RANKS, SUBDEGREES, SUBORBITAL GRAPHS AND CYCLE INDICES ASSOCIATED WITH THE PRODUCT ACTION OF AFFINE GROUPS

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DECLARATION

This research project is my original work and has not been presented for a degree award in any other university or for any other award.

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DEDICATION

To my mum, Jane siahi, fiancee, Christine, Siblings, Brian, Jackline, Mozart, Derrick and Laurine.

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ABBREVIATIONS AND NOTATIONS

$Aff(q)$	-	Affine group over $GF(q)$
A_r	-	Alternating group of degree r
C_r	-	The monogeniuos group of cardinality r
D_r	-	The dihedral group of cardinality $2r$
$Fix(\gamma)$	-	Set of points fixed by γ
$\pi(\gamma)$	-	Total points that are fixed by γ
$K \leq G$	-	The subgroup K of G
$GF(q)$	-	Galois field of q elements
$G_1 \times G_2 \times \ldots \times G_n$	-	External cartesian product of affine groups.
$\|A\|$	-	The cardinal number of set A
$m\|n$	-	m divides n
O_i	-	i^{th} suborbital of G
$Orb_G(a)$	-	Orbit of G containing a
$\phi(d)$	-	Euler's phi function
$C_{Q_{[1,2]}}$	-	External cartesian product of cyclic groups $C_{(q_1-1)} \times C_{(q_2-1)}$
$C_{Q_{[1,2,3]}}$	-	External cartesian product of cyclic groups $C_{(q_1-1)} \times C_{(q_2-1)} \times C_{(q_3-1)}$.
$R(G)$	-	Rank of the action of Affine group G on Ω
S_r	-	Symmetric group with degree r
$Stab_G(\alpha)$	-	The stabilizer of a point α in an Affine group G
Δ	-	The orbit of G_a
Δ^*	-	The G_a orbit which is paired with Δ
Γ_i	-	i^{th} suborbital graph corresponding to O_i
(s, r)	-	The g.c.d of points s and r
$[s, r]$	-	The l.c.m of points s and r
Γ	-	A graph
$Z(G)$	-	Cycle index of a group G
$\Omega_1 \times \Omega_2 \times \ldots \times \Omega_n$	-	Cartesian product of the sets $\Omega_1, \Omega_2, \ldots, \Omega_3$

ABSTRACT

Many scholars have studied the ranks, subdegrees, cycle index and graphs of the action of the groups C_n, D_n and $Aff(q)$ on a set X, where $X = \{1, 2, \ldots, n\}$ leaving out product actions. Recently, Kangogo (2015) studied the action of affine group over Galois field. The action of $Aff(q_1) \times Aff(q_2)$ on $GF(q_1) \times GF(q_2)$ and $Aff(q_1) \times Aff(q_2) \times Aff(q_3)$ on $GF(q_1) \times GF(q_2) \times GF(q_3)$ has not been studied. Using the definition of product action of orbits, the properties of the action of $Aff(q_1) \times Aff(q_2)$ on $GF(q_1) \times GF(q_2)$ and $Aff(q_1) \times Aff(q_2) \times Aff(q_3)$ on $GF(q_1) \times GF(q_2) \times GF(q_3)$ were studied and the rank was found to be 2^k, where $k = 2, 3$ is the number of affine groups in the cross product. The subdegrees were found to be $1, (q_1 - 1), (q_2 - 1), (q_1 - 1)(q_2 - 1)$ and $1, (q_1 - 1), (q_2 - 1), (q_3 - 1), (q_1 - 1)(q_2 - 1), (q_1 - 1)(q_3 - 1), (q_2 - 1)(q_3 - 1), (q_1 - 1)(q_2 - 1)(q_3 - 1)$ respectively. The corresponding non trivial graphs of $Aff(q_1) \times Aff(q_2)$ on $GF(q_1) \times GF(q_2)$ and $Aff(q_1) \times Aff(q_2) \times Aff(q_3)$ on $GF(q_1) \times GF(q_2) \times GF(q_3)$ were constructed using Sim's procedure and were found to have a girth of $0, 3, 6$ and $0, 3, 4, 6$ respectively. Finally, cycle index were determined by first determining the cycle index of $Aff(q)$ acting on $GF(q)$ and then using multiplication of monomials to get the cycle index of the product action. The cycle indices have applications in chemistry when counting isomers.The graphs constructed provide useful information to graph theorist. Connectivity in graphs helps biologists to explain how the different parts of the brain are connected. The results have been represented in form of theorems and graphs.

CHAPTER 1

INTRODUCTION

1.1 Background information

An affine group over Galois field of q elements is a group of all transformations represented as $\alpha x+\beta, \forall \alpha, \beta \in GF(q), \alpha \neq 0$. The affine group can be viewed as matrices of the form $\left(\begin{smallmatrix}\alpha & \beta\\ 0 & 1\end{smallmatrix}\right)$ with a cardinality of $q(q-1)$ represented by$Aff(q)$. Let $G_1, G_2, \ldots, G_k$ be groups with respect to the binary operations$(\star_1, \star_2, \ldots, \star_k)$, then the external direct product $G_1 \times G_2 \times G_3 \times \ldots \times G_k$ is defined by $\{(g_1, g_2, g_3, \ldots, g_k) : g_i \in G_i\}$. In addition,it follows that $(g_1, g_2, \ldots, g_k)(h_1, h_2, \ldots, h_k) = (g_1 \star_1 h_1, g_2 \star_2 h_2, \ldots, g_k \star_k h_k)$. The cardinality of the group $(G_1 \times G_2 \times G_3 \ldots \times G_k)$ is given by $|G_1||G_2||G_3| \ldots |G_k|$. If G_1 acts on Ω_1, G_2 acts on Ω_2 and G_3 acts on Ω_3. Then the product action of $G_1 \times G_2 \times G_3$ on $\Omega_1 \times \Omega_2 \times \Omega_3$, is given by the rule, $(g_1, g_2, g_3)(\alpha, \beta, \gamma) = (g_1(\alpha), g_2(\beta), g_3(\gamma))$. [see (Cameron *et al.*, 2007)].

1.2 Definitions of terms and preliminary results

Definition 1.2.1. *A group G acts on the left of the set Ω if and only if for every element $a \in \Omega$,the mapping $G \times \Omega \mapsto \Omega$, $(g, \alpha) \mapsto g\alpha, \forall g_1, g_2 \in G$, we have $(g_1 g_2)\alpha = g_1(g_2\alpha)$ and $e\alpha = \alpha$, where e is the identity in G.*

Definition 1.2.2. *If G acts on Ω, and partitions it into equivalence classes called orbits having $\alpha \in \Omega$. Then $Orb_{(G)}(\alpha)$ is the orbit having $\alpha \in \Omega$.*

Definition 1.2.3. *Suppose that G acts on a set $\Omega, g \in G$. The set $Fix(g) = \{\alpha \in \Omega : g\alpha = \alpha\}$ is the fixed point set of g.*

Definition 1.2.4. *Suppose the group G acts on Ω. Then the stabilizer of a is the set $Stab_G(a) = \{ga = a : g \in G\}$.*

Theorem 1.2.5. *(Armstrong, 2013) The $(G_1 \times G_2)-$ orbit having $(\alpha, \beta) \in \Omega_1 \times \Omega_2$ is represented as $Orb_{G_1}(\alpha) \times Orb_{G_2}(\beta)$ and the stabilizer of (α, β) in $G_1 \times G_2$ is given by $Stab_{G_1}(\alpha) \times Stab_{G_2}(\beta)$.*

Theorem 1.2.6 (Orbit-Stabilizer Theorem)**.** *(Rose, 1978) Suppose the group G acts on Ω. For all $\alpha \in \Omega$ where Ω is finite, then*

$$|Orb_G(\alpha)| = |G : G_\alpha| \tag{1.1}$$

Definition 1.2.7. *Given that $g \in G$ and $g\alpha = \beta$. Then $\forall \alpha, \beta \in \Omega$, the action of G on Ω is transitive if it has only one orbit.*

Definition 1.2.8. *Suppose a group G acts on Ω transitively. If G_α is the stabilizer of $\alpha \in \Omega$, then the suborbits of G on Ω is given by $\Delta_0 = \{\alpha\}, \Delta_1, \Delta_2, ..., \Delta_{(r-1)}$. The subdegrees of the group G on Ω are given by the lengths of the Δ_i, the rank of G acting on Ω denoted by $R(G)$ is the number of the suborbits.*

Definition 1.2.9. *Consider an orbit Δ of G_a acting on Ω where by G acts transitively on Ω. The set $\Delta^\star = \{ga : g \in G, a \in g\Delta\}$ is a G_a-orbit called a suborbit of G on Ω paired with Δ. The sets $\Delta^\star$ and Δ have the same sizes. If $\Delta = \Delta^\star$, then Δ is self-paired.*

Definition 1.2.10. *Suppose the action of a group G on finite set Ω is transitive.Then for each $g \in G$, $g\Psi = \Psi$ or $g\Psi \bigcap \Psi = \emptyset$,is the block for the action.*

Definition 1.2.11. *The group G is said to act primitively if the transitive action of G on a non-empty set Ω has no non trivial blocks.*

Definition 1.2.12. *A diagram comprising of nodes and edges is called a graph denoted by $\Gamma(V, E)$. If a graph has no edges,then it is referred to as a null graph.*

Definition 1.2.13. *A graph $\Gamma(V, E)$ consisting of finite sequence of edges of the form $a_1 \to a_2, a_2 \to a_3, \ldots, a_{(r-1)} \to a_r$ is called a walk of length r. If the finite sequence of edges has distinct points, then it is called a path but if it has distinct edges then its called a trail. A circuit or cycle is a path that is closed.*

Definition 1.2.14. *The length of the shortest circuit in Γ is called the girth.*

Definition 1.2.15. *The distance between two points α and β, represented by $d(\alpha, \beta)$ is the minimum length of a path from α to β in $\Gamma(V, E)$. If no path exists from u to v then $d(u, v) = \infty$. The diameter $dim(\Gamma(V, E))$ is, $max\{d(\alpha, \beta) : \alpha, \beta \in V\}$.*

Definition 1.2.16. *A graph $\Gamma(V, E)$ is said to be connected if for every pair of vertices $x_1, x_2 \in X$,there exist a path from x_1 to x_2,otherwise it is disconnected. Hence the diameter of a connected graph is finite. A connected subgraph of a group G that is maximal with respect to connectedness is called a component of $\Gamma(V, E)$.*

Definition 1.2.17. *The least number of colours the points or vertices of a graph with no two adjoining points having similar colour in the graph is referred to as chromatic number.*

Definition 1.2.18. *A graph $\Gamma(V, E)$ is bipartite if V can be divided into two subsets such that the edges connect the two vertices from distinct subsets with no* *edge connecting vertices of same subset. It follows from Definition* 1.2.18, *that the chromatic number of a bipartite graph is 2.*

Definition 1.2.19. *A transitive group G on the cartesian product set $\Omega \times \Omega$ is represented by $g(\alpha, \gamma) = (g\alpha, g\gamma), g \in G$ and $\alpha, \gamma \in \Omega$. Then $G-$ orbits resulting from this action are referred to as suborbitals of G on Ω denoted by $O(\alpha, \gamma)$.*

Definition 1.2.20. *A suborbital graph $\Gamma(V, E)$ corresponding to the suborbital $O \subseteq \Omega \times \Omega$ is a graph whose vertex is Ω and ab is a directed edge if and only if $(a, b) \in O$.*

Definition 1.2.21. *Let $\Gamma(V, E)$ be a suborbital graph corresponding to the suborbital $O \subseteq \Omega \times \Omega$. Then the vertex degree is the number of edges that are connected to each vertex.*

Theorem 1.2.22. *(Wilson and Watkins, 1990) Suppose Γ is undirected graph that is connected. Γ is said to be Eurelian if and only if every vertex is of even degree.*

Theorem 1.2.23. *(Jefferson* et al.*, 2018) Let Γ be a suborbital graph of a transitive action. Then all disconnected components of Γ are isomorphic.*

Definition 1.2.24. *The conjugacy class of an element $a \in G$ is the set of elements that conjugate to it given by $\{yay^{-1} : y \in G\}$.*

Definition 1.2.25. *Suppose G acts on Ω. Given that G is a finite group $\forall g \in G$ and $|\Omega| = r$ having cycle type $(\alpha_1, \alpha_2, \ldots, \alpha_r)$, the monomial of g is defined as $mon(g) = s_1^{\alpha_1} s_2^{\alpha_2} \ldots s_r^{\alpha_r}$ where $s_1, s_2, \ldots, s_r$ are distinct indeterminates that commutes.*

Definition 1.2.26. *Suppose G acts on Ω. The cycle index of the action is the polynomial in $r_1, r_2, \cdots, r_n$(over the field of rational Q) given by $Z(G) =$*

$\frac{1}{|G|}\sum mon(g)$. *If conjugacy classes of* G *are* $K_1, K_2, \cdots, K_m$ *with* $g_i \in K_i$*, then* $Z(G) = \frac{1}{|G|}\sum_{n=1}^{m} |K_i| mon(g_i)$.

Theorem 1.2.27. *(Muthoka* et al.*, 2019) Let* $Aff(q)$ *act on* $GF(q)$ *and* q *be prime,then cycle index is given by,*

$$Z_{(G)} = \frac{1}{G}[t_1^q + (q-1)t_p^{\frac{q}{p}} + q\sum_{(1\neq d|q-1)} \phi(d)t_1 t_d^{\frac{q-1}{d}}]. \tag{1.2}$$

Theorem 1.2.28. *(Higman, 1967) A non-trivial suborbital graph is connected if and only if each of the transitive action is primitive.*

Theorem 1.2.29. *(Cameron* et al.*, 2007) Let* s *and* r *-cycle in a permutation* $g_1 \in G_1$ *and* $g_2 \in G_2$ *respectively. Then* (g_1, g_2) *acts on the product of these two cycle as gcd* (s, r) *cycles of length lcm* (s, r)*. Thus the cycle index of* $G_1 \times G_2$ *can be determined by defining* $t_s^{\alpha} \circ t_r^{\beta} = (t_{lcm(s,r)})^{(\alpha\beta(gcd(s,r))}$*,which can be protracted by product of monomial and finally by addition to polynomials. Therefore,* $Z(G_1 \times G_2) = Z(G_1) \circ Z(G_2)$*.*

Theorem 1.2.30. *(Sims, 1967) Let* G *act transitively on* Ω *and the suborbit* $\Delta_j, (j = 0, 1, \ldots, r-1)$ *be represented by suborbital* O_j*.The suborbital graph* Γ_j *is said to be undirected whenever* Δ_j *is self-paired otherwise it is said to be directed.*

1.3 Problem statement and justification

Over a period of years, a lot has been studied on the ranks, subdegrees of alternating groups and symmetric groups acting on ordered and unordered subsets of X. Kangogo (2015), investigated the properties of the action of $Aff(q)$ on $GF(q)$.

However, very little has been done on the product action of affine groups. This study is set to investigate the action of $Aff(q_1) \times Aff(q_2)$ on $GF(q_1) \times GF(q_2)$ and $Aff(q_1) \times Aff(q_2) \times Aff(q_3)$ on $GF(q_1) \times GF(q_2) \times GF(q_3)$ respectively, by determining ranks, subdegrees, suborbital graphs and cycle indices corresponding to each action.

1.4 Objectives of the study

1.4.1 General objective

To study the action of $Aff(q_1) \times Aff(q_2)$ on $GF(q_1) \times GF(q_2)$ and $Aff(q_1) \times Aff(q_2) \times Aff(q_3)$ on $GF(q_1) \times GF(q_2) \times GF(q_3)$ respectively.

1.4.2 Specific objectives

i To test for transitivity of each action.

ii To compute the ranks and subdegrees for each action.

iii To construct suborbital graphs and give properties arising from each action.

iv To derive cycle index formula of $Aff(q_1) \times Aff(q_2)$ on $GF(q_1) \times GF(q_2)$.

1.5 Significance of the study

Cycle index formula is very useful in combinatorics. For instance, in chemistry they are used when counting the number of isomers in chemical compounds. Ranks are useful in computation of characters in representation theory. Both ranks and subdegrees are important in construction of suborbital graphs. Suborbital graphs constructed will provide valuable information to graph theorists. For instance, connected graphs can serve as models in the construction of electric circuits by electricians. Graphs can also be used to determine the shortest distance on the surface of the earth. This will help in electrification and designing of transport network of a given area. Suborbital graphs can also be applied in backtrack search, which is applied in permutation groups and cosets for search . The method is executed in MAGMA and GAP as a technique in analyzing problems in computing groups and intersection of cosets, partition stabilizers and set, normalizers and centralizers.

1.6 Outline of the study

The project has been partitioned into six chapters. In the second chapter the literature on the ranks, subdegrees, the subrbital graphs and finally cycle index formulas is given. In chapter three, transitivity is tested and then the ranks and subdegrees of $Aff(q_1) \times Aff(q_2)$ on $GF(q_1) \times GF(q_2)$ and $Aff(q_1) \times Aff(q_2) \times Aff(q_3)$ on $GF(q_1) \times GF(q_2) \times GF(q_3)$ is determined respectively. Then in chapter four, the suborbital graphs of the action and properties of these graphs are determined. In chapter five, the cycle index formula of $Aff(q_1) \times Aff(q_2)$ on $GF(q_1) \times GF(q_2)$ is derived . Finally, chapter six gives the conclusion and recommendations.

CHAPTER 2

LITERATURE REVIEW

2.1 Introduction

This chapter reviews the study that have already been done by other scholars on ranks,subdegrees,suborbital graphs and cycle indices. The chapter is divided into four sections. In section 2.2, we review the ranks and subdegrees, section 2.3, we review the suborbital graphs while in section 2.4 we review the cycle index.

2.2 Ranks and subdegrees

Higman (1970) studied the subdegrees and the rank of symmetric group S_r and proved that S_r acting on unordered pairs from $X = \{1, 2, \ldots, r\}$, where $r \geqslant 4$, has a rank 3 while the subdegrees are 1, $(2r - 4)$ and $\begin{pmatrix} r-2 \\ \\ 2 \end{pmatrix}$.

Kamuti (2006) studied the properties of primitive permutation representation of $G = PGL(2, q)$. The computation of the rank was done when q is odd and found to be $(q + 3)$, when q is even it was determined to be $\frac{1}{2}(q + 2)$. The subdegrees were also determined when q is odd and found to be $1, \frac{1}{2}(q + 2), (q - 1), 2(q - 1)$, on the other hand when q is even the subdegrees were found to be $1, (q - 1), 2(q - 1)$.

Nyaga *et al.* (2011) carried a study of S_r acting on $X^{[s]}$ and the action was transitive, the rank is $s+1, n \geqslant 2s$ and the subdegrees are $1, \binom{n-s}{s-1}, \binom{s}{2}, \binom{n-s}{s-2}, \binom{s}{3}, \binom{n-s}{s-3}, \binom{s}{s-1}, \binom{n-s}{1}, \binom{n-s}{s}$.

Rimberia *et al.* (2013) proved that the symmetric group $S_r, r \geqslant 2s$, acts transitively on s -ordered subset from $X = \{1, 2, 3, \ldots, r\}$ and its rank is

$$\begin{aligned} R(G) =& s! + s^2(s-1)! + \frac{s^2(s-1)^2(s-2)!}{2!2!} + \frac{s^2(s-1)^2(s-2)^2(s-3)!}{3!3!} \\ &+ ... + \frac{s^2(s-1)^2}{2!} + s^2 + 1. \end{aligned} \tag{2.1}$$

Kimani *et al.* (2014) used the table of marks to identify the rank of $S_r (r = 5, 6, 7)$ acting on ordered pairs from $X = \{1, 2, \ldots, r\}$ and it was found to be 7.

Gachago *et al.* (2015) proved that the alternating group $A_r (r = 5, 6, 7)$ acts transitively on unordered and ordered quadruples of $X = \{1, 2, \ldots, r\}$.

Rotich (2016) studied the group action of $G = PSL(2, q)$ on a cyclic group $C_{(q-1)}$. It was established that the rank is $q + 4$ and subdegrees are $1, 1$ and $q - 1$.

Magero (2015) studied the group action of $G = PSL(2, q)$ on $\frac{C_{(q-1)}}{r}$, when $r = (q-1, 2)$. For q even the rank is 1 and $q+4$ while when q is odd the rank is $2(q+3)$. Thus when q is even the subdegrees were found to be $1, (q-1), (q-1), \ldots, (q-1)$, until $(q+2)$ times. When q is odd it was found to be $1, \frac{(q-1)}{2}, \frac{(q-1)}{2}, \ldots, \frac{(q-1)}{2}, 2(q+2)$ times.

2.3 Suborbital graphs

The concept of suborbital graphs of non-trivial suborbits of a permutation group G was introduced by Sims (1967).

Kamuti (1992) studied the action of disconnected cycle structures of $G = PGL(2, q)$ and $PSL(2, q)$ on maximal dihedral groups that provided another technique of constructing coxeter graphs initially introduced by Coxeter (1983).

Akbas (2001) proved that suborbital graph of a $G = PSL(2, Z)$ is a forest with no triangle. This was after studying action of modular group on the rational projective line and the corresponding suborbital graphs.

Kamuti *et al.* (2012) investigated some properties of $Stab_{\Gamma}(\infty)$ the modular group,acting on the set of integers and the research showed that the action is transitive and imprimitive. It was proved that the suborbital graph $\Gamma(0,x)$ has $|x|$ components and it is paired with $\Gamma(x,0)$.

Rimberia *et al.* (2013) investigated the action of the group S_r on the subset of ordered s -elements $X^{[s]}$. It was found that all the graphs are disconnected. The girth of the graph corresponding to the self-paired orbits of $G_{[1,2,\ldots,S]}$ on $X^{[s]}$ with exactly s elements from $B = \{1,2,\ldots,s\}$ was found to be zero and the paired orbits with exactly s -elements from B was found to be three. In case γ contains no elements from B, then the girth is three if and only if $n \geqslant 3s$.

Magero (2015) investigated the properties of $G = PSL(2,q)$ acting on its cyclic subgroup $\frac{C_(q-1)}{k}$, where $k = (q-1,2)$. The action was found to be imprimitive and the number of the self-paired suborbital when $p = 2, q \cong 1 mod 4$ and $q \cong 1 mod 4$ is $1+q, 2+q$ and $3+q$ respectively.

2.4 Cycle index

Redfield (1927) introduced the idea of cycle index of permutation groups. During the study, the links between combinatorial analysis and permutation groups was emphasized. Every permutation group was associated with a group of reduced function.

Fel and Zimmels (2009) renamed the group reduction function to cycle indices. This cycle index was used to count graphs and chemical compounds using a devised powerful general method called Polya's Enumeration Theorem. This theorem enabled cycle index to become a very powerful tool in enumeration and therefore a need arose to determine cycle indices of the corresponding permutation group.

Kamuti (1992) studied the disjoint cycle structures of components of $G = PGL(2, q)$ and $G = PSL(2, q)$ for all primitive permutation representation of these groups and derived the general formula for the cycle indices of these representations.

Kamuti and Obon'go (2002) investigated the properties of ordered triple group $S_r^{[3]}$ and later Kamuti and Njuguna (2004) extended this study to cycle index of the reduced ordered t-group $S_r^{[t]}$.

Muthoka *et al.* (2015) investigated the properties of a group D_r on unordered pairs. It was found that the cycle index formula is given by,

$$Z(D_{2r}, X^{(2)}) = \frac{1}{2r}\left[\sum_{d|r} \phi(d) t_d^{\frac{r^2-r}{2d}} + r t_1^{\frac{r-1}{2}} t_2^{\frac{r^2-2r+1}{4}}\right]. \tag{2.2}$$

for odd r and

$$Z(D_{2r}, X^{(2)}) = \frac{1}{2r}\left[\sum_{d|r,\ 2|r} \phi(d) t_{\frac{d}{2}}^{\frac{r}{d}} t_d^{\frac{r^2-2r}{2d}} + \sum_{d|r,\ 2\nmid d} \phi(d) t_d^{\frac{r^2-r}{2d}} + r t_1^{\frac{r}{2}} t_2^{\frac{r^2-2r}{4}}\right] \tag{2.3}$$

for even r.

Muthoka *et al.* (2019) studied the properties of the action of Affine (q) group on Z_p. The resulting cycle index is,

$$Z_{(G)} = \frac{1}{|G|}[t_1^q + (q-1)t_p^{\frac{q}{p}} + q \sum_{(1\neq d|q-1)} \phi(d) t_1 t_d^{\frac{q-1}{d}}]. \tag{2.4}$$

From the above review, it is clear that a lot has not been done for $Aff(q_1) \times Aff(q_2)$ and $Aff(q_1) \times Aff(q_2) \times Aff(q_3)$ on $GF(q_1) \times GF(q_2)$ and $GF(q_1) \times GF(q_2) \times GF(q_3)$ respectively. Therefore, this study investigates the properties of affine groups by studying the ranks. The subdegrees and corresponding suborbital graphs are constructed and finally determination of cycle index of the action.

CHAPTER 3

RANKS AND SUBDEGREES OF $Aff(q_1) \times Aff(q_2)$ ON $GF(q_1) \times GF(q_2)$ AND $Aff(q_1) \times Aff(q_2) \times Aff(q_3)$ ON $GF(q_1) \times GF(q_2) \times GF(q_3)$

3.1 Introduction

This chapter is partitioned into section 3.2 which tests the transitivity of the action while section 3.3 which computes the ranks and subdegrees of the action.

3.2 Transitivity

Lemma 3.2.1. *Let the group* $G = Aff(q)$ *act on the set* $\Omega = (0, \alpha, \alpha^2, \alpha^3, \ldots, \alpha^{(q-1)})$. *Then the* $Stab_G(0)$ *is a group* $C_{(q-1)}$, *which is cyclic .*

Proof. Let $G = Aff(q)$ act on $GF(q)$. Then,

$$\begin{aligned} Stab_G(0) &= \{hx + k : h(0) + k = 0, h \neq 0 \in GF(q), k \in GF(q)\} \\ &= \{hx + k : k = 0\} \\ &= \{hx : h \in GF(q)^\star\} = C_{(q-1)}. \end{aligned}$$

□

This can also be proved using the method applied by $(Kangogo, 2015)$.

Proposition 3.2.1. *Let* $Aff(q_1)\times Aff(q_2)$ *act on* $GF(q_1)\times GF(q_2)$ *and* $Aff(q_1)\times Aff(q_2)\times Aff(q_3)$ *act on* $GF(q_1)\times GF(q_2)\times GF(q_3)$. *Then* $Stab_G(0,0)$*is isomorphic to* $C_{Q_{[1,2]}} = C_{(q_1-1)}\times C_{(q_2-1)}$ *while* $Stab_G(0,0,0)$ *is isomorphic to* $C_{Q_{[1,2,3]}} = C_{(q_1-1)}\times C_{(q_2-1)}\times C_{(q_3-1)}$.

Proof. By Lemma 3.2.1, the $Stab_{(G_1)}(0), Stab_{(G_2)}(0)$ and $Stab_{(G_3)}(0)$ are cyclic groups $C_{(q_1-1)}, C_{(q_2-1)}$ and $C_{(q_3-1)}$ respectively. It follows from Theorem 1.2.5,

$$\begin{aligned} Stab_G(0,0) =& Stab_{(G_1)}(0)\times Stab_{(G_2)}(0) \\ \implies\ & C_{Q_{[1,2]}} = C_{(q_1-1)}\times C_{(q_2-1)}. \end{aligned} \tag{3.1}$$

and

$$\begin{aligned} Stab_G(0,0,0) &= Stab_{G_1}(0)\times Stab_{(G_2)}(0)\times Stab_{(G_3)}(0) \\ \implies\ C_{Q_{[1,2,3]}} &= C_{(q_1-1)}\times C_{(q_2-1)}\times C_{(q_3-1)}. \end{aligned} \tag{3.2}$$

□

Theorem 3.2.2. *The action of* $Aff(q_1)\times Aff(q_2)$ *on* $GF(q_1)\times GF(q_2)$ *is transitive.*

Proof. Let $G = Aff(q_1)\times Aff(q_2)$ act on $\Omega = GF(q_1)\times GF(q_2)$. Then $Stab_G(0,0) = Stab_{G_1}(0)\times Stab_{G_2}(0)$ and by proposition 3.2.1,

$$\begin{aligned} |Stab_G(0,0)| =& |C_{(q_1-1)}|\times |C_{(q_2-1)}| \\ &= (q_1-1)(q_2-1). \end{aligned} \tag{3.3}$$

It follows from Theorem 1.2.6,

$$\begin{aligned}|Orb_G(0,0)| =& \frac{|G|}{|Stab_G(0,0)|} = \frac{q_1(q_1-1)\times q_2(q_2-1)}{(q_1-1)\times(q_2-1)} \\ &= q_1 \times q_2 = |\Omega|.\end{aligned} \tag{3.4}$$

Hence the action is transitive. □

Theorem 3.2.3. *The action of* $Aff(q_1)\times Aff(q_2)\times Aff(q_3)$ *on* $GF(q_1)\times GF(q_2)\times GF(q_3)$ *is transitive.*

Proof. Let $G = Aff(q_1)\times Aff(q_2)\times Aff(q_3)$ act on $\Omega = GF(q_1)\times GF(q_2)\times GF(q_3)$. Then $Stab_G(0,0,0) = Stab_{G_1}(0) \times Stab_{G_2}(0) \times Stab_{G_3}(0)$ and by proposition 3.2.1,

$$\begin{aligned}|Stab_G(0,0,0)| =& |C_{(q_1-1)}| \times |C_{(q_2-1)}| \times |C_{(q_3-1)}| \\ &= (q_1-1)(q_2-1)(q_3-1).\end{aligned} \tag{3.5}$$

It follows from Theorem 1.2.6,

$$\begin{aligned}|Orb_G(0,0,0)| =& \frac{|G|}{|Stab_G(0,0,0)|} = \frac{q_1(q_1-1)\times q_2(q_2-1)\times q_3(q_3-1)}{(q_1-1)\times(q_2-1)\times(q_3-1)} \\ &= q_1 \times q_2 \times q_3 = |\Omega|.\end{aligned} \tag{3.6}$$

Hence the action is transitive. □

3.3 Ranks and Subdegrees

Example 3.3.1. *Let* $G = G_1 \times G_2 = Aff(2) \times Aff(2)$ *act on* $\Omega = \Omega \times \Omega = GF(2) \times GF(2)$*. Then the rank is 4 and subdegrees are* $1, 2, 2$ *and* 4.

Proof. Let $A = Stab_{G_1}(0), B = Stab_{G_2}(0)$ and $H = Stab_G(0,0)$. Then by Definition 1.2.2, $A-$ orbits on $GF(2)$ are $\{0\}$ and $\{1\}$ and $B-$orbits on $GF(2)$ are $\{0\}$ and $\{1\}$.

By Theorem 1.2.5, the suborbits of G on Ω are:

$$\Delta_0 = Orb_H\Big(0,0\Big) = Orb_A(0) \times Orb_B(0) = \{0\} \times \{0\} = \{(0,0)\}. \tag{3.7}$$

Therefore, $|\Delta_0| = 1$.

$$\Delta_1 = Orb_H\Big(0,1\Big) = Orb_A(0) \times Orb_B(1) = \{0\} \times \{1\} = \{(0,1)\}. \tag{3.8}$$

Therefore, $|\Delta_1| = 1$.

$$\Delta_2 = Orb_H\Big(1,0\Big) = Orb_A(1) \times Orb_B(0) = \{1\} \times \{0\} = \{(1,0)\}. \tag{3.9}$$

Therefore, $|\Delta_2| = 1$.

$$\Delta_3 = Orb_H\Big(1,1\Big) = Orb_A(1) \times Orb_B(1) = \{1\} \times \{1\} = \{(1,1)\}. \tag{3.10}$$

Therefore, $|\Delta_3| = 1$.

The number of elements in all the four $G-$ orbits is $1+1+1+1 = 4 = |\Omega|$. Thus the rank is 4 and the subdegrees are $1, 1, 1$ and 1 . $\square$

Example 3.3.2. *Let* $G = G_1 \times G_2 = Aff(3) \times Aff(3)$ *act on* $\Omega = \Omega_1 \times \Omega_2 = GF(3) \times GF(3)$. *Then the rank is* 4 *and subdegree are* $1, 2, 2$ *and* 4.

Proof. Let $A = Stab_{G_1}(0), B = Stab_{G_2}(0)$ and $H = Stab_G(0,0)$. Then by Definition 1.2.2, $A-$ orbits on $GF(3)$ are $\{0\}$ and $\{1,2\}, B-$orbits on $GF(3)$ are $\{0\}$ and $\{1,2\}$.

By Theorem 1.2.5 , the suborbits of G on Ω are:

$$\Delta_0 = Orb_H\Big(0,0\Big) = Orb_A(0) \times Orb_B(0) = \{0\} \times \{0\} = \{(0,0)\}. \tag{3.11}$$

Therefore, $|\Delta_0| = 1$.

$$\Delta_1 = Orb_H\Big(0,2\Big) = Orb_A(0) \times Orb_B(2) = \{0\} \times \{1,2\} = \{(0,1),(0,2)\}. \tag{3.12}$$

Therefore, $|\Delta_1| = 2$.

$$\Delta_2 = Orb_H\Big(1,0\Big) = Orb_A(1) \times Orb_B(0) = \{1,2\} \times \{0\} = \{(1,0),(2,0)\}. \tag{3.13}$$

Therefore, $|\Delta_2| = 2$.

$$\begin{aligned} \Delta_3 =& Orb_H\Big(1,2\Big) = Orb_A(1) \times Orb_B(2) = \{1,2\} \times \{1,2\} \\ &= \{(1,2),(1,1),(2,2),(2,1)\}. \end{aligned} \tag{3.14}$$

Therefore, $|\Delta_3| = 4$.

The number of elements in all the four $G-$ orbits is $1 + 2 + 2 + 4 = 9 = |\Omega|$. Thus the rank is 4 and the subdegrees are $1, 2, 2$ and 4 . □

Example 3.3.3. *Let* $G = G_1 \times G_2 = Aff(2) \times Aff(3)$ *act on* $\Omega = \Omega_1 \times \Omega_2 = GF(2) \times GF(3)$. *Then the rank is* 4 *and subdegrees are* $1, 1, 2$ *and* 2.

Proof. Let $A = Stab_{G_1}(0), B = Stab_{G_2}(0)$ and $H = Stab_G(0,0)$. Then by Definition 1.2.2, $A-$ orbits on $GF(2)$ are $\{0\}$ and $\{1\}$ and $B-$orbits on $GF(3)$ are $\{0\}$ and $\{1,2\}$. By by Theorem 1.2.5, the suborbits of G on Ω are:

$$\Delta_0 = Orb_H\Big(0,0\Big) = Orb_A(0) \times Orb_B(0) = \{0\} \times \{0\} = \{(0,0)\}. \tag{3.15}$$

Therefore, $|\Delta_0| = 1$.

$$\Delta_1 = Orb_H\Big(0,2\Big) = Orb_A(0) \times Orb_B(2) = \{0\} \times \{1,2\} = \{(0,1),(0,2)\}. \quad (3.16)$$

Therefore, $|\Delta_1| = 2$.

$$\Delta_2 = Orb_H\Big(1,0\Big) = Orb_A(1) \times Orb_B(0) = \{1\} \times \{0\} = \{(1,0)\}. \quad (3.17)$$

Therefore, $|\Delta_2| = 1$.

$$\Delta_3 = Orb_H\Big(1,2\Big) = Orb_A(1) \times Orb_B(2) = \{1\} \times \{1,2\} = \{(1,1),(1,2)\}. \quad (3.18)$$

Therefore, $|\Delta_3| = 2$.

The number of elements in all the four $G-$ orbits is $1+1+2+2 = 6 = |\Omega|$. Thus the rank is 4 and the subdegrees are $1, 2, 2$ and 2. □

Theorem 3.3.4. *Suppose* $G = G_1 \times G_2 = Aff(q_1) \times Aff(q_2)$ *acting on* $\Omega = \Omega_1 \times \Omega_2 = GF(q_1) \times GF(q_2)$. *Then the rank is* 2^k, *where* $k = 2$ *is the number of affine groups in the product action and subdegree is* $1, (q_1-1), (q_2-1)$ *and* $(q_1-1)(q_2-1)$.

Proof. Let $A = Stab_{G_1}(0), B = Stab_{G_2}(0), C = Stab_{G_3}(0)$ and $H = Stab_G(0,0,0)$. α is a primitive element in Ω_1 and β a primitive element in Ω_2. Then by Definition 1.2.2, $A-$ orbits on Ω_1 are $\{0\}$ and $\{\alpha, \alpha^2, \ldots, \alpha^{(q_1-1)}\}$ and $B-$orbits on Ω_2 are $\{0\}$

and $\{\beta, \beta^2, \ldots, \beta^{(q_2-1)}\}$. By Theorem 1.2.5, the suborbits of G on Ω are:

$$\Delta_0 = Orb_H\Big(0,0\Big) = Orb_A(0) \times Orb_B(0) = \{0\} \times \{0\} = \{(0,0)\}. \tag{3.19}$$

Therefore, $|\Delta_0| = 1$.

$$\begin{aligned}\Delta_1 =& Orb_H\Big(0,\beta\Big) = Orb_A(0) \times Orb_B(\beta) = \{0\} \times \{\beta, \beta^2, \beta^3, \ldots, \beta^{(q_2-1)}\} \\ &= \{(0,\beta), (0,\beta^2), (0,\beta^3), \ldots, (0, \beta^{(q_2-1)})\}.\end{aligned} \tag{3.20}$$

Therefore, $|\Delta_1| = (q_2 - 1)$.

$$\begin{aligned}\Delta_2 &= Orb_H\Big(\alpha,0\Big) = Orb_A(\alpha) \times Orb_B(0) = \{\alpha, \alpha^2, \alpha^3, \ldots, \alpha^{q_1-1)}\} \times \{0\} \\ &= \{(\alpha,0), (\alpha^2,0), (\alpha^3,0), \ldots, (\alpha^{(q_1-1)}, 0)\}.\end{aligned} \tag{3.21}$$

Therefore, $|\Delta_2| = (q_1 - 1)$.

$$\begin{aligned}\Delta_3 &= Orb_H\Big(\alpha,\beta\Big) = Orb_A(\alpha) \times Orb_B(\beta) \\ &= \{\alpha, \alpha^2, \alpha^3, \ldots, \alpha^{(q_1-1)}\} \times \{\beta, \beta^2, \beta^3, \beta^4, \ldots, \beta^{(q_2-1)}\} \\ &= |(q_1-1)| \times |(q_2-1)|.\end{aligned} \tag{3.22}$$

Therefore, $|\Delta_3| = (q_1 - 1)(q_2 - 1)$.

The number of elements in all the four $G-$ orbits is $1+(q_1-1)+(q_2-1)+(q_1-1)(q_2-1) = q_1q_2 = |\Omega|$. Thus the rank is $2^2 = 4$ and the subdegrees are $1, (q_1-1), (q_2-1)$ and $(q_1-1)(q_2-1)$. $\square$

Example 3.3.5. *Suppose* $G = G_1 \times G_2 \times G_3 = Aff(3) \times Aff(3) \times Aff(3)$ *act on* $\Omega = \Omega_1 \times \Omega_2 \times \Omega_3 = GF(3) \times GF(3) \times GF(3)$. *Then the rank is* 8 *and subdegrees are* $1, 2, 2, 2, 4, 4, 4$ *and* 8.

Proof. Let $A = Stab_{G_1}(0), B = Stab_{G_2}(0), C = Stab_{G_3}(0)$ and $H = Stab_G(0, 0, 0)$. Then by Definition 1.2.2, $A-$ orbits on $GF(3)$ are $\{0\}$ and $\{1, 2\}$, $B-$ orbits on $GF(3)$ are $\{0\}$ and $\{1, 2\}$ and C -orbits on $GF(3)$ are $\{0\}$ and $\{1, 2\}$. By Theorem 1.2.5, the suborbits of G on Ω are:

$$\begin{aligned} \Delta_0 =& Orb_H\Big(0,0,0\Big) = Orb_A(0) \times Orb_B(0) \times Orb_C(0) \\ &= \{0\} \times \{0\} \times \{0\} = \{(0,0,0)\}. \end{aligned} \tag{3.23}$$

Therefore, $|\Delta_0| = 1$.

$$\begin{aligned} \Delta_1 =& Orb_H\Big(0,0,2\Big) = Orb_A(0) \times Orb_B(0) \times Orb_C(2) \\ &= \{0\} \times \{0\} \times \{1,2\} = \{(0,0,1),(0,0,2)\}. \end{aligned} \tag{3.24}$$

Therefore, $|\Delta_1| = 2$.

$$\begin{aligned} \Delta_2 =& Orb_H\Big(0,2,0\Big) = Orb_A(0) \times Orb_B(2) \times Orb_C(0) \\ &= \{0\} \times \{1,2\} \times \{0\} = \{(0,1,0),(0,2,0)\}. \end{aligned} \tag{3.25}$$

Therefore, $|\Delta_2| = 2$.

$$\begin{aligned} \Delta_3 =& Orb_H\Big(2,0,0\Big) = Orb_A(2) \times Orb_B(0) \times Orb_C(0) \\ &= \{1,2\} \times \{0\} \times \{0\} = \{(2,0,0),(1,0,0)\}. \end{aligned} \tag{3.26}$$

Therefore, $|\Delta_3| = 2$.

$$\begin{aligned}\Delta_4 =& Orb_H\Big(0,2,2\Big) = Orb_A(0) \times Orb_B(2) \times Orb_C(2)\\ &= \{0\} \times \{1,2\} \times \{1,2\} = \{(0,2,2),(0,2,1),(0,1,2),(0,1,1)\}.\end{aligned} \tag{3.27}$$

Therefore, $|\Delta_4| = 4$.

$$\begin{aligned}\Delta_5 =& Orb_H\Big(2,0,2\Big)\\ &= Orb_A(2) \times Orb_B(0) \times Orb_C(2)\\ &= \{2,1\} \times \{0\} \times \{2,1\} = \{(2,0,2),(2,0,1),(1,0,2),(1,0,1)\}.\end{aligned} \tag{3.28}$$

Therefore, $|\Delta_5| = 4$.

$$\begin{aligned}\Delta_6 =& Orb_H\Big(2,2,0\Big)\\ &= Orb_A(2) \times Orb_B(2) \times Orb_C(0)\\ &= \{2,1\} \times \{2,1\} \times \{0\} = \{(2,2,0),(2,1,0),(1,2,0),(1,1,0)\}.\end{aligned} \tag{3.29}$$

Therefore, $|\Delta_6| = 4$.

$$\begin{aligned}\Delta_7 =& Orb_H\Big(2,2,2\Big)\\ &= Orb_A(2) \times Orb_B(2) \times Orb_C(2)\\ &= \{2,1\} \times \{2,1\} \times \{2,1\}\\ &= \{(2,2,2),(2,2,1),(2,1,2),(2,1,1),(1,2,2),(1,2,1),(1,1,2),(1,1,1)\}.\end{aligned} \tag{3.30}$$

Therefore, $|\Delta_7| = 8$.

The number of elements in all the four $G-$ orbits is $1+2+2+2+4+4+4+8 = 27 = |\Omega|$.

Thus the rank is 8 and the subdegrees are $1, 2, 2, 2, 4, 4, 4$ and 8. □

Example 3.3.6. *Suppose* $G = G_1 \times G_2 \times G_3 = Aff(2) \times Aff(3) \times Aff(5)$ *act on* $\Omega = GF(2) \times GF(3) \times GF(5)$. *Then the rank is* 8 *and subdegrees are* $1, 1, 2, 2, 4, 4, 8$ *and* 8.

Proof. Let $A = Stab_{G_1}(0), B = Stab_{G_2}(0), C = Stab_{G_3}(0)$ and $H = Stab_G(0, 0, 0)$. Then by Definition 1.2.2, $A-$orbits on $GF(2)$ are $\{0\}$ and $\{1\}$, $B-$ orbits on $GF(3)$ are $\{0\}$ and $\{1, 2\}$ and $C-$orbits on $GF(5)$ are$\{0\}$ and $\{1, 2, 3, 4\}$. By Theorem 1.2.5, the suborbits of G on Ω are:

$$\begin{aligned}\Delta_0 =&Orb_H\Big(0,0,0\Big) = Orb_A(0) \times Orb_B(0) \times Orb_C(0)\\ =&\{0\} \times \{0\} \times \{0\} = \{(0,0,0)\}.\end{aligned} \tag{3.31}$$

Therefore, $|\Delta_0| = 1$.

$$\begin{aligned}\Delta_1 =&Orb_H\Big(0,0,3\Big) = Orb_A(0) \times Orb_B(0) \times Orb_C(3)\\ =&\{0\} \times \{0\} \times \{1,2,3,4\} = \{(0,0,1),(0,0,2),(0,0,3),(0,0,4)\}.\end{aligned} \tag{3.32}$$

Therefore, $|\Delta_1| = 4$.

$$\begin{aligned}\Delta_2 =&Orb_H\Big(0,2,0\Big) = Orb_A(0) \times Orb_B(2) \times Orb_C(0)\\ =&\{0\} \times \{1,2\} \times \{0\} = \{(0,1,0),(0,2,0)\}.\end{aligned} \tag{3.33}$$

Therefore, $|\Delta_2| = 2$.

$$\begin{aligned}\Delta_3 =& Orb_H\Big(1,0,0\Big) = Orb_A(1) \times Orb_B(0) \times Orb_C(0)\\ =& \{1\} \times \{0\} \times \{0\} = \{(1,0,0)\}.\end{aligned} \tag{3.34}$$

Therefore,$|\Delta_3| = 1$.

$$\begin{aligned}\Delta_4 =& Orb_H\Big(0,2,3\Big) = Orb_A(0) \times Orb_B(2) \times Orb_C(3)\\ =& \{0\} \times \{1,2\} \times \{1,2,3,4\}\\ =& \begin{matrix}\{(0,1,1),(0,1,2),(0,1,3),(0,1,4),\\ (0,2,1),(0,2,2),(0,2,3),(0,2,4)\}.\end{matrix}\end{aligned} \tag{3.35}$$

Therefore, $|\Delta_4| = 8$.

$$\begin{aligned}\Delta_5 =& Orb_H\Big(1,0,3\Big) = Orb_A(1) \times Orb_B(0) \times Orb_C(3)\\ =& \{1\} \times \{0\} \times \{1,2,3,4\}\\ =& \{(1,0,1),(1,0,2),(1,0,3),(1,0,4)\}.\end{aligned} \tag{3.36}$$

Therefore,$|\Delta_5| = 4$.

$$\begin{aligned}\Delta_6 =& Orb_H\Big(1,2,0\Big) = Orb_A(1) \times Orb_B(2) \times Orb_C(0)\\ =& \{1\} \times \{1,2\} \times \{0\} = \{(1,1,0),(1,2,0)\}.\end{aligned} \tag{3.37}$$

Therefore, $|\Delta_6| = 2$.

$$\begin{aligned}\Delta_7 =& Orb_H\Big(1,2,3\Big) = Orb_A(1) \times Orb_B(2) \times Orb_C(3)\\ =&\{1\} \times \{1,2\} \times \{1,2,3,4,\}\\ =&\{(1,1,1),(1,1,2),(1,1,3),(1,1,4),\\ &(1,2,1),(1,2,2),(1,2,3),(1,2,4)\}.\end{aligned} \tag{3.38}$$

Therefore, $|\Delta_7| = 8$.

The number of elements in all the four $G-$ orbits is $1{+}1{+}2{+}2{+}4{+}4{+}8{+}8 = 30 = |\Omega|$.

Thus the rank is 8 and the subdegrees are $1, 1, 2, 2, 4, 4, 8$ and 8. $\square$

Theorem 3.3.7. *Suppose* $G = G_1 \times G_2 \times G_3 = Aff(q_1) \times Aff(q_2) \times Aff(q_3)$ *act on* $\Omega = \Omega_1 \times \Omega_2 \times \Omega_3 = GF(q_1) \times GF(q_2) \times GF(q_3)$. *Then the rank is* 2^k, *where* $k = 3$ *is the number of affine groups in the product action and subdegrees are* $1, (q_1-1), (q_2-1), (q_3-1), (q_1-1)(q_2-1), (q_1-1)(q_3-1), (q_2-1)(q_3-1), (q_1-1)(q_2-1)(q_3-1)$.

Proof. Let $A = Stab_{G_1}(0), B = Stab_{G_2}(0), C = Stab_{G_3}(0)$ and $H = Stab_G(0,0,0)$. Also let α, β and γ be primitive elements in Ω_1, Ω_2 and Ω_3 respectively. Then by Definition 1.2.2, $A-$ orbits on Ω_1 are $\{0\}$ and $\{\alpha, \alpha^2, ..., \alpha^{(q_1-1)}\}$, $B-$orbits on Ω_2 are $\{0\}$ and $\{\beta, \beta^2, ..., \beta^{(q_2-1)}\}$ and $C-$orbits on Ω_3 are $\{0\}$ and $\{\gamma, \gamma^2, ..., \gamma^{(q_3-1)}\}$. By Theorem 1.2.5, the suborbits of G on Ω are:

$$\begin{aligned}\Delta_0 =& Orb_H\Big(0,0,0\Big) = Orb_A(0) \times Orb_B(0) \times Orb_C(0)\\ =&\{0\} \times \{0\} \times \{0\} = \{(0,0,0)\}.\end{aligned} \tag{3.39}$$

Therefore, $|\Delta_0| = 1$.

$$\begin{aligned}\Delta_1 =& Orb_H\Big(0,0,\gamma\Big) = Orb_A(0)\times Orb_B(0)\times Orb_C(\gamma)\\ =&\{0\}\times\{0\}\times\{\gamma,\gamma^2,\gamma^3,\dots,\gamma^{(q_3-1)}\}\\ =&\{(0,0,\gamma),(0,0,\gamma^2),(0,0,\gamma^3),\dots,(0,0,\gamma^{(q_3-1))}\}.\end{aligned} \tag{3.40}$$

Therefore,$|\Delta_1| = (q_3-1)$.

$$\begin{aligned}\Delta_2 =& Orb_H\Big(0,\beta,0\Big) = Orb_A(0)\times Orb_B(\beta)\times Orb_C(0)\\ =&\{0\}\times\{\beta,\beta^2,\beta^3,...,\beta^{(q_2-1)}\}\times\{0\}\\ =&\{(0,\beta,0),(0,\beta^2,0),(0,\beta^3,0),...,(0,\beta^{(q_2-1)},0)\}.\end{aligned} \tag{3.41}$$

Therefore, $|\Delta_2| = (q_2-1)$.

$$\begin{aligned}\Delta_3 =& Orb_H\Big(\alpha,0,0\Big) = Orb_A(\alpha)\times Orb_B(0)\times Orb_C(0)\\ =&\{\alpha,\alpha^2,\alpha^3,\dots,\alpha^{(q_1-1)}\}\times\{0\}\times\{0\}\\ =&\{(\alpha,0,0),(\alpha^2,0,0),(\alpha^3,0,0),\dots,(\alpha^{(q_1-1)},0,0)\}.\end{aligned} \tag{3.42}$$

Therefore, $|\Delta_3| = (q_1-1)$.

$$\begin{aligned}\Delta_4 =& Orb_H\Big(\alpha,\beta,0\Big) = Orb_A(\alpha)\times Orb_B(\beta)\times Orb_C(0)\\ =&\{\alpha,\alpha^2,\alpha^3,\dots,\alpha^{(q_1-1)}\}\times\{\beta,\beta^2,\beta^3,\dots,\beta^{(q_2-1)}\}\times\{0\}\\ =&|(q_1-1)|\times|(q_2-1)|\times|1|.\end{aligned} \tag{3.43}$$

Therefore, $|\Delta_4| = (q_1 - 1)(q_2 - 1)$.

$$\begin{aligned} \Delta_5 =& Orb_H\Big(\alpha, 0, \gamma\Big) = Orb_A(\alpha) \times Orb_B(0) \times Orb_C(\gamma) \\ =& \{\alpha, \alpha^2, \alpha^3, \dots, \alpha^{(q_1-1)}\} \times \{0\} \times \{\gamma, \gamma^2, \gamma^3, \cdots, \gamma^{(q_3-1)}\} \\ =& |(q_1 - 1)| \times |1| \times |(q_3 - 1)|. \end{aligned} \tag{3.44}$$

Therefore, $|\Delta_5| = (q_1 - 1)(q_3 - 1)$.

$$\begin{aligned} \Delta_6 =& Orb_H\Big(0, \beta, \gamma\Big) = Orb_A(0) \times Orb_B(\beta) \times Orb_C(\gamma) \\ =& \{0\} \times \{\beta, \beta^2, \beta^3, \cdots, \beta^{(q_2-1)}\} \times \{\gamma, \gamma^2, \gamma^3, ..., \gamma^{(q_3-1)}\} \\ =& |1| \times |(q_2 - 1)| \times |(q_3 - 1)|. \end{aligned} \tag{3.45}$$

Therefore, $|\Delta_6| = (q_2 - 1)(q_3 - 1)$.

$$\begin{aligned} \Delta_7 =& Orb_H\Big(\alpha, \beta, \gamma\Big) = Orb_A(\alpha) \times Orb_B(\beta) \times Orb_C(\gamma) \\ =& \{\alpha, \alpha^2, \alpha^3, \dots, \alpha^{(q_1-1)}\} \times \{\beta, \beta^2, \beta^3, \dots, \beta^{(q_2-1)}\} \times \{\gamma, \gamma^2, \gamma^3, ..., \gamma^{(q_3-1)}\} \\ =& |(q_1 - 1)| \times |(q_2 - 1)| \times |(q_3 - 1)|. \end{aligned} \tag{3.46}$$

Therefore, $|\Delta_7| = (q_1 - 1)(q_2 - 1)(q_3 - 1)$.

The number of elements in all the four $G-$ orbits is $1 + (q_1 - 1) + (q_2 - 1) + (q_3 - 1) + (q_1 - 1)(q_2 - 1) + (q_1 - 1)(q_3 - 1) + (q_2 - 1)(q_3 - 1) + (q_1 - 1)(q_2 - 1)(q_3 - 1) = q_1 q_2 q_3 = |\Omega|$.

Thus the rank is $2^3 = 8$ and the subdegrees are $1, (q_1 - 1), (q_2 - 1), (q_3 - 1), (q_1 - 1)(q_2 - 1), (q_1 - 1)(q_3 - 1), (q_2 - 1)(q_3 - 1), (q_1 - 1)(q_2 - 1)(q_3 - 1)$. □

CHAPTER 4

SUBORBITAL GRAPHS OF $\boldsymbol{Aff(q_1) \times Aff(q_2)}$ ON $\boldsymbol{GF(q_1) \times GF(q_2)}$ AND $\boldsymbol{Aff(q_1) \times Aff(q_2) \times Aff(q_3)}$ ON $\boldsymbol{GF(q_1) \times GF(q_2) \times GF(q_3)}$

4.1 Introduction

Having computed the ranks and subdegrees of $Aff(q_1)\times Aff(q_2)$ on $GF(q_1)\times GF(q_2)$ and $Aff(q_1) \times Aff(q_2) \times Aff(q_3)$ on $GF(q_1) \times GF(q_2) \times GF(q_3)$, it is important to now investigate some of the properties of suborbital graphs corresponding to these actions. This chapter has been divided into three major sections. Section 4.2 examines the suborbital graphs of $Aff(q_1) \times Aff(q_2)$ acting on $GF(q_1) \times GF(q_2)$ while section 4.3 deals with the suborbital graphs of $Aff(q_1) \times Aff(q_2) \times Aff(q_3)$ acting on $GF(q_1) \times GF(q_2) \times GF(q_3)$.

4.2 Suborbital graphs of $\boldsymbol{Aff(q_1)\times Aff(q_2)}$ acting on $\boldsymbol{GF(q_1)\times GF(q_2)}$

Suppose that $Stab_{G_1\times G_2}(a,b)$ acts on $\Omega_1 \times \Omega_2$ and Δ is an orbit of $Stab_{G_1\times G_2}(a,b)$ on $\Omega_1 \times \Omega_2$. Then, the suborbital O corresponding to Δ is given by,

$$\begin{aligned} O &= \{\Big((g_1,g_2)(a,b),(g_1,g_2)(\alpha,\beta)\Big) : (\alpha,\beta) \in \Delta, (g_1,g_2) \in G_1 \times G_2\}. \\ &= \{\Big((g_1a,g_2b),(g_1\alpha,g_2\beta)\Big) : (\alpha,\beta) \in \Delta, (g_1,g_2) \in G_1 \times G_2\}. \qquad (4.1) \end{aligned}$$

The graph Γ of suborbital O is formed by taking $\Omega_1 \times \Omega_2$ as the vertex set and directed edges from (μ_1,μ_2) to (ω_1,ω_2) if and only if $((\mu_1,\mu_2),(\omega_1,\omega_2)) \in O$. The graph that corresponds to Δ_0 is referred to as the null graph.

Example 4.2.1. *Suborbital graphs of $Aff(2) \times Aff(2)$ acting on $GF(2) \times GF(2)$.*

By Equations (3.8) to (3.10), $(0,1) \in \Delta_1, (1,0) \in \Delta_2, (1,1) \in \Delta_3$. The suborbital O_i that corresponds to suborbit Δ_i is given by,

$$\begin{aligned} O_1 =&\{\Big((g_1,g_2)(0,0),(g_1,g_2)(0,1)\Big) : (g_1,g_2) \in G_1 \times G_2\} \\ &= \{\Big((g_1(0),g_2(0)),(g_1(0),g_2(1)\Big) : (g_1,g_2) \in G_1 \times G_2\} \\ &= \{(\mu_1,\mu_2),(\omega_1,\omega_2) : \mu_1 = \omega_1, \mu_2 \neq \omega_2)\}. \end{aligned} \tag{4.2}$$

$$\begin{aligned} O_2 =&\{\Big((g_1,g_2)(0,0),(g_1,g_2)(1,0)\Big) : (g_1,g_2) \in G_1 \times G_2\} \\ &= \{\Big((g_1(0),g_2(0)),(g_1(1),g_2(0)\Big) : (g_1,g_2) \in G_1 \times G_2\} \\ &= \{(\mu_1,\mu_2),(\omega_1,\omega_2) : \mu_1 \neq \omega_1, \mu_2 = \omega_2)\}. \end{aligned} \tag{4.3}$$

$$\begin{aligned} O_3 =&\{\Big(g_1,g_2)(0,0),(g_1,g_2)(1,1)\Big) : (g_1,g_2) \in G_1 \times G_2\} \\ &= \{\Big((g_1(0),g_2(0)),(g_1(1),g_2(1)\Big) : (g_1,g_2) \in G_1 \times G_2\} \\ &= \{(\mu_1,\mu_2),(\omega_1,\omega_2) : \mu_1 \neq \omega_1, \mu_2 \neq \omega_2)\}. \end{aligned} \tag{4.4}$$

Using Equation (4.2), we obtain Γ_1 as Figure 4.1.

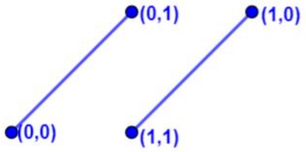


Figure 4.1: Suborbital graph Γ_1 corresponding to the action of $Aff(2) \times Aff(2)$ on $GF(2) \times GF(2)$

Using Equation (4.3), we obtain Γ_2 as Figure 4.2.

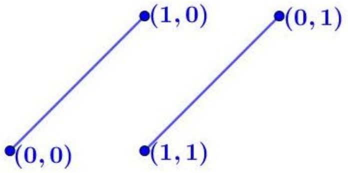


Figure 4.2: Suborbital graph Γ_2 corresponding to the action of $Aff(2) \times Aff(2)$ on $GF(2) \times GF(2)$

Using Equation (4.4),we obtain Γ_3 as Figure 4.3.

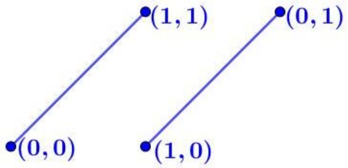


Figure 4.3: Suborbital graph Γ_3 corresponding to the action of $Aff(2) \times Aff(2)$ on $GF(2) \times GF(2)$

The suborbital graph Γ_1, Γ_2 and Γ_3 are all undirected, regular of degree 1,each has a girth of 0 and they are disconnected each with 2 components, each component has diameter 1.

Example 4.2.2. *Suborbital graphs of* $Aff(3) \times Aff(3)$ *acting on* $GF(3) \times GF(3)$.

By Equations (3.12) to (3.14), $(0,2) \in \Delta_1, (1,0) \in \Delta_2$ and $(1,2) \in \Delta_3$. The suborbital O_i that corresponds to suborbit Δ_i is given by,

$$\begin{aligned} O_1 =& \{\Big((g_1,g_2)(0,0),(g_1,g_2)(0,2)\Big) : (g_1,g_2) \in G_1 \times G_2\} \\ &= \{\Big((g_1(0),g_2(0)),(g_1(0),g_2(2)\Big) : (g_1,g_2) \in G_1 \times G_2\} \\ &= \{(\mu_1,\mu_2),(\omega_1,\omega_2) : \mu_1 = \omega_1, \mu_2 \neq \omega_2)\}. \end{aligned} \tag{4.5}$$

$$\begin{aligned} O_2 =& \{\Big((g_1,g_2)(0,0),(g_1,g_2)(1,0)\Big) : (g_1,g_2) \in G_1 \times G_2\} \\ &= \{\Big((g_1(0),g_2(0)),(g_1(1),g_2(0)\Big) : (g_1,g_2) \in G_1 \times G_2\} \\ &= \{(\mu_1,\mu_2),(\omega_1,\omega_2) : \mu_1 \neq \omega_1, \mu_2 = \omega_2)\}. \end{aligned} \tag{4.6}$$

$$\begin{aligned} O_3 =& \{\Big((g_1,g_2)(0,0),(g_1,g_2)(1,2)\Big) : (g_1,g_2) \in G_1 \times G_2\} \\ &= \{\Big((g_1(0),g_2(0)),(g_1(1),g_2(2)\Big) : (g_1,g_2) \in G_1 \times G_2\} \\ &= \{(\mu_1,\mu_2),(\omega_1,\omega_2) : \mu_1 \neq \omega_1, \mu_2 \neq \omega_2)\}. \end{aligned} \tag{4.7}$$

Applying Equations (4.5), we obtain Γ_1 as Figure 4.4.

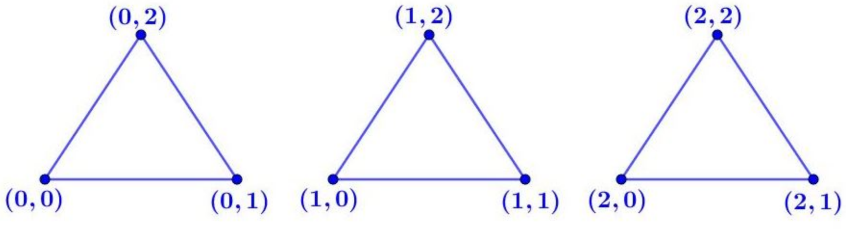


Figure 4.4: Suborbital graph Γ_1 corresponding to the action of $Aff(3) \times Aff(3)$ on $GF(3) \times GF(3)$

Γ_1 is undirected, has girth 3, regular of degree 2, disconnected graph. It has 3 connected components each with a diameter of 1.

Applying Equation (4.6), we obtain Γ_2 as Figure 4.5.

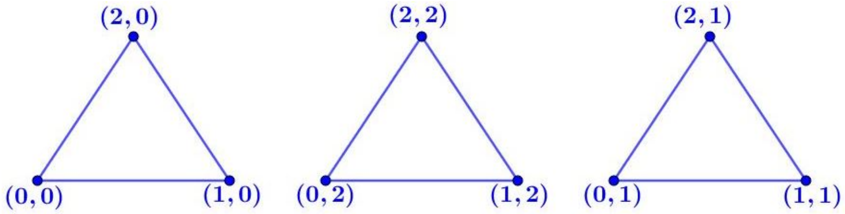


Figure 4.5: Suborbital graph $\boldsymbol{\Gamma_2}$ corresponding to the action of $\boldsymbol{Aff(3) \times Aff(3)}$ on $\boldsymbol{GF(3) \times GF(3)}$

Γ_2 is undirected, has girth of 3 and it is a disconnected graph,regular of degree 2. It has 3 connected components each with a diameter of 1 .

Applying Equations(4.7), we obtain Γ_3 as Figure 4.6.

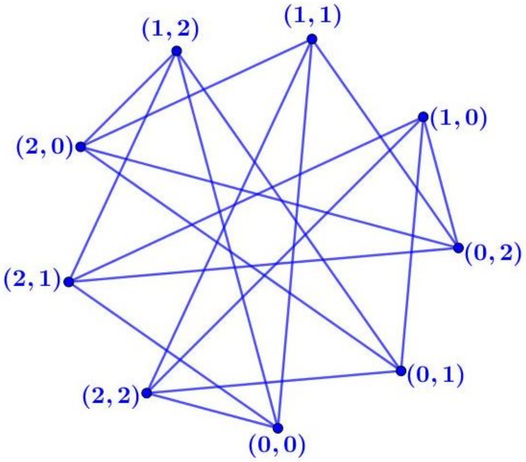


Figure 4.6: Suborbital graph Γ_3 corresponding to the action of $Aff(3) \times Aff(3)$ on $GF(3) \times GF(3)$

Γ_3 is undirected,has girth of 3, connected, regular of degree 4 hence Eurelian, and has a diameter of 2.

Example 4.2.3. *Suborbital graphs of $Aff(2) \times Aff(3)$ acting on $GF(2) \times GF(3)$.*

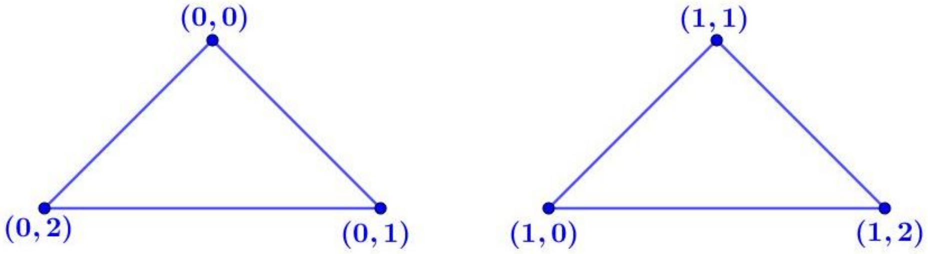
By Equations (3.16) to (3.18), $(0,1) \in \Delta_1, (1,0) \in \Delta_2$ and $(1,1) \in \Delta_3$. The suborbital O_i that corresponds to suborbit Δ_i is given by,

$$\begin{aligned} O_1 =& \{\Big((g_1,g_2)(0,0),(g_1,g_2)(0,1)\Big) : (g_1,g_2) \in G_1 \times G_2\} \\ &= \{\Big((g_1(0),g_2(0)),(g_1(0),g_2(1)\Big) : (g_1,g_2) \in G_1 \times G_2\} \\ &= \{(\mu_1,\mu_2),(\omega_1,\omega_2) : \mu_1 = \omega_1, \mu_2 \neq \omega_2)\}. \end{aligned} \tag{4.8}$$

$$\begin{aligned} O_2 =& \{\Big((g_1,g_2)(0,0),(g_1,g_2)(1,0)\Big) : (g_1,g_2) \in G_1 \times G_2\} \\ &= \{\Big((g_1(0),g_2(0)),(g_1(1),g_2(0)\Big) : (g_1,g_2) \in G_1 \times G_2\} \\ &= \{(\mu_1,\mu_2),(\omega_1,\omega_2) : \mu_1 \neq \omega_1, \mu_2 = \omega_2)\}. \end{aligned} \tag{4.9}$$

$$\begin{aligned} O_3 =& \{\Big((g_1,g_2)(0,0),(g_1,g_2)(1,1)\Big) : (g_1,g_2) \in G_1 \times G_2\} \\ &= \{\Big((g_1(0),g_2(0)),(g_1(1),g_2(1)\Big) : (g_1,g_2) \in G_1 \times G_2\} \\ &= \{(\mu_1,\mu_2),(\omega_1,\omega_2) : \mu_1 \neq \omega_1, \mu_2 \neq \omega_2)\}. \end{aligned} \tag{4.10}$$

Applying Equations (4.8) we obtain Γ_1 as Figure 4.7.



.

Figure 4.7: Suborbital graph Γ_1 corresponding to the action of $Aff(2) \times Aff(3)$ on $GF(2) \times GF(3)$

Γ_1 is undirected, has girth 3, regular of degree 2, disconnected with 2 connected components each with a diameter of 1.

Applying Equation (4.9), we obtain Γ_2 as Figure 4.8.

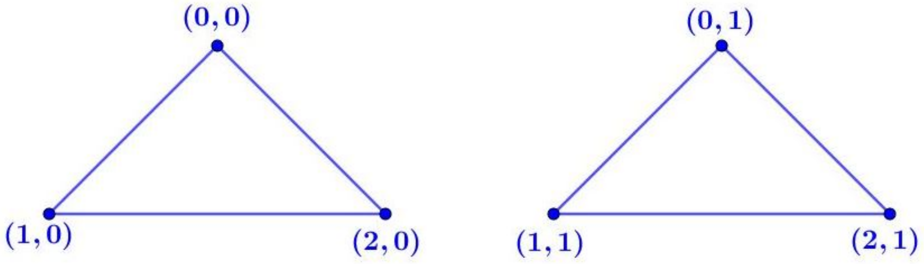


Figure 4.8: Suborbital graph Γ_2 corresponding to the action of $Aff(2) \times Aff(3)$ on $GF(2) \times GF(3)$

Γ_1 is undirected, has girth 3, regular of degree 2, disconnected with 2 connected components each with a diameter of 1.

Applying Equation (4.10), we obtain Γ_3 as Figure 4.9.

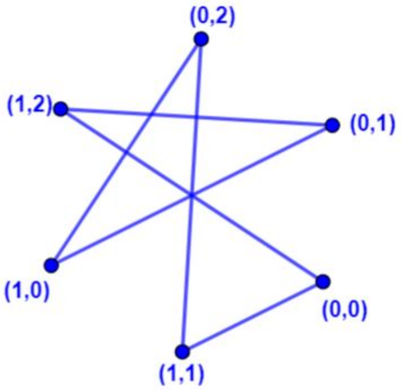


Figure 4.9: Suborbital graph Γ_3 corresponding to the action of $Aff(2) \times Aff(3)$ on $GF(2) \times GF(3)$

Γ_3 is undirected, has girth 6 and a diameter of 2. The graph is connected and Eurelian since it is regular with degree 2.

Example 4.2.4. *Suborbital graphs of $Aff(q_1) \times Aff(q_2)$ acting on $GF(q_1) \times GF(q_2)$.*

By Equations (3.19), (3.20), (3.21) and (3.22),it follows that $(0,0) \in \Delta_0, (0,\beta) \in \Delta_1, (\alpha,0) \in \Delta_2$ and $(\alpha,\beta) \in \Delta_3$. By application of Equation (4.19), we obtain,

$$\begin{aligned} O_1 =& \{\Big((g_1,g_2)(0,0),(g_1,g_2)(0,\beta\Big) : (g_1,g_2) \in G_1 \times G_2\} \\ &= \{\Big((g_1(0),g_2(0)),(g_1(0),g_2\beta)\Big) : (g_1,g_2) \in G_1 \times G_2\} \\ &= \{((\mu_1,\mu_2),(\omega_1,\omega_2)) : \mu_1 = \omega_1, \mu_2 \neq \omega_2\}. \end{aligned} \tag{4.11}$$

$$\begin{aligned} O_2 =&\{\Big((g_1,g_2)(0,0),(g_1,g_2)(\alpha,0)\Big):(g_1,g_2)\in G_1\times G_2\} \\ &=\{\Big((g_1(0),g_2(0)),(g_1\alpha,g_2(0))\Big):(g_1,g_2)\in G_1\times G_2\} \\ &=\{((\mu_1,\mu_2),(\omega_1,\omega_2)):\mu_1\neq\omega_1,\mu_2=\omega_2\}. \end{aligned} \tag{4.12}$$

$$\begin{aligned} O_3 =&\{\Big((g_1,g_2)(0,0),(g_1,g_2)(\alpha,\beta)\Big):(g_1,g_2)\in G_1\times G_2\} \\ &=\{\Big((g_1(0),g_2(0)),(g_1\alpha,g_2\beta)\Big)\} \\ &=\{((\mu_1,\mu_2),(\omega_1,\omega_2)):\mu_1\neq\omega_1,\mu_2\neq\omega_2\}. \end{aligned} \tag{4.13}$$

Theorem 4.2.5. *All the non trivial graphs corresponding to the action of* $Aff(q_1)\times Aff(q_2)$ *on* $GF(q_1)\times GF(q_2)$ *are undirected.*

Proof. By Equations (4.11) to (4.13), if $((\mu_1,\mu_2),(\omega_1,\omega_2))\in O_i,(i=1,2,3)$ then $(\omega_1,\omega_2),(\mu_1,\mu_2))\in O_i$. □

Theorem 4.2.6. *The suborbital graph* Γ_3 *is Eulerian for an odd* q.

Proof. Suppose that q is odd,then $|\Delta_3|=(q-1)^2$ represent the degree of each vertex the graph. Therefore by Theorem 1.2.22, Γ_3 is Eulerian. □

Lemma 4.2.7. *Let* $Comp_{\Gamma_i(0,0)},(i=1,2,3)$ *represent all the nodes in the component having* $(0,0)$ *in the suborbital graph* Γ_i. *Then,*

$$Comp_{\Gamma_1(0,0)} =\{(0,u):u\in GF(q_2)\}, \tag{4.14}$$

$$Comp_{\Gamma_2(0,0)} =\{(u,0):u\in GF(q_1)\} \tag{4.15}$$

and

$$Comp_{\Gamma_3(0,0)} = \{GF(q_1) \times GF(q_2)\}. \tag{4.16}$$

Proof. Let $(h,k) \in Comp_{\Gamma_1}(0,0)$. Then there exist a path $(0,0) \to (h_1,k_1) \to \ldots \to (h_{i-1},k_{i-1}) = (h,k)$. It follows From Equation (4.11) that, $0 = h_1 = h_2 = \ldots = h$. Therefore $(h,k) \in \{(0,u) : u \in GF(q_2)\}$. This implies that,

$$Comp_{\Gamma_1}(0,0) \subseteq \{(0,u) : u \in GF(q_2)\} \tag{4.17}$$

Let $(0,k) \in \{(0,u) : u \in GF(q_2)\}$. If $k = 0$, then $(0,k) = (0,0) \in Comp_{\Gamma_1}(0,0)$. If $k \neq 0$, then by Equation (4.11), there exist an edge $(0,0) \to (0,k) \in \Gamma_1$. Therefore, $(0,k) \in Comp_{\Gamma_1}(0,0)$. This implies that,

$$\{(0,u) : u \in GF(q_2)\} \subseteq Comp_{\Gamma_1}(0,0). \tag{4.18}$$

From Equations (4.17) and (4.18), we get Equation (4.14). Equation (4.15) is proved in a similar way as Equation (4.14).

Let $(u,v) \in GF(q_1) \times GF(q_2)$. Consider the vertex (h,k), where $u \neq h \in GF(q_1)^\star \setminus \{h\}$ and $v \neq k \in GF(q_2)^\star \setminus \{k\}$. By Equation (4.13), (h,k) is adjacent to both $(u,v)$and $(0,0)$. This implies that $(u,v) \in Comp_{\Gamma_3}(0,0)$, hence $GF(q_1) \times GF(q_2) \subseteq Comp_{\Gamma_3}(0,0)$. But $GF(q_1) \times GF(q_2)$ is the set of all the vertices in Γ_3, thus $Comp_{\Gamma_3}(0,0) \subseteq GF(q_1) \times GF(q_2)$. Thus we get Equation (4.16). $\square$

Theorem 4.2.8. *The graphs* Γ_1, Γ_2 *and* Γ_3 *have* q_1, q_2 *and* 1 *components respectively,where* $q_1, q_2 > 2$.

Proof. By Theorem 1.2.23, all the components in a suborbital graph are isomorphic. Applying Lemma 4.2.7,

$$|Comp_{\Gamma_1}(0,0)| = q_2, |Comp_{\Gamma_2}(0,0)| = q_1, |Comp_{\Gamma_3}(0,0)| = q_1q_2.$$

Since each of the graphs has q_1q_2 vertices, there are q_1 components for Γ_1, q_2 components for Γ_2 and 1 component for Γ_3. □

Lemma 4.2.9. *For the action of* $Aff(2) \times Aff(q_2)$ *on* $GF(2) \times GF(q_2)$*, where* $q_2 \geqslant 3, \Gamma_3$ *is bipartite.*

Proof. Let $V(\Gamma_3)$ be the vertices corresponding to Γ_3, then the vertices of the graph G are divided into two $V_1 = \{(0,k) : k \in GF(q_2)\}$ and $V_2 = \{(1,m) : m \in GF(q_1)\}$. Thus from Equation (4.13), the edges move from a vertex V_1 to a vertex V_2 and no edge in Γ_3 joining vertices in the same subset. Hence from Definition 1.2.18, Γ_3 is bipartite. □

Corollary 4.2.10. *The chromatic number of* Γ_3 *in* $Aff(2) \times Aff(q_2)$ *acting on* $GF(2) \times GF(q_2)$*, where* $q_2 \geqslant 3$ *is* 2.

Proof. By Lemma 4.2.9, Γ_3 is bipartite and using Definition 1.2.18, chromatic number of a bipartite graph is 2. □

Theorem 4.2.11. Γ_3 *corresponding to the action of* $Aff(2) \times Aff(q_2)$ *on* $GF(2) \times GF(q_2)$ *has a girth of* 4*, when* $q_2 > 3$ *and a girth of* 6 *when* $q_2 = 3$.

Proof. Since Γ_3 is bipartite, all circuits are of even lengths. Suppose $h \in GF(q_2)$ is a primitive element, if $q_2 > 3$, then $(0,0) \to (1,h) \to (0,h^2) \to (1,h^3) \to (0,0)$ is a circuit in Γ_3 by Equation (4.13). Thus the girth is 4. Also, if $q_2 = 3$, then by Equation (4.13), we obtain a circuit $(0,0) \to (1,h) \to (0,h^2) \to (1,h^3) \to (1,h^4) \to (1,h^5) \to (0,0)$. Thus the girth is 6. □

Theorem 4.2.12. *The suborbital graph Γ_1 and Γ_2 corresponding to the action of $Aff(q_1) \times Aff(q_2)$ on $GF(q_1) \times GF(q_2)$, has a girth of 0 when $q_1 = q_2 = 2$ and a girth of 3 when $q_1 = q_2 > 2$.*

Proof. Let the elements $h \in GF(q_2)$, and $k \in GF(q_3)$ be primitive elements. Then if $q_1 = q_2 = 2$, using Equations(4.11) to (4.13), we obtain,

$(0,0) \to (0,k)$ and $(h,0) \to (h,k) \in \Gamma_1$,

$(0,0) \to (h,0)$ and $(0,k) \to (h,k) \in \Gamma_2$,

$(0,0) \to (h,k)$ and $(h,0) \to (0,k) \in \Gamma_3$. Thus girth 0.

Suppose $q_1 = q_2 > 2$, using Equations(4.11) to (4.13),we get,

$(0,0) \to (0,k) \to (0,k^2) \to (0,0) \in \Gamma_1$,

$(0,0) \to (h,0) \to (h^2,0) \to (0,0) \in \Gamma_2$,

$(0,0) \to (h,k) \to (h^2,k^2) \to (0,0) \in \Gamma_3$, . Thus the girth is 3. □

Corollary 4.2.13. *The action of $Aff(q_1) \times Aff(q_2)$ on $GF(q_1) \times GF(q_2)$ is imprimitive.*

Proof. From the Theorem 4.2.8, there are some non-trivial suborbital graphs which are disconnected. Thus from Theorem 1.2.28, the action is imprimitive. □

4.3 Suborbital graphs of $\boldsymbol{Aff(q_1) \times Aff(q_2) \times Aff(q_3)}$ acting on $\boldsymbol{GF(q_1) \times GF(q_2) \times GF(q_3)}$

Suppose that $Stab_{G_1\times G_2\times G_3}(a,b,c)$ acts on $\Omega_1 \times \Omega_2 \times \Omega_3$ and Δ is an orbit of $Stab_{G_1\times G_2\times G_3}(a,b,c)$ on $\Omega_1 \times \Omega_2 \times \Omega_3$ respectively. Then,

$$\begin{aligned} O &= \{\Big((g_1,g_2,g_3)(a,b,c),(g_1,g_2,g_3)(\alpha,\beta,\gamma)\Big) : (\alpha,\beta,\gamma) \in \Delta, \\ & \quad (g_1,g_2,g_3) \in G_1 \times G_2 \times G_3\}. \\ &= \{\Big((g_1a,g_2b,g_3c),(g_1\alpha,g_2\beta,g_3\gamma\Big) : (\alpha,\beta,\gamma) \in \Delta, \\ & \quad (g_1,g_2,g_3) \in G_1 \times G_2 \times G_3\}. \end{aligned} \tag{4.19}$$

The graph Γ of suborbital O is formed by taking $\Omega_1\times\Omega_2\times\Omega_3$ as the vertex set and directed edges from (μ_1,μ_2,μ_3) to $(\omega_1,\omega_2,\omega_3)$, if and only if $((\mu_1,\mu_2,\mu_3),(\omega_1,\omega_2,\omega_3)) \in O$. The graph that corresponds to Δ_0 is referred to as the null graph.

Example 4.3.1. *Suborbital graphs of $Aff(3)\times Aff(3)\times Aff(3)$ acting on $GF(3)\times GF(3)\times GF(3)$.*

By Equations (3.40) to (3.46), $(0,0,2) \in \Delta_1, (0,2,0) \in \Delta_2, (2,0,0) \in \Delta_3, (2,2,0) \in \Delta_4, (2,0,2) \in \Delta_5, (0,2,2) \in \Delta_6$ and $(2,2,2) \in \Delta_7$. The suborbital O_i that corresponds to suborbit Δ_i is given by,

$$\begin{aligned} O_1 =& \{\Big((g_1, g_2, g_3)(0,0,0), (g_1, g_2, g_3)(0,0,2)\Big) : g_1 \in G_1, g_2 \in G_2, g_3 \in G_3\} \\ &= \{\Big((g_1(0), g_2(0), g_3(0), g_1(0), g_2(0), g_2(2)\Big) : g_1 \in G_1, g_2 \in G_2, g_3 \in G_3\} \\ &= \{(\mu_1, \mu_2, \mu_3), (\omega_1, \omega_2, \omega_3) : \mu_1 = \omega_1, \mu_2 = \omega_2, \mu_3 \neq \omega_3\}. \end{aligned} \tag{4.20}$$

Using Equation (4.20), we obtain Γ_1 as Figure 4.10.

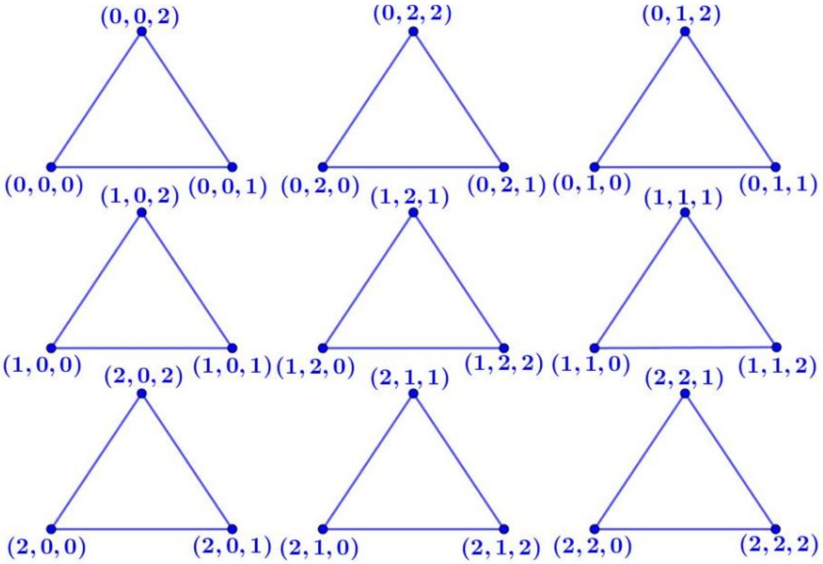


Figure 4.10: Suborbital graph Γ_1 corresponding to the action of $Aff(3) \times Aff(3) \times Aff(3)$ on $GF(3) \times GF(3) \times GF(3)$

Γ_1 is undirected, has girth of 3, regular of degree 2, it is disconnected with 9 connected components each with a diameter of 1.

$$\begin{aligned} O_2 =& \{\Big((g_1, g_2, g_3)(0,0,0), (g_1, g_2, g_3)(0,2,0)\Big) : g_1 \in G_1, g_2 \in G_2, g_3 \in G_3\} \\ &= \{\Big((g_1(0), g_2(0), g_3(0), g_1(0), g_2(2), g_2(0)\Big) : g_1 \in G_1, g_2 \in G_2, g_3 \in G_3\} \\ &= \{(\mu_1, \mu_2, \mu_3), (\omega_1, \omega_2, \omega_3) : \mu_1 = \omega_1, \mu_2 \neq \omega_2, \mu_3 = \omega_3\}. \end{aligned} \tag{4.21}$$

Using Equation (4.21), we obtain Γ_2 as Figure 4.11.

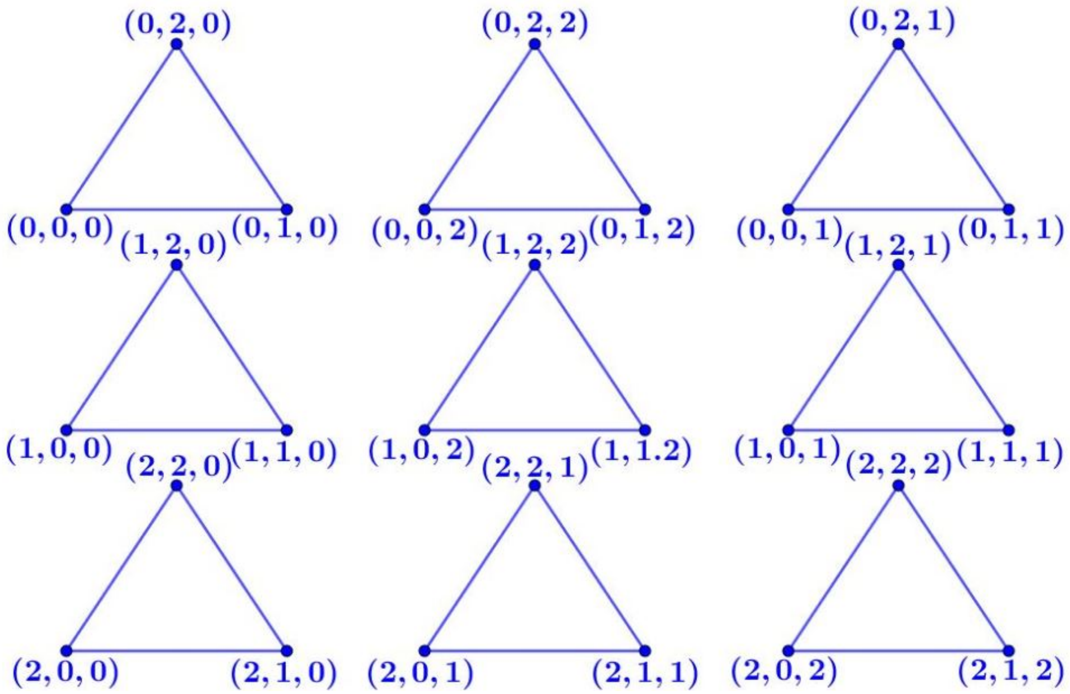


Figure 4.11: Suborbital graph Γ_2 corresponding to the action of $Aff(3) \times Aff(3) \times Aff(3)$ on $GF(3) \times GF(3) \times GF(3)$

Γ_2 is undirected, has girth 3, disconnected graph, regular of degree 2. It has 9 connected components each with a diameter of 1, hence component is complete.

$$\begin{aligned} O_3 =& \{\Big((g_1, g_2, g_3)(0,0,0), (g_1, g_2, g_3)(2,0,0)\Big) : g_1 \in G_1, g_2 \in G_2, g_3 \in G_3\} \\ &= \{\Big((g_1(0), g_2(0), g_3(0), g_1(2), g_2(0), g_2(0)\Big) : g_1 \in G_1, g_2 \in G_2, g_3 \in G_3\} \\ &= \{(\mu_1, \mu_2, \mu_3), (\omega_1, \omega_2, \omega_3) : \mu_1 \neq \omega_1, \mu_2 = \omega_2, \mu_3 = \omega_3\}. \end{aligned} \tag{4.22}$$

Using Equation (4.22), we obtain Γ_3 as Figure 4.12.

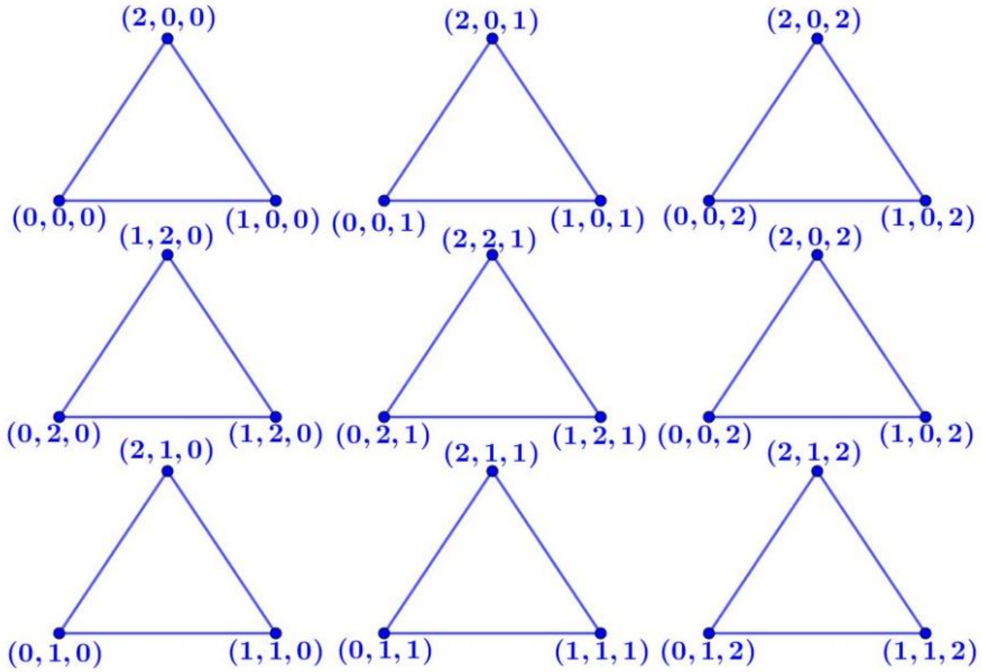


Figure 4.12: Suborbital graph $\boldsymbol{\Gamma_3}$ corresponding to the action of $\boldsymbol{Aff(3) \times Aff(3) \times Aff(3)}$ on $\boldsymbol{GF(3) \times GF(3) \times GF(3)}$

Γ_3 is undirected graph, has girth 3, regular of degree 2,disconnected with 9 connected components each with a diameter of 1. Each component is complete.

$$\begin{aligned} O_4 =&\{\Big((g_1,g_2,g_3)(0,0,0),(g_1,g_2,g_3)(2,2,0)\Big) : g_1 \in G_1, g_2 \in G_2, g_3 \in G_3\} \\ &= \{\Big((g_1(0),g_2(0),g_3(0),g_1(2),g_2(2),g_2(0)\Big) : g_1 \in G_1, g_2 \in G_2, g_3 \in G_3\} \\ &= \{(\mu_1,\mu_2,\mu_3),(\omega_1,\omega_2,\omega_3) : \mu_1 \neq \omega_1, \mu_2 \neq \omega_2, \mu_3 = \omega_3\}. \end{aligned} \tag{4.23}$$

Using Equation (4.23), we obtain Γ_6 as Figure 4.13.

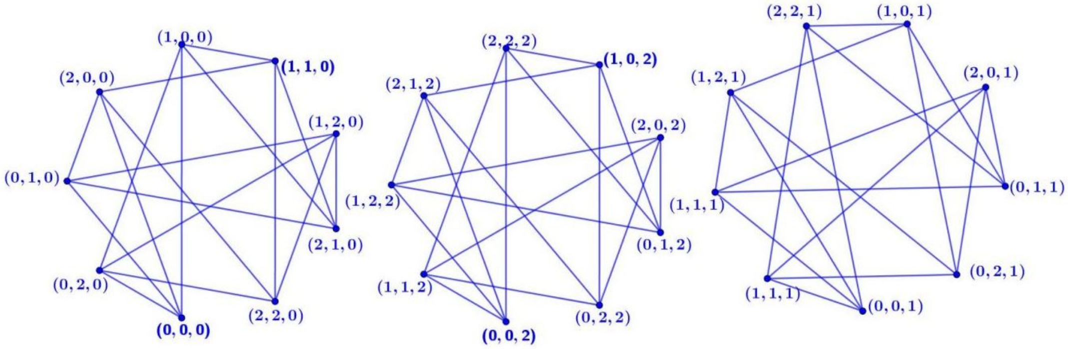


Figure 4.13: Suborbital graph Γ_4 corresponding to the action of $Aff(3) \times Aff(3) \times Aff(3)$ on $GF(3) \times GF(3) \times GF(3)$

Γ_4 is undirected, has girth 3, disconnected graph,regular of degree 4,with 3 connected components each with a diameter of 2.

$$\begin{aligned} O_5 =&\{\Big((g_1,g_2,g_3)(0,0,0),(g_1,g_2,g_3)(2,0,2)\Big) : g_1 \in G_1, g_2 \in G_2, g_3 \in G_3\} \\ &= \{\Big((g_1(0),g_2(0),g_3(0),g_1(2),g_2(0),g_2(2)\Big) : g_1 \in G_1, g_2 \in G_2, g_3 \in G_3\} \\ &= \{(\mu_1,\mu_2,\mu_3),(\omega_1,\omega_2,\omega_3) : \mu_1 \neq \omega_1, \mu_2 = \omega_2, \mu_3 \neq \omega_3\} \end{aligned} \tag{4.24}$$

Using Equation (4.24), we obtain Γ_5 as Figure 4.14.

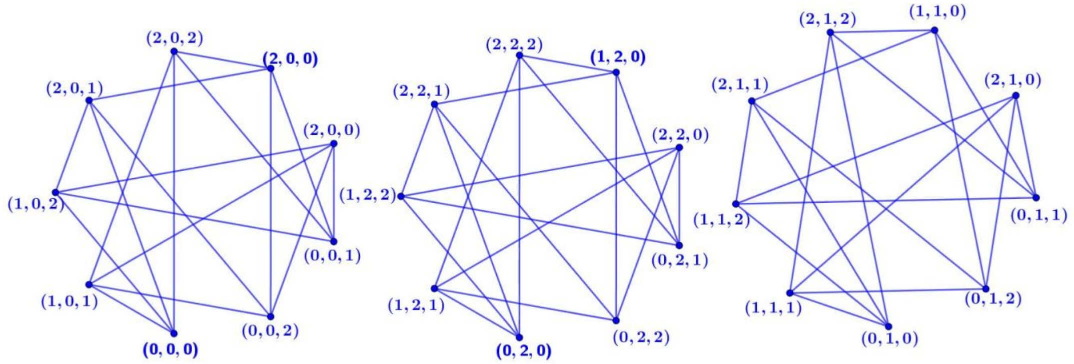


Figure 4.14: Suborbital graph $\boldsymbol{\Gamma_5}$ corresponding to the action of $\boldsymbol{Aff(3) \times Aff(3) \times Aff(3)}$ on $\boldsymbol{GF(3) \times GF(3) \times GF(3)}$

Γ_5 is undirected, has girth 3, disconnected graph,regular of degree 4 with 3 connected components each with a diameter of 2 .

$$\begin{aligned} O_6 =&\{\Big((g_1,g_2,g_3)(0,0,0),(g_1,g_2,g_3)(0,2,2)\Big) : g_1 \in G_1, g_2 \in G_2, g_3 \in G_3\} \\ &= \{\Big((g_1(0),g_2(0),g_3(0),g_1(0),g_2(2),g_2(2)\Big) : g_1 \in G_1, g_2 \in G_2, g_3 \in G_3\} \\ &= \{(\mu_1,\mu_2,\mu_3),(\omega_1,\omega_2,\omega_3) : \mu_1 = \omega_1, \mu_2 \neq \omega_2, \mu_3 \neq \omega_3\}. \end{aligned} \tag{4.25}$$

Using Equation (4.25), we obtain Γ_6 as Figure 4.15.

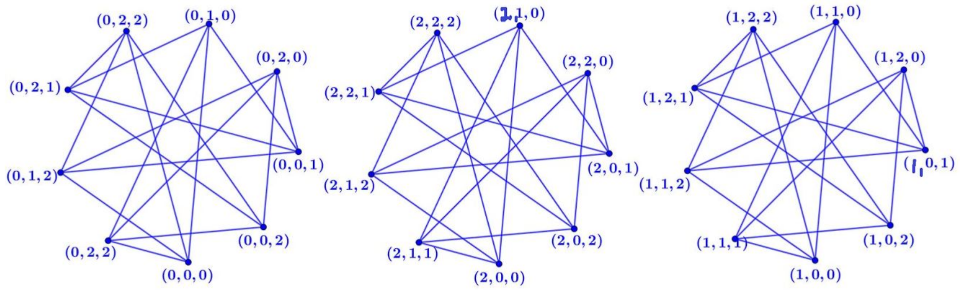


Figure 4.15: Suborbital graph $\boldsymbol{\Gamma_6}$ corresponding to the action of $\boldsymbol{Aff(3) \times Aff(3) \times Aff(3)}$ on $\boldsymbol{GF(3) \times GF(3) \times GF(3)}$

Γ_6 is undirected,has girth 3, disconnected, regular of degree 4 with 3 connected components each with a diameter of 2.

$$\begin{aligned} O_7 =&\{\Big((g_1, g_2, g_3)(0,0,0), (g_1, g_2, g_3)(2,2,2)\Big) : g_1 \in G_1, g_2 \in G_2, g_3 \in G_3\} \\ &= \{\Big((g_1(0), g_2(0), g_3(0), g_1(2), g_2(2), g_2(2)\Big) : g_1 \in G_1, g_2 \in G_2, g_3 \in G_3\} \\ &= \{(\mu_1, \mu_2, \mu_3), (\omega_1, \omega_2, \omega_3) : \mu_1 \neq \omega_1, \mu_2 \neq \omega_2, \mu_3 \neq \omega_3\}. \end{aligned} \tag{4.26}$$

Using Equation (4.26), we obtain Γ_7 as Figure 4.16.

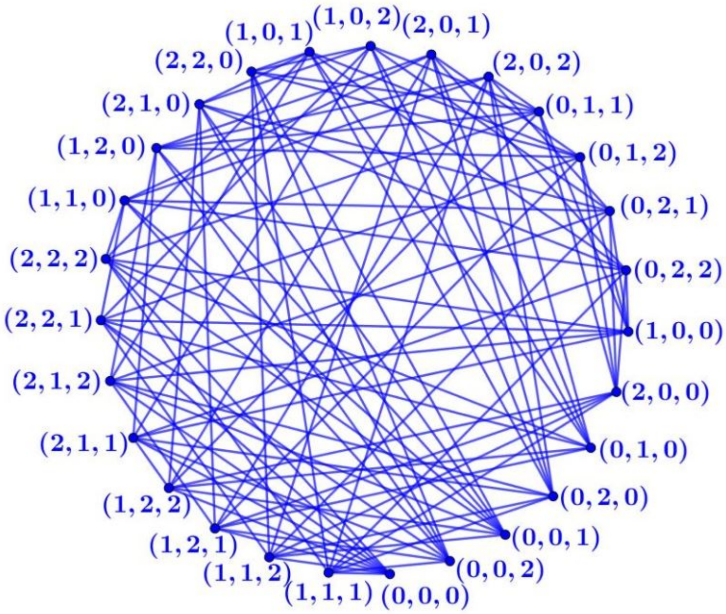


Figure 4.16: Suborbital graph Γ_7 corresponding to the action of $Aff(3) \times Aff(3) \times Aff(3)$ on $GF(3) \times GF(3) \times GF(3)$

Γ_7 is undirected, Eurelian since it is a regular graph of degree 8, has girth 3. It is connected with a diameter of 2 .

Example 4.3.2. *Suborbital graphs of $Aff(2) \times Aff(3) \times Aff(5)$ acting on $GF(2) \times GF(3) \times GF(5)$.*

By Equations (3.32) to (3.38), $(0,0,3) \in \Delta_1, (0,2,0) \in \Delta_2, (1,0,0) \in \Delta_3, (0,2,3) \in \Delta_4, (1,0,3) \in \Delta_5, (1,2,0) \in \Delta_6$ while $(1,2,3) \in \Delta_7$. The suborbital O_i that corresponds to suborbit Δ_i is given by,

$$\begin{aligned} O_1 =& \{\Big((g_1,g_2,g_3)(0,0,0),(g_1,g_2,g_3)(0,0,3)\Big) : g_1 \in G_1, g_2 \in G_2, g_3 \in G_3\} \\ &= \{\Big((g_1(0),g_2(0),g_3(0),g_1(0),g_2(0),g_2(3)\Big) : g_1 \in G_1, g_2 \in G_2, g_3 \in G_3\} \\ &= \{(\mu_1,\mu_2,\mu_3),(\omega_1,\omega_2,\omega_3) : \mu_1 = \omega_1, \mu_2 = \omega_2, \mu_3 \neq \omega_3\}. \end{aligned} \tag{4.27}$$

Using Equation (4.27), we obtain Γ_1 as Figure 4.17.

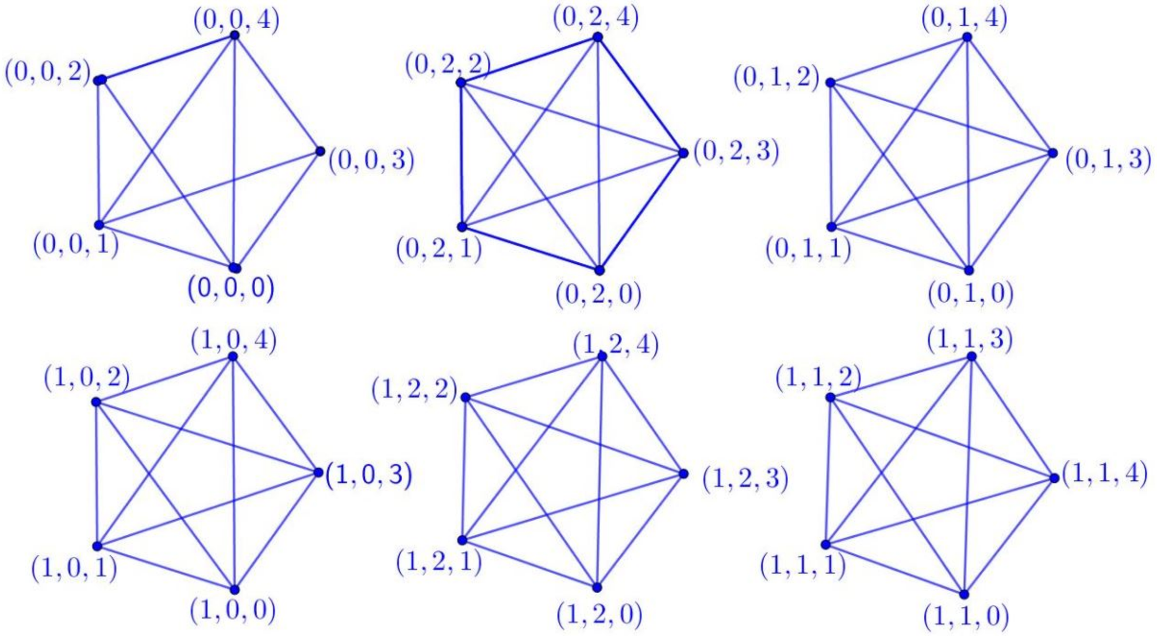


Figure 4.17: Suborbital graph Γ_1 corresponding to the action of $Aff(2) \times Aff(3) \times Aff(5)$ on $GF(2) \times GF(3) \times GF(5)$

Γ_1 is undirected, has girth 3, disconnected graph,regular of degree 4 with 6 connected components each of diameter of 2.

$$\begin{aligned} O_2 =& \{\Big((g_1, g_2, g_3)(0,0,0), (g_1, g_2, g_3)(0,2,0)) : g_1 \in G_1, g_2 \in G_2, g_3 \in G_3\} \\ &= \{\Big(g_1(0), g_2(0), g_3(0), g_1(0), g_2(2), g_2(0)\Big) : g_1 \in G_1, g_2 \in G_2, g_3 \in G_3\} \qquad (4.28) \\ &= \{(\mu_1, \mu_2, \mu_3), (\omega_1, \omega_2, \omega_3) : \mu_1 = \omega_1, \mu_2 \neq \omega_2, \mu_3 = \omega_3\}. \end{aligned}$$

Using Equation (4.28), we obtain Γ_2 as Figure 4.18.

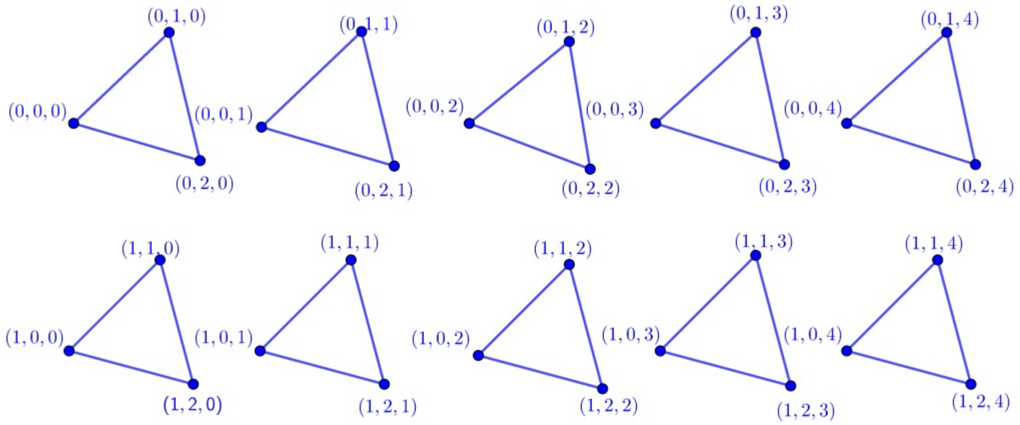


Figure 4.18: Suborbital graph Γ_2 corresponding to the action of $Aff(2) \times Aff(3) \times Aff(5)$ on $GF(2) \times GF(3) \times GF(5)$

Γ_2 is undirected, has girth 3, disconnected graph, regular of degree 2 with 10 connected components each with a diameter of 1.

$$\begin{aligned} O_3 =&\{\Big((g_1, g_2, g_3)(0,0,0), (g_1, g_2, g_3)(1,0,0)\Big) : g_1 \in G_1, g_2 \in G_2, g_3 \in G_3\} \\ &= \{\Big(g_1(0), g_2(0), g_3(0), g_1(1), g_2(0), g_2(0)\Big) : g_1 \in G_1, g_2 \in G_2, g_3 \in G_3\} \\ &= \{(\mu_1, \mu_2, \mu_3), (\omega_1, \omega_2, \omega_3) : \mu_1 \neq \omega_1, \mu_2 = \omega_2, \mu_3 = \omega_3\}. \end{aligned} \tag{4.29}$$

Using Equation (4.29), we obtain Γ_3 as Figure 4.19.

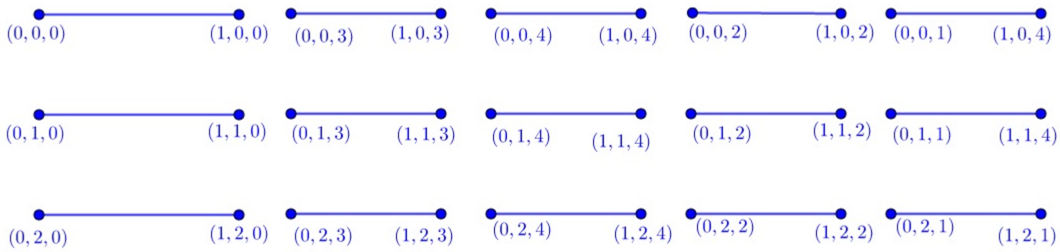


Figure 4.19: Suborbital graph Γ_3 corresponding to the action of $Aff(2) \times Aff(3) \times Aff(5)$ on $GF(2) \times GF(3) \times GF(5)$

Γ_3 is undirected,has a girth of 0,regular of degree 1 and it is disconnected with 15 connected components each of diameter of 1.

$$\begin{aligned} O_4 =&\{\Big((g_1, g_2, g_3)(0,0,0), (g_1, g_2, g_3)(0,2,3)\Big) : g_1 \in G_1, g_2 \in G_2, g_3 \in G_3\} \\ &= \{\Big(g_1(0), g_2(0), g_3(0), g_1(0), g_2(0), g_2(3)\Big) : g_1 \in G_1, g_2 \in G_2, g_3 \in G_3\} \\ &= \{(\mu_1, \mu_2, \mu_3), (\omega_1, \omega_2, \omega_3) : \mu_1 = \omega_1, \mu_2 \neq \omega_2, \mu_3 \neq \omega_3\}. \end{aligned} \tag{4.30}$$

Using Equation (4.30), we obtain Γ_4 as Figure 4.20.

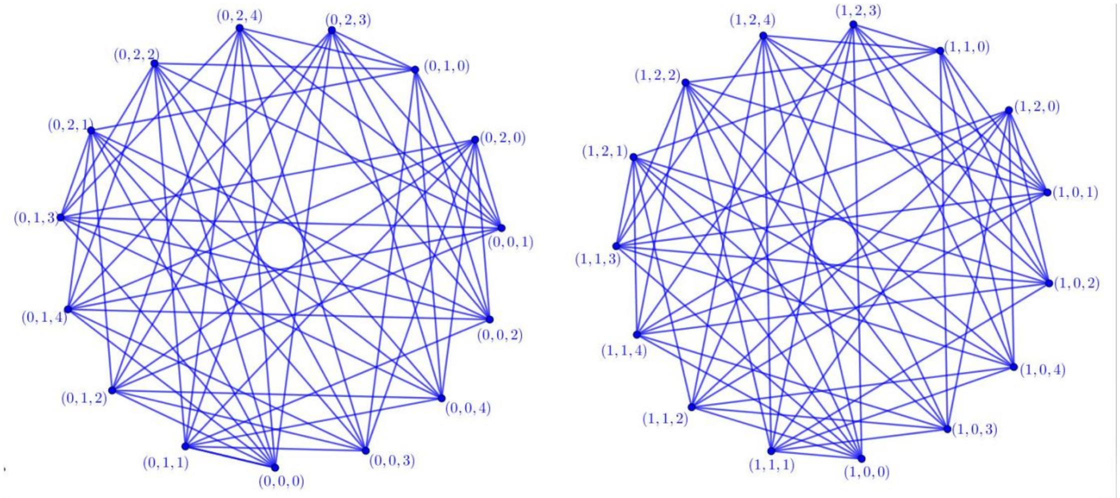


Figure 4.20: Suborbital graph Γ_4 corresponding to the action of $Aff(2) \times Aff(3) \times Aff(5)$ on $GF(2) \times GF(3) \times GF(5)$

Γ_4 is undirected, has girth 3, disconnected graphs,regular of degree 8 with 2 connected components each of diameter 2.

$$\begin{aligned} O_5 &= \{\Big((g_1, g_2, g_3)(0,0,0), (g_1, g_2, g_3)(1,0,3)\Big) : g_1 \in G_1, g_2 \in G_2, g_3 \in G_3\} \\ &= \{\Big(g_1(0), g_2(0), g_3(0), g_1(1), g_2(0), g_2(3)\Big) : g_1 \in G_1, g_2 \in G_2, g_3 \in G_3\} \\ &= \{((\mu_1, \mu_2, \mu_3), (\omega_1, \omega_2, \omega_3) : \mu_1 \neq \omega_1, \mu_2 = \omega_2, \mu_3 \neq \omega_3\}. \end{aligned} \tag{4.31}$$

Using Equation (4.31), we obtain Γ_5 as Figure 4.21.

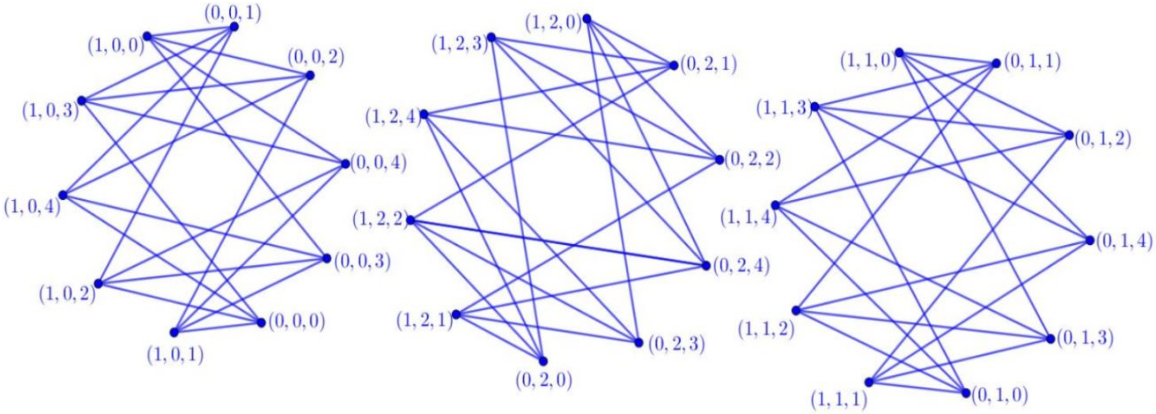


Figure 4.21: Suborbital graph Γ_5 corresponding to the action of $Aff(2) \times Aff(3) \times Aff(5)$ on $GF(2) \times GF(3) \times GF(5)$

Γ_5 is undirected, has girth 4 ,disconnected graph,regular of degree 4 with 3 connected components each of diameter 3.

$$\begin{aligned} O_6 =&\{\Big((g_1, g_2, g_3)(0,0,0), (g_1, g_2, g_3)(1,2,0)\Big) : g_1 \in G_1, g_2 \in G_2, g_3 \in G_3\} \\ &= \{\Big(g_1(0), g_2(0), g_3(0), g_1(1), g_2(2), g_2(0)\Big) : g_1 \in G_1, g_2 \in G_2, g_3 \in G_3\} \\ &= \{(\mu_1, \mu_2, \mu_3), (\omega_1, \omega_2, \omega_3) : \mu_1 \neq \omega_1, \mu_2 \neq \omega_2, \mu_3 = \omega_3\} \end{aligned} \tag{4.32}$$

Using Equation (4.32), we obtain Γ_6 as Figure 4.22.

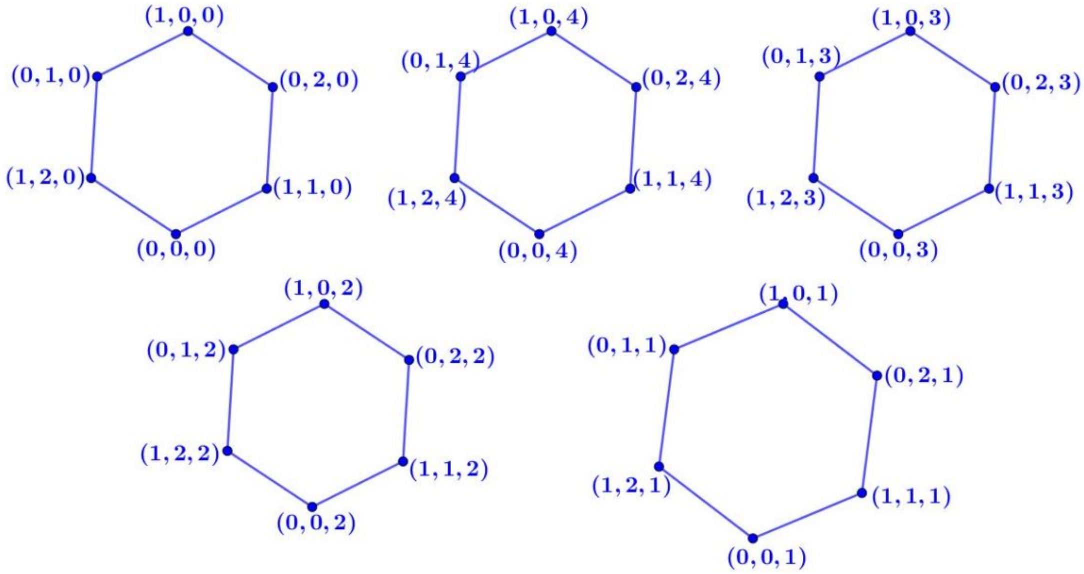


Figure 4.22: Suborbital graph Γ_6 corresponding to the action of $Aff(2) \times Aff(3) \times Aff(5)$ on $GF(2) \times GF(3) \times GF(5)$

Γ_6 is undirected, has girth 6, disconnected graph, regular of degree 2 with 5 connected components each of diameter 3.

$$\begin{aligned} O_7 =& \{\Big((g_1, g_2, g_3)(0,0,0), (g_1, g_2, g_3)(1,2,3)\Big) : g_1 \in G_1, g_2 \in G_2, g_3 \in G_3\} \\ &= \{\Big(g_1(0), g_2(0), g_3(0), g_1(1), g_2(2), g_2(3)\Big) : g_1 \in G_1, g_2 \in G_2, g_3 \in G_3\} \\ &= \{(\mu_1, \mu_2, \mu_3), (\omega_1, \omega_2, \omega_3) : \mu_1 \neq \omega_1, \mu_2 \neq \omega_2, \mu_3 \neq \omega_3\} \end{aligned} \tag{4.33}$$

Using Equation (4.33), we obtain Γ_7 as Figure 4.23.

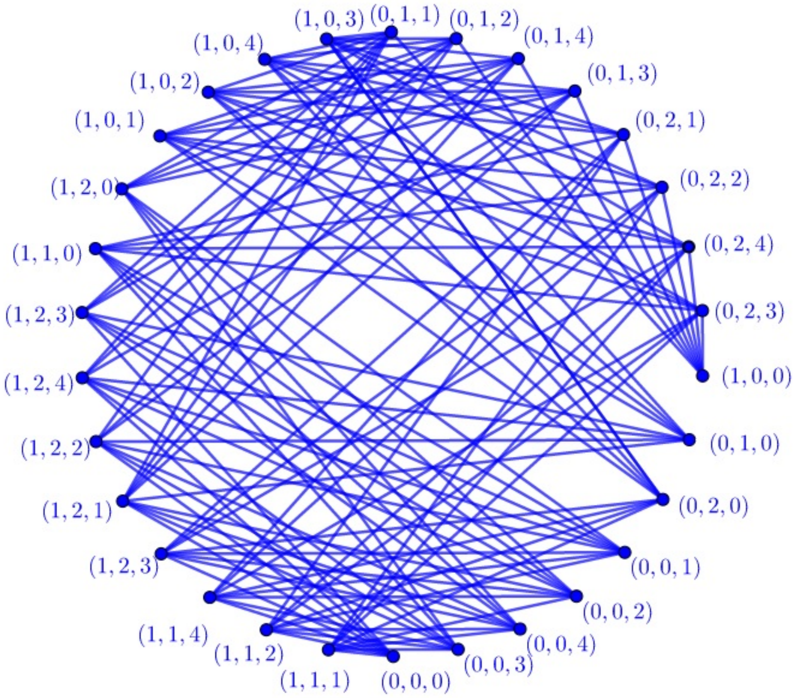


Figure 4.23: Suborbital graph Γ_7 corresponding to the action of $Aff(2) \times Aff(3) \times Aff(5)$ on $GF(2) \times GF(3) \times GF(5)$

Γ_7 is undirected, has girth4, regular of degree 8 thus Eurelian, connected graph of diameter 2.

Example 4.3.3. *Suborbital graphs of $Aff(q_1) \times Aff(q_2) \times Aff(q_3)$ acting on $GF(q_1) \times GF(q_2) \times GF(q_3)$.*

Using equations (3.40), (3.41), (3.42), (3.43), (3.44), (3.45) and (3.46), we have that $(0,0,0) \in \Delta_0, (0,0,\gamma) \in \Delta_1, (0,\beta,0) \in \Delta_2, (\alpha,0,0) \in \Delta_3, (\alpha,\beta,0) \in \Delta_4, (\alpha,0,\gamma) \in \Delta_5, (0,\beta,\gamma) \in \Delta_6$ and $(\alpha,\beta,\gamma) \in \Delta_7$.

Then by application of Equation (4.19), we obtain,

$$\begin{aligned} O_0 =&\{\Big((g_1, g_2, g_3)(0,0,0)), (g_1, g_2, g_3)(0,0,0)\Big) : g_1 \in G_1, g_2 \in G_2, g_3 \in G_3\} \\ &= \{\Big((g_1(0), g_2(0), g_3(0), g_1(0), g_2(0), g_2(0)\Big) : g_1 \in G_1, g_2 \in G_2, g_3 \in G_3\} \\ &= \{((\mu_1, \mu_2, \mu_3), (\omega_1, \omega_2, \omega_3) : \mu_1 = \omega_1, \mu_2 = \omega_2, \mu_3 = \omega_3\}. \end{aligned} \tag{4.34}$$

$$\begin{aligned} O_1 =&\{\Big((g_1, g_2, g_3)(0,0,0)), (g_1, g_2, g_3)(0,0,\gamma)\Big) : g_1 \in G_1, g_2 \in G_2, g_3 \in G_3\} \\ &= \{\Big((g_1(0), g_2(0), g_3(0), g_1(0), g_2(0), g_2(\gamma)\Big) : g_1 \in G_1, g_2 \in G_2, g_3 \in G_3\} \\ &= \{((\mu_1, \mu_2, \mu_3), (\omega_1, \omega_2, \omega_3) : \mu_1 = \omega_1, \mu_2 = \omega_2, \mu_3 \neq \omega_3\}. \end{aligned} \tag{4.35}$$

$$\begin{aligned} O_2 =&\{\Big((g_1, g_2, g_3)(0,0,0)), (g_1, g_2, g_3)(0,\beta,0)\Big) : g_1 \in G_1, g_2 \in G_2, g_3 \in G_3\} \\ &= \{\Big((g_1(0), g_2(0), g_3(0), g_1(0), g_2\beta), g_2(0)\Big) : g_1 \in G_1, g_2 \in G_2, g_3 \in G_3\} \\ &= \{((\mu_1, \mu_2, \mu_3), (\omega_1, \omega_2, \omega_3) : \mu_1 = \omega_1, \mu_2 \neq \omega_2, \mu_3 = \omega_3\}. \end{aligned} \tag{4.36}$$

$$\begin{aligned} O_3 =&\{\Big((g_1, g_2, g_3)(0,0,0), (g_1, g_2, g_3)(\alpha,0,0)\Big) : g_1 \in G_1, g_2 \in G_2, g_3 \in G_3\} \\ &= \{\Big(g_1(0), g_2(0), g_3(0), g_1\alpha, g_2(0), g_2(0)\Big) : g_1 \in G_1, g_2 \in G_2, g_3 \in G_3\} \\ &= \{((\mu_1, \mu_2, \mu_3), (\omega_1, \omega_2, \omega_3) : (\mu_1 \neq \omega_1), \mu_2 = \omega_2, \mu_3 = \omega_3\}. \end{aligned} \tag{4.37}$$

$$\begin{aligned} O_4 =&\{\Big((g_1,g_2,g_3)(0,0,0),(g_1,g_2,g_3)(\alpha,\beta,0)\Big) : g_1 \in G_1, g_2 \in G_2, g_3 \in G_3\} \\ &= \{\Big(g_1(0),g_2(0),g_3(0),g_1\alpha,g_2\beta),g_2(0)\Big) : g_1 \in G_1, g_2 \in G_2, g_3 \in G_3\} \\ &= \{(\mu_1,\mu_2,\mu_3),(\omega_1,\omega_2,\omega_3) : \mu_1 \neq \omega_1, \mu_2 \neq \omega_2, \mu_3 = \omega_3\} \end{aligned} \tag{4.38}$$

$$\begin{aligned} O_5 =&\{\Big((g_1,g_2,g_3)(0,0,0),(g_1,g_2,g_3)(\alpha,0,\gamma)\Big) : g_1 \in G_1, g_2 \in G_2, g_3 \in G_3\} \\ &= \{\Big((g_1(0),g_2(0),g_3(0),g_1\alpha,g_2(0),g_2(\gamma)\Big) : g_1 \in G_1, g_2 \in G_2, g_3 \in G_3\} \\ &= \{(\mu_1,\mu_2,\mu_3),(\omega_1,\omega_2,\omega_3) : (\mu_1 \neq \omega_1), \mu_2 = \omega_2, \mu_3 \neq \omega_3\}. \end{aligned} \tag{4.39}$$

$$\begin{aligned} O_6 =&\{\Big((g_1,g_2,g_3)(0,0,0)),(g_1,g_2,g_3)(0,\beta,\gamma)\Big) : g_1 \in G_1, g_2 \in G_2, g_3 \in G_3\} \\ &= \{\Big((g_1(0),g_2(0),g_3(0),g_1(0),g_2\beta),g_2(\gamma)\Big) : g_1 \in G_1, g_2 \in G_2, g_3 \in G_3\} \\ &= \{((\mu_1,\mu_2,\mu_3),(\omega_1,\omega_2,\omega_3) : \mu_1 = \omega_1, (\mu_2 \neq \omega_2), \mu_3 \neq \omega_3\}. \end{aligned} \tag{4.40}$$

$$\begin{aligned} O_7 =&\{\Big((g_1,g_2,g_3)(0,0,0),(g_1,g_2,g_3)(\alpha,\beta,\gamma)\Big) : g_1 \in G_1, g_2 \in G_2, g_3 \in G_3\} \\ &= \{\Big((g_1(0),g_2(0),g_3(0),g_1\alpha,g_2\beta),g_2(\gamma)\Big) : g_1 \in G_1, g_2 \in G_2, g_3 \in G_3\} \\ &= \{((\mu_1,\mu_2,\mu_3),(\omega_1,\omega_2,\omega_3) : \mu_1 \neq \omega_1, \mu_2 \neq \omega_2, \mu_3 \neq \omega_3\}. \end{aligned} \tag{4.41}$$

Theorem 4.3.4. *All the non trivial graphs corresponding to the action of* $Aff(q_1) \times Aff(q_2) \times Aff(q_3)$ *on* $GF(q_1) \times GF(q_2) \times GF(q_3)$ *are undirected.*

Proof. From Equations (4.34) to (4.41), if $((\mu_1,\mu_2,\mu_3),(\omega_1,\omega_2,\omega_3)) \in O_i$, $(i = 0,1,\ldots,7)$, then $((\omega_1,\omega_2,\omega_3),(\mu_1,\mu_2,\mu_3)) \in O_i$, respectively. □

Theorem 4.3.5. *The suborbital graph* Γ_7 *is Eulerian for odd* q.

Proof. Suppose that q is odd, then $|\Delta_7| = (q-1)^3$ represent the degree of each vertex the graph. Therefore by Theorem 1.2.22, Γ_7 is Eulerian. □

Lemma 4.3.6. *Let* $Comp_{\Gamma_i}(0,0,0), (i = 1,2,3,...,7)$ *represent the vertices of the component having* $(0,0,0)$ *in the suborbital graph* Γ_i*. Then*

$$Comp_{\Gamma_1}(0,0,0) =\{(0,0,u) : u \in GF(q_3)\}, \tag{4.42}$$

$$Comp_{\Gamma_2}(0,0,0) = \{(0,u,0) : u \in GF(q_2)\}, \tag{4.43}$$

$$Comp_{\Gamma_3}(0,0,0) = \{(u,0,0) : u \in GF(q_1)\}, \tag{4.44}$$

$$Comp_{\Gamma_4}(0,0,0) = \{(u,v,0) : u \in GF(q_1), v \in GF(q_2)\}, \tag{4.45}$$

$$Comp_{\Gamma_5}(0,0,0) = \{(u,0,w) : u \in GF(q_1), w \in GF(q_3)\}, \tag{4.46}$$

$$Comp_{\Gamma_6}(0,0,0) =\{(0,v,w) : v \in GF(q_2), w \in GF(q_3)\} \tag{4.47}$$

and

$$Comp_{\Gamma_7}(0,0,0) =GF(q_1) \times GF(q_2) \times GF(q_3). \tag{4.48}$$

Proof. Let $(h,k,l) \in Comp_{\Gamma_1}(0,0,0)$. Then there exist a finite path $(0,0,0) \rightarrow (h_1,k_1,l_1) \rightarrow (h_2,k_2,l_2) \rightarrow \ldots \rightarrow (h_{i-1},k_{i-1},l_{i-1}) = (h,k,l)$. It follows From Equation (4.35), $0 = h_1 = h_2 = \ldots = h$ and $0 = k_1 = k_2 = \ldots = k$. Therefore $(h,k,l) \in \{(0,0,u) : u \in GF(q_3)\}$. This implies that,

$$Comp_{\Gamma_1}(0,0,0) \subseteq \{(0,0,u) : u \in GF(q_3)\} \tag{4.49}$$

Let $(0,0,l) \in \{(0,0,u) : u \in GF(q_3)\}$. If $l = 0$, $(0,0,l) = (0,0,0) \in Comp_{\Gamma_1}(0,0,0)$. If $l \neq 0$, by Equation (4.35), there exist an edge $(0,0,0) \rightarrow (0,0,l)$. Therefore $(0,0,l) \in Comp_{\Gamma_1}(0,0,0)$. This implies that,

$$\{(0,0,u) : u \in GF(q_3)\} \subseteq Comp_{\Gamma_1}(0,0,0). \tag{4.50}$$

From Equations (4.49) and (4.50) we get Equation (4.42). Equation (4.43) and (4.44) are proved in a similar way as Equation (4.42).

Let $(h,k,l) \in Comp_{\Gamma_4}(0,0,0)$. Then there exist a finite path $(0,0,0) \rightarrow (h_1,k_1,l_1) \rightarrow (h_2,k_2,l_2) \ldots \rightarrow (h_{i-1},k_{i-1},l_{i-1}) = (h,k,l)$ of length i. It follows from Equation (4.38) that, $0 = l_1 = l_2 = \ldots = l$. Therefore $(h,k,l) \in \{(u,v,0) : u \in GF(q_1), v \in GF(q_2)\}$. This implies that,

$$Comp_{\Gamma_4}(0,0,0) \subseteq \{(u,v,0) : u \in GF(q_1), v \in GF(q_2)\} \tag{4.51}$$

Let $(u,v,0) \in \{((u,v,0) : u \in GF(q_1), v \in GF(q_2))\}$. Vertex $(h,k,0)$, where $h \in GF(q_1)^\star \setminus \{u\}$ and $k \in GF(q_2)^\star \setminus \{v\}$ is adjacent to both $(0,0,0)$ and $(u,v,0)$.This implies that,

$$\{(u,v,0) : u \in GF(q_1), v \in GF(q_2)\} \subseteq Comp_{\Gamma_4}(0,0,0). \tag{4.52}$$

From Equations (4.51) and (4.52) we get Equation (4.45). Equation (4.46) and (4.47) are proved in a similar way as Equation (4.45).

Let $(u, v, w) \in GF(q_1) \times GF(q_2)) \times GF(q_3)$ space. Consider the vertex (h, k, l), where $u \neq h \in GF(q_1)^\star \setminus \{h\}$ space , $v \neq k \in GF(q_2)^\star \setminus \{k\}$ space and $w \neq l \in GF(q_3)^\star \setminus \{l\}$ space . By Equation (4.48), (h, k, l) is adjacent to both $(0, 0, 0)$ and (u, v, w). This implies that $(u, v, w) \in Comp_{\Gamma_7}(0, 0, 0)$, hence $GF(q_1) \times GF(q_2) \times GF(q_3) \subseteq Comp_{\Gamma_7}(0, 0, 0)$. But $GF(q_1) \times GF(q_2) \times GF(q_3)$ is the set of all the vertices in Γ_7, thus $Comp_{\Gamma_7}(0, 0, 0) \subseteq GF(q_1) \times GF(q_2) \times GF(q_3)$. Thus we get Equation (4.48). □

Theorem 4.3.7. *Each of the suborbital graphs* Γ_1, Γ_2 *to* Γ_7*, has* $q_1q_2, q_1q_3, q_2q_3, q_3, q_2, q_1$ *and* 1 *components respectively,where* $q_1, q_2, q_3 > 2$*.*

Proof. By Theorem 1.2.23, all the components in a suborbital graph are isomorphic to one another. Applying Lemma 4.3.6, we have

$|Comp_{\Gamma_1}(0, 0, 0)| = q_3, \qquad |Comp_{\Gamma_2}(0, 0, 0)| = q_2.$

$|Comp_{\Gamma_3}(0, 0, 0)| = q_1, \qquad |Comp_{\Gamma_4}(0, 0, 0)| = q_1q_2.$

$|Comp_{\Gamma_5}(0, 0, 0)| = q_1q_3, \qquad |Comp_{\Gamma_6}(0, 0, 0)| = q_2q_3$

and $|Comp_{\Gamma_7}(0, 0, 0)| = q_1q_2q_3$. Since each of these graphs have $q_1q_2q_3$ vertices, the results follows. When $q_1 = q_2 = q_3 = q = 2$, then there are q^2 components in each suborbital graph which are all isomorphic to one another. □

Lemma 4.3.8. *For the action of* $Aff(2) \times Aff(q_2) \times Aff(q_3)$ *acting on* $GF(2) \times GF(q_2) \times GF(q_3)$*, where* $q_2, q_3 \geqslant 3$*,* Γ_7 *is bipartite.*

Proof. Let $V(\Gamma_7)$ be the vertices corresponding to Γ_7,then the vertices of Γ_7 can be partitioned into two subsets V_1 and V_2 such that, $V_1 = \{(0, k, l) : k \in GF(q_2), l \in$

$GF(q_3)\}$ and $V_2 = \{(1, m, n) : m \in GF(q_2), n \in GF(q_3)\}$. Thus from Equation (4.41), there exist an edge in Γ_7 moving from each vertex V_1 to V_2 and no edge in Γ_7 joining vertices in the same subset. Hence from Definition 1.2.18, Γ_7 is bipartite. $\square$

Corollary 4.3.9. *The chromatic number of* Γ_7 *of* $Aff(2)\times Aff(q_2)\times Aff(q_3)$ *acting on* $GF(2) \times GF(q_2) \times GF(q_3)$*, where* $q_2, q_3 \geqslant 3$ *is* 2.

Proof. By Lemma 4.2.9, Γ_7 is bipartite and using Definition 1.2.18, chromatic number of a bipartite graph is 2. $\square$

Theorem 4.3.10. Γ_7 *corresponding to the action of* $Aff(2) \times Aff(q_2) \times Aff(q_3)$ *on* $GF(2) \times GF(q_2) \times GF(q_3)$*, where* $q_2, q_3 \geqslant 3$ *has girth* 4.

Proof. Since Γ_7 is bipartite, all circuits are of even lengths. Suppose elements $h \in GF(q_2)$, and $k \in GF(q_3)$ are primitive elements. If $q_2, q_3 \geqslant 3$, then, by Equation (4.41), $(0, 0, 0) \to (h, k, l) \to (h^2, k^2, l^2) \to (h^3, k^3, l^3) \to (0, 0, 0) \in \Gamma_7$ is a circuit in Γ_7 with a girth of 4. $\square$

Theorem 4.3.11. *The suborbital graph* $\Gamma_1, \Gamma_2, ..., \Gamma_6$ *corresponding to the action of* $Aff(q_1) \times Aff(q_2) \times Aff(q_3)$ *on* $GF(q_1) \times GF(q_2) \times GF(q_3)$*, has a girth of* 0 *when* $q_1 = q_2 = q_3 = 2$ *and a girth of* 3 *when* $q_1 = q_2 = q_3 > 2$.

Proof. Suppose that $q_1 = q_2 = q_3 = 2$, then there does not exist a path that connects the vertices of the graph and hence a girth of zero.

If $q_1 = q_2 = q_3 > 2$, let the elements $h \in GF(q_1), k \in GF(q_2)$ and $l \in GF(q_3)$. By Equations (4.35) and (4.40), we obtain,

$(0, 0, 0) \to (0, 0, l) \to (0, 0, l^2) \to (0, 0, 0) \in \Gamma_1$,

$(0,0,0) \to (0,k,0) \to (0,k^2,0) \to (0,0,0) \in \Gamma_2$,

$(0,0,0) \to (h,k,0) \to (h^2,k^2,0) \to (0,0,0) \in \Gamma_3$,

$(0,0,0) \to (h,0,0) \to (h^2,0,0) \to (0,0,0) \in \Gamma_4$,

$(0,0,0) \to (h,0,l) \to (h^2,0,l^2) \to (0,0,0) \in \Gamma_5$,

$(0,0,0) \to (0,k,l) \to (0,k^2,l^2) \to (0,0,0) \in \Gamma_6$,

$(0,0,0) \to (h,k,l) \to (h^2,k^2,l^2) \to (0,0,0) \in \Gamma_7$. Thus the girth is 3. □

Corollary 4.3.12. *The action of* $Aff(q_1)\times Aff(q_2)\times Aff(q_3)$ *on* $GF(q_1)\times GF(q_2)\times GF(q_3)$ *is imprimitive.*

Proof. From the Theorem 4.3.7, there are some non-trivial suborbital graphs which are disconnected. Thus from Theorem 1.2.28, the action is imprimitive. □

CHAPTER 5

CYCLE INDEX FORMULAE OF $Aff(q_1) \times Aff(q_2)$ ON $GF(q_1) \times GF(q_2)$

5.1 Introduction

This chapter investigates the properties of the product action of $Aff(q_1) \times Aff(q_2)$ on $GF(q_1) \times GF(q_2)$,where $q = p^r$, by deriving the cycle index formula.

5.2 Cycle index formula of $Aff(q_1) \times Aff(q_2)$ on $GF(q_1) \times GF(q_2)$

The cycle index formula of affine groups $Aff(q_1) \times Aff(q_2)$ on $GF(q_1) \times GF(q_2)$ is derived as shown below.

Theorem 5.2.1. *The cycle index formula of $Aff(q_1) \times Aff(q_2)$ on $GF(q_1) \times GF(q_2)$ is,*

$$
\begin{aligned}
Z(G) = Z(G_1) \circ Z(G_2) \quad = \quad & \frac{1}{q_1 q_2 (q_1 - 1)(q_2 - 1)} \Bigg[t_1^{q_1 q_2} + (q_1 - 1) t_{p_1}^{\frac{q_1 q_2}{p_1}} + (q_2 - 1) t_{p_2}^{\frac{q_1 q_2}{p_2}} \\
& + q_1 \sum_{(1 \neq d_1 | q_1 - 1)} \phi(d_1) t_1^{q_2} t_{d_1}^{\frac{q_2(q_1 - 1)}{d_1}} + q_2 \sum_{(1 \neq d_2 | q_2 - 1)} \phi(d_2) t_1^{q_1} t_{d_2}^{\frac{q_1(q_2 - 1)}{d_2}} \\
& + (q_1 - 1)(q_2 - 1) t_{p_1 p_2}^{\frac{q_1 q_2}{p_1 p_2}} \\
& + q_1 (q_2 - 1) \sum_{(1 \neq d_1 | q_1 - 1)} \phi(d_1) t_{p_2}^{\frac{q_2}{p_2}} t_{p_2 d_1}^{\frac{q_2(q_1 - 1)}{p_2 d_1}} \\
& + q_2 (q_1 - 1) \sum_{(1 \neq d_2 | q_2 - 1)} \phi(d_2) t_{p_1}^{\frac{q_1}{p_1}} t_{p_1 d_2}^{\frac{q_1(q_2 - 1)}{p_1 d_2}} \\
& + q_1 q_2 \sum_{(1 \neq d_1 | q_1 - 1, 1 \neq d_2 | q_2 - 1)} \phi(d_1) \phi(d_2) t_1^1 t_{d_1}^{\frac{q_1 - 1}{d_1}} t_{d_2}^{\frac{q_2 - 1}{d_2}} t_{d_1 d_2}^{\frac{(q_1 - 1)(q_2 - 1)}{d_1 d_2}} \Bigg].
\end{aligned}
$$

Proof. From Theorem 1.2.27 and Theorem 1.2.29, that the cycle index of $Aff(q_1) \times Aff(q_2)$ acting on $GF(q_1) \times GF(q_2)$ is,

$Z(G) = Z(G_1) \circ Z(G_2)$. Since,

$$Z(G_1) = \frac{1}{q_1(q_1-1)}[t_1^{q_1} + (q_1-1)t_{p_1}^{\frac{q_1}{p_1}} + q_1 \sum_{(1\neq d_1|q_1-1)} \phi(d_1)t_1 t_{d_1}^{\frac{q_1-1}{d_1}}] \tag{5.1}$$

$$Z(G_2) = \frac{1}{q_2(q_2-1)}[t_1^{q_2} + (q_2-1)t_{p_2}^{\frac{q_2}{p_2}} + q_2 \sum_{(1\neq d_2|q_2-1)} \phi(d_2)t_1 t_{d_2}^{\frac{q_2-1}{d_2}}]. \tag{5.2}$$

and

$$\begin{aligned}
Z(G_1)\circ Z(G_2) &= \frac{1}{q_1q_2(q_1-1)(q_2-1)}\left[t_1^{q_1} \circ \left(t_1^{q_2} + (q_2-1)t_{p_2}^{\frac{q_2}{p_2}}\right.\right. \\
&\quad \left. +q_2 \sum_{(1\neq d_2|q_2-1)} \phi(d_2)t_1^1 t_{d_2}^{\frac{q_2-1}{d_2}}\right) + (q_1-1)t_{p_1}^{\frac{q_1}{p_1}} \circ \left(t_1^{q_2} + (q_2-1)t_{p_2}^{\frac{q_2}{p_2}}\right. \\
&\quad \left. +q_2 \sum_{(1\neq d_2|q_2-1)} \phi(d_2)t_1^1 t_{d_2}^{\frac{q_2-1}{d_2}}\right) \\
&\quad +q_1 \sum_{(1\neq d_1|q_1-1)} \phi(d_1)t_1^1 t_{d_1}^{\frac{q_1-1}{d_1}} \circ \left(t_1^{q_2} + (q_2-1)t_{p_2}^{\frac{q_2}{p_2}}\right. \\
&\quad \left.\left. +q_2 \sum_{(1\neq d_2|q_2-1)} \phi(d_2)t_1^1 t_{d_2}^{\frac{q_2-1}{d_2}}\right)\right] \\
&= \frac{1}{q_1q_2(q_1-1)(q_2-1)}\left[(t_1^{q_1} \circ t_1^{q_2}) + (q_2-1)(t_1^{q_1} \circ t_{p_2}^{\frac{q_2}{p_2}})\right. \\
&\quad +q_2 \sum_{(1\neq d_2|q_2-1)} \phi(d_2)(t_1^{q_1} \circ t_1^1 t_{d_2}^{\frac{q_2-1}{d_2}}) + (q_1-1)(t_{p_1}^{\frac{q_1}{p_1}} \circ t_1^{q_2}) \\
&\quad +(q_1-1)(q_2-1)(t_{p_1}^{\frac{q_1}{p_1}} \circ t_{p_2}^{\frac{q_2}{p_2}}) \\
&\quad +q_2(q_1-1) \sum_{(1\neq d_2|q_2-1)} \phi(d_2)(t_{p_1}^{\frac{q_1}{p_1}} \circ t_1^1 t_{d_2}^{\frac{q_2-1}{d_2}}) \\
&\quad +q_1 \sum_{(1\neq d_1|q_1-1)} \phi(d_1)(t_1^1 t_{d_1}^{\frac{q_1-1}{d_1}} \circ t_1^{q_2}) \\
&\quad +q_1(q_2-1) \sum_{(1\neq d_1|q_1-1)} \phi(d_1)(t_1^1 t_{d_1}^{\frac{q_1-1}{d_1}} \circ t_{p_2}^{\frac{q_2}{p_2}}) \\
&\quad \left. +q_1q_2 \sum_{(1\neq d_1|q_1-1, 1\neq d_2|q_2-1)} \phi(d_1)\phi(d_2)(t_1^1 t_{d_1}^{\frac{q_1-1}{d_1}} \circ t_1^1 t_{d_2}^{\frac{q_2-1}{d_2}})\right].
\end{aligned}$$

$$
\begin{aligned}
= \quad & \frac{1}{q_1 q_2 (q_1-1)(q_2-1)} \Big[t_1^{q_1 q_2} + (q_2-1) t_{p_2}^{\frac{q_1 q_2}{p_2}} \\
& + q_2 \sum_{(1 \neq d_2 | q_2-1)} \phi(d_2) t_1^{q_1} t_{d_2}^{\frac{q_1(q_2-1)}{d_2}} \\
& + (q_1-1) t_{p_1}^{\frac{q_1 q_2}{p_1}} + (q_1-1)(q_2-1) t_{p_1 p_2}^{\frac{q_1 q_2}{p_1 p_2}} \\
& + q_2(q_1-1) \sum_{(1 \neq d_2 | q_2-1)} \phi(d_2) t_{p_1}^{\frac{q_1}{p_1}} t_{p_1 d_2}^{\frac{q_1(q_2-1)}{p_1 d_2}} \\
& + q_1 \sum_{(1 \neq d_1 | q_1)} \phi(d_1) t_1^{q_2} t_{d_1}^{\frac{q_2(q_1-1)}{d_1}} \\
& + q_1(q_2-1) \sum_{(1 \neq d_1 | q_1-1)} \phi(d_1) t_{p_2}^{\frac{q_2}{p_2}} t_{p_2 d_1}^{\frac{q_2(q_1-1)}{p_2 d_1}} \\
& + q_1 q_2 \sum_{(1 \neq d_1 | q_1-1, 1 \neq d_2 | q_2-1)} \phi(d_1)\phi(d_2) t_1^1 t_{d_1}^{\frac{q_1-1}{d_1}} t_{d_2}^{\frac{q_2-1}{d_2}} t_{d_1 d_2}^{\frac{(q_1-1)(q_2-1)}{d_1 d_2}} \Big] \\
= \quad & \frac{1}{q_1 q_2 (q_1-1)(q_2-1)} \Big[t_1^{q_1 q_2} + (q_1-1) t_{p_1}^{\frac{q_1 q_2}{p_1}} + (q_2-1) t_{p_2}^{\frac{q_1 q_2}{p_2}} \\
& + q_1 \sum_{(1 \neq d_1 | q_1-1)} \phi(d_1) t_1^{q_1} t_{d_1}^{\frac{q_2(q_1-1)}{d_1}} + q_2 \sum_{(1 \neq d_2 | q_2-1)} \phi(d_2) t_1^{q_1} t_{d_2}^{\frac{q_1(q_2-1)}{d_2}} \\
& + (q_1-1)(q_2-1) t_{p_1 p_2}^{\frac{q_1 q_2}{p_1 p_2}} \\
& + q_1(q_2-1) \sum_{(1 \neq d_1 | q_1-1)} \phi(d_1) t_{p_2}^{\frac{q_2}{p_2}} t_{p_2 d_1}^{\frac{q_2(q_1-1)}{p_2 d_1}} \\
& + q_2(q_1-1) \sum_{(1 \neq d_2 | q_2-1)} \phi(d_2) t_{p_1}^{\frac{q_1}{p_1}} t_{p_1 d_2}^{\frac{q_1(q_2-1)}{p_1 d_2}} \\
& + q_1 q_2 \sum_{(1 \neq d_1 | q_1-1, 1 \neq d_2 | q_2-1)} \phi(d_1)\phi(d_2) t_1^1 t_{d_1}^{\frac{q_1-1}{d_1}} t_{d_2}^{\frac{q_2-1}{d_2}} t_{d_1 d_2}^{\frac{(q_1-1)(q_2-1)}{d_1 d_2}} \Big].
\end{aligned} \tag{5.3}
$$

□

Example 5.2.2. *Cycle index formula of* $Aff(4) \times Aff(5)$ *on* $GF(4) \times GF(5)$*. Now* $q_1 = 4, q_2 = 5, |G| = 240, p_1 = 2,$ *and* $p_2 = 5,$ *values of* d_1 *are* $\{1, 3\}$ *while those of* d_2 *are* $\{1, 2, 4\}$*. Substituting in Theorem 5.2.1, we have,*

$$\begin{aligned} Z(G) = Z(G_1) \circ Z(G_2) &= \frac{1}{240}\left[t_1^{20} + 3t_2^{10} + 4t_5^4 + 4(2)t_1^5t_3^5 + 5[(1)t_1^4t_2^8\right. \\ &\quad +(2)t_1^4t_4^4] + 12t_{10}^2 + 16(2)t_5^1t_{15}^1 + 15[(1)t_2^2t_2^8 + (2)t_2^2t_4^4] \\ &\quad \left.+20[(2)(1)t_1^1t_3^1t_2^2t_6^2 + (2)(2)t_1^1t_3^1t_4^1t_{12}^1]\right]. \\ &= \frac{1}{240}\left[t_1^{20} + 3t_2^{10} + 4t_5^4 + 8t_1^5t_3^5 + 5t_1^4t_2^8 + 10t_1^4t_4^4 + 12t_{10}^2\right. \\ &\quad \left.+32t_5^1t_{15}^1 + 15t_2^2t_2^8 + 30t_2^2t_4^4 + 40t_1^1t_3^1t_2^2t_6^2 + 80t_1^1t_3^1t_4^1t_{12}^1\right]. \\ &= \frac{1}{240}\left[t_1^{20} + 18t_2^{10} + 4t_5^4 + 8t_1^5t_3^5 + 5t_1^4t_2^8 + 10t_1^4t_4^4 + 12t_{10}^2\right. \\ &\quad \left.+32t_5^1t_{15}^1 + 30t_2^2t_4^4 + 40t_1^1t_3^1t_2^2t_6^2 + 80t_1^1t_3^1t_4^1t_{12}^1\right]. \qquad (5.4) \end{aligned}$$

Corollary 5.2.3. *The cycle index formula of* $Aff(q) \times Aff(q)$ *on* $GF(q) \times GF(q)$ *is,*

$$\begin{aligned} Z(G_1) \circ Z(G_2) &= \frac{1}{q^2(q-1)^2}\left[t_1^{q^2} + (q^2-1)t_p^{\frac{q^2}{p}} + 2q \sum_{(1 \neq d_1 | q-1)} \phi(d_1) t_1^q t_{d_1}^{\frac{q(q-1)}{d_1}}\right. \\ &\quad +2q(q-1) \sum_{(1 \neq d_2 | q-1)} \phi(d_2) t_p^{\frac{q}{p}} t_{pd_2}^{\frac{q(q-1)}{pd_2}} \\ &\quad \left.+q^2 \sum_{(1 \neq d_1 | q-1, 1 \neq d_2 | q-1)} [\phi(d_1)][\phi(d_2)] t_1^1 t_{d_1}^{\frac{q-1}{d_1}} t_{d_2}^{\frac{q-1}{d_2}} t_{d_1 d_2}^{\frac{(q-1)^2}{d_1 d_2}}\right]. \end{aligned}$$

Proof. Note that when $q_1 = q_2 = q$ then $p_1 = p_2 = p$ and values of d_1 are equal to values of d_2 . Also, $[p, p] = p$ and $(p, p) = 1$. Using Theorem 5.2.1,we obtain,

$$
\begin{aligned}
Z(G) = Z(G_1) \circ Z(G_2) \quad = \quad & \frac{1}{q^2(q-1)^2} \Bigg[t_1^{q^2} + (q-1)^2 t_p^{\frac{q^2}{p}} \\
& + q \sum_{(1 \neq d_2 | q-1)} \phi(d_2) t_1^q t_{d_2}^{\frac{q(q-1)}{d_2}} \\
& + (q-1) t_p^{\frac{q^2}{p}} + (q-1)^2 t_p^{\frac{q^2}{p}} \\
& + q(q-1) \sum_{(1 \neq d_2 | q-1)} \phi(d_2) t_p^{\frac{q}{p}} t_{pd_2}^{\frac{q(q-1)}{pd_2}} \\
& + q \sum_{(1 \neq d_1 | q-1)} \phi(d_1) t_1^q t_{d_1}^{\frac{q(q-1)}{d_1}} \\
& + q(q-1) \sum_{(1 \neq d_1 | q-1)} \phi(d_1) t_p^{\frac{q}{p}} t_{pd_1}^{\frac{q(q-1)}{pd_1}} \\
& + q^2 \sum_{(1 \neq d_1, d_2 | q-1)} \phi(d_1)\phi(d_2) t_1^1 t_{d_1}^{\frac{q-1}{d_1}} t_{d_2}^{\frac{q-1}{d_2}} t_{[d_1,d_2]}^{\frac{(q-1)^2}{[d_1,d_2]}} \Bigg]
\end{aligned}
\tag{5.5}
$$

$$
\begin{aligned}
= \quad & \frac{1}{q^2(q-1)^2} \Bigg[t_1^{q^2} + (q^2-1) t_p^{\frac{q^2}{p}} + 2q \sum_{(1 \neq d_1 | q-1)} \phi(d_1) t_1^q t_{d_1}^{\frac{q(q-1)}{d_1}} \\
& + 2q(q-1) \sum_{(1 \neq d_2 | q-1)} \phi(d_2) t_p^{\frac{q}{p}} t_{pd_2}^{\frac{q(q-1)}{pd_2}} \\
& + q^2 \sum_{(1 \neq d_1 | q-1, 1 \neq d_2 | q-1)} [\phi(d_1)][\phi(d_2)] t_1^1 t_{d_1}^{\frac{q-1}{d_1}} t_{d_2}^{\frac{q-1}{d_2}} t_{d_1 d_2}^{\frac{(q-1)^2}{d_1 d_2}} \Bigg] .
\end{aligned}
\tag{5.6}
$$

□

Example 5.2.4. *Cycle index formula of $Aff(5) \times Aff(5)$ on $GF(5) \times GF(5)$.*

Now $q = 5, |G| = 400$ and $p = 5$ values of d_1 is $\{1, 2, 4\}$ while those of d_2 are $\{1, 2, 4\}$.

Substituting in Theorem 5.2.1, we have,

$$\begin{aligned}
Z(G_1) \circ Z(G_2) &= \frac{1}{400}\left[t_1^{25} + 24t_5^5 + 10[(1)t_1^5t_2^{10} + (2)t_1^5t_4^5] + 40[(1)t_5^1t_{10}^2 + (2)t_5^1t_{10}^2]\right. \\
&\quad \left. +25[(1)(1)t_1^1t_2^2t_2^2t_2^8 + (2)(2)t_1^1t_4^1t_4^1t_4^4 + (1)(2)t_1^1t_2^2t_4^1t_4^4]\right]. \\
&= \frac{1}{400}\left[t_1^{25} + 24t_5^5 + 10t_1^5t_2^{10} + 20t_1^5t_4^5 + 40t_5^1t_{10}^2 + 80t_5^1t_{10}^2 + 25t_1^1t_2^2t_2^2t_2^8\right. \\
&\quad \left. +100t_1^1t_4^1t_4^1t_4^4 + 50t_1^1t_2^2t_4^1t_4^4\right]. \\
&= \frac{1}{400}\left[t_1^{25} + 24t_5^5 + 10t_1^5t_2^{10} + 20t_1^5t_4^5 + 40t_5^1t_{10}^2 + 80t_5^1t_{10}^2 + 25t_1^1t_2^{12}\right. \\
&\quad \left. +100t_1^1t_4^6 + 50t_1^1t_2^2t_4^5\right].
\end{aligned}$$

CHAPTER 6

CONCLUSION AND RECOMMENDATIONS

6.1 Conclusion

The action of $Aff(q_1) \times Aff(q_2)$ on $GF(q_1) \times GF(q_2)$ and $Aff(q_1) \times Aff(q_2) \times Aff(q_3)$ on $GF(q_1) \times GF(q_2) \times GF(q_3)$ has been investigated,the corresponding graphs constructed and cycle index determined. The results of this study are outlined in chapters 3, 4 and 5. In chapter 3,the transitivity of the product action of $Aff(q_1) \times Aff(q_2)$ on $GF(q_1) \times GF(q_2)$ and $Aff(q_1) \times Aff(q_2) \times Aff(q_3)$ on $GF(q_1) \times GF(q_2) \times GF(q_3)$ was determined and all the actions were transitive. The rank of $Aff(q_1) \times Aff(q_2)$ on $GF(q_1) \times GF(q_2)$ was also found to be 4 while the subdegrees were found to be $1, (q_1 - 1), (q_2 - 1)$ and $(q_1 - 1)(q_2 - 1)$.

On the other hand the rank of $Aff(q_1) \times Aff(q_2) \times Aff(q_3)$ on $GF(q_1) \times GF(q_2) \times GF(q_3)$ was found to be 8 while subdegrees were $1, (q_1 - 1), (q_2 - 1), (q_3 - 1), (q_1 - 1)(q_2 - 1), (q_1 - 1)(q_3 - 1), (q_2 - 1)(q_3 - 1)$ and $(q_1 - 1)(q_2 - 1)(q_3 - 1)$. In chapter 4, non-trivial suborbital graphs were constructed and all are undirected graphs. Some were disconnected while others were connected and in particular suborbital graphs Γ_3 and Γ_7 are connected when $q > 3$. In chapter 5, we derived the cycle index formula for the direct product of $Aff(q_1) \times Aff(q_2)$ on $GF(q_1) \times GF(q_2)$ and the result given as in Theorems 5.2.1. It was further simplified for the case when $q_1 = q_2$ as given in Corollary 5.2.3.

6.2 Recommendation

Having considered the product action of $Aff(q_1) \times Aff(q_2)$ on $GF(q_1) \times GF(q_2)$ and $Aff(q_1) \times Aff(q_2) \times Aff(q_3)$ on $GF(q_1) \times GF(q_2) \times GF(q_3)$ one can extend by considering the product action of $Aff(q_1) \times Aff(q_2) \times, \ldots, \times Aff(q_i)$ on $GF(q_1) \times GF(q_2) \times, \ldots, \times GF(q_i)$, where $q = p^r$. One can also determine the cycle index formula of $Aff(q_1) \times Aff(q_2) \times Aff(q_3)$ on $GF(q_1) \times GF(q_2) \times GF(q_3)$.

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