{55} = (1,9,7)(2,18,6)(3,19,15)(4,11,16)(5,10,17)(12,20,14)$ |
| | | $240^\circ$ | $g_{56} = (1,7,9)(2,6,18)(3,15,19)(4,16,11)(5,17,10)(12,14,20)$ |
| 9. | $v_9 \longleftrightarrow v_{14}$ | $120^\circ$ | $g_{57} = (1,11,17)(2,19,7)(3,20,6)(4,13,15)(5,12,16)(8,10,18)$ |
| | | $240^\circ$ | $g_{58} = (1,17,11)(2,7,19)(3,6,20)(4,15,13)(5,16,12)(8,18,10)$ |
| 10. | $v_{10} \longleftrightarrow v_{15}$ | $120^\circ$ | $g_{59} = (1,12,18)(2,11,9)(3,19,8)(4,20,7)(5,13,17)(6,14,16)$ |
| | | $240^\circ$ | $g_{60} = (1,18,12)(2,9,11)(3,8,19)(4,7,20)(5,17,13)(6,16,14)$ |

**Remarks 6.**

- *Let the group of rotational symmetries of a dodecahedron be denoted by* $\mathbb{G}_r$*. It is composed of all the preceding* $g_i, i = 1, 2, ....60$*. i.e.* $|\mathbb{G}_r| = 60$.

### 5.1.2 Reflection-al Group Symmetry of a Dodecahedron

The dodecahedron has 15 unique reflection planes by figure **5.3** below.

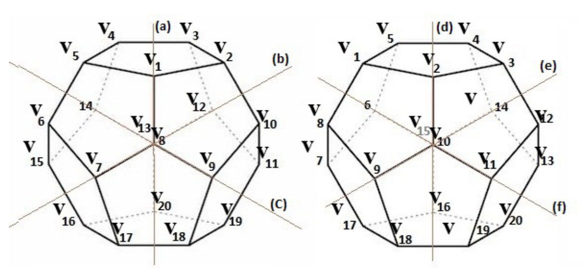



Figure 5.3: Planes of reflection of a dodecahedron

Table 5.9: Reflectional Permutations of a Dodecahedron

| | Permutation | $\wp$ |
|---|---|---|
| 1. | $g_{61} = (1,2)(3,5)(6,12)(7,11)(8,10)(13,15)(16,20)(17,19)$ | |
| 2. | $g_{62} = (1,3)(4,5)(6,14)(7,13)(8,12)(9,11)(17,20)(18,19)$ | **(d)** |
| 3. | $g_{63} = (1,4)(2,3)(7,15)(8,14)(9,13)(10,12)(16,17)(18,20)$ | |
| 4. | $g_{64} = (1,5)(2,4)(6,8)(9,15)(10,14)(11,13)(16,18)(19,20)$ | |
| 5. | $g_{65} = (1,6)(2,15)(3,14)(7,8)(9,17)(10,16)(11,20)(12,13)$ | |
| 6. | $g_{66} = (1,7)(2,17)(3,16)(4,15)(5,6)(10,18)(11,19)(12,20)$ | **(c)** |
| 7. | $g_{67} = (1,8)(2,9)(3,18)(4,17)(5,7)(12,19)(13,20)(14,16)$ | **(f)** |
| 8. | $g_{68} = (1,9)(2,10)(3,11)(4,19)(5,18)(6,17)(14,20)(15,16)$ | **(b)** |
| 9. | $g_{69} = (1,10)(4,12)(5,11)(6,19)(7,18)(8,9)(13,14)(15,20)$ | |
| 10. | $g_{70} = (1,13)(2,12)(5,14)(6,15)(7,16)(8,20)(9,19)(10,11)$ | |
| 11. | $g_{71} = (1,19)(2,11)(3,12)(4,13)(5,20)(6,16)(7,17)(8,18)$ | **(e)** |
| 12. | $g_{72} = (1,16)(2,20)(3,13)(4,14)(5,15)(8,17)(9,18)(10,19)$ | |
| 13. | $g_{73} = (2,5)(3,4)(6,10)(7,9)(11,15)(12,14)(16,19)(17,18)$ | **(a)** |
| 14. | $g_{74} = (2,8)(3,7)(4,6)(9,10)(11,18)(12,17)(13,16)(14,15)$ | |
| 15. | $g_{75} = (3,10)(4,9)(5,8)(6,7)(11,12)(13,19)(14,18)(15,17)$ | |

**Note 2.** *From figure **5.3**and table **5.1.2** we obtain permutations of* $6$ *reflectional planes **(a), (b), (c), (d), (e)** and **(f)**. The rest of the permutations can be obtained in a similar way.*

**Remarks 7.** *Let the group of reflectional symmetries of a dodecahedron be denoted by* $\mathbb{G}_f$*. We note that the reflectional symmetries given above are only* $15$*, with the help of Lim (2008) we can observe that the missing permutations can be obtained by multi-reflections of these reflections. i.e.* $|\mathbb{G}_f| = 60$.

### 5.1.3 Full Symmetry Group of a Dodecahedron

The full symmetry group of a dodecahedron denoted by $\mathbb{G}$ is composed of both its rotational symmetries $\mathbb{G}_r$, and reflection-al symmetries $\mathbb{G}_f$. i.e. $|\mathbb{G}| = 120$.

## 5.2 The Action of $\mathbb{G}$ on the Vertices of a Dodecahedron

**Theorem 5.2.1.** *Let the set of vertices of a dodecahedron be denoted by* $V = \{v_i\}, i = 1, 2, ..., 20$*, then action of* $\mathbb{G}$ *on the set* $V$ *is well defined.*

*Proof.* From Theorem **3.2.1** we observe that through the action $g_1 \in \mathbb{G}$ fixes all the elements of $V$. Moreover any other $2$ elements of $\mathbb{G}$ will in turn permute any vertex accordingly. Thus the action is well-defined. □

**Example 11.**

*i. Taking* $g_1 = I \in \mathbb{G}$ *and* $v_{12} \in V$ *then* $g_1(v_{12}) = v_{12}$

*ii. Taking* $g_8 = (1, 19, 4, 9, 13)(2, 11, 3, 10, 12)(5, 18, 14, 8, 20)(6, 17, 15, 7, 16)$*,*
$g_{67} = (1, 8)(2, 9)(3, 18)(4, 17)(5, 7)(12, 19)(13, 20)(14, 16) \in \mathbb{G}$*, we have*

$$\begin{aligned}(g_8 * g_{67})v_{12} &= (1, 20)(2, 13, 5, 16, 8, 19)(3, 14, 6, 17, 9, 11)(4, 15, 7, 18, 10, 12) * v_{12} \\ &= v_4\end{aligned}$$

$On\ the\ other\ hand,$

$$\begin{aligned}g_8\{g_{67} * v_{12}\} &= g_{16}\{(1, 8)(2, 9)(3, 18)(4, 17)(5, 7)(12, 19)(13, 20)(14, 16) * v_{12}\} \\ &= (1, 19, 4, 9, 13)(2, 11, 3, 10, 12)(5, 18, 14, 8, 20)(6, 17, 15, 7, 16) * v_{19} \\ &= v_4\end{aligned}$$

**Lemma 5.2.1.** *The order of a stabilizer of* $v_i \in V$ *in* $\mathbb{G}$ *is* $6$.

*Proof.* Let $X$ be the dodecahedron. The non-identity element of $\mathbb{G}$ fixing the point $v_i \in V$ is $g_j$ i.e. rotations through that vertex. Any rotational axis going through a vertex and the center of gravity of $X$, must go through the vertex of the opposite side. Thus, the angle of rotation that leaves $X$ invariant is $\{\frac{2\pi}{3}\}$. Consequently, the order of $g_j$ is $3$. Additionally, the reflections of $X$ that leave $X$ invariant are the reflection mirror that go through the vertex, which are of order $3$. Hence, the order of the stabilizer of $V$ in $\mathbb{G}$ is $6$. □

**Example 12.** *Take* $v_{13} \in V$*, then using Definition* ***1.2.9*** *we obtain the following result:-*

$$\begin{aligned}
Stab_{\mathbb{G}}(v_9) &= \{g_1, g_{47}, g_{56}, g_{61}, g_{66}, g_{71}\} \\
Stab_{\mathbb{G}}(v_9) &= \left\{\begin{array}{l}
g_1 = I, \\
g_{57} = (1,11,17)(2,19,7)(3,20,6)(4,13,15)(5,12,16)(8,10,18), \\
g_{58} = (1,17,11)(2,7,19)(3,6,20)(4,15,13)(5,16,12)(8,18,10), \\
g_{61} = (1,2)(3,5)(6,12)(7,11)(8,10)(13,15)(16,20)(17,19), \\
g_{66} = (1,7)(2,17)(3,16)(4,15)(5,6)(10,18)(11,19)(12,20), \\
g_{71} = (1,19)(2,11)(3,12)(4,13)(5,20)(6,16)(7,17)(8,18).
\end{array}\right\} \\
i.e.\ |Stab_{\mathbb{G}}(v_9)| &= 6.
\end{aligned}$$

**Theorem 5.2.2.** *The action of* $\mathbb{G}$ *on the set of vertices* $V$ *of a dodecahedron is transitive.*

*Proof.* From Definition **1.2.10** and Lemma **5.2.1**, we observe that

$$Orb_{\mathbb{G}}(v_i) = V, \forall\ i = 1, 2, ..., 20.$$

Thus the action is transitive. □

**Theorem 5.2.3.** *The action of $\mathbb{G}$ on the set of vertices $V$ of a dodecahedron is Faithful.*

*Proof.* We observe that only $g_1(v_i) = v_i, g_1 = I$ fixes any of the vertices while the rest of the elements of $\mathbb{G}$ maps the vertices to different points. Then by Definition **1.2.11** above the action of $\mathbb{G}$ on $V$ is indeed faithful. □

**Theorem 5.2.4.** *The action of $\mathbb{G}$ on the set of vertices $V$ of a dodecahedron is not regular.*

*Proof.* By Definition **1.2.10** and Definition **1.2.11** the action of $\mathbb{G}$ on $V$ is transitive but not regular since it is not semi-regular. i.e. $|Stab_{\mathbb{G}(v_i)}| \neq 1$. □

**Theorem 5.2.5.** *The action of $\mathbb{G}$ on the set of vertices $V$ of a dodecahedron is not primitive.*

*Proof.* From Lemma **5.2.1**, any isotropy subgroup $\mathbb{G}_{vi}$ has an order of $6$. The index,

$$|Orb_{\mathbb{G}}(v_i)| = |\mathbb{G} : \mathbb{G}_{v_i}| = \frac{120}{6} = 60, not\ prime$$

Hence by Theorem **1.3.2**, $\mathbb{G}_{v_i}$ is not maximal and the action is not primitive. □

**Theorem 5.2.6.** *The action of $\mathbb{G}$ on the set of vertices $V$ of a dodecahedron has sub-degrees $2^{(1)}, 2^{(3)}$ and $2^{(6)}$. Thus the rank is $6$.*

*Proof.* Let $X$ be the dodecahedron, and $v_i \longleftrightarrow v_j$ be an axis through $v_i$ to the vertex directly opposite it, $v_j$ while $X$ is held upright on either of this vertices.

By Lemma **5.2.1**, the order of the isotropy subgroup $\mathbb{G}_{v_i}$ on $V$ is $6$. The action of $\mathbb{G}_{v_i}$ has $6$ orbits namely;

- trivial $\delta_i = v_i$,

- trivial $\Delta_i = v_j$,
- 2 non-trivial $\Lambda_i$ and $\lambda_i$ each of order 6,
- 2 non-trivial $\varrho$ and $\eta$ each of order 3,

Where $\delta_i$ the element $v_i$ being acted upon by $\mathbb{G}_{v_i}$ while $\Delta_i$ denotes the element $v_j$ directly opposite the vertex $v_i$ being acted upon.

Consider $\mathbb{G}_9$, the orbits of that action are $\delta_9 = v_9, \Delta_9 = v_{14}$, $\lambda_9 = \{v_1, v_2, v_7, v_{11}, v_{17}, v_{19}\}, \Lambda_9 = \{v_3, v_5, v_6, v_{12}, v_{16}, v_{20}\}, \varrho = \{v_4, v_{13}, v_{15}\}$ and $\eta = \{v_8, v_{10}, v_{18}\}$.

Now, with the results above and Definition **1.2.12** then we can conclude that the sub-degrees are $2^{(1)}, 2^{(3)}$ and $2^{(6)}$. Hence the rank $R(G)$ is 6. □

## 5.3 Symmetry Group of an Icosahedron

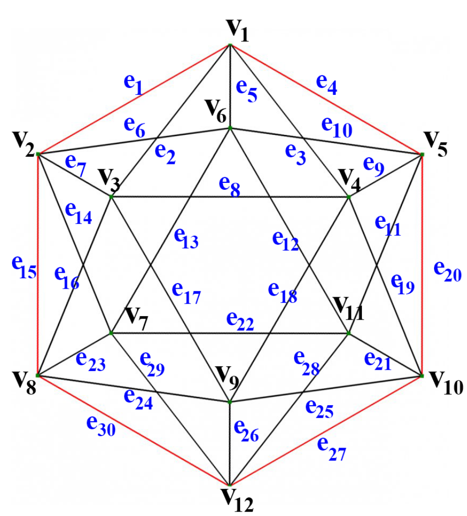



Figure 5.4: Face, edges and vertices of an icosahedron

### 5.3.1 Rotational Group of Symmetry of an Icosahedron

The group of rotational symmetries in $\mathbb{R}^3$ leaving the icosahedron invariant will be denoted by $\mathbb{G}_r$. Its elements are

- The identity, $g_1 = I$
- Rotation of an axis joining $2$ directly opposite vertices about $72°, 144°, 216°$ and $288°$
- Rotation of an axis through the midpoints of $2$ directly opposite edges about $180°$
- Rotations of the centre of the $2$ directly opposite faces about $120°$ and $240°$

Using the denotations of faces and edges as described in figure **5.4** we can illustrate the above elements using figure **5.3.1** below

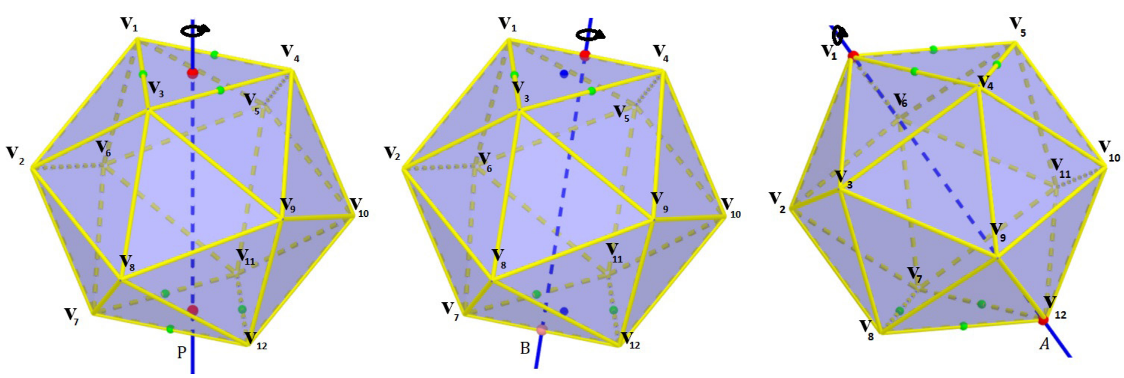



Figure 5.5: Rotational axes of an icosahedron

Table 5.10: Rotation of axis $\boldsymbol{A}$ passing through directly opposite vertices

| Axis | $\theta$ | Permutation | Axis | $\theta$ | Permutation |
|---|---|---|---|---|---|
| $v_1 \longleftrightarrow v_{12}$ | $72^\circ$ | $g_2 = (2,3,4,5,6)(7,8,9,10,11)$ | $v_2 \longleftrightarrow v_{10}$ | $72^\circ$ | $g_{14} = (1,6,7,8,3)(4,5,11,12,9)$ |
| | $144^\circ$ | $g_3 = (2,4,6,3,5)(7,9,11,8,10)$ | | $144^\circ$ | $g_{15} = (1,7,3,6,8)(4,11,9,5,12)$ |
| | $216^\circ$ | $g_4 = (2,5,3,6,4)(7,10,8,11,9)$ | | $216^\circ$ | $g_{16} = (1,8,5,3,7)(4,12,5,9,11)$ |
| | $288^\circ$ | $g_5 = (2,6,5,4,3)(7,11,10,9,8)$ | | $288^\circ$ | $g_{17} = (1,3,8,7,6)(4,9,12,11,5)$ |
| $v_3 \longleftrightarrow v_{11}$ | $72^\circ$ | $g_6 = (1,2,8,9,4)(5,6,7,12,10)$ | $v_4 \longleftrightarrow v_7$ | $72^\circ$ | $g_{18} = (1,3,9,10,5)(2,8,12,11,6)$ |
| | $144^\circ$ | $g_7 = (1,8,4,2,9)(5,7,10,6,12)$ | | $144^\circ$ | $g_{19} = (1,9,5,3,10)(2,12,6,8,11)$ |
| | $216^\circ$ | $g_8 = (1,9,2,4,8)(5,12,6,10,7)$ | | $216^\circ$ | $g_{20} = (1,10,3,5,9)(2,11,8,6,12)$ |
| | $288^\circ$ | $g_9 = (1,4,9,8,2)(5,10,12,7,6)$ | | $288^\circ$ | $g_{21} = (1,5,10,9,3)(2,6,11,12,8)$ |
| $v_5 \longleftrightarrow v_8$ | $72^\circ$ | $g_{10} = (1,4,10,11,6)(2,3,9,12,7)$ | $v_6 \longleftrightarrow v_9$ | $72^\circ$ | $g_{22} = (1,5,11,7,2)(3,4,10,12,8)$ |
| | $144^\circ$ | $g_{11} = (1,10,6,4,11)(2,9,7,3,12)$ | | $144^\circ$ | $g_{23} = (1,11,2,5,7)(3,10,8,4,12)$ |
| | $216^\circ$ | $g_{12} = (1,11,4,6,10)(2,12,3,7,9)$ | | $216^\circ$ | $g_{24} = (1,7,5,2,11)(3,12,4,8,10)$ |
| | $288^\circ$ | $g_{13} = (1,6,11,10,4)(2,7,12,9,3)$ | | $288^\circ$ | $g_{25} = (1,2,7,11,5)(3,8,12,10,4)$ |

Table 5.11: Rotation of axis $\boldsymbol{B}$ through the midpoints of directly opposite edges

| | Axis | $180^\circ$ |
|---|---|---|
| 1. | $e_1 \longleftrightarrow e_{27}$ | $g_{26} = (1,2)(3,6)(4,7)(5,8)(9,11)(10,12)$ |
| 2. | $e_2 \longleftrightarrow e_{28}$ | $g_{27} = (1,3)(2,4)(5,8)(6,9)(7,10)(11,12)$ |
| 3. | $e_3 \longleftrightarrow e_{29}$ | $g_{28} = (1,4)(2,10)(3,5)(6,9)(7,12)(8,11)$ |
| 4. | $e_4 \longleftrightarrow e_{30}$ | $g_{29} = (1,5)(2,10)(3,11)(4,6)(7,9)(8,12)$ |
| 5. | $e_5 \longleftrightarrow e_{26}$ | $g_{30} = (1,6)(2,5)(3,11)(4,7)(8,10)(9,12)$ |
| 6. | $e_7 \longleftrightarrow e_{21}$ | $g_{31} = (1,8)(2,3)(4,7)(5,12)(6,9)(10,11)$ |
| 7. | $e_{15} \longleftrightarrow e_{20}$ | $g_{32} = (1,12)(2,8)(3,7)(4,11)(5,10)(6,9)$ |
| 8. | $e_{14} \longleftrightarrow e_{19}$ | $g_{33} = (1,12)(2,7)(3,11)(4,10)(5,9)(6,8)$ |
| 9. | $e_6 \longleftrightarrow e_{25}$ | $g_{34} = (1,7)(2,6)(3,11)(4,12)(5,8)(9,10)$ |
| 10. | $e_8 \longleftrightarrow e_{22}$ | $g_{35} = (1,9)(2,10)(3,4)(5,8)(6,12)(7,11)$ |
| 11. | $e_{17} \longleftrightarrow e_{12}$ | $g_{36} = (1,12)(2,10)(3,9)(4,8)(5,7)(6,11)$ |
| 12. | $e_{16} \longleftrightarrow e_{11}$ | $g_{37} = (1,12)(2,9)(3,8)(4,7)(5,11)(6,10)$ |
| 13. | $e_9 \longleftrightarrow e_{23}$ | $g_{38} = (1,10)(2,12)(3,11)(4,5)(6,9)(7,8)$ |
| 14. | $e_{18} \longleftrightarrow e_{13}$ | $g_{39} = (1,12)(2,11)(3,10)(4,9)(5,8)(6,7)$ |
| 15. | $e_{10} \longleftrightarrow e_{24}$ | $g_{40} = (1,11)(2,10)(3,12)(4,7)(5,6)(8,9)$ |

Table 5.12: Rotations of axis ***P*** centre of directly opposite faces

| | Directly opposite faces | $\theta$ | Permutation |
|---|---|---|---|
| 1. | $f_1 \longleftrightarrow f_{20}$ | 120° | $g_{41} = (1,2,3)(4,6,8)(5,7,9)(10,11,12)$ |
| | | 240° | $g_{42} = (1,3,2)(4,8,6)(5,9,7)(10,12,11)$ |
| 2. | $f_2 \longleftrightarrow f_{16}$ | 120° | $g_{43} = (1,3,4)(2,9,5)(6,8,10)(7,12,11)$ |
| | | 240° | $g_{44} = (1,4,3)(2,5,9)(6,10,8)(7,11,12)$ |
| 3. | $f_3 \longleftrightarrow f_{17}$ | 120° | $g_{45} = (1,4,5)(2,11,9)(3,10,6)(7,8,12)$ |
| | | 240° | $g_{46} = (1,5,4)(2,9,11)(3,6,10)(7,12,8)$ |
| 4. | $f_4 \longleftrightarrow f_{18}$ | 120° | $g_{47} = (1,5,6)(2,4,11)(3,10,7)(8,9,12)$ |
| | | 240° | $g_{48} = (1,6,5)(2,11,4)(3,7,10)(8,12,9)$ |
| 5. | $f_5 \longleftrightarrow f_{19}$ | 120° | $g_{49} = (1,6,2)(3,5,7)(4,11,8)(9,10,12)$ |
| | | 240° | $g_{50} = (1,2,6)(3,7,5)(4,8,11)(9,12,10)$ |
| 6. | $f_8 \longleftrightarrow f_{13}$ | 120° | $g_{51} = (1,7,9)(2,8,3)(4,6,12)(5,11,10)$ |
| | | 240° | $g_{52} = (1,9,7)(2,3,8)(4,12,6)(5,10,11)$ |
| 7. | $f_9 \longleftrightarrow f_{14}$ | 120° | $g_{53} = (1,7,10)(2,12,4)(3,8,9)(5,6,11)$ |
| | | 240° | $g_{54} = (1,10,7)(2,4,12)(3,9,8)(5,11,6)$ |
| 8. | $f_{10} \longleftrightarrow f_{15}$ | 120° | $g_{55} = (1,8,10)(2,12,5)(3,9,4)(6,7,11)$ |
| | | 240° | $g_{56} = (1,10,8)(2,5,13)(3,4,9)(6,11,7)$ |
| 9. | $f_{11} \longleftrightarrow f_6$ | 120° | $g_{57} = (1,11,8)(2,6,7)(3,5,12)(4,10,9)$ |
| | | 240° | $g_{58} = (1,8,11)(2,7,6)(3,12,5)(4,9,10)$ |
| 10. | $f_{12} \longleftrightarrow f_7$ | 120° | $g_{59} = (1,11,9)(2,7,8)(3,6,12)(4,5,10)$ |
| | | 240° | $g_{60} = (1,9,11)(2,8,7)(3,12,6)(4,10,5)$ |

**Remarks 8.** *Let the group of rotational symmetries of an icosahedron be denoted by* $\mathbb{G}_r$ *then it is composed of all the immediate preceding elements* $g_i, i = 1, 2, ..., 60$.

### 5.3.2 Reflection-al Group of Symmetry of an Icosahedron

The Icosahedron has $15$ unique permutations in the reflection-al group of symmetry $\mathbb{G}_f$. This can be illustrated by figure **5.3.2** below.

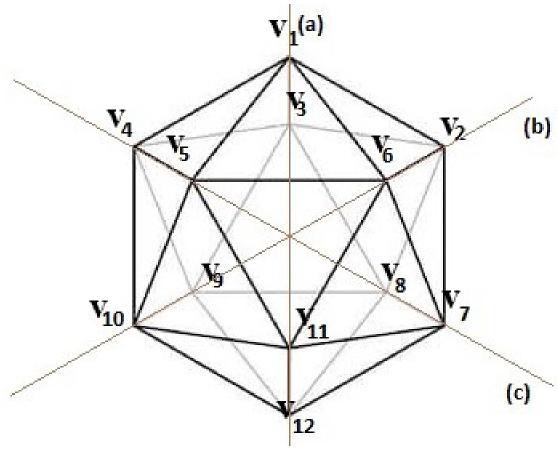



Table 5.13: Planes of Reflection of an Icosahedron

| | Permutation | $\wp$ | | Permutation | $\wp$ |
|---|---|---|---|---|---|
| 1. | $g_{61} = (2,5)(3,4)(7,11)(8,10)$ | | 9. | $g_{69} = (1,6)(3,7)(4,11)(9,12)$ | |
| 2. | $g_{62} = (3,6)(4,5)(7,8)(9,11)$ | | 10. | $g_{70} = (1,9)(2,8)(5,10)(6,12)$ | |
| 3. | $g_{63} = (2,4)(5,6)(7,10)(8,9)$ | **(a)** | 11. | $g_{71} = (1,4)(2,9)(6,10)(7,12)$ | |
| 4. | $g_{64} = (2,6)(3,5)(8,11)(9,10)$ | | 12. | $g_{72} = (1,2)(4,8)(5,7)(10,12)$ | |
| 5. | $g_{65} = (2,3)(4,6)(7,9)(10,11)$ | | 13. | $g_{73} = (1,5)(2,11)(3,10)(8,12)$ | |
| 6. | $g_{66} = (1,8)(4,9)(5,12)(6,7)$ | | 14. | $g_{74} = (1,10)(2,12)(3,9)(6,11)$ | **(c)** |
| 7. | $g_{67} = (1,7)(3,8)(4,12)(5,11)$ | **(b)** | 15. | $g_{75} = (1,11)(2,7)(3,12)(4,10)$ | |
| 8. | $g_{68} = (1,3)(5,9)(6,8)(11,12)$ | | | | |

**Note 3.** *From figure* ***5.3.2*** *we obtain permutations of* $3$ *reflectional planes* ***(a), (b)*** *and* ***(c)****. The rest of the permutations can be obtained in a similar way.*

**Remarks 9.** *Let the group of reflectional symmetries of an icosahedron be denoted by* $\mathbb{G}_f$*. We note that the reflectional symmetries given above are only* $15$*, with the help of Lim (2008) we can observe that the missing permutations can be obtained by multi-reflections of these reflections. i.e.* $|\mathbb{G}_f| = 60$.

### 5.3.3 Full Symmetry Group of an Icosahedron

Let the full symmetry group of an icosahedron be denoted by $\mathbb{G}$, then it is composed of the elements in the rotational and reflection-al groups of symmetry of an icosahedron. i.e $\mathbb{G} = \mathbb{G}_r + \mathbb{G}_f$.

## 5.4 The Action of $\mathbb{G}$ on the Vertices of an Icosahedron

**Theorem 5.4.1.** *The action of $\mathbb{G}$ on the set of vertices $V$ of a icosahedron is well defined.*

*Proof.* From Definition **3.2.1** we observe that through the action $g_1 \in \mathbb{G}$ fixes all the elements of $V$. Moreover any other 2 elements of $\mathbb{G}$ will in turn permute any vertex accordingly. Thus the action is well-defined. □

**Example 13.**

*i. Taking $g_1 = I$ and $v_{11}$ then $g_1(v_{11}) = v_{11}$*

*ii. Taking $g_{61} = (2,5)(3,4)(7,11)(8,10), g_{19} = (1,10,6,4,11)(2,9,7,3,12) \in \mathbb{G}$, we have;*

$$\begin{aligned}(g_{61} * g_{19})v_{11} &= \{(2,5)(3,4)(7,11)(8,10) * (1,10,6,4,11)(2,9,7,3,12)\}v_{11}\\ &= \{(1,8,10,6,3,12,5,2,9,11)\}v_{11}\\ &= v_1\end{aligned}$$

*On the other hand,*

$$\begin{aligned}g_{61}(g_{19} * v_{11}) &= g_{61}\{(1,10,6,4,11)(2,9,7,3,12) * v_{11}\}\\ &= (2,5)(3,4)(7,11)(8,10) * v_1\\ &= v_1\end{aligned}$$

**Lemma 5.4.1.** *Let $v_i \in V$ where $V$ is the set of vertices of a dodecahedron. Then the orbit of $v_i$ is the length $20$.*

*Proof.* By Definition **1.2.8**, we get,

$$\begin{aligned} Orb_{\mathbb{G}}(v_i) &= V, \\ and\ since\ |V| &= 20 \\ i.e.\ |\ Orb_{\mathbb{G}}(v_i)\ | &= 20. \Rightarrow Hence\ the\ orbits\ are\ of\ length\ 20. \end{aligned}$$

□

**Example 14.** *Take $v_{14}$ in the set of vertices of a dodecahedron then its corresponding orbit in $\mathbb{G}$ is given by*

$$\begin{aligned} Orb_{\mathbb{G}}(v_{14}) &= \{v_1, v_2, v_3, v_4, \ldots, v_{19}, v_{20}\} = V \\ Orb_{\mathbb{G}}(v_{14}) &= V \\ i.e. |Orb_G(v_{14})| &= 20 \end{aligned}$$

**Lemma 5.4.2.** *Let $v_i \in V$ where $V$ is the set of vertices of a icosahedron. Then the orbit of $v_i$ is the length $12$.*

*Proof.* By Definition **1.2.8**, we get,

$$\begin{aligned} Orb_{\mathbb{G}}(v_i) &= V, \\ and\ since\ |V| &= 12 \\ i.e.\ |\ Orb_{\mathbb{G}}(v_i)\ | &= 12. \Rightarrow Hence\ the\ orbits\ are\ of\ length\ 12. \end{aligned}$$

□

**Example 15.** *Take $v_5$ in the set of vertices of an icosahedron then its corresponding orbit in $\mathbb{G}$ is given by*

$$\begin{aligned} Orb_{\mathbb{G}}(v_5) &= \{v_1, v_2, v_3, v_4, v_5, v_6, v_7, v_8, v_9, v_{10}, v_{11}, v_{12}\} = V \\ Since\ |V| &= 12 \\ i.e.|Orb_G(v_5)| &= 12 \end{aligned}$$

**Lemma 5.4.3.** *The order of a stabilizer of $v_i$ of an icosahedron in $\mathbb{G}$ is $10$.*

*Proof.* Let $X$ be the icosahedron. The non-identity element of $\mathbb{G}$ fixing the point $v_i \in V$ is $g_j$ i.e. rotations through that vertex. Any rotational axis going through a vertex and the center of gravity of $X$, must go through the vertex of the opposite side. Thus, the angle of rotation that leaves $X$ invariant is $\{\frac{2\pi}{5}\}$. Consequently, the order of $g_j$ is $5$. Additionally, the reflections of $X$ that leave $X$ invariant are the reflection mirror that go through the vertex, which are of order $5$. Hence, the order of the stabilizer of $V$ in $\mathbb{G}$ is $10$. □

**Example 16.** *Take $v_5$ in the set of vertices of an icosahedron then using Definition **1.2.9** we obtain the following result:-*

$$\begin{array}{rcl} Stab_{\mathbb{G}}(v_5) & = & \{g_1, g_{18}, g_{19}, g_{20}, g_{21}, g_{65}, g_{69}, g_{71}, g_{74}, g_{75}\} \\ Stab_{\mathbb{G}}(v_5) & = & \left\{ \begin{array}{l} g_1 = I, g_{18} = (1,4,10,11,6)(2,3,9,12,7), \\ g_{19} = (1,10,6,4,11)(2,9,7,3,12), \\ g_{20} = (1,11,4,6,10)(2,12,3,7,9), \\ g_{21} = (1,6,11,10,4)(2,7,12,9,3), \\ g_{65} = (2,3)(4,6)(7,9)(10,11), \\ g_{69} = (1,6)(3,7)(4,11)(9,12), \\ g_{71} = (1,4)(2,9)(6,10)(7,12), \\ g_{74} = (1,10)(2,12)(3,9)(6,11), \\ g_{75} = (1,11)(2,7)(3,12)(4,10). \end{array} \right\} \\ i.e.\ |Stab_{\mathbb{G}}(v_5)| & = & 10. \end{array}$$

**Theorem 5.4.2.** *The action of $\mathbb{G}$ on the set of vertices $V$ of an icosahedron $V$ is transitive.*

*Proof.* From Definition **1.2.10** and Lemma **5.4.2**, we observe that

$$Orb_{\mathbb{G}}(v_i) = V, \ \ \forall\, i = 1, 2, ..., 12.$$

Thus the action is transitive. □

**Theorem 5.4.3.** *The action of $\mathbb{G}$ on the set of vertices $V$ of an icosahedron is Faithful.*

*Proof.* We observe that only $g_1(v_i) = v_i, g_1 = I$ fixes any of the vertices while the rest of the elements of $\mathbb{G}$ maps the vertices to different points. Then by Definition **1.2.11** above the action of $\mathbb{G}$ on $V$ is indeed faithful. □

**Theorem 5.4.4.** *The action of $\mathbb{G}$ on the set of vertices $V$ of an icosahedron is not regular.*

*Proof.* By Definition **1.2.10** and Definition **1.2.13** the action of $\mathbb{G}$ on $V$ is transitive but not regular since it is not semi-regular. i.e. $|Stab_{\mathbb{G}(v_i)}| \neq 1$. □

**Theorem 5.4.5.** *The action of $\mathbb{G}$ on the set of vertices $V$ of an icosahedron is not primitive.*

*Proof.* From Lemma **5.4.3**, any isotropy subgroup $\mathbb{G}_{vi}$ has an order of $10$. The index,

$$|Orb_{\mathbb{G}}(v_i)| = |\mathbb{G} : \mathbb{G}_{v_i}| = \frac{120}{10} = 12, \textit{not prime}$$

Hence by Theorem **1.3.2**, $\mathbb{G}_{v_i}$ is not maximal and the action is not primitive. □

**Theorem 5.4.6.** *The action of $\mathbb{G}$ on the set of vertices $V$ of an icosahedron has sub-degrees $2^{(1)}$ and $2^{(5)}$. Thus the rank is $4$.*

*Proof.* Let $X$ be the icosahedron, and $v_i \longleftrightarrow v_j$ be an axis through $v_i$ to the vertex directly opposite it, $v_j$ while $X$ is held upright on either of this vertices.

By Lemma **5.4.3**, the order of the isotropy subgroup $\mathbb{G}_{v_i}$ on $V$ is $10$. The action of $\mathbb{G}_{v_i}$ has $4$ orbits namely;

- trivial $\delta_i = v_i$,
- trivial $\Delta_i = v_j$,
- 2 non-trivial $\lambda_i$ and $\Lambda$ of order 5,

Where $\delta_i$ denotes the element of $V$ being acted upon by $\mathbb{G}_{v_i}$ and $\Delta_i$ denotes $v_j$.

Consider $\mathbb{G}_{v_8}$, the orbits its action with elements of $V$ are $\delta_8 = v_8, \Delta_8 = v_5, \lambda_8 = \{v_1, v_4, v_6, v_{10}, v_{11}\}$ and $\Lambda_8 = \{v_2, v_3, v_7, v_9, v_{12}\}$.

Now, with the results above and Definition **1.2.12** then we can conclude that the sub-degrees are $2^{(1)}$ and $2^{(5)}$. Hence the rank $R(G)$ is $4$. $\square$

# Chapter 6

# CONCLUSION AND RECOMMENDATIONS

## 6.1 Conclusion

In this project we have analysed the properties of the action of the full symmetry groups of Platonic Solids namely; tetrahedron, cube, octahedron, dodecahedron and icosahedron acting on their respective vertices.

The group actions on the set of vertices of the Platonic Solids were seen to be transitive, faithful and they were neither regular nor primitive. Furthermore we explored the ranks and sub-degrees and the results are as shown in table **6.1** below.

Table 6.1: The ranks and sub-degrees of various Platonic Solids

| | Platonic Solid | Sub-degrees | Ranks |
|---|---|---|---|
| 1. | Tetrahedron | $1^{(1)}, 1^{(3)}$ | $R(G) = 2$ |
| 2. | Cube | $2^{(1)}, 2^{(3)}$ | $R(G) = 4$ |
| 3. | Octahedron | $2^{(1)}, 1^{(4)}$ | $R(G) = 3$ |
| 4. | Dodecahedron | $2^{(1)}, 2^{(3)}, 2^{(6)}$ | $R(G) = 6$ |
| 5. | Icosahedron | $2^{(1)}, 2^{(5)}$ | $R(G) = 4$ |

## 6.2 Recommendations and Areas of Further Research

Having explored the group action of the symmetry groups acting on the respective vertices of a Platonic Solids using conventional permutation cycles, it is also possible to find the suborbital graphs of the symmetry groups of Platonic Solids acting on their respective vertices.

# References


Atiyah, M. and Sutcliffe, P. (2003). Polyhedra in physics, chemistry and geometry. *Milan Journal of Mathematics*, 71(1):33–58. 6

Cheung, K. Y. (2014). Symmetry group of platonic solids. Accesible on http://hkumath.hku.hk/course/temp/Symmetry%20group%20of%20platonic%20solids.pdf. 11

Cornford, F. M. (2014). *Plato's cosmology: the Timaeus of Plato*. Routledge. x, 2

Coxeter, H. S. M. (1973). *Regular polytopes*. Dover Publications, Inc. 8

Cromwell, P. R. (1997). *Polyhedra. 1997.* Cambridge University Press, Cambridge. 9

Foster, L. (1990). On the symmetry group of the dodecahedron. *Mathematics Magazine*, 63. x, 5, 9

Fraleigh, J. B. (2003). *A first course in abstract algebra*. Pearson Education India. 5

Heath, T. L. et al. (1956). *The thirteen books of Euclid's Elements*. Courier Corporation. x, 8

Homans, C. D. (2016). On the group-theoretic properties of the automorphism groups of various graphs. Obtainable on http://math.uchicago.edu/ may/REU2016/REUPapers/Homans.pdf. 11

Huisinga, M. (2012). Polyas counting theory. Accessible on https://www.whitman.edu/Documents/Academics/Mathematics/Huisinga.pdf. 10

Kamuti, I. N. (1992). *Combinatorial formulas, invariants and structures associated with primitive permutation representations of PGL(2,q) and PSL(2,q)*. Ph.d, Mathematical studies. 4

Lim, Y. (2008). Symmetry groups of platonic solids. 3, 8, 9, 15, 24, 31, 42, 50

Morandi, P. J. (2004). Computing the symmetry groups of the platonic solids with the help of maple. *Resonance*, 9(08). x, 9

Rose, J. S. (1978). *A course on group theory*. Cambridge University Press, Cambridge. 4, 5

Schwartz, R. (2007). Symmetry groups of platonic solids. Available on https://www.math.brown.edu/ res/MFS/handout3.pdf. 9

Silas, J. (2011). Symmetry groups of the platonic solids. Available on https://www.math.wisc.edu/ shamgar/RT-P1-Solutions.pdf. 10

Wenninger, M. J. (1974). *Polyhedron models*. Cambridge University Press. 9

Wielandt, H., Booker, H., DeClaris, N., and Bromley, D. A. (1964). *Finite permutation groups*. Academic Press, London. 5

Williams, K. and Xavier, J. (2011). Nexus 2010: Relationships between architecture and mathematics. *Nexus Network Journal*, 13:1–4. 2

Winger, S. (2012). Symmetry groups of the platonic solids. Obtained from https://swingeth.files.wordpress.com/2012/03/platonic-symmetry-groups.pdf. 10