SUBDEGREES AND SUBORBITAL GRAPHS OF SYMMETRY GROUPS OF PLATONIC SOLIDS ACTING ON THE EDGES OF THE RESPECTIVE SOLIDS

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A research project submitted in partial fulfillment of the requirements for the award of the degree of Masters of Science (Pure Mathematics) in the School of Pure and Applied Sciences, Kenyatta University

March 2021

DECLARATION

Declaration by the Candidate

This project is my original work and has not been published or presented for a degree award in any other university or any other award.

Signature.............................

Date.....................................

John Kiprop Tanui

I56/37476/2017

Declaration by the Supervisor

This project has been submitted for examination with my approval as the University supervisor.

Signature................................

Date.......................................

Prof. Ireri Kamuti

Mathematics Department

Kenyatta University,

DEDICATION

This work is dedicated to the community of researchers, in the field of combinatorics. It is my hope that this work will be used in improving understanding on the fascinating platonic solids.

ACKNOWLEDGEMENT

I sincerely acknowledge my friends and family for their unrelenting support during the course of my studies. I am also greatly indebted to my supervisor, Professor Ireri Kamuti, who tirelessly supported and guided me through this research. His insights, encouragement and availability for consultations have enabled me to not only complete my research, but also acquire valuable intellectual values which I will strive to maintain in the future. I would also like to extend my gratitude to the Chairperson of the Mathematics Department, together with other lecturers in the department who have rightly imparted knowledge on me.

Above all, I thank God for giving me strength, endurance, encouragement and provision during the entire period of my studies.

ABSTRACT

The action of the symmetry groups of platonic solids on the edges of the respective solids is studied. The symmetry groups of platonic solids have previously been studied by Benson and Grove (1971) and Mokami (2011). In particular, the cycle indices of the action have been investigated. In this research, we employ other methods or otherwise quote these findings for completeness purposes. However, the corresponding ranks, subdegrees and suborbital graphs have not been investigated. Thus, this project deals with the computation of the ranks, subdegrees and the construction of the corresponding suborbital graphs and their properties. The ranks and subdegrees are determined using algebraic concepts such as the Burnside's lemma and the stabilizer of the edges. The corresponding suborbital graphs are constructed using methods developed by Sims (1967), while the properties are determined using concepts from Graph Theory, such as determining the number of graphs using the rank computed, computing the girth of the graphs and determining whether the graphs are connected or disconnected. In particular, using the rank obtained and the suborbitals corresponding to the action on each solid, the suborbital graphs are constructed and their properties analyzed. The ranks of the tetrahedron, cube, octahedron, dodecahedron and icosahedron have been found to be 4, 7, 7, 16 and 16 respectively. The subdegrees of the tetrahedron have been found to be 1,1,2,2 while those of the cube and octahedron have been found to be 1,1,2,2,2,2,2. The subdegrees of the dodecahedron and icosahedron have been found to be 1,1,2,2,2,2,2,2,2,2,2,2,2,2,2,2. It has been found that the number of self-paired graphs of the tetrahedron, cube, octahedron, dodecahedron and icosahedron have been found to be 2, 5, 5, 8 and 8 respectively. The directed suborbital graphs of the tetrahedron, cube and octahedron have a girth of three. Also, based on concepts developed by Sims (1967), the disconnected graphs on all solids show that the action is imprimitive.

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NOTATIONS AND ABBREVIATIONS

Γ - The suborbital graph corresponding to the suborbit Δ

$Stab_D(y)$ or D_y – Stabilizer of a point y in D

{1, 2, ..., r} - Unordered r-element subset

Δ - Suborbit of H on Y

O - Suborbital of H on $Y \times Y$

$|A_n|$ - number of elements in the alternative group of degree n.

$\emptyset(t)$ - the number of positive integers $\leq t$ and relatively prime to t

$|Fix(h)|$ -Number of elements fixed by $h \epsilon D$

1. INTRODUCTION

1.1 Background information

The Platonic solids are regular, convex polyhedral where the faces of each solid is congruent and regular. There are only five platonic solids; the cube, tetrahedron, octahedron, dodecahedron and icosahedron.

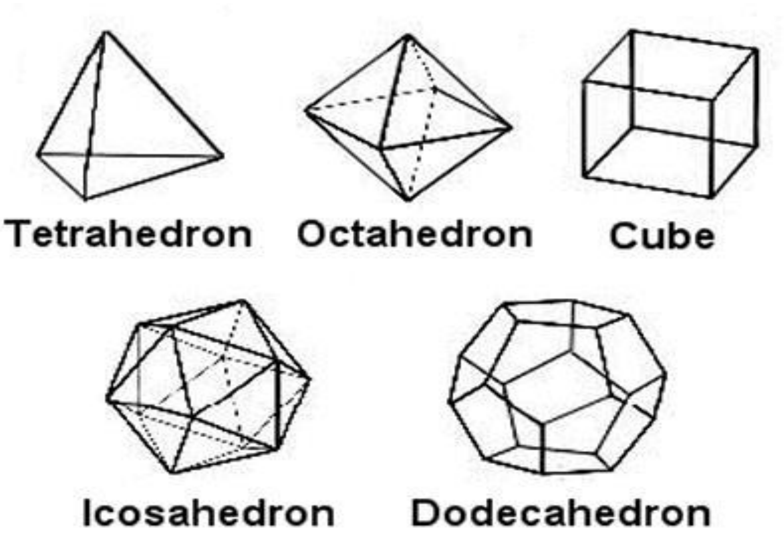

Figure 1.1: The Platonic Solids

Name	Number of Edges	Number of Vertices	Number of Faces
Tetrahedron	6	4	4
Cube	12	8	6
Octahedron	12	6	8
Dodecahedron	30	20	12
Icosahedron	30	12	20

Table 1.1: Properties of Platonic Solids

Platonic solids have been studied for thousands of years. The physical applications of these solids are also an ancient pursuit. The solids were named after Plato who at around 400 B.C was so moved by the perfect form of the solids that in his dialogue, Timaeus, created associations between the five solids and what at the time was thought to be the basic elements of the world: fire, water, ether, earth and air. Later, Euclid in his book, the Elements, constructed the solids such as to circumscribe each one by a sphere. About 300 hundred years later, William Hamilton pioneered work linking Euclidean geometry, on the solids, to modern non-commutative algebra. Shortly after, Felix Klein fully explained the connection in terms of the rotation groups of the solids, where each is a subgroup of the rotation group of the sphere (Smith, 2012). Around 1912, Henri Poincare used the cube and octahedron to introduce and explain basic concepts of algebraic topology. The works of other individuals are subsequent developments or related to the works of Euclid, Hamilton and Poincare.

In this project, we compute the ranks, subdegrees and construct the suborbital graphs and of the action of symmetry groups of platonic solids on the edges of the solids. We also determine the properties of the suborbital graphs.

1.2 Statement of the problem

The symmetry groups of platonic solids have been studied by several authors since their discovery by Plato. The action of the symmetry groups of platonic solids on the edges, vertices and faces of the solids has been studied by Benson and Grove (1971). However, the ranks, subdegrees and suborbital graphs of the action of the symmetry groups of platonic solids on the edges, vertices and faces of the solids has not been investigated. Also, the properties of the suborbital graphs have not been investigated. In this research project, we investigate the ranks, subdegrees and suborbital graphs of the action of the symmetry groups of platonic solids on the edges of the solids. Also, we will determine the properties of the suborbital graphs.

1.3 Objectives of the research study

1.3.1 General objective

The general objective of this research is to determine the ranks, suborbits, subdegrees and suborbital graphs of the action of the symmetry groups of platonic solids acting on the edges of the solids.

1.3.2 Specific objectives

The specific objectives are:

(i) To determine the stabilizer and transitivity of the action of the symmetry groups of platonic solids on the edges of the solids.

(ii) To determine the primitivity of the action of the symmetry groups of platonic solids on the edges of the solids.

(iii) To determine the ranks, suborbits and subdegrees of the action of the symmetry groups of platonic solids on the edges of the solids.

(iv) To determine the corresponding suborbital graphs of the action of the symmetry groups of platonic solids on the edges of the solids and their properties.

1.4 Significance of the study

Through this research, we anticipate to come up with knowledge which is valuable to researchers in the field of combinatorics by advancing knowledge on the action of symmetry groups of platonic solids. This knowledge has several real-world applications - for instance, the symmetries are used in chemistry to study the minimizing of energy of point particles (Atiyah & Sutcliffe, 2003).

1.5 Definition of terms

Definition 1.5.1

If Y is a finite set; say $Y = \{1, 2, . . ., n\}$ then the group of all permutations of Y is called the symmetric group on n elements (or of degree n) denoted by S_n and is of order $n!$ .The symmetry group of a geometric object is the group of all transformations under which the object is invariant, endowed with the group operation of composition.

Definition 1.5.2

A permutation of a finite set is either even or odd based on whether it can be expressed as the product of an even or odd number of 2-cycles (transpositions).

Definition 1.5.3

A regular solid in $\mathbb{R}^3$ is called a platonic solid.

Definition 1.5.4

Let Y be a non-empty set and D, be a group. It is said that D acts on the left of Y if for each $y \in Y$ and $d \in D$, there corresponds a unique element $dy \in Y$ such that for all $y \in Y$ and $d_1, d_2 \in D$

$(d_1 d_2) y = d_1 (d_2 y)$.

$1.y = y$, where 1 is the identity of D.

Definition 1.5.5

If a finite group H acts on a set Y containing r elements, for each $h \epsilon H$ there corresponds a permutation α of Y, which can be expressed as a product of disjoint cycles. If α contains α_1 cycles of length 1, α_2 of length 2, …., α_n of length n, it is said that α and hence h is of cycle type $(\alpha_1, \alpha_2, \ldots, \alpha_n)$.

Definition 1.5.6

A permutation group G acting on a set X is said to be transitive if for all $x, y \in X$ there exists an element $g \in G$ such that $gx = y$. Alternatively, if the action of the group G on the set X has one orbit, then it is said that G acts transitively on X.

Definition 1.5.7

The orbit of G containing a point $y \in Y$ is the set $Orb_G(y) = \{gy: g \in G\}$.

Definition 1.5.8

Let G act on a set X, then the stabilizer G_y of a point $y \epsilon X$ consists of entirely those elements $h \epsilon G$ for which $hy = y$, that is $G_y = \{h \epsilon G: hy = y\}$.

Definition 1.5.9

Suppose a group D acts transitively on the finite set Y, for each subset T of Y and each $d \in D$, let $dT = \{dt: t \in T\} \subseteq Y$. A subset T of Y is said to be a block for the action if, for each $d \in D$, either $dT = T$ or $dT \cap T = \emptyset$. Particularly, $\emptyset$, Y and all 1-element subsets of Y are blocks. They are called trivial blocks. The action is said to be primitive if there are only trivial blocks.

Definition 1.5.10

Let H be transitive on a set Y and let Hy be the stabilizer in H of a point $y \in Y$. The orbits $\Delta_0 = \{y\}$, $\Delta_1, \cdots, \Delta_{r-1}$ of Hy on Y are called the suborbits of H. The rank of H in this case is r.

Definition 1.5.11

The sizes $n_i = |\Delta_i|$, ($i = 0, 1, 2, \ldots., r -1$) often called the 'lengths' of the suborbits are known as subdegrees of H.

Definition 1.5.12

A proper subgroup H of D is called a maximal subgroup of D if there does not exist any subgroup F such that $H<F<D$. If D is a finite group, then D contains at least one maximal subgroup. Any subgroup of a finite group is contained in some maximal subgroup.

Definition 1.5.13

A graph is a diagram made of a set V, whose elements are called points, nodes or vertices and a set E of unordered pairs of vertices known as the edges or lines.

Such a graph is denoted by $G (V, E)$ or G if no ambiguity of V or E exists. From the definition, a graph is finite and does not have loops or multiple edges.

Definition 1.5.14

A graph $D (V', E')$ is a subgraph of $G (V, E)$ if $V' \subseteq V$ and $E' \subseteq E$. D is a spanning subgraph of G if it is a subgraph of G and $V' = V$.

Definition 1.5.15

Two vertices of a graph r and y are said to be adjacent if there is an edge joining them. They are denoted as $\{r, y\}$ or ry.

Definition 1.5.16

A vertex r of a graph is said to be incident to an edge $t = \{r, b\}$ if r is an end vertex of t. Two edges t and t' are said to be incident if they share an end vertex. Otherwise, t and t' are non-incident.

Definition 1.5.17

A trivial graph is one where the vertex set contains only one vertex.

Definition 1.5.18

A multigraph is a graph which has multiple edges, joining two distinct vertices, but no loops, which are edges joining vertices to themselves. If both loops and multiple edges are allowed, then we have a pseudograph.

Definition 1.5.19

A simple graph is a graph with no loops or multiple edges.

Definition 1.5.20

For a graph $G= G(V, E)$, a walk is defined as a sequence of alternating vertices and edges such as $v_0, e_1, e_2, \ldots, e_k, v_k$ where each edge $e_i=\{v_{i-1}, v_i\}$. The length of this walk is k. A trail is defined as a walk with no repeated edges. A path is defined as an open trail with no repeated vertices. A cycle is defined as a closed trail where no other vertices are repeated apart from the start/end vertex.

The girth of a graph is the length of the shortest cycle contained in the graph. The diameter of a connected graph is the maximum eccentricity of a vertex contained in the graph. The valency $d_G (v)$ or degree of a vertex v contained in a graph $G (V, E)$ refers to the number of edges incident to v or the number of vertices of G adjacent to v.

Definition 1.5.21

A regular graph is a graph where each vertex has the same number of neighboring vertices.

Definition 1.5.22

A graph $G (V, E)$ is connected if every pair of vertices of G is joined by a path otherwise, G is disconnected.

Definition 1.5.23

The compliment $\bar{G}(V, E)$ of the graph $G (V, E)$ is a graph with $V(G)$ as its vertex set. Two vertices are adjacent in $\bar{G}(V, E)$ if and only if they are not adjacent in $G (V, E)$. Hence, $G (V, E)$ and $\bar{G}(V, E)$ are complimentary graphs.

Definition 1.5.24

A directed graph, also called a digraph, is made up of a finite non-empty vertex set $V(G)$ and a prescribed collection F of ordered pairs of distinct vertices. The elements belonging to F are directed edges. A directed graph is called strongly connected if and only if there is a path from u to v and from v to u whenever u and v are vertices in the graph. A directed graph is called weakly connected if and only if there is a path between any two vertices in the underlying undirected graph.

Definition 1.5.25

Two graphs having the same number of graph vertices which are connected in the same way are isomorphic.

Definition 1.5.26

Let $(H_{1,} Y_1)$ and $(H_{2,} Y_2)$ be finite permutation groups (i.e. H_i acts on Y_i, $i = 1,2$). To say that $(H_{1,} Y_1) \cong (H_{2,} Y_2)$ (permutation isomorphism) means that there exists a group isomorphism φ: $H_1 \rightarrow H_2$ and a bijection θ: $Y_1 \rightarrow Y_2$ so that $\theta(hy) = \varphi\mathrm{h}\ (\theta(y))$ for all $h \in H_1$, $y \in Y_1$ or $\theta h = \varphi h\theta$ for all $h \in H_1$.

Definition 1.5.27

Let H be a transitive permutation group acting on a set Y. Then H acts on $Y \times Y$ by $h(x,y) = (hx,hy)$, $h \in H$, $x,y \in Y$.

If $O \subseteq Y \times Y$ is a H-orbit, then for a fixed $x \in Y$,

$\Delta = \{y \in Y\colon (x, y) \in O\}$

is a H_x-orbit. On the other hand, if $\Delta \subseteq Y$ is a H_x -orbit, then $O = \{(hx,hy) | h \in H, y \in \Delta\}$ is a H-orbit on $Y \times Y$. We say Δ corresponds to O. The H_x -orbits on Y are called suborbits and H-orbits on $Y \times Y$ are called suborbitals.

The orbit containing (x,y) is denoted by $O(x,y)$. From $O(x,y)$, we can form a suborbital graph $\Gamma(x,y)$; its vertices are the elements of Y, and there is a directed edge from γ to δ if $(\gamma,\delta) \in O(x,y)$. Clearly, $O(y,x)$ is also a suborbital, and it is either equal or disjoint from $O(x,y)$. In the former case $\Gamma(y,x) = \Gamma(x,y)$ and the graph consists of pairs of oppositely directed edges. It is convenient to replace each such pair by a single undirected edge, so that we have an undirected graph which we call self-paired. In the latter case, $\Gamma(y,x)$ is just $\Gamma(x,y)$ with arrows reversed, and we call $\Gamma(x,y)$ and $\Gamma(y,x)$ paired suborbital graphs. Also, if $x = y$, then $O(x, x) = \{(x,x) : x \in Y\}$ is the diagonal of $Y \times Y$. The corresponding suborbital graph $H(x, x)$, called the trivial suborbital graph, is self-paired and consists of a loop based at each vertex $x \in Y$.

Theorem 1.5.28 (Cromwell, 2002)

The cube, tetrahedron, dodecahedron, icosahedron and octahedron are the only regular solids.

Theorem 1.5.29 (Orbit-Stabilizer Theorem-Rose, 1978)

Let D act on the set Y, and let $y \epsilon Y$. Then,

$|Orb_D(y)| = |D: Stab_D(y)|$

Theorem 1.5.30 (Benson & Grove, 1971)

Let G be a finite group acting on a finite set X, then the number of G- orbits is

$\frac{1}{|G|}\sum_{h \varepsilon G}|fix(h)|$, where $Fix(h) = \{y \epsilon X: hy = y\}$.

Theorem 1.5.31 (Wielandt, 1964)

Let $y \in Y$, $|Y|>1$. A transitive group D on Y is primitive if and only if D_y is a maximal subgroup of D.

Theorem 1.5.32 (Sims, 1967)

Let H be transitive on Y. Then H is primitive if and only if each suborbital graph Γ_x $(x = 1, 2, \ldots, r-1)$ is connected.

Theorem 1.5.33 (Cameron, 1975)

Let *G* be a transitive group on *X*. The number of self-paired suborbits $is\ \frac{1}{|G|}\sum_{g\in G}|Fix\ (g^2)|$

2. LITERATURE REVIEW

2.1 Platonic Solids

The symmetries of objects in Group Theory have some powerful applications in chemistry such as the study of isomer enumeration which uses the Redfield-Polya Theorem (Atiyah & Sutcliffe, 2003). The story of Platonic solids can be traced back to Classical Greek thought of Pythagoras and Euclid. Kepler in 1956 attributed cosmic importance to the Platonic solids. In Kepler's book, Mysterium Cosmographicum, the solar system is presented as a nested version of the Platonic solids with the radius of many concentric spheres intertwined with the orbits of planets (Smith, 2012). Wenninger (1974) established a number of geometric principles, construction and design of complex polyhedral solids. In 1976, Pugh discovered simple visual representations of regular and semi-regular polyhedral solids. In 1991, Holden, formed the structures of the regular polyhedrons. The structure composed of up to 10 linking cubes and the sketch of the structure was based on the isometry of the polyhedrons. Cromwell (2002) proved that the cube, tetrahedron, icosahedron, dodecahedron and octahedron are the only platonic solids. Mokami (2011) investigated the cycle types and cycle indices of the action of the group of symmetries of platonic solids on the edges, vertices and faces of the solids.

2.2 Ranks and Subdegrees

The rank and subdegrees of permutations groups have been investigated by several mathematicians. In the great little monograph authored by Wielandt (1964), finite permutation groups were investigated. Specifically, Wielandt showed that a primitive group of degree 2s, where s is prime has a rank of less than or equal to 3. Higman (1964) defined the rank of a transitive permutation group as the total number of orbits of a stabilizer at a given point. In 1970, Higman characterized the families of rank 3 permutation groups by subdegrees. It was shown that the symmetric group S_h on $R = \{1,2,\ldots, h\}$, $h \geq 4$ acts as a rank 3 group on the set of $\binom{h}{2}$ 2-elements which is a subset of R, with the subdegrees 1, $2(h-2)$, $\binom{h-2}{2}$.

2.3 Suborbital Graphs

The idea of suborbital graphs of a permutation group G acting on a set X, which is possibly directed, with vertex-set X and on which G induces automorphisms, was introduced by Sims (1967). Graphs like the Petersen graph, the Coxeter graph and the Biggs- Smith graph can be seen as suborbital graphs of a group acting on a given set (Neumann 1977; Kamuti 1992; Bon & Cohen, 1989). Suborbital graphs of subgroups of the modular group PSL (2, $\mathbb{Z}$) acting on the rational projective line $\mathbb{Q} = \mathbb{Q} \cup \{\infty\}$ have been studied by several scholars (Keskin & Demirtürk, 2009; Guler et al., 2008; Keskin, 2004; Akbas & Baskan, 1996).

A method of calculating the subdegrees of permutation transitive groups through a table of marks was devised in 1983 (Ivano et al., 1983). Finite permutation groups of small suborbits were investigated by Cai et al. (2004), founded on the classification of Wang (1992) and Quiring (1971), the investigation yielded an exact list of primitive permutation groups with suborbits of length 4. Kamuti (1992) devised a method for constructing some of the suborbital graphs of PSL(2, q) acting on the cosets of its maximal dihedral subgroup of order q - 1. This method gave an alternative way of constructing the Coxeter graph. This is a non-Hamiltonian cubic graph on 28 vertices and 42 edges with girth 7.

Cai et al. (2004) analyzed the orbital graphs of primitive permutation groups with a suborbit of length 3 or of length 4. They obtained a complete classification of vertex-primitive arc-transitive graphs of valency 3 and valency 4, and proved that there exist no vertex-primitive half-arc-transitive graphs of valency less than 10. Finally, they concluded by constructing vertex-primitive half-arc-transitive graphs of valency 2k for infinitely many integers k, with 14 as the smallest valency.

In this project, we intend to compute the ranks, subdegrees and construct the suborbital graphs of the action of the symmetry groups of platonic solids acting on the edges of the solids. We will also investigate the properties of the suborbital graphs. We will mainly use concepts investigated by Mokami (2011) in computing the action of the symmetry groups on the edges while using concepts introduced by Sims (1967), and developed further by the discussed individuals, to compute the subdegrees and suborbital graphs.

3. FINDINGS

Using findings from Benson and Grove (1971), we shall compute the stabilizer of the action of symmetry groups of platonic solids on the edges of the solids. Also, using concepts from Benson and Grove (1971), we shall compute the transitivity and primitivity of the action of platonic solids on the edges of the solids. Consequently, using the Burnside's lemma and the stabilizer of the edges of the solids, we shall compute the ranks and subdegrees of the action of the symmetry groups of platonic solids on the edges of the solids. Also, using Sims (1967) concepts we shall construct the suborbital graphs. Finally, using concepts from Graph Theory, we shall determine the properties of the suborbital graphs.

3.1 TETRAHEDRON

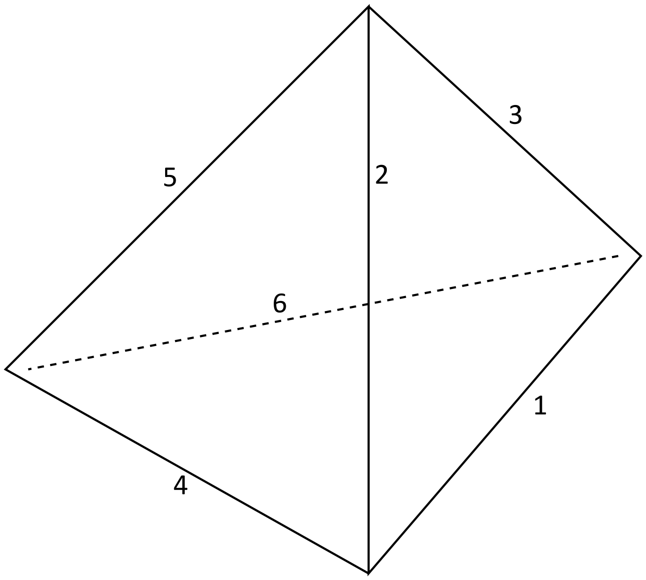

Figure 3.1.1: A tetrahedron with labeled edges

In order to determine the permutations, cycle types, subdegrees and suborbits of the action of the symmetry group of a tetrahedron on its edges, we design a model of a tetrahedron labeled identically to **Figure 3.1.1** which we use to observe the position of each numbered edge during each rigid motion.

3.1.1 Group of symmetries of a tetrahedron

Suppose that a tetrahedron is situated such that its centre is at the origin in $\mathbb{R}^3$. The subgroup of the rotations in $\mathbb{R}^3$ which leave the tetrahedron invariant is denoted by $\daleth$. The elements of $\daleth$ are:

I. Two rotations through angles of 120° and 240° about each of the four axes joining the vertices to the centre of opposite faces.
II. A rotation through the 180° about each of the three axes joining the midpoints of opposite edges.
III. The identity.

Thus, $|\daleth| = 4.2 + 3.1 + 1 = 12$. The group of symmetries of a tetrahedron is isomorphic to A_4.

3.1.2 Cycle type, stabilizer, transitivity and primitivity of A_4 acting on the edges of a tetrahedron

Table 3.1

Rotations	**Permutations**	**Number of Permutations**	**Cycle Type**
Rotations about each of the four axes joining the vertices with centres of opposite faces through 120° and 240°	(1 4 6) (2 5 3)	8	(0,0,2,0,0,0)

Rotation through the 180° about each of the three axes joining the midpoints of opposite edges	(1) (5) (3 4) (2 6)	3	(2,2,0,0,0,0)
The Identity	I	1	(6,0,0,0,0,0)

Theorem 3.1.1

The stabilizer of the action of the symmetry group of a tetrahedron on its edges is {I, 180°}.

Proof: The identity rotation stabilizes all the edges. Moreover, the rotation through 180° about each of the three axis joining the midpoints of opposite edges stabilizes opposite edges of a tetrahedron. Consequently, the stabilizer of the edges of a tetrahedron is {I, 180°}, which is a cyclic group of order 2.

Theorem 3.1.2

The action of the symmetry group of a tetrahedron on the edges of the solid is transitive.

Proof: From **Table 3.1**, we determine the transitivity by first obtaining $|Fix\ (g)|$ for each $g \in A_4$ then using **Theorem 1.5.30** we find the number of orbits when A_4 acts on the edges of the tetrahedron. The equation is given by;

$$\frac{1}{12}(8.0 + 3.2 + 1.6) = 1.$$

Therefore, from **Definition 1.5.6**, A_4 acts transitively on the edges of a tetrahedron.

Alternatively, using **Theorem 1.5.29**, we find that $|A_4 : C_2| = 12/2 = 6$; which is the number of edges of a tetrahedron, hence the action of A_4 on the edges of a tetrahedron is transitive.

3.1.3 Primitivity of A_4 acting on the edges of a tetrahedron

Theorem 3.1.3

The action of the symmetry group of a tetrahedron on its edges is imprimitive.

Proof: The stabilizer of the edges of a tetrahedron is a cyclic group of order 2. From **Definition 1.5.12,** C_2 is not a maximal subgroup of A_4. Hence, from **Theorem 1.5.31,** A_4 acts imprimitively on the edges of a tetrahedron.

3.1.4 Rank and subdegrees of the symmetry group acting on the edges of a tetrahedron

Theorem 3.1.4

The rank of the action of the symmetry group of a tetrahedron on its edges is 4 and the subdegrees are 1, 1, 2, 2.

Proof: Using **Definition 1.5.10** and **Theorem 1.5.30** we can compute the rank of the action of the symmetry group on the edges of a tetrahedron whereby;

$$\frac{1}{2}(6+2) = \frac{8}{2} = 4,$$

where 6 is the number of edges fixed by the identity and 2 is the number of edges fixed by the 180° rotation about the axes joining the midpoints of opposite edges.

Using the rank obtained, the stabilizer, **Definition 1.5.10** and **Definition 1.5.11,** we can obtain the corresponding subdegrees of the action of $\daleth$ on the edges of a tetrahedron. From **Figure 3.1.1** with the axis of rotation joining the midpoints of edges labelled 1 and 5, under the 180° rotation we obtain the following permutation (1) (5) (3 4) (2 6) whereby, the suborbits are $\Delta_0 = \{1\}$, $\Delta_1 = \{5\}$, $\Delta_2 = \{3, 4\}$, $\Delta_3 = \{2, 6\}$ and the corresponding subdegrees are 1, 1, 2, 2.

3.1.5 Suborbital graphs of A_4 acting on the edges of a tetrahedron

After computing the subdegrees, we proceed to analyze the suborbits using **Definition 1.5.27.** The rank is equal to the total number of suborbital graphs, so we need to find 4 suborbitals. See Appendix I for the suborbital graphs.

3.1.6 Properties of the suborbital graphs

3.1.6.1 Properties of graph Γ_1

The graph is directed and strongly connected. The girth is 3 and the diameter is 3.

3.1.6.2 Properties of graph Γ_2

The graph is directed and strongly connected. The girth is 3 and the diameter is 3.

3.1.6.3 Properties of graph Γ_3

The graph is undirected and disconnected. The graph has 3 connected components, the girth is 0.

From the properties of the graphs and **Theorem 1.5.32,** we can verify the previous finding that the action of the symmetry group of a tetrahedron on its edges is imprimitive since Γ_3 is disconnected.

We can also confirm the number of self-paired graphs using **Theorem 1.5.33** and **Table 3.1**.

We have the equation;

$$\frac{1}{12}(6.3 + 6.1) = \frac{24}{12} = 2.$$

where each of the 3 permutations of 180° rotation about the axis joining the midpoints of opposite edges becomes the identity rotation when squared hence fixing 6 edges. The identity rotation fixes 6 edges.

Using **Definition1.5.27** we have the following self-paired graphs; Γ_0 and Γ_3 (see Appendix I).

3.2 CUBE

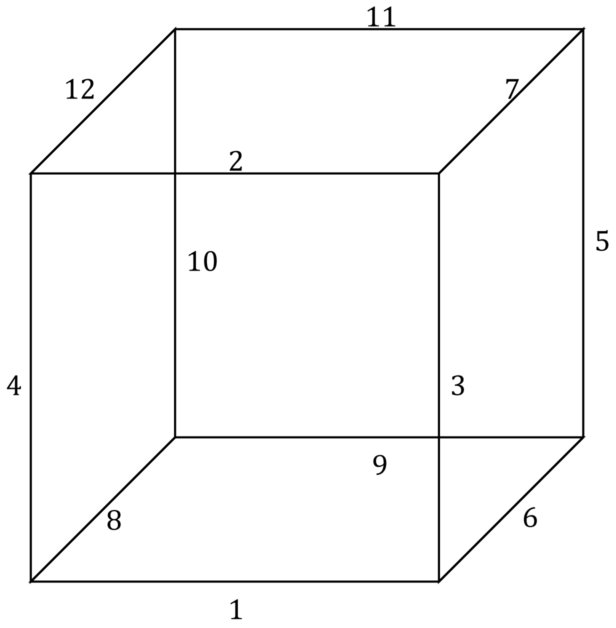

Figure 3.2.1: A cube with labeled edges

In order to determine the permutations, cycle types, subdegrees and suborbits of the action of the symmetry group of a cube on its edges, we design a model of a cube labeled identically to **Figure 3.2.1** that we use to observe the position of each numbered edge during each rigid motion.

3.2.1 Group of symmetries of a cube

Suppose that a cube is situated such that its centre is at the origin in $\mathbb{R}^3$. The subgroup of the rotations in $\mathbb{R}^3$ which leave the cube invariant is denoted by $\daleth$. The elements of $\daleth$ are:

I. Two rotations of 120° and 240° about each of the four axes joining pairs of opposite vertices.

II. A rotation through the 180° about each of the six axes joining the midpoints of pairs of opposite edges.

III. Three rotations through 90°, 270° or 180° about the 3 axes joining the centres of three pairs of opposite faces.

IV. The identity.

Thus, $|\daleth| = 3.3 + 6.1 + 4.2 + 1 = 24$. The group of symmetries of a cube is isomorphic to $S_{4.}$

3.2.2 Cycle type, stabilizer, transitivity and primitivity of S_4 acting on the edges of a cube

Table 3.2

Rotations	**Permutations**	**Number of Permutations**	**Cycle Type**
Rotations about the axes joining three pairs of opposite faces through I. 90° or 270°	(1 8 9 6) (3 4 10 5) (7 2 12 11)	6	(0,0,0,3,0,0,0,0,0,0,0,0)
II. 180°	(1 9) (8 6) (3 10) (4 5) (7 12) (11 2)	3	(0,6,0,0,0,0,0,0,0,0,0,0)
Rotations about the axes joining the centres of 6 pair of opposite edges through 180°	(1) (11) (2 9) (3 8) (4 6) (5 12) (7 10)	6	(2,5,0,0,0,0,0,0,0,0,0,0)
120° and 240° rotations about the axes joining the four points of opposite vertices	(1 4 8) (2 10 6) (3 12 9) (5 7 11)	8	(0,0,4,0,0,0,0,0,0,0,0,0)
Identity	I	1	(12,0,0,0,0,0,0,0,0,0,0,0)

Theorem 3.2.1

The stabilizer of the action of the symmetry group of a cube on its edges is {I, 180°}.

Proof: The identity rotation stabilizes all the edges. Moreover, the rotation through 180° about each of the three axis joining the midpoints of opposite edges stabilizes opposite edges of a cube. Consequently, the stabilizer of the edges of a cube is {I, 180°}, which is a cyclic group of order 2.

Theorem 3.2.2

The action of the symmetry group of a cube on its edges is transitive

Proof: From **Table 3.2**, we determine the transitivity by first obtaining |*Fix (g)*| for each $g \in S_4$ then using **Theorem 1.5.30** we find the number of orbits when S_4 acts on the edges of the cube to be

$$\frac{1}{24}(6.0 + 3.0 + 6.2 + 8.0 + 1.12) = 1.$$

Therefore, from **Definition 1.5.6**, S_4 acts transitively on the edges of a cube.

Alternatively, using **Theorem 1.5.29**, we find that $|S_4 : C_2| = 24/2 = 12$; which is the number of edges of a cube, hence the action of S_4 on the edges of a cube is transitive.

3.2.3 Primitivity of S_4 acting on the edges of a cube

Theorem 3.2.3

The action of the symmetry group of a cube on its edges is imprimitive

Proof: The stabilizer of the action of S_4 on the edges of a cube is a cyclic group of order 2. From **Definition 1.5.12,** C_2 is not a maximal subgroup of S_4. Hence, from **Theorem 1.5.31** S_4 acts imprimitively on the edges of a cube.

3.2.4 Rank and subdegrees of the symmetry group acting on the edges of a cube

Theorem 3.2.4

The rank of the action of the symmetry group of a cube on its edges is 7 and the subdegrees are 1,1,2,2,2,2,2.

Proof: Using **Definition 1.5.10** and **Theorem 1.5.30** we can compute the rank of the action of the symmetry group on the edges of a cube whereby;

$$\frac{1}{2}(12+2) = \frac{14}{2} = 7,$$

where 12 is the number of edges fixed by the identity and 2 is the number of edges fixed by the 180° rotation about the axes joining the midpoints of opposite edges.

Using the rank obtained, the stabilizer, **Definition 1.5.10** and **Definition 1.5.11,** we can obtain the corresponding subdegrees of the action of ד on the edges of a cube. From **Figure 3.2.1** with the axis of rotation joining the midpoints of edges labelled 1 and 11, under the 180° rotation we obtain the following permutation (1) (11) (2 9) (3 8) (4 6) (5 12) (7 10) whereby, the suborbits are $\Delta_0 = \{1\}$, $\Delta_1 = \{11\}$, $\Delta_2 = \{2, 9\}$, $\Delta_3 = \{3, 8\}$, $\Delta_4 = \{4, 6\}$, $\Delta_5 = \{5, 12\}$, $\Delta_6 = \{7, 10\}$ and the corresponding subdegrees are 1,1,2,2,2,2,2.

3.2.5 Suborbital graphs of the symmetry group acting on the edges of a cube

After computing the subdegrees, we proceed to analyze the suborbits using **Definition 1.5.27**. The rank is equal to the total number of suborbital graphs, so we need to find 7 suborbitals. See the Appendix II for the suborbital graphs.

3.2.6 Properties of the suborbital graphs

3.2.6.1 Properties of graph Γ_1

The graph is undirected and disconnected. The graph has 3 connected components and its girth is 4.

3.2.6.2 Properties of graph Γ_2

The graph is directed and strongly connected. The girth is 3 and the diameter is 4.

3.2.6.3 Properties of graph Γ_3

The graph is directed and strongly connected. The girth is 3 and the diameter is 4.

3.2.6.4 Properties of graph Γ_4

The graph is undirected and disconnected. The graph has 4 connected components and its girth is 3.

3.2.6.5 Properties of graph Γ_5

The graph is undirected and disconnected. The graph has 4 connected components and its girth is 3.

3.2.6.6 Properties of graph Γ_6

The graph is undirected and disconnected. The graph has 6 connected components and its girth is 0.

From the properties of the graphs and **Theorem 1.5.32,** we can verify our previous finding that the action of the symmetry group of a cube acting on its edges is imprimitive since Γ_6, Γ_5 and Γ_4 are disconnected.

We can also confirm the number of self-paired graphs using **Theorem 1.5.33** and **Table 3.2**. We have;

$$\frac{1}{24}(12.3 + 12.6 + 12.1) = \frac{120}{24} = 5,$$

where each of the 3 permutations of 180° rotation about the axis joining the centres of opposite faces becomes the identity rotation when squared fixing 12 edges. Also, each of the 6 permutations of 180° rotation about the axis joining centres of opposite edges becomes the identity rotation when squared fixing 12 edges. Finally, the identity rotation fixes 12 edges.

Using **Definition1.5.27** we have the following self-paired graphs; Γ_0, Γ_1, Γ_4, Γ_5 and Γ_6 (See Appendix II).

3.3 OCTAHEDRON

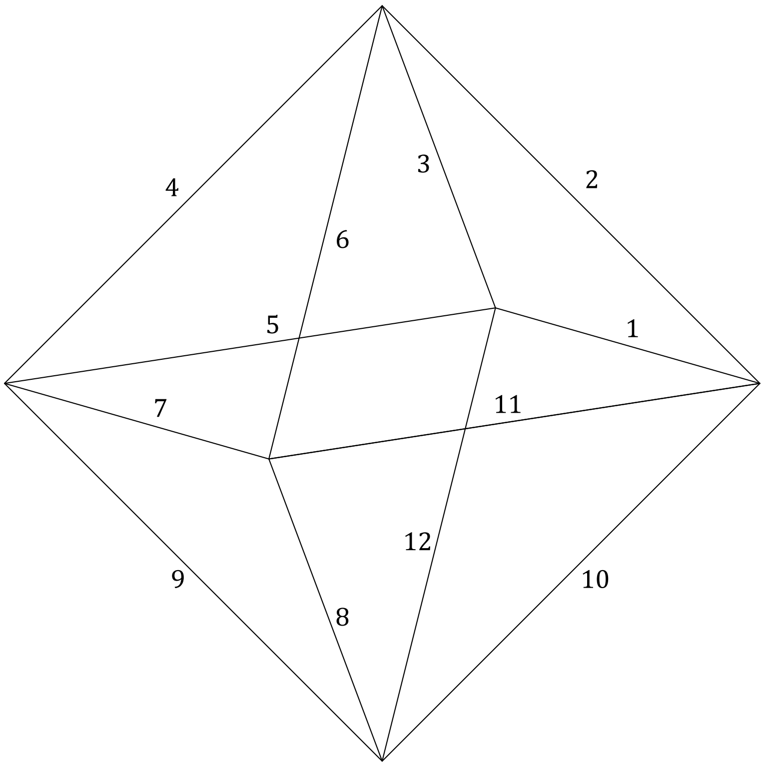

Figure 3.3.1: An octahedron with labelled edges

In order to determine the permutations, cycle types, subdegrees and suborbits of the action of the symmetry group of an octahedron on its edges, we design a model of an octahedron labeled identically to **Figure 3.3.1** that we use to observe the position of each numbered edge during each rigid motion.

3.3.1 Group of symmetries of an octahedron

Suppose that an octahedron is situated such that its centre is at the origin in $\mathbb{R}^3$. The subgroup of the rotations in $\mathbb{R}^3$ which leave the octahedron invariant is denoted by ד. The elements of ד include:

I. Two rotations of 120° and 240° about each of the four axes joining pairs of opposite faces.

II. A rotation through 180° about each of the six axes joining the midpoints of pairs of opposite edges.

III. Three rotations through 90° and 270° or 180° about the axes joining centres of three pairs of opposite vertices.

IV. The identity.

Thus, $|ד| = 3.3 + 6.1 + 4.2 + 1 = 24$. The group of symmetries of an octahedron is isomorphic to S_4.

3.3.2 Cycle type, stabilizer, transitivity and primitivity of S_4 acting on the edges

Table 3.3

Rotations	**Permutations**	**Number of Permutations**	**Cycle Type**
Rotations about the three pairs of opposite vertices through I. 90° or 270°	(7 5 1 11) (4 3 2 6) (8 9 12 10)	6	(0,0,0,3,0,0,0,0,0,0,0,0)
II. 180°	(1 7) (2 4) (3 6) (5 11) (8 12) (9 10)	3	(0,6,0,0,0,0,0,0,0,0,0,0)
Rotations about the axes joining the centre of 6 pairs of opposite edges through 180°	(2) (9) (1 6) (3 11) (4 10) (5 8) (7 12)	6	(2,5,0,0,0,0,0,0,0,0,0,0)
120° and 240° rotations about the four axes joining pairs of opposite faces	(1 2 3) (4 12 11) (6 5 10) (9 8 7)	8	(0,0,4,0,0,0,0,0,0,0,0,0)
Identity	I	1	(12,0,0,0,0,0,0,0,0,0,0,0)

Theorem 3.3.1

The stabilizer of the action of the symmetry group of an octahedron on its edges is

{I, 180°}

Proof: The identity rotation stabilizes all the edges. Moreover, the rotation through 180° about each of the three axis joining the midpoints of opposite edges stabilizes opposite edges of an octahedron. Consequently, the stabilizer of the edges of an octahedron is {I, 180°}, which is a cyclic group of order 2.

Theorem 3.3.2

The action of the symmetry group of an octahedron on its edges is transitive.

Proof: From **Table 3.3**, we determine the transitivity by first obtaining $|Fix\ (g)|$ for each $g \in S_4$ then using **Theorem 1.5.30** we find the number of orbits when S_4 acts on the edges of an octahedron to be

$$\frac{1}{24}(6.0 + 3.0 + 6.2 + 8.0 + 1.12) = 1.$$

Therefore, from **Definition 1.5.6**, S_4 acts transitively on the edges of an octahedron.

Alternatively, using **Theorem 1.5.29**, we find that $|S_4 : C_2| = 24/2 = 12$; which is the number of edges of an octahedron, hence the action of S_4 on the edges of an octahedron is transitive.

3.3.3 Primitivity of S_4 acting on the edges

Theorem 3.3.3

The action of the symmetry group of an octahedron on its edges is imprimitive.

Proof: The stabilizer of the action of S_4 acting on the edges of an octahedron is a cyclic group of order 2; C_2. From **Definition 1.5.12,** C_2 is not a maximal subgroup of S_4. Thus, from **Theorem 1.5.31** S_4 acts imprimitively on the edges of an octahedron.

3.3.4 Rank and subdegrees of the symmetry group acting on the edges

Theorem 3.3.4

The rank of the action of the symmetry group of an octahedron on its edges is 7 and the subdegrees are 1,1,2,2,2,2,2.

Proof: Using **Definition 1.5.10** and **Theorem 1.5.30** we can compute the rank of the action of the symmetry group on the edges of an octahedron whereby;

$$\frac{1}{2}(12+2) = \frac{14}{2} = 7,$$

where 12 is the number of edges fixed by the identity and 2 is the number of edges fixed by the rotation of 180° about the axes joining the midpoints of opposite edges.

Using the rank obtained, the stabilizer, **Definition 1.5.10** and **Definition 1.5.11,** we can obtain the corresponding subdegrees of the action of $\daleth$ on the edges of an octahedron. From **Figure 3.3.1** with the axis of rotation joining the midpoints of edges labelled 9 and 12, under 180° rotation we obtain the following permutation (9) (2) (1 6) (3 11) (4 10) (5 8) (7 12) whereby, the suborbits are $\Delta_0 = \{9\}$, $\Delta_1 = \{2\}$, $\Delta_2 = \{1, 6\}$, $\Delta_3 = \{3, 11\}$, $\Delta_4 = \{4, 10\}$, $\Delta_5 = \{5, 8\}$, $\Delta_6 = \{7, 12\}$ and the subdegrees are 1,1,2,2,2,2,2.

3.3.5 Suborbital graphs of the symmetry group acting on the edges

After computing the subdegrees, we proceed to analyze the suborbits using **Definition 1.5.27**. The rank is equal to the total number of suborbital graphs, so we need to find 7 suborbitals. See Appendix III for the suborbital graphs.

3.3.6 Properties of the suborbital graphs

3.3.6.1 Properties of graph Γ_1

The graph is directed and strongly connected. The girth is 3 and the diameter is 4.

3.3.6.2 Properties of graph Γ_2

The graph is directed and strongly connected. The girth is 3 and the diameter is 4.

3.3.6.3 Properties of graph Γ_3

The graph is undirected and disconnected. The graph has 4 connected components and its girth is 3.

3.3.6.4 Properties of graph Γ_4

The graph is undirected and disconnected. The graph has 3 connected components and its girth is 4.

3.3.6.5 Properties of graph Γ_5

The graph is undirected and disconnected. The graph has 4 connected components and its girth is 3.

3.3.6.6 Properties of graph Γ_6

The graph is undirected and disconnected. The graph has 6 connected components and its girth is 0.

From the properties of the graphs and **Theorem 1.5.32,** we can verify our previous findings that the action of the symmetry group of an octahedron on its edges is imprimitive as Γ_6, Γ_5 and Γ_3 are disconnected.

We can also confirm the number of self-paired graphs using **Theorem 1.5.33** and **Table 3.3**. We have;

$$\frac{1}{24}(12.3 + 12.6 + 12.1) = \frac{120}{24} = 5.$$

where each of the 3 permutations of 180° rotation about the axis joining opposite vertices becomes the identity rotation when squared fixing 12 edges. Also, each of the 6 permutations of 180° rotation about the axis joining centres of opposite edges becomes the identity rotation when squared fixing 12 edges. Finally, the identity rotation fixes 12 edges.

Using **Definition1.5.27** we have the following self-paired graphs; Γ_0, Γ_3, Γ_4, Γ_5 and Γ_6 (See Appendix III).

3.4 DODECAHEDRON

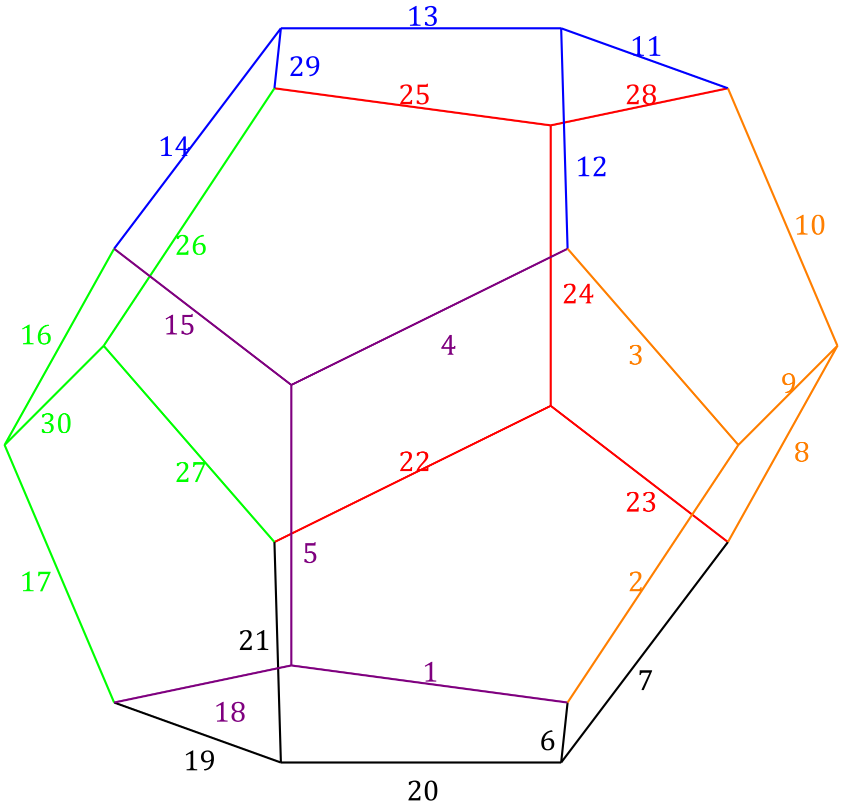

Figure 3.4.1: A dodecahedron with labeled edges

In order to determine the permutations, cycle types, subdegrees and suborbits of the action of the symmetry group of a dodecahedron on its edges, we design a model of a dodecahedron labeled identically to **Figure 3.4.1** that we use to observe the position of each numbered edge during each rigid motion.

3.4.1 Group of symmetries of a dodecahedron

Suppose that a dodecahedron is situated such that its centre is at the origin in $\mathbb{R}^3$. The subgroup of the rotations in $\mathbb{R}^3$ which leave the dodecahedron invariant is denoted by $\daleth$. The elements of $\daleth$ include:

I. Four rotations of 72°, 144°, 216° and 288° about the axes joining the centres of 6 pairs of opposite faces.

II. A rotation of 180° about the axes joining the centres of 15 pairs of opposite edges.

III. Two rotations through 120° and 240° about the axes joining the centres of 10 pairs of opposite vertices.

IV. The identity.

Thus, $|\daleth| = 4.6 + 1.15 + 2.10 + 1 = 60$. The group of symmetries of a dodecahedron is isomorphic to A_5.

3.4.2 Cycle type, stabilizer, transitivity and primitivity of A_5 acting on the edges of a dodecahedron

Table 3.4

Rotations	**Permutations**	**Number of Permutations**	**Cycle Type**
Rotation about the axes joining the centres of 6 pairs of opposite faces through 72°.	(1 4 14 30 19) (7 9 11 25 22) (8 10 28 24 23) (12 29 27 20 2) (13 26 21 6 3) (15 16 17 18 5)	24	(0,0,0,0,6,0,0,0,0,0,0,0,0,0,0,0,0,0,0,0,0,0,0,0,0,0,0,0,0,0)
Rotations about the axes joining the centres of 15 pairs of opposite edges through 180°	(2) (26) (1 9) (3 6) (4 7) (5 8) (10 18) (11 19) (12 20) (13 21) (14 22) (16 24) (25 30) (17 28) (23 15) (27 29)	15	(2,14,0,0,0,0,0,0,0,0,0,0,0,0,0,0,0,0,0,0,0,0,0,0,0,0,0,0,0,0)
120° and 240° rotations about the axes joining the	(1 5 18) (2 15 19) (3 16 20) (7 12 30)	20	(0,0,10,0,0,0,0,0,0,0,0,0,0,0,0,0,0,0,0,0,0,0,0,0,0,0,0,0,0,0)

10 pairs of opposite vertices	(8 13 27) (9 14 21) (10 29 22) (11 26 23) (24 28 25)		
Identity	I	1	(30,0,0,0,0,0,0,0,0,0,0,0,0,0, 0,0,0,0,0,0,0,0,0,0,0,0,0,0,0,0)

Theorem 3.4.1

The stabilizer of the action of the symmetry group of a dodecahedron on its edges is {I, 180°}.

Proof: The identity rotation stabilizes all the edges. Moreover, the rotation through 180° about each of the 15 axis joining the midpoints of opposite edges stabilizes opposite edges of a dodecahedron. Consequently, the stabilizer of the edges of a dodecahedron is {I, 180°}, which is a cyclic group of order 2.

Theorem 3.4.2

The action of the symmetry group of a dodecahedron on its edges is transitive.

Proof: From **Table 3.4**, we determine the transitivity by first obtaining $|Fix\ (g)|$ for each $g \in A_5$ then using **Theorem 1.5.30**, we find the number of orbits when A_5 acts on the edges of a dodecahedron is given by;

$$\frac{1}{60}(24.0 + 15.2 + 20.0 + 1.30) = 1.$$

Therefore, from **Definition 1.5.6**, A_5 acts transitively on the edges of a dodecahedron.

Alternatively, using **Theorem 1.5.29**, we find that | $A_5 : C_2$| = 60/2 = 30; which is the number of edges of a dodecahedron, hence the action of A_5 on the edges of a dodecahedron is transitive.

3.4.3 Primitivity of A_5 acting on the edges of a dodecahedron

Theorem 3.4.3

The action of the symmetry group of a dodecahedron on its edges is imprimitive.

Proof: The stabilizer of the action of A_5 on the edges of a dodecahedron is a cyclic group of order 2. From **Definition 1.5.12,** C_2 is not a maximal subgroup of A_5 hence, from **Theorem 1.5.31** A_5 acts imprimitively on the edges of a dodecahedron.

3.4.4 Rank and subdegrees of the symmetry group acting on the edges

Theorem 3.4.4

The rank of the action of the symmetry group of a dodecahedron on its edges is 16 and the subdegrees are 1,1,2,2,2,2,2,2,2,2,2,2,2,2,2,2.

Proof: Using **Definition 1.5.10** and **Theorem 1.5.30** we can compute the rank of the action of the symmetry group on the edges of a dodecahedron whereby;

$$\frac{1}{2}(30 + 2) = \frac{32}{2} = 16,$$

where 30 is the number of edges fixed by the identity and 2 is the number of edges fixed by the rotation of 180° about the axes joining the midpoints of opposite edges.

Using the rank obtained, the stabilizer, **Definition 1.5.10** and **Definition 1.5.11,** we can obtain the corresponding subdegrees of the action of ך on the edges of a dodecahedron. From **Figure 3.4.1** with the axis of rotation joining the midpoint of edges labelled 2 and 26, under the 180° rotation we obtain the following permutation (2) (26) (1 9) (3 6) (4 7) (5 8) (10 18) (11 19) (12 20)

(13 21) (14 22) (16 24) (25 30) (17 28) (15 23) (27 29) whereby, the suborbits are $\Delta_0 = \{2\}$, $\Delta_1 = \{26\}$, $\Delta_2 = \{1, 9\}$, $\Delta_3 = \{3, 6\}$, $\Delta_4 = \{4, 7\}$, $\Delta_5 = \{5, 8\}$, $\Delta_6 = \{10, 18\}$, $\Delta_7 = \{11, 19\}$, $\Delta_8 = \{12, 20\}$, $\Delta_9 = \{13, 21\}$, $\Delta_{10} = \{14, 22\}$, $\Delta_{11} = \{16, 24\}$, $\Delta_{12} = \{25, 30\}$, $\Delta_{13} = \{17, 28\}$, $\Delta_{14} = \{15, 23\}$, $\Delta_{15} = \{27, 29)$ and the corresponding subdegrees are 1,1,2,2,2,2,2,2,2,2,2,2,2,2,2,2.

3.4.5 Suborbital graphs of the symmetry group acting on the edges

After computing the subdegrees, we proceed to analyze the suborbits using **Definition 1.5.27** and **Figure 3.4.1.** The rank is equal to the total number of suborbital graphs, so we need to find 16 suborbitals. See Appendix IV for the suborbital graphs.

3.4.6 Properties of the suborbital graphs

3.4.6.1 Properties of graphs $\Gamma_1, \Gamma_4, \Gamma_8$ and Γ_{12}

The graph is directed and strongly connected. The girth is 3 and the diameter is 7.

3.4.6.2 Properties of graphs Γ_2 and Γ_3

The graph is directed and strongly connected. The girth is 5 and the diameter is 6.

3.4.6.3 Properties of graphs $\Gamma_5, \Gamma_7, \Gamma_{13}$ and Γ_{15}

The graph is undirected and disconnected. The graph has 6 connected components and the girth is 5.

3.4.6.4 Properties of graphs Γ_6 and Γ_{10}

The graph is directed and disconnected. The graph has 5 connected components and its girth is 3.

3.4.6.5 Properties of graphs Γ_9 and Γ_{11}

The graph is undirected and disconnected. The graph has 10 connected components and its girth is 3.

3.4.6.6 Properties of graphs Γ_{14}

The graph is undirected and disconnected. The graph has 15 connected components and its girth is 0.

From the properties of the graphs and **Theorem 1.5.32,** we can verify our previous findings that the action of the symmetry group of a dodecahedron on its edges is imprimitive as Γ_6, Γ_5, Γ_7, Γ_{13}, Γ_{15}, Γ_{10}, Γ_9, Γ_{11} and Γ_{14} are disconnected.

We can also confirm the number of self-paired graphs using **Theorem 1.5.33** and **Table 3.4**. We have;

$$\frac{1}{60}(30.15+30.1)=\frac{480}{60}=8,$$

where each of the 15 permutations of 180° rotation about the axis joining the centres of opposite edges becomes the identity rotation when squared fixing 30 edges and the identity rotation fixes 30 edges.

Using **Definition1.5.27** we have the following self-paired graphs; Γ_0, Γ_{14}, Γ_{11}, Γ_9, Γ_5, Γ_7, Γ_{13} and Γ_{15} (See Appendix IV).

3.5 ICOSAHEDRON

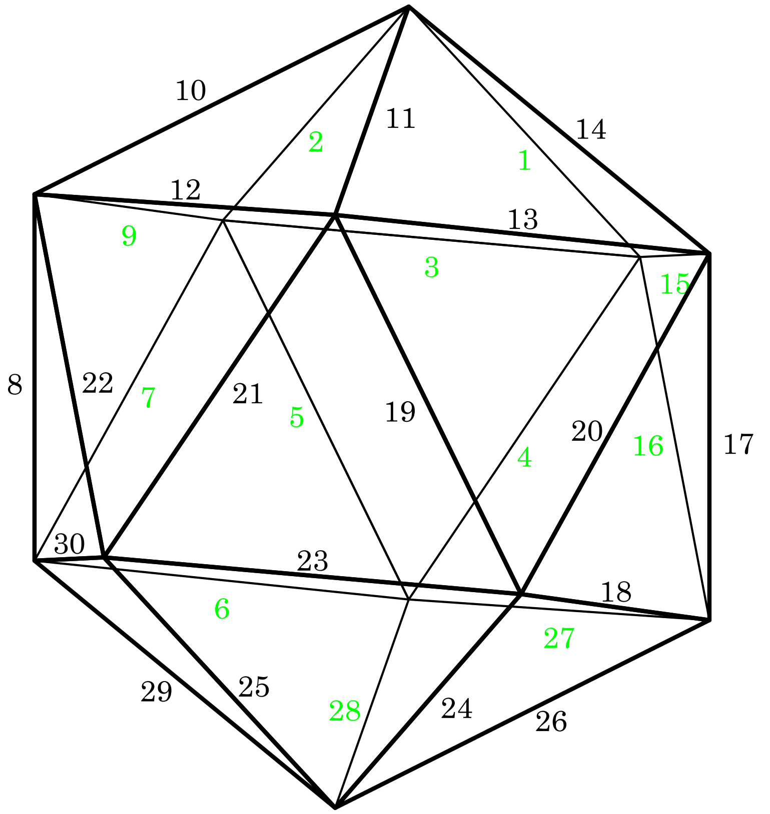

Figure 3.5.1: An icosahedron with labeled edges

In order to determine the permutations, cycle types, subdegrees and suborbits of the action of the symmetry group of an icosahedron on its edges, we design a model of an icosahedron labeled identically to **Figure 3.5.1** that we use to observe the position of each numbered edge during each rigid motion.

3.5.1 Group of symmetries of an icosahedron

Suppose that an icosahedron is situated such that its centre is at the origin in $\mathbb{R}^3$. The subgroup of the rotations in $\mathbb{R}^3$ which leave the icosahedron invariant is denoted by ℸ. The elements of ℸ include:

I. Ten rotations of 120° and 240° about the axes joining the centres of 10 pairs of opposite faces.

II. A rotation of 180° about the axes joining the centres of 15 pairs of opposite edges.

III. Four rotations of 72°, 144°, 216° and 288° about the axes joining the centres of 6 pairs of opposite vertices.

IV. The identity.

Thus, $|\daleth| = 2.10 + 1.15 + 4.6 + 1 = 60$. The group of symmetries of an icosahedron is isomorphic to A_5.

3.5.2 Cycle type, stabilizer, transitivity and primitivity of A_5 acting on the edges of an icosahedron

Table 3.5

Rotations	**Permutations**	**Number of Permutations**	**Cycle Type**
Rotation about the axes joining the centres of 6 pairs of opposite vertices through $72°, 144°, 216°, 280°$	(1 4 6 8 10) (2 3 5 7 9) (21 13 17 26 25) (11 15 27 29 22) (12 14 16 28 30) (18 24 23 19 20)	24	(0,0,0,0,6,0,0,0,0,0,0,0,0,0, 0,0,0,0,0,0,0,0,0,0,0,0,0,0,0,0)

Rotations about the axes joining the centres of 15 pairs of opposite edges through 180°	(2) (24) (1 9) (3 10) (4 12) (5 11) (6 13) (7 14) (21 27) (8 15) (16 22) (17 30) (18 25) (19 28) (20 29) (23 26)	15	(2,14,0,0,0,0,0,0,0,0,0,0,0,0,0,0,0,0,0,0,0,0,0,0,0,0,0,0,0,0)
Rotation about the axes joining the centres of 10 pairs of opposite faces through 120° and 240°.	(1 3 2) (4 9 14) (5 10 15) (6 12 17) (21 18 29) (7 16 11) (8 27 13) (19 26 30) (23 25 24) (20 28 22)	20	(0,0,10,0,0,0,0,0,0,0,0,0,0,0,0,0,0,0,0,0,0,0,0,0,0,0,0,0,0,0)
Identity	I	1	(30,0,0,0,0,0,0,0,0,0,0,0,0,0,0,0,0,0,0,0,0,0,0,0,0,0,0,0,0,0)

Theorem 3.5.1

The stabilizer of the action of the symmetry group of an icosahedron on its edges is

{I, 180°}.

Proof: The identity rotation stabilizes all the edges. Moreover, the rotation through 180° about each of the 15 axis joining the midpoints of opposite edges stabilizes opposite edges of an icosahedron. Consequently, the stabilizer of an edge of an icosahedron is {I, 180°}, which is a cyclic group of order 2.

Theorem 3.5.2

The action of the symmetry group of an icosahedron on its edges is transitive

Proof: From **Table 3.5**, we determine the transitivity by first obtaining $|Fix\ (g)|$ for each $g \in A_5$ then using **Theorem 1.5.30**, we find the number of orbits when A_5 acts on the edges of an icosahedron. The equation is given by;

$$\frac{1}{60}(24.0 + 15.2 + 20.0 + 1.30) = 1.$$

Therefore, from **Definition 1.5.6**, A_5 acts transitively on the edges of an icosahedron.

Alternatively, using **Theorem 1.5.29**, we find that $|A_5 : C_2|$ = 60/2 = 30; which is the number of edges of an icosahedron, hence the action of A_5 on the edges of an icosahedron is transitive.

3.5.3 Primitivity of A_5 acting on the edges of an icosahedron

Theorem 3.5.3

The action of the symmetry group of an icosahedron on its edges is imprimitive.

Proof: The stabilizer of the action of A_5 on the edges of an icosahedron is a cyclic group of order 2. From **Definition 1.5.12,** C_2 is not a maximal subgroup of A_5 then from **Theorem 1.5.31** A_5 acts imprimitively on the edges of an icosahedron.

3.5.4 Rank and subdegrees of the symmetry group acting on the edges

Theorem 3.5.4

The rank of the action of the symmetry group of an icosahedron on its edges is 16 and the subdegrees are 1,1,2,2,2,2,2,2,2,2,2,2,2,2,2,2.

Proof: Using **Definition 1.5.10** and **Theorem 1.5.30** we can compute the rank of the action of the symmetry group on the edges of an icosahedron whereby;

$$\frac{1}{2}(30+2) = \frac{32}{2} = 16,$$

where 30 is the number of edges fixed by the identity and 2 is the number of edges fixed by the rotation of 180° about the axes joining the midpoints of opposite edges.

Using the rank obtained, the stabilizer, **Definition 1.5.10** and **Definition 1.5.11,** we can obtain the corresponding subdegrees of the action of ℸ on the edges of an icosahedron. From **Figure 3.5.1** with the axis of rotation joining the midpoint of edges labelled 2 and 24, under the 180° rotation we obtain the following permutation (2) (24) (1 9) (3 10) (4 12) (5 11) (6 13) (7 14) (21 27) (8 15) (16 22) (17 30) (18 25) (19 28) (20 29) (23 26) whereby, the suborbits are Δ_0 = {2}, Δ_1 = {24}, Δ_2 = {1, 9}, Δ_3 = {3, 10}, Δ_4 = {4, 12}, Δ_5 = {5, 11}, Δ_6 = {6, 13}, Δ_7 = {7, 14}, Δ_8 = {21, 27}, Δ_9 = {8, 15}, Δ_{10} = {16, 22}, Δ_{11} = {17, 30}, Δ_{12} = {18, 25}, Δ_{13} = {19, 28}, Δ_{14} = {20, 29} and Δ_{15} = {23, 26) and the subdegrees are 1,1,2,2,2,2,2,2,2,2,2,2,2,2,2,2.

3.5.5 Suborbital graphs of the symmetry group acting on the edges

After computing the subdegrees, we proceed to analyze the suborbits using **Definition 1.5.27** and **Figure 3.5.1.** The rank is equal to the total number of suborbital graphs, so we need to find 16 suborbitals. See Appendix V for the suborbital graphs.

3.5.6 Properties of the suborbital graphs

3.5.6.1 Properties of graphs Γ_1, Γ_2, Γ_7 and Γ_8

The graphs are directed and strongly connected. The girth is 3 and the diameter is 7.

3.5.6.2 Properties of graphs Γ_5 and Γ_3

The graphs are directed and strongly connected. The girth is 5 the diameter is 6.

3.5.6.3 Properties of graphs Γ_6, Γ_4, Γ_{12} and Γ_{14}

The graphs are undirected and disconnected. The graph has 6 connected components and the girth is 5.

3.5.6.4 Properties of graphs Γ_9 and Γ_{10}

The graphs are directed and disconnected. The graph has 5 connected components and its girth is 3.

3.5.6.5 Properties of graphs Γ_{15} and Γ_{11}

The graphs are undirected and disconnected. The graph has 10 connected components and its girth is 3.

3.5.6.6 Properties of graph Γ_{13}

The graph is undirected and disconnected. The graph has 15 connected components and its girth is 0.

From the properties of the graphs and **Theorem 1.5.32,** we can verify our previous findings that the action of the symmetry group of an icosahedron acting on its edges is imprimitive as Γ_6, Γ_4, Γ_{13}, Γ_{15}, Γ_{10}, Γ_{12}, Γ_9, Γ_{11} and Γ_{14} are disconnected.

We can also confirm the number of self-paired graphs using **Theorem 1.5.33** and **Table 3.5**. We have;

$$\frac{1}{60}(30.15 + 30.1) = \frac{480}{60} = 8,$$

where each of the 15 permutations of 180° rotation about the centre of opposite edges becomes the identity rotation when squared fixing 30 edges and the identity rotation fixes 30 edges.

Using **Definition1.5.27** we have the following self-paired graphs; $\Gamma_{0,}$ $\Gamma_{14,}$ $\Gamma_{11,}$ $\Gamma_{6,}$ $\Gamma_{4,}$ $\Gamma_{15,}$ Γ_{13} and Γ_{12} (See Appendix V).

4. CONCLUSION AND RECOMMENDATIONS

4.1 Conclusion

In this project, we have successfully computed the stabilizers of the edges of the platonic solids using **Definition 1.5.8**. In **Chapter 3** under each subsection we have shown that the actions are transitive using **Tables 3.1.1, 3.2.1, 3.3.1, 3.4.1** and **3.5.1,** and **Theorems 1.5.30** and **1.5.29**. We have also shown that the actions are imprimitive using **Definition 1.5.12** and **Theorem 1.5.31**. Furthermore, we have computed the ranks using **Definition 1.5.10** and **Theorem 1.5.30,** and computed subdegrees using the **rank, stabilizer, Definition 1.5.10** and **Definition 1.5.11.** Also, we have constructed the corresponding suborbital graphs using **Definition 1.5.27.**

We have also investigated the properties of each graph using **Definition 1.5.20, Definition 1.5.22, Definition 1.5.24** and **Definition 1.5.27,** and confirmed previous findings on imprimitivity of the actions using **Theorem 1.5.32**. Lastly, we have computed the number of self-paired graphs using **Theorem 1.5.33**.

4.2 Recommendations for further research

Having investigated the subdegrees and suborbital graphs of the action of the symmetry groups of platonic solids acting on the edges of the solids, one can also investigate the subdegrees and suborbital graphs of the action of the symmetry groups of platonic solids acting on the faces and vertices of the solids.

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6. APPENDICES

Appendix I: Suborbitals and suborbital graphs of a tetrahedron

Using **Definition 1.5.27** we compute the following suborbitals;

X×X = {(1, 1), (1, 2), (1, 3), (1, 4), (1, 5), (1, 6), (2, 1), (2, 2), (2, 3), (2, 4), (2, 5), (2, 6), (3, 1), (3, 2), (3, 3), (3, 4), (3, 5), (3, 6), (4, 1), (4, 2), (4, 3), (4, 4), (4, 5), (4, 6), (5, 1), (5, 2), (5, 3), (5, 4), (5, 5), (5, 6), (6, 1), (6, 2), (6, 3), (6, 4), (6, 5), (6, 6)}

$\mathrm{Orb}_G(1, 1) = O_0$ = {(1, 1), (2, 2), (3, 3), (4, 4), (5, 5), (6, 6)}

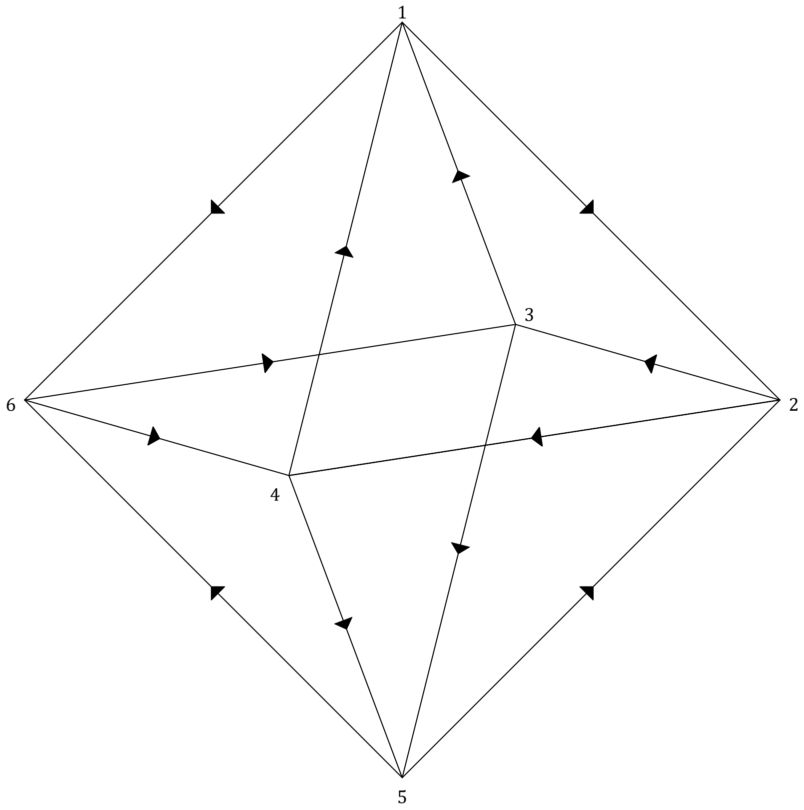
$\mathrm{Orb}_G(1, 2) = O_1$ = {(1, 2), (6, 3), (4, 5), (2, 3), (3, 1), (2, 4), (4, 1), (6, 4), (3, 5), (1, 6), (5, 2), (5, 6)}

$\mathrm{Orb}_G(2, 1) = O_2$ = {(2, 1), (3, 6), (5, 4), (3, 2), (1, 3), (4, 2), (1, 4), (4, 6), (5, 3), (6, 1), (2, 5), (6, 5)}

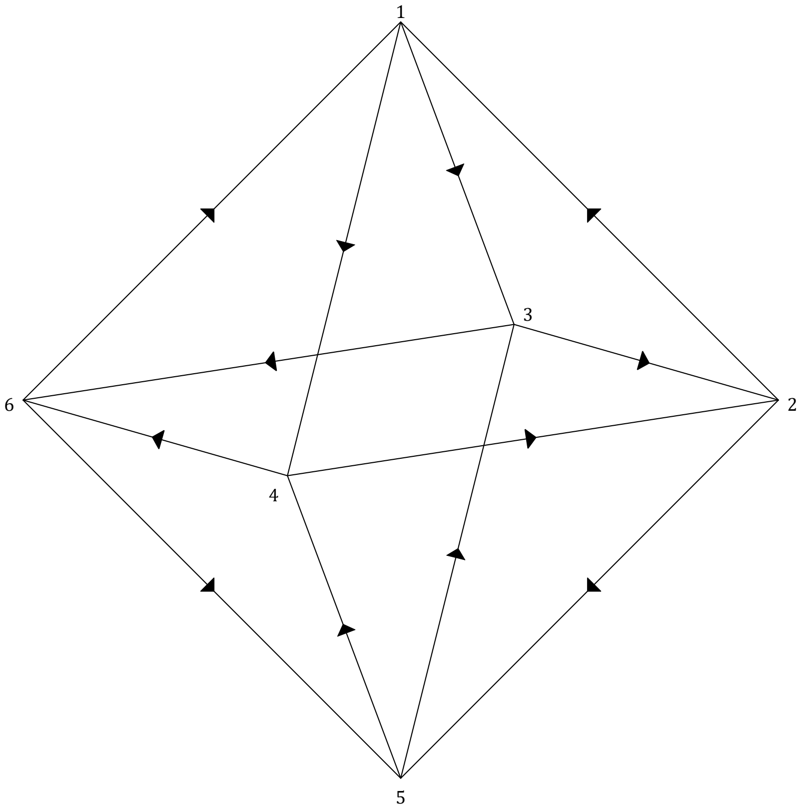
$\mathrm{Orb}_G(1, 5) = O_3$ = {(1, 5), (6, 2), (4, 3), (2, 6), (3, 4), (2, 6), (4, 3), (6, 2), (3, 4), (1, 5), (5, 1), (5, 1)}

Next we construct the corresponding non-trivial suborbital graphs.

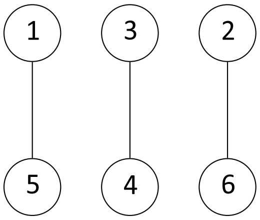
Γ_1:



Γ_2:



Γ_3:



Appendix II: Suborbitals and suborbital graphs of a cube

Using **Definition 1.5.27** we compute the following suborbitals,

X×X = {(1,1), (1,2), (1,3), (1,4), (1,5), (1,6), (1,7), (1,8), (1,9), (1,10), (1,11), (1,12), (2,1), (2,2), (2,3), (2,4), (2,5), (2,6), (2,7), (2,8), (2,9), (2,10), (2,11), (2,12), (3,1), (3,2), (3,3), (3,4), (3,5), (3,6), (3,7), (3,8), (3,9), (3,10), (3,11), (3,12), (4,1), (4,2), (4,3), (4,4), (4,5), (4,6), (4,7), (4,8), (4,9), (4,10), (4,11), (4,12), (5,1), (5,2), (5,3), (5,4), (5,5), (5,6), (5,7), (5,8), (5,9), (5,10), (5,11), (5,12), (6,1), (6,2), (6,3), (6,4), (6,5), (6,6), (6,7), (6,8), (6,9), (6,10), (6,11), (6,12), (7,1), (7,2), (7,3), (7,4), (7,5), (7,6), (7,7), (7,8), (7,9), (7,10), (7,11), (7,12), (8,1), (8,2), (8,3), (8,4), (8,5), (8,6), (8,7), (8,8), (8,9), (8,10), (8,11), (8,12), (9,1), (9,2), (9,3), (9,4), (9,5), (9,6), (9,7), (9,8), (9,9), (9,10), (9,11), (9,12), (10,1), (10,2), (10,3), (10,4), (10,5), (10,6), (10,7), (10,8), (10,9), (10,10), (10,11), (10,12), (11,1), (11,2), (11,3), (11,4), (11,5), (11,6), (11,7), (11,8), (11,9), (11,10), (11,11), (11,12), (12,1), (12,2), (12,3), (12,4), (12,5), (12,6), (12,7), (12,8), (12,9), (12,10), (12,11), (12,12)}

Orb_G (1, 1) = O_0 = {(1, 1), (2, 2), (3, 3), (4, 4), (5, 5), (6, 6), (7,7), (8,8), (9,9), (10,10), (11,11), (12,12)}

Orb_G (1, 2) = O_1 = {(1,2), (3,4), (2,1), (4,3), (9,1), (11,9), (2,11), (8,12), (9,11), (6,7), (1,9), (11,2), (12,8), (7,6), (5,10), (10,5), (4,10), (8,6), (6,8), (3,5), (5,3), (12,7), (7,12), (10,4)}

Orb_G (1, 3) = O_2 = {(1,3), (8,4), (9,10), (6,5), (3,2), (2,4), (4,1), (9,6), (11,5), (2,7), (1,8), (10,11), (5,9), (12,10), (7,3), (11,12), (4,12), (8,9), (6,1), (3,6), (10,8), (7,11), (12,2), (5,7)}

Orb_G (1, 4) = O_3 = {(1,4), (3,1), (2,3), (4,2), (9,8), (11,10), (2,12), (8,10), (9,5), (6,3), (1,6), (10,9), (5,11), (12,4), (7,5), (11,7), (4,8), (8,1), (6,9), (3,7), (10,12), (7,2), (12,11),(5,6)}

$\mathrm{Orb}_G(1, 5) = O_4$ = {(1,5), (3,11), (2,10), (4,9), (9,7), (11,3), (2,6), (8,3), (9,4), (6,10), (1,12), (10,2), (5,1), (12,5), (7,4), (11,8), (4,7), (8,11), (6,2), (3,8), (10,6), (7,9), (12,1), (5,12)}

$\mathrm{Orb}_G(1, 7) = O_5$ = {(1,7), (3,12), (2,8), (4,6), (9,3), (11,6), (2,5), (8,2), (9,12), (6,11), (1,10), (10,7), (5,8), (12,9), (7,1), (11,4), (4,11), (8,5), (6,4), (3,9), (10,1), (7,10), (12,3), (5,2)}

$\mathrm{Orb}_G(1, 11) = O_6$ = {(1,11), (3,10), (2,9), (4,5), (9,2), (11,1), (2,9), (8,7), (9,2), (6,12), (1,11), (10,3), (5,4), (12,6), (7,8), (11,1), (4,5), (8,7), (6,12), (3,10), (10,3), (7,8), (12,6), (5,4)}

Next we construct the corresponding non-trivial suborbital graphs.

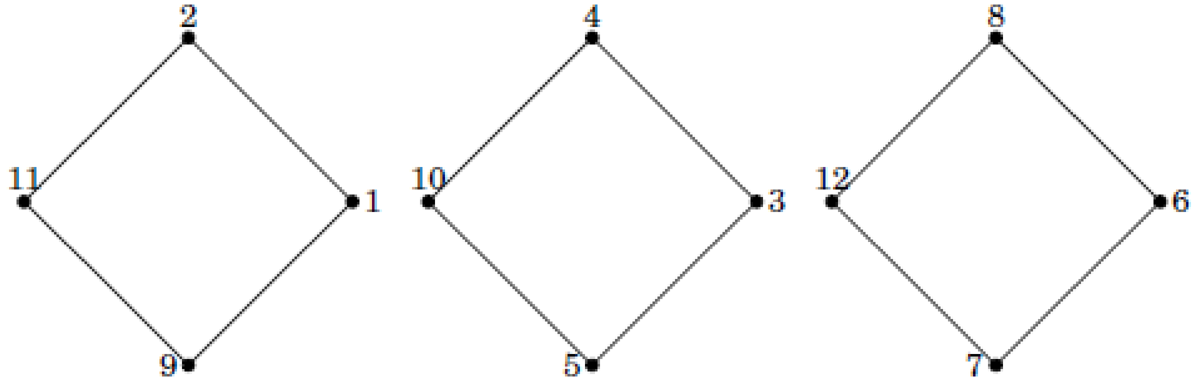


Figure 4.2.2: Graph Γ_1

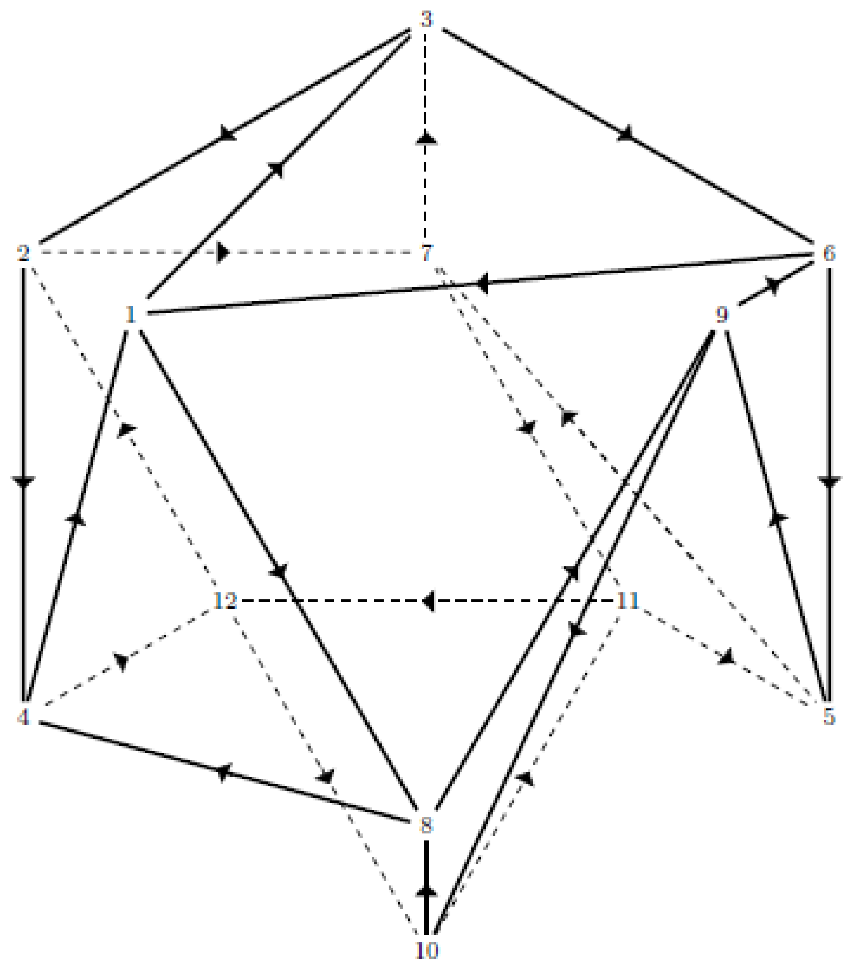


Figure 4.2.3: Graph Γ_2

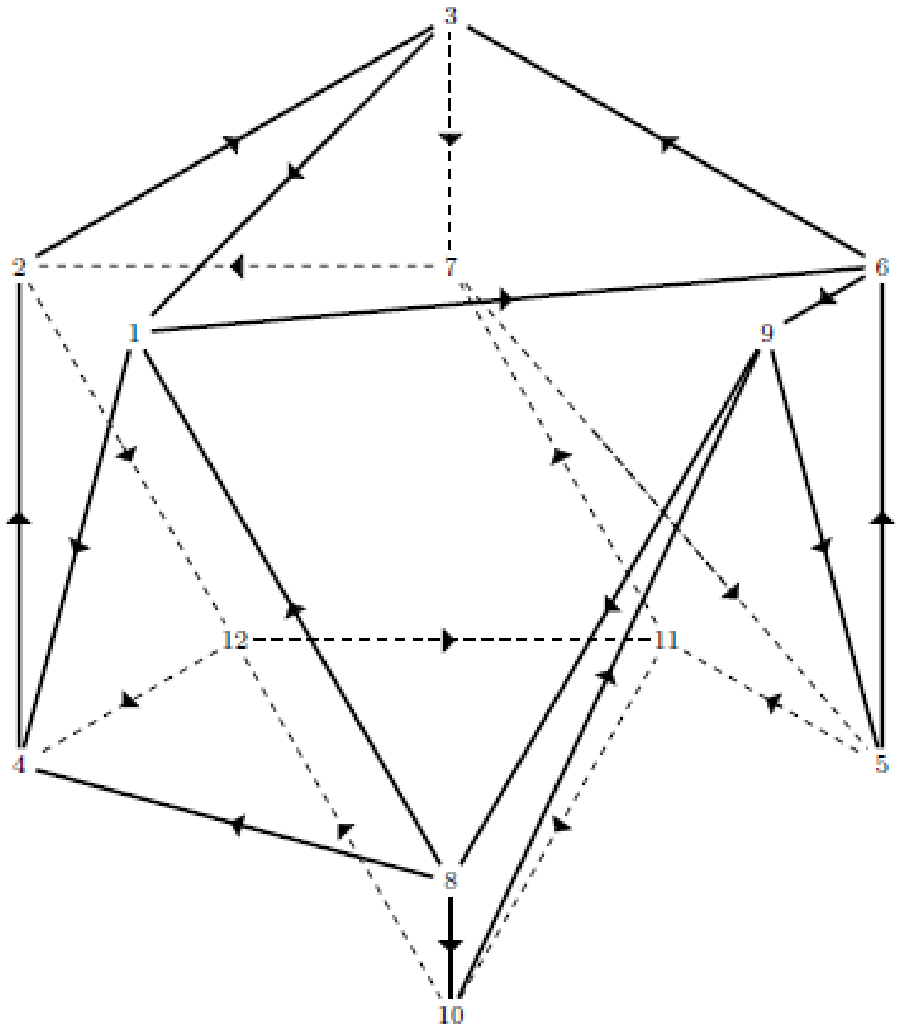


Figure 4.2.4: Graph Γ_3

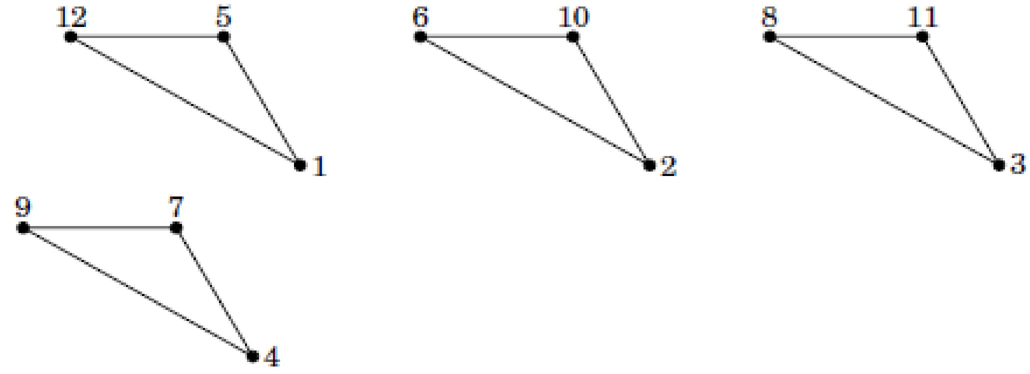


Figure 4.2.5: Graph Γ_4

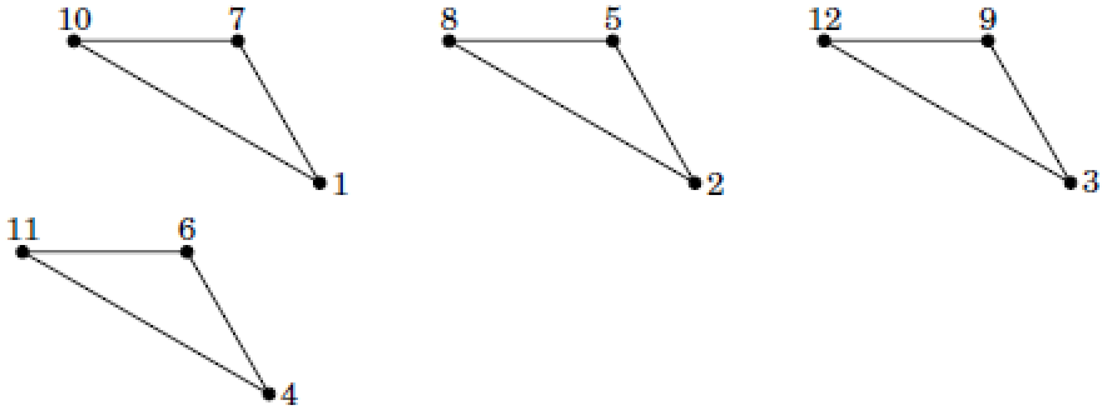


Figure 4.2.6: Graph Γ_5

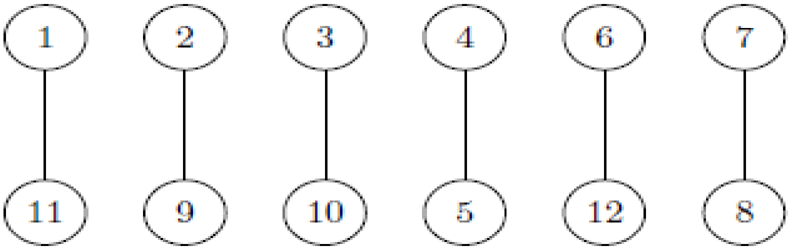


Figure 4.2.7: Graph Γ_6

Appendix III: Suborbitals and suborbital graphs of an octahedron

Using **Definition 1.5.27** we compute the following suborbitals,

X×X = {(1,1), (1,2), (1,3), (1,4), (1,5), (1,6), (1,7), (1,8), (1,9), (1,10), (1,11), (1,12), (2,1), (2,2), (2,3), (2,4), (2,5), (2,6), (2,7), (2,8), (2,9), (2,10), (2,11), (2,12), (3,1), (3,2), (3,3), (3,4), (3,5), (3,6), (3,7), (3,8), (3,9), (3,10), (3,11), (3,12), (4,1), (4,2), (4,3), (4,4), (4,5), (4,6), (4,7), (4,8), (4,9), (4,10), (4,11), (4,12), (5,1), (5,2), (5,3), (5,4), (5,5), (5,6), (5,7), (5,8), (5,9), (5,10), (5,11), (5,12), (6,1), (6,2), (6,3), (6,4), (6,5), (6,6), (6,7), (6,8), (6,9), (6,10), (6,11), (6,12), (7,1), (7,2), (7,3), (7,4), (7,5), (7,6), (7,7), (7,8), (7,9), (7,10), (7,11), (7,12), (8,1), (8,2), (8,3), (8,4), (8,5), (8,6), (8,7), (8,8), (8,9), (8,10), (8,11), (8,12), (9,1), (9,2), (9,3), (9,4), (9,5), (9,6), (9,7), (9,8), (9,9), (9,10), (9,11), (9,12), (10,1), (10,2), (10,3), (10,4), (10,5), (10,6), (10,7), (10,8), (10,9), (10,10), (10,11), (10,12), (11,1), (11,2), (11,3), (11,4), (11,5), (11,6), (11,7), (11,8), (11,9), (11,10), (11,11), (11,12), (12,1), (12,2), (12,3), (12,4), (12,5), (12,6), (12,7), (12,8), (12,9), (12,10), (12,11), (12,12)}

Orb_G (1, 1) = O_0 = {(1, 1), (2, 2), (3, 3), (4, 4), (5, 5), (6, 6), (7,7), (8,8), (9,9), (10,10), (11,11), (12,12)}

Orb_G (1, 2) = O_1 = {(1, 2), (3, 1), (2, 3), (9, 12), (6, 7), (12, 5), (10, 8), (4, 6), (8, 11), (6, 2), (4, 5), (8, 9), (1, 12), (7, 8), (9, 7), (11, 6), (7, 4), (5, 3), (12, 10), (5, 9), (3, 4), (10, 1), (11, 10), (2, 11)}

Orb_G (1, 3) = O_2 = {(1, 3), (3, 2), (2, 1), (9, 5), (6, 4), (12, 9), (10, 11), (4, 7), (8, 10), (6, 11), (4, 3), (8, 7), (1, 10), (7, 9), (9, 8), (11, 2), (7, 6), (5, 4), (3, 5), (5, 12), (12, 1), (10, 12), (11, 8), (2, 6)}

Orb_G (1, 4) = O_3 ={(1, 4), (3, 11), (2, 12), (9, 3), (6, 5), (12, 7), (10, 6), (4, 8), (8, 1), (6, 10), (4, 1), (8, 4), (1, 8), (7, 12), (9, 11), (11, 3), (7, 2), (5, 6), (12, 2), (5, 10), (3, 9), (10, 5), (11, 9), (2, 7)}

Orb_G (1, 5) = O_4 = {(1, 5), (3, 6), (2,10), (9, 4), (6, 3), (12, 8), (10, 2), (4, 9), (8, 12), (6, 8), (4, 2), (8, 6), (1, 11), (7, 5), (9, 10), (11, 1), (7, 11), (5, 7), (12, 3), (5, 1), (3, 12), (10, 9), (11, 7), (2, 4)}

Orb_G (1, 6) = O_5 = {(1,6), (3, 10), (2, 5), (9, 1), (6, 9), (12, 4), (10, 7), (4, 11), (8, 2), (6, 1), (4, 12), (8, 5), (1, 9), (7, 10), (9, 6), (11, 4), (7, 3), (5, 2), (12, 11), (5, 8), (3, 7), (10, 3), (11, 12), (2, 8)}

Orb_G (1, 7) = O_6 = {(1, 7), (3, 8), (2, 9), (9, 2), (6, 12), (12, 6), (10, 4), (4, 10), (8, 3), (6, 12), (4, 10), (8, 3), (1, 7), (7, 1), (9, 2), (11, 5), (7, 1), (5, 11), (12, 6), (5, 11), (3, 8), (10, 4), (11, 5), (2, 9)}

Next we construct the corresponding non-trivial suborbital graphs.

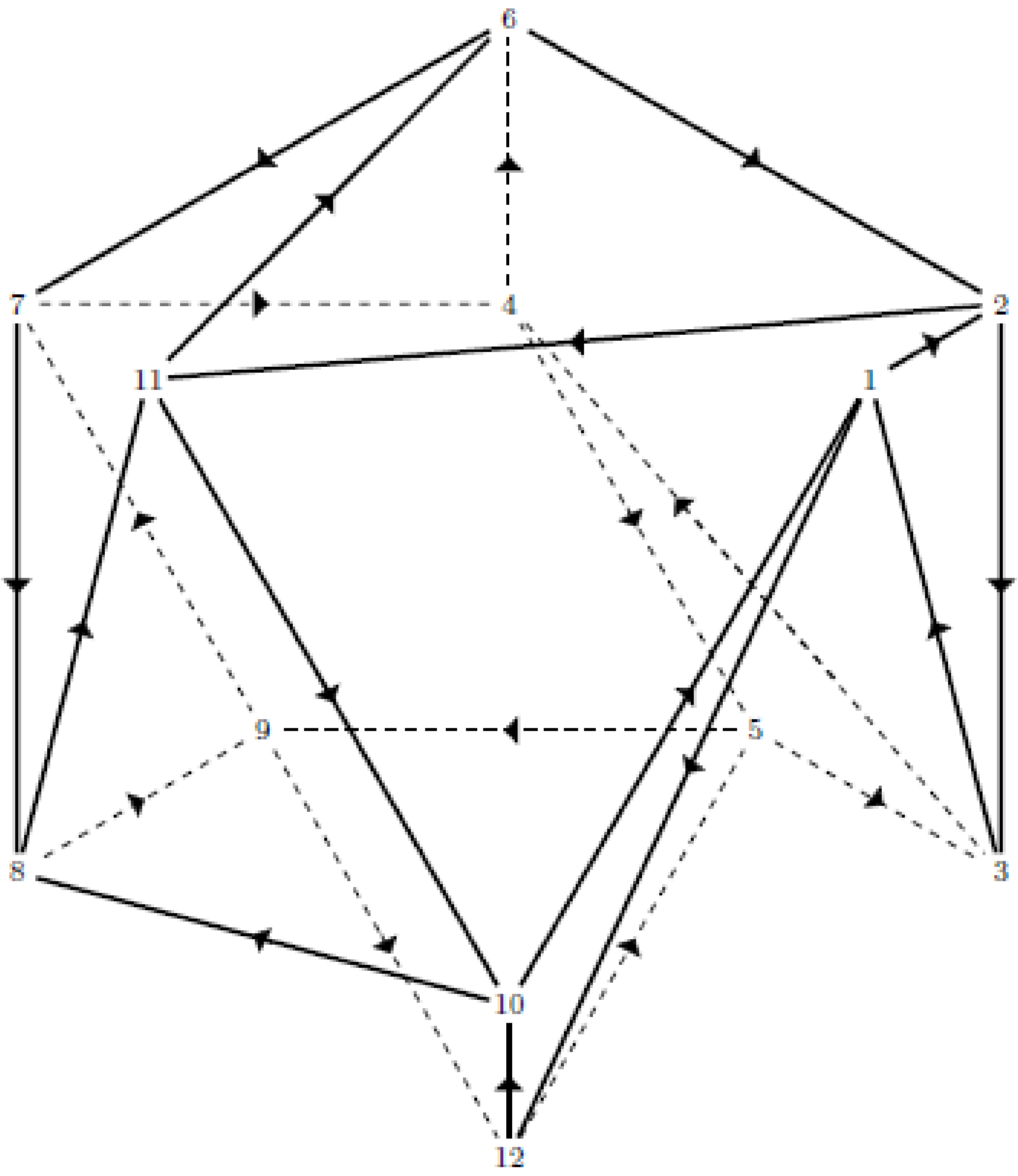


Figure 4.3.2: Graph Γ_1

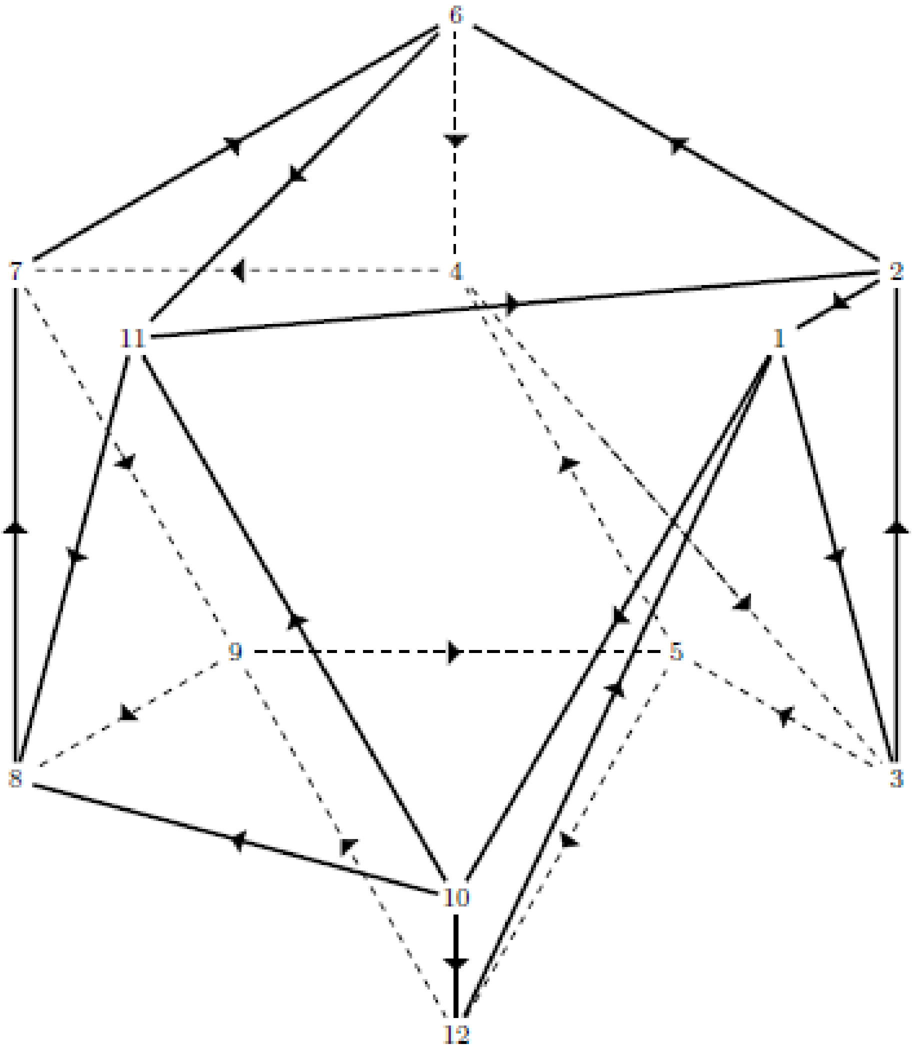


Figure 4.3.3: Graph Γ_2

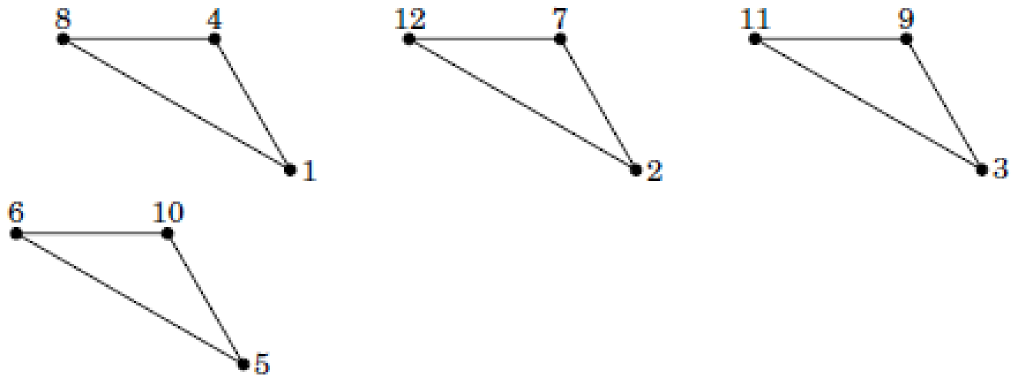


Figure 4.3.4: Graph Γ_3

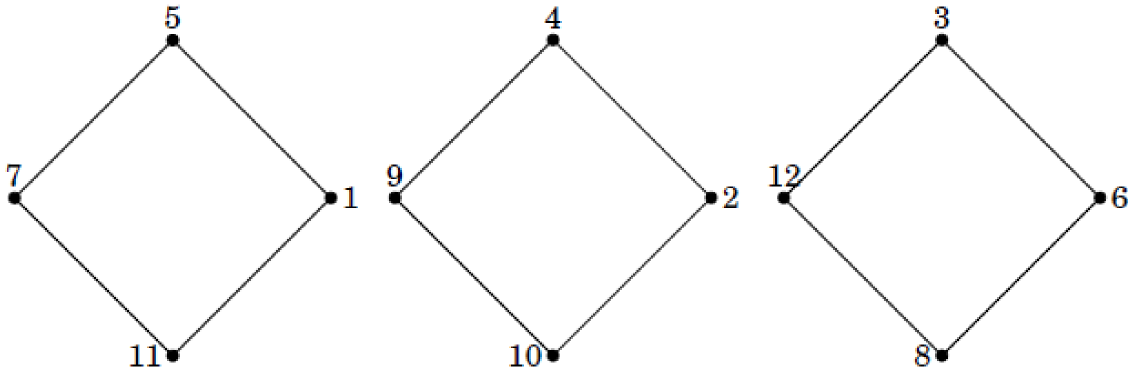


Figure 4.3.5: Graph Γ_4

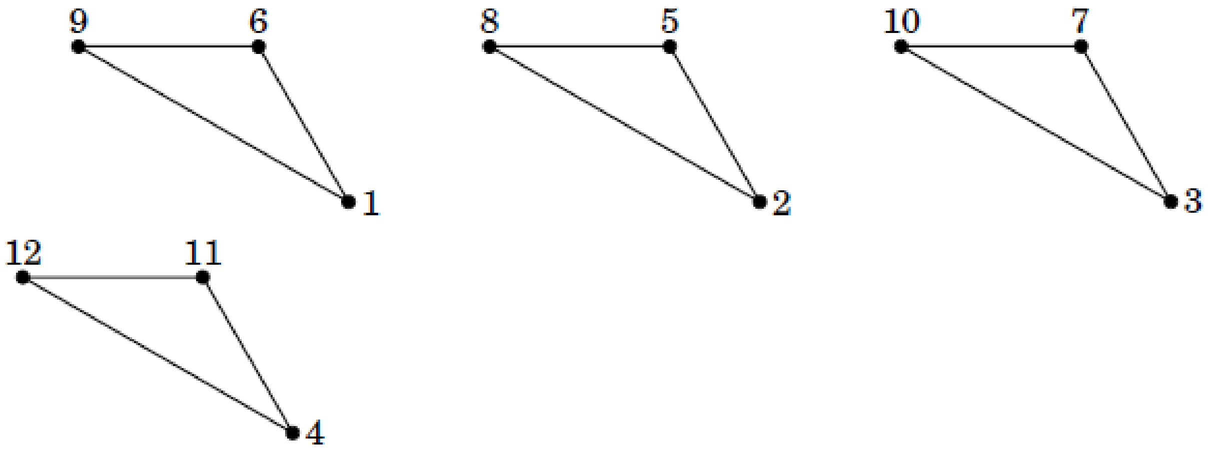


Figure 4.3.6: Graph Γ_5

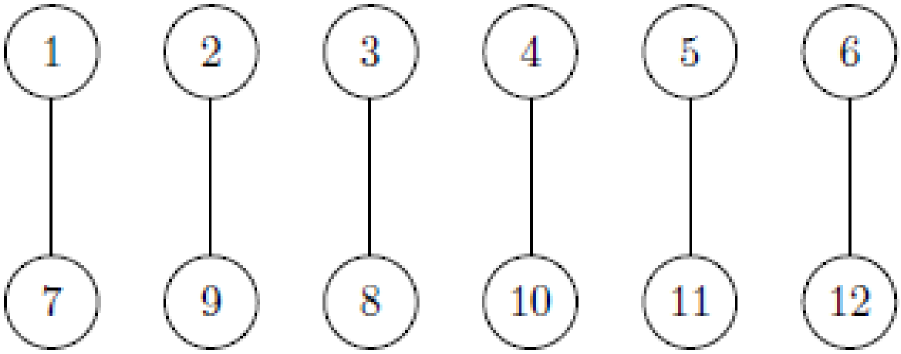


Figure 4.3.7: Graph Γ_6

Appendix IV: Suborbitals and suborbital graphs of a dodecahedron

Using **Definition 1.5.27** we compute the following suborbitals,

X×X = { (1,1), (1,2), (1,3), (1,4), (1,5), (1,6), (1,7), (1,8), (1,9), (1,10), (1,11), (1,12), (1,13), (1,14), (1,15), (1,16), (1,17), (1,18), (1,19), (1,20), (1,21), (1,22), (1,23), (1,24), (1,25), (1,26), (1,27), (1,28), (1,29), (1,30), (2,1), (2,2), (2,3), (2,4), (2,5), (2,6), (2,7), (2,8), (2,9), (2,10), (2,11), (2,12), (2,13), (2,14), (2,15), (2,16), (2,17), (2,18), (2,19), (2,20), (2,21), (2,22), (2,23), (2,24), (2,25), (2,26), (2,27), (2,28), (2,29), (2,30), (3,1), (3,2), (3,3), (3,4), (3,5), (3,6), (3,7), (3,8), (3,9), (3,10), (3,11), (3,12), (3,13), (3,14), (3,15), (3,16), (3,17), (3,18), (3,19), (3,20), (3,21), (3,22), (3,23), (3,24), (3,25), (3,26), (3,27), (3,28), (3,29), (3,30), (4,1), (4,2), (4,3), (4,4), (4,5), (4,6), (4,7), (4,8), (4,9), (4,10), (4,11), (4,12), (4,13), (4,14), (4,15), (4,16), (4,17), (4,18), (4,19), (4,20), (4,21), (4,22), (4,23), (4,24), (4,25), (4,26), (4,27), (4,28), (4,29), (4,30), (5,1), (5,2), (5,3), (5,4), (5,5), (5,6), (5,7), (5,8), (5,9), (5,10), (5,11), (5,12), (5,13), (5,14), (5,15), (5,16), (5,17), (5,18), (5,19), (5,20), (5,21), (5,22), (5,23), (5,24), (5,25), (5,26), (5,27), (5,28), (5,29), (5,30), (6,1), (6,2), (6,3), (6,4), (6,5), (6,6), (6,7), (6,8), (6,9), (6,10), (6,11), (6,12), (6,13), (6,14), (6,15), (6,16), (6,17), (6,18), (6,19), (6,20), (6,21), (6,22), (6,23), (6,24), (6,25), (6,26), (6,27), (6,28), (6,29), (6,30), (7,1), (7,2), (7,3), (7,4), (7,5), (7,6), (7,7), (7,8), (7,9), (7,10), (7,11), (7,12), (7,13), (7,14), (7,15), (7,16), (7,17), (7,18), (7,19), (7,20), (7,21), (7,22), (7,23), (7,24), (7,25), (7,26), (7,27), (7,28), (7,29), (7,30), (8,1), (8,2), (8,3), (8,4), (8,5), (8,6), (8,7), (8,8), (8,9), (8,10), (8,11), (8,12), (8,13), (8,14), (8,15), (8,16), (8,17), (8,18), (8,19), (8,20), (8,21), (8,22), (8,23), (8,24), (8,25), (8,26), (8,27), (8,28), (8,29), (8,30), (9,1), (9,2), (9,3), (9,4), (9,5), (9,6), (9,7), (9,8), (9,9), (9,10), (9,11), (9,12), (9,13), (9,14), (9,15), (9,16), (9,17), (9,18), (9,19), (9,20), (9,21), (9,22), (9,23), (9,24), (9,25), (9,26), (9,27), (9,28), (9,29), (9,30), (10,1), (10,2), (10,3), (10,4), (10,5), (10,6), (10,7), (10,8), (10,9), (10,10), (10,11), (10,12), (10,13), (10,14), (10,15), (10,16), (10,17), (10,18), (10,19), (10,20), (10,21), (10,22), (10,23), (10,24), (10,25), (10,26), (10,27), (10,28), (10,29), (10,30), (11,1), (11,2), (11,3), (11,4), (11,5), (11,6), (11,7), (11,8), (11,9), (11,10), (11,11), (11,12), (11,13), (11,14), (11,15), (11,16), (11,17), (11,18), (11,19), (11,20), (11,21), (11,22), (11,23), (11,24), (11,25), (11,26), (11,27), (11,28), (11,29), (11,30), (12,1), (12,2), (12,3), (12,4), (12,5), (12,6), (12,7), (12,8), (12,9), (12,10), (12,11), (12,12), (12,13), (12,14), (12,15), (12,16), (12,17), (12,18), (12,19), (12,20), (12,21), (12,22), (12,23), (12,24), (12,25), (12,26), (12,27), (12,28), (12,29), (12,30), (13,1), (13,2), (13,3), (13,4), (13,5), (13,6), (13,7), (13,8), (13,9), (13,10), (13,11), (13,12), (13,13), (13,14), (13,15), (13,16), (13,17), (13,18), (13,19), (13,20), (13,21), (13,22), (13,23), (13,24), (13,25), (13,26), (13,27), (13,28), (13,29), (13,30), (14,1), (14,2), (14,3), (14,4), (14,5), (14,6), (14,7), (14,8), (14,9), (14,10), (14,11), (14,12), (14,13), (14,14), (14,15), (14,16), (14,17), (14,18), (14,19), (14,20), (14,21), (14,22), (14,23), (14,24), (14,25), (14,26), (14,27), (14,28), (14,29), (14,30), (15,1), (15,2), (15,3), (15,4), (15,5), (15,6), (15,7), (15,8), (15,9), (15,10), (15,11), (15,12), (15,13), (15,14), (15,15), (15,16), (15,17), (15,18), (15,19), (15,20), (15,21), (15,22), (15,23), (15,24), (15,25), (15,26), (15,27), (15,28), (15,29), (15,30), (16,1), (16,2), (16,3), (16,4), (16,5), (16,6), (16,7), (16,8), (16,9), (16,10), (16,11), (16,12), (16,13), (16,14), (16,15), (16,16), (16,17), (16,18), (16,19), (16,20), (16,21), (16,22), (16,23), (16,24), (16,25), (16,26), (16,27), (16,28), (16,29), (16,30), (17,1), (17,2), (17,3), (17,4), (17,5), (17,6), (17,7), (17,8), (17,9), (17,10), (17,11), (17,12), (17,13), (17,14), (17,15), (17,16), (17,17), (17,18), (17,19), (17,20), (17,21), (17,22), (17,23), (17,24), (17,25), (17,26), (17,27), (17,28), (17,29), (17,30), (18,1), (18,2), (18,3), (18,4), (18,5), (18,6), (18,7), (18,8), (18,9), (18,10), (18,11), (18,12), (18,13), (18,14), (18,15), (18,16), (18,17), (18,18), (18,19), (18,20), (18,21), (18,22), (18,23), (18,24), (18,25), (18,26), (18,27), (18,28), (18,29), (18,30), (19,1), (19,2), (19,3), (19,4), (19,5), (19,6), (19,7), (19,8), (19,9), (19,10), (19,11), (19,12), (19,13), (19,14), (19,15), (19,16), (19,17), (19,18), (19,19), (19,20), (19,21), (19,22), (19,23), (19,24), (19,25), (19,26), (19,27), (19,28), (19,29), (19,30), (20,1), (20,2), (20,3), (20,4), (20,5), (20,6), (20,7), (20,8), (20,9), (20,10), (20,11), (20,12), (20,13), (20,14), (20,15), (20,16), (20,17), (20,18), (20,19), (20,20), (20,21), (20,22), (20,23), (20,24), (20,25), (20,26), (20,27), (20,28), (20,29), (20,30), (21,1), (21,2), (21,3), (21,4), (21,5), (21,6), (21,7), (21,8), (21,9), (21,10), (21,11), (21,12), (21,13), (21,14), (21,15), (21,16), (21,17), (21,18), (21,19), (21,20), (21,21), (21,22), (21,23), (21,24), (21,25), (21,26), (21,27), (21,28), (21,29), (21,30), (22,1), (22,2), (22,3), (22,4), (22,5), (22,6), (22,7), (22,8), (22,9), (22,10), (22,11), (22,12), (22,13), (22,14), (22,15), (22,16), (22,17), (22,18), (22,19), (22,20), (22,21), (22,22), (22,23), (22,24), (22,25), (22,26), (22,27), (22,28), (22,29), (22,30), (23,1), (23,2), (23,3), (23,4), (23,5), (23,6), (23,7), (23,8), (23,9), (23,10), (23,11), (23,12), (23,13), (23,14), (23,15), (23,16), (23,17), (23,18), (23,19), (23,20), (23,21), (23,22), (23,23), (23,24), (23,25), (23,26), (23,27), (23,28), (23,29), (23,30), (24,1), (24,2), (24,3), (24,4), (24,5), (24,6), (24,7), (24,8), (24,9), (24,10), (24,11), (24,12), (24,13), (24,14), (24,15), (24,16), (24,17), (24,18), (24,19), (24,20), (24,21), (24,22), (24,23), (24,24), (24,25), (24,26), (24,27), (24,28), (24,29), (24,30), (25,1), (25,2), (25,3), (25,4), (25,5), (25,6), (25,7), (25,8), (25,9), (25,10), (25,11), (25,12), (25,13), (25,14), (25,15), (25,16), (25,17), (25,18), (25,19), (25,20), (25,21), (25,22), (25,23), (25,24), (25,25), (25,26), (25,27), (25,28), (25,29), (25,30), (26,1), (26,2), (26,3), (26,4), (26,5), (26,6), (26,7), (26,8), (26,9), (26,10), (26,11), (26,12), (26,13), (26,14), (26,15), (26,16), (26,17), (26,18), (26,19), (26,20), (26,21), (26,22), (26,23), (26,24), (26,25), (26,26), (26,27), (26,28), (26,29), (26,30), (27,1), (27,2), (27,3), (27,4), (27,5), (27,6), (27,7), (27,8), (27,9), (27,10), (27,11), (27,12), (27,13), (27,14), (27,15), (27,16), (27,17), (27,18), (27,19), (27,20), (27,21), (27,22), (27,23), (27,24), (27,25), (27,26), (27,27), (27,28), (27,29), (27,30), (28,1), (28,2), (28,3), (28,4), (28,5), (28,6), (28,7), (28,8), (28,9), (28,10), (28,11), (28,12), (28,13), (28,14), (28,15), (28,16), (28,17), (28,18), (28,19), (28,20), (28,21), (28,22), (28,23), (28,24), (28,25), (28,26), (28,27), (28,28), (28,29), (28,30), (29,1), (29,2), (29,3), (29,4), (29,5), (29,6), (29,7), (29,8), (29,9), (29,10), (29,11), (29,12), (29,13), (29,14), (29,15), (29,16), (29,17), (29,18), (29,19), (29,20), (29,21), (29,22), (29,23), (29,24), (29,25), (29,26), (29,27), (29,28), (29,29), (29,30), (30,1), (30,2), (30,3), (30,4), (30,5), (30,6), (30,7), (30,8), (30,9), (30,10), (30,11), (30,12), (30,13), (30,14), (30,15), (30,16), (30,17), (30,18), (30,19), (30,20), (30,21), (30,22), (30,23), (30,24), (30,25), (30,26), (30,27), (30,28), (30,29), (30,30)}

Orb_G (1, 1) = O_0 = {(1, 1), (2, 2), (3, 3), (4, 4), (5, 5), (6, 6), (7,7), (8,8), (9,9), (10,10), (11,11), (12,12), (13, 13), (14, 14), (15, 15), (16, 16), (17, 17), (18, 18), (19, 19), (20, 20), (21, 21), (22,22), (23,23), (24,24), (25,25), (26, 26), (27,27), (28,28), (29,29), (30,30)}

Orb_G (1, 2) = O_1 = {(1,2), (2,3), (3,4), (4,5), (5,1), (20,6), (23,7), (10,8), (3,9), (17,18), (26,30), (28,25), (9,10), (18,5), (19,17), (20,21), (6,7), (7,8), (24,28), (29,13), (15,4), (4,12), (14,29), (30,27), (19,20), (9,2), (13,14), (11,12), (1,18), (15,16), (27,26), (17,30), (25,29), (22,27), (24,22), (29,26), (28,10), (25,24), (26,25), (7,20), (5,15), (18,19), (12,13), (16,17), (23,24), (30,16), (6,1), (2,6), (14,15), (10,11), (12,3), (8,9), (22,23), (13,11), (11,28), (27,21), (21,19), (8,23), (16,14), (21,22)}

Orb_G (1, 3) = O_2 = {((1,3), (18,20), (5,16), (16,18), (12,14), (30,14), (23,28), (6,18), (2,7), (14,4), (10,12), (12,9), (8,2), (22,7), (13,28), (11,25), (27,19), (21,17), (8,24), (16,29), (21,23), (9,6), (13,15), (11,3), (1,19), (15,17), (27, 25), (17,27), (25,13), (22,26), (24,27), (29,30), (28,8), (25,22), (26,24), (7,21), (2,4), (3,5), (4,1), (5,2), (20,1), (23,20), (10,23), (3,10), (17,5), (26,16), (28,29),

(9,11), (18,15), (19,30), (20,22), (6,8), (7,9), (24,10), (29,11), (15,12), (4,13), (14,26), (30,21), (19,6)}

Orb_G (1, 4) = O_3 = {(1,4), (4,14), (14,30), (30,19), (19,1), (7,2), (24,8), (29,28), (15,13), (18,16), (19,27), (20,23), (6,9), (17,15), (26,14), (28,13), (9,12), (20,18), (23,21), (10,24), (3,11), (2,5), (3,1), (4,2), (5,3), (9,7), (13,4), (11,9), (1,20), (15,18), (27,24), (17,21), (25,11), (22,25), (24,26), (29,16), (28,23), (25,27), (26,22), (7,22), (18,6), (5,17), (16,5), (12,15), (30,29), (23,10), (6,19), (2,8), (14,12), (10,3), (12,10), (8,6), (22,20), (13,25), (11,29), (27,17), (21,30), (8,28), (16,26), (21,7)}

Orb_G (1, 5) = O_4 = {(1,5), (5,4), (4,3), (3,2), (2,1), (10,28), (3,12), (20,19), (23,22), (28,11), (9,3), (26,29), (17,16), (20,7), (6,2), (18,17), (19,21), (29,25), (15,14), (7,6), (24,23), (30,17), (19,18), (14, 16), (4,15), (9,8), (13,12), (11,10), (1,6), (15,5), (27,22), (28,24), (17,19), (25,28), (22,24), (29,14), (24,25), (25,26), (26,27), (7,23), (18,1), (5,18), (16,15), (12,4), (30,26), (23,8), (6,20), (2,9), (14,13), (10,9), (12,11), (8,7), (22,21), (13,29), (11,13), (27,30), (21,27), (8,10), (16,30), (21,20)}

Orb_G (1, 7) = O_5 = {(1,7), (4,9), (14,11), (30,25), (19,22), (7,24), (24,29), (29,15), (15,1), (18,2), (19,5), (20,17), (6,21), (17,20), (26,21), (28,22), (9,23), (20,8), (23,9), (10,2), (3,6), (2,10), (3,13), (4,16), (5,19), (9,4), (13,26), (11,14), (1,15), (15,29), (27,16), (28,12), (17,14), (25,30), (22,19), (24,7), (29,24), (25,10), (26,13), (7,1), (18,30), (5,12), (16,27), (12,28), (30,18), (23,27), (6,3), (2,18), (14,17), (10,25), (12,5), (8,11), (22,28), (13,3), (11,8), (27,23), (21,6), (8,20), (16,4), (21,26)}

Orb_G (1, 8) = O_6 ={(1,8), (4,10), (14,28), (30,24), (19,23), (7,28), (24,13), (29,4), (15,2), (18,3), (19,15), (20,30), (6,22), (17,6), (26,19), (28,27), (9,24), (20,9), (23,2), (10,6), (3,7), (2,11), (3,14), (4,17), (5,20), (9,5), (13,30), (11,15), (1,16), (15,26), (27,14), (28,3), (17,29), (25,16), (22,17), (24,20), (29,22), (25,8), (26,11), (7,18), (18,27), (5,13), (16,21), (12,25), (30,5), (23,26), (6,4), (2,19), (14,18), (10,29), (12,1), (8,12), (22,10), (13,9), (11,23), (27,7), (21,1), (8,21), (16,12), (21,25)}

Orb_G (1, 9) = O_7 = {(1,9), (4,11), (14,25), (30,22), (19,7), (7,10), (24,11), (29,12), (15,3), (18,4), (19,16), (20,27), (6,23), (17,1), (26,17), (28,26), (9,28), (20,2), (23,6), (10,7), (3,8), (2,12), (3,15), (4,18), (5,6), (9,1), (13,16), (11,4), (1,17), (15,30), (27,29), (28,9), (17,26), (25,14), (22,30), (24,21), (29,27), (25,23), (26,28), (7,19), (18,21), (5,14), (16,19), (12,29), (30,15), (23,25), (6,5), (2,20), (14,5), (10,13), (12,2), (8,3), (22,8), (13,10), (11,24), (27,20), (21,18), (8,22), (16,13), (21,24)}

Orb_G (1, 10) = O_8 = {(1,10), (4,28), (14,24), (30,23), (19,8), (7,11), (24,12), (29,3), (15,9), (18,12), (19,14), (20,26), (6,24), (17,2), (26,18), (28,30), (9,25), (20,3), (23,1), (10,20), (3,23), (2,13), (3.16), (4,19), (5,7), (9,18), (13, 17), (11,5), (1,30), (15,27), (27,13), (28,2), (17,25), (25,15), (22,16), (24,19), (29,21), (25,7), (26,10), (7,17), (18,22), (5,29), (16,20), (12,26), (30,4), (23,29), (6,15), (2,21), (14,1), (10,14), (12,6), (8,4), (22,9), (13,8), (11,22), (27,6), (21,5), (8,27), (16,11), (21,28)}

Orb_G (1, 11) = O_9 ={(1,11), (4,25), (14,22), (30,7), (19,9), (7,12), (24,3), (29,9), (15,10), (18,13), (19,29), (20,25), (6,28), (17,3), (26,5), (28,16), (9,29), (20,4), (23,18), (10,21), (3,24), (2,14), (3,17), (4,20), (5,8), (9,19), (13,18), (11,1), (1,27), (15,21), (27,11), (28,6), (17,24), (25,4), (22,14), (24,17), (29,19), (25,20), (26,8), (7,30), (18,23), (5,26), (16,6), (12,30), (30,12), (23,13), (6,16), (2,22), (14,2), (10,15), (12,7), (8,5), (22,2), (13,23), (11,27), (27,1), (21,15), (8,26), (16,28), (21,10)}

$\mathrm{Orb}_G(1, 12) = O_{10}$ ={(1,12), (4,29), (14,27), (30,20), (19,2), (7,3), (24,9), (29,10), (15,11), (18,14), (19,26), (20,24), (6,10), (17,4), (26,15), (28,14), (9,13), (20,5), (23,19), (10,22), (3,28), (2,15), (3,18), (4,6), (5,9), (9,20), (13,5), (11,2), (1,21), (15,19), (27,28), (28,7), (17,22), (25,12), (22,29), (24,30), (29,17), (25,21), (26,23), (7,27), (18,7), (5,30), (16,1), (12,16), (30,13), (23,11), (6,17), (2,23), (14,3), (10,4), (12,8), (8,1), (22,6), (13,24), (11,26), (27,18), (21,16), (8,25), (16,25), (21,8)}

$\mathrm{Orb}_G(1, 13) = O_{11}$ = {(1,13), (4,26), (14,21), (30,6), (19,3), (7,4), (24,2), (29,8), (15,28), (18,29), (19,25), (20,28), (6,11), (17,12), (26,4), (28,15), (9,14), (20,15), (23,17), (10,27), (3,25), (2,16), (3,19), (4,7), (5,10), (9,21), (13,1), (11,6), (1,22), (15,20), (27,10), (28,20), (17,23), (25,3), (22,13), (24,16), (29,18), (25,19), (26,7), (7,26), (18,8), (5,27), (16,2), (12,17), (30,11), (23,12), (6,30), (2,24), (14,9), (10,5), (12,23), (8,18), (22,1), (13,22), (11,30), (27,5), (21,14), (8,29), (16,24), (21,9)}

$\mathrm{Orb}_G(1, 14) = O_{12}$ = {(1,14), (4,30), (14,19), (30,1), (19,4), (7,5), (24,6), (29,23), (15,25), (18,26), (19,24), (20,10), (6,12), (17,13), (26,12), (28,4), (9,15), (20,16), (23,30), (10,26), (3,29), (2,17), (3,20), (4,8), (5,11), (9,22), (13,2), (11,7), (1,23), (15,6), (27,8), (28,21), (17,7), (25,9), (22,11), (24,14), (29,5), (25,17), (26,20), (7,25), (18,9), (5,21), (16,3), (12,18), (30,28), (23,3), (6,27), (2,28), (14,10), (10,1), (12,24), (8,19), (22,18), (13,27), (11,16), (27,15), (21,29), (8,13), (16,22), (21,2)}

$\mathrm{Orb}_G(1, 24) = O_{13}$ = {(1,24), (4,23), (14,8), (30,10), (19,28), (7,29), (24,15), (29,1), (15,7), (18,10), (19,12), (20,14), (6,26), (17,8), (26,6), (28,19), (9,27), (20,11), (23,4), (10,18), (3,21), (2,25), (3,26), (4,27), (5,22), (9,16), (13,21), (11,17), (1,29), (15,24), (27,4), (28,5), (17,11), (25,18), (22,5), (24,1), (29,7), (25,2), (26,3), (7,15), (18,25), (5,28), (16,23), (12,22), (30,2), (23,16), (6,13),

(2,30), (14,20), (10,30), (12,19), (8,14), (22,12), (13,6), (11,20), (27,9), (21,3), (8,17), (16,9), (21,13)}

Orb_G (1, 25) = O_{14} = {(1,25), (4,22), (14,7), (30,9), (19,11), (7,14), (24,5), (29,6), (15,23), (18,28), (19,11), (20,13), (6,29), (17,10), (26,2), (28,18), (9,30), (20,13), (23,15), (10,17), (3,27), (2,26), (3,27), (4,22), (5,24), (9,30), (13,20), (11,19), (1,25), (15,23), (27,3), (28,18), (17,10), (25,1), (22,4), (24,5), (29,6), (25,1), (26,2), (7,14), (18,28), (5,24), (16,8), (12,21), (30,9), (23,15), (6,29), (2,26), (14,7), (10,17), (12,21), (8,16), (22,4), (13,20), (11,19), (27,3), (21,12), (8,16), (16,8), (21,12)}

Orb_G (1, 26) = O_{15} = {(1,26), (4,21), (14,6), (30,3), (19,13), (7,16), (24,18), (29,20), (15,22), (18,24), (19,10), (20,12), (6,14), (17,28), (26,9), (28,1), (9,17), (20,29), (23,14), (10,16), (3,30), (2,27), (3,22), (4,24), (5,25), (9,26), (13,7), (11,21), (1,28), (15,8), (27,2), (28,17), (17,9), (25,6), (22,3), (24,4), (29,2), (25,5), (26,1), (7,13), (18,11), (5,23), (16,10), (12,20), (30,8), (23,5), (6,25), (2,29), (14,23), (10,19), (12,27), (8,30), (22,15), (13,19), (11,18), (27,12), (21,11), (8,15), (16,7), (21,4)}

Next we construct the corresponding non-trivial suborbital graphs,

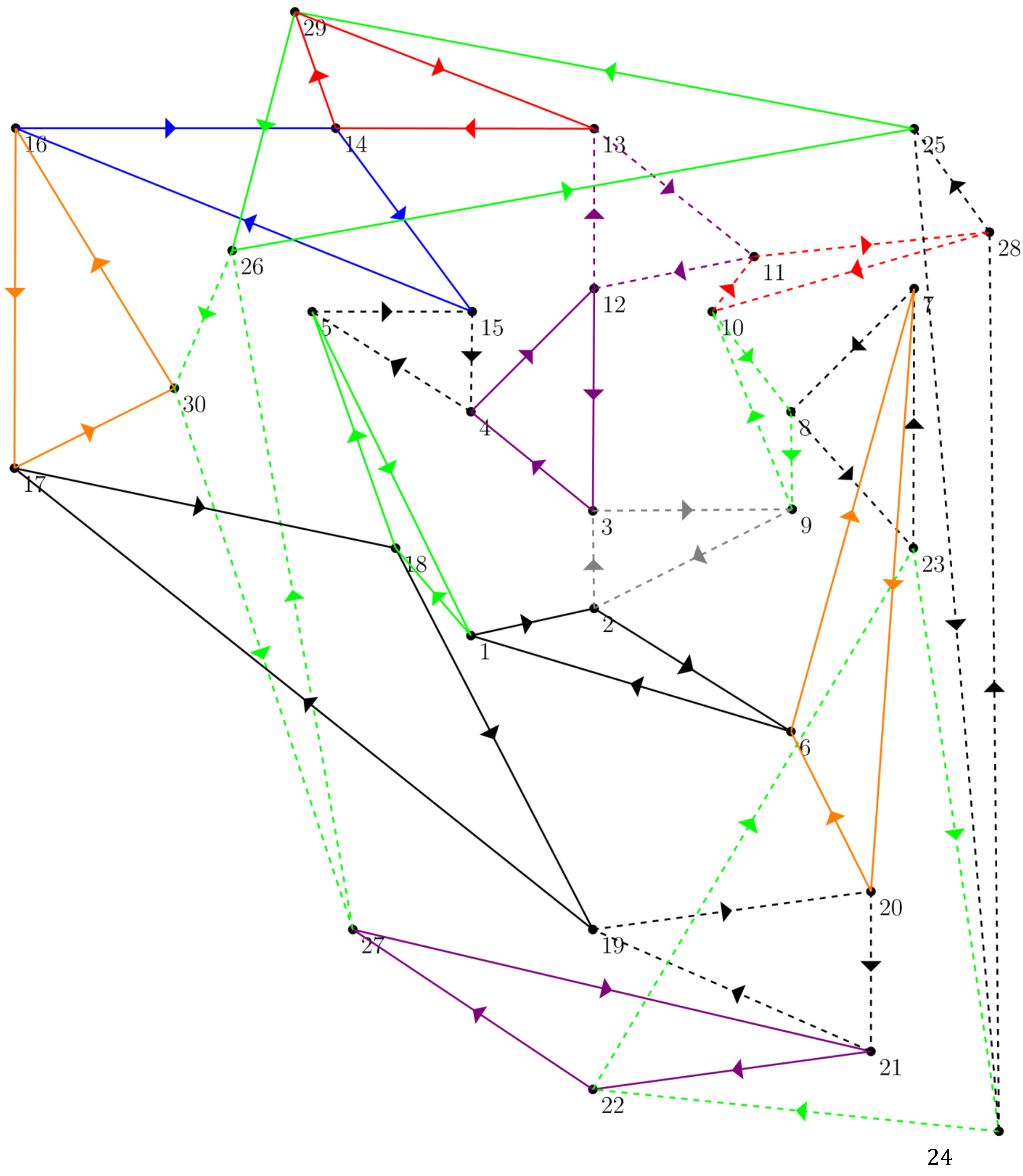


Figure 4.4.2: Graph Γ₁

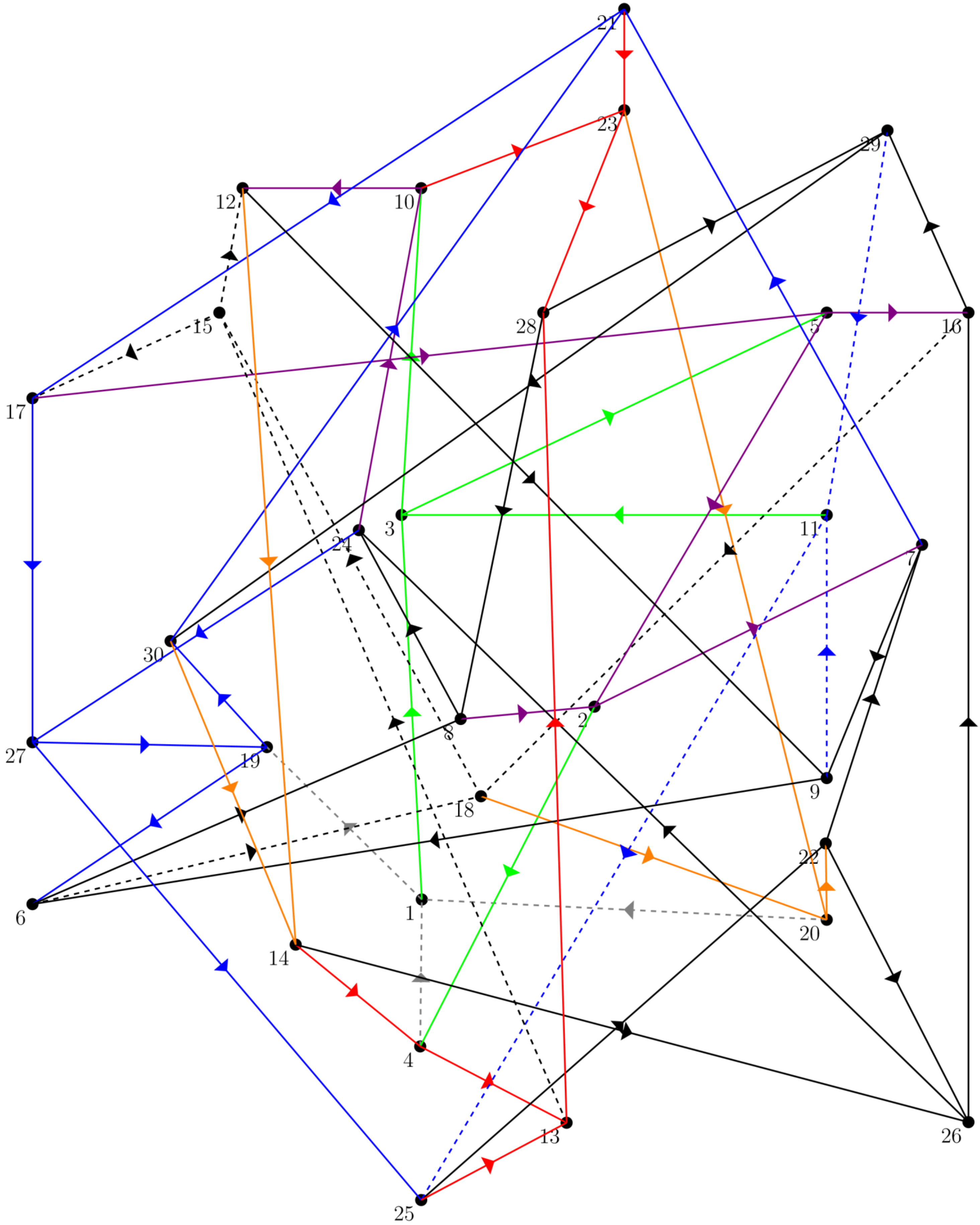


Figure 4.4.3: Graph Γ_2

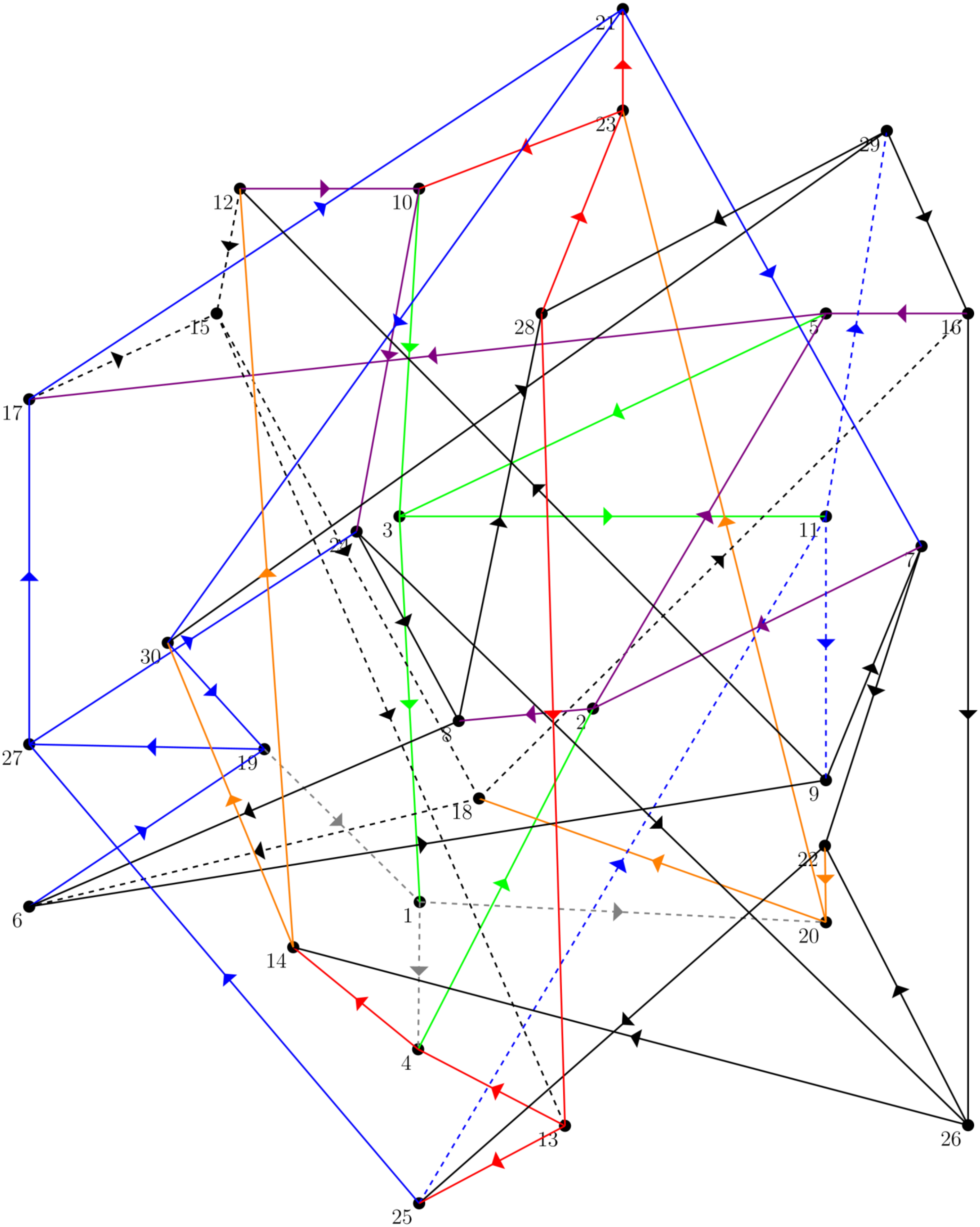


Figure 4.4.4: Graph Γ_3

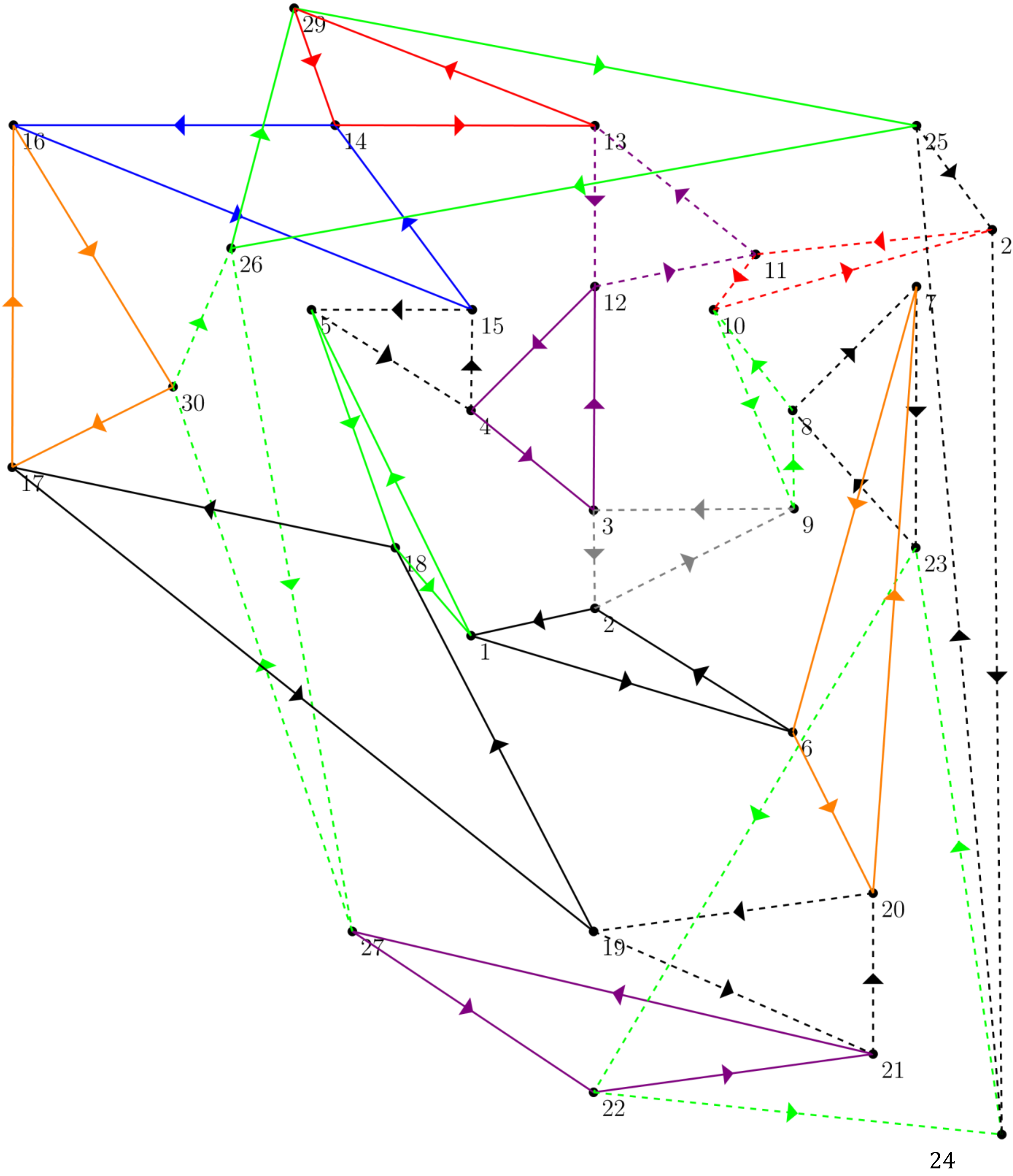


Figure 4.4.5: Graph Γ_4

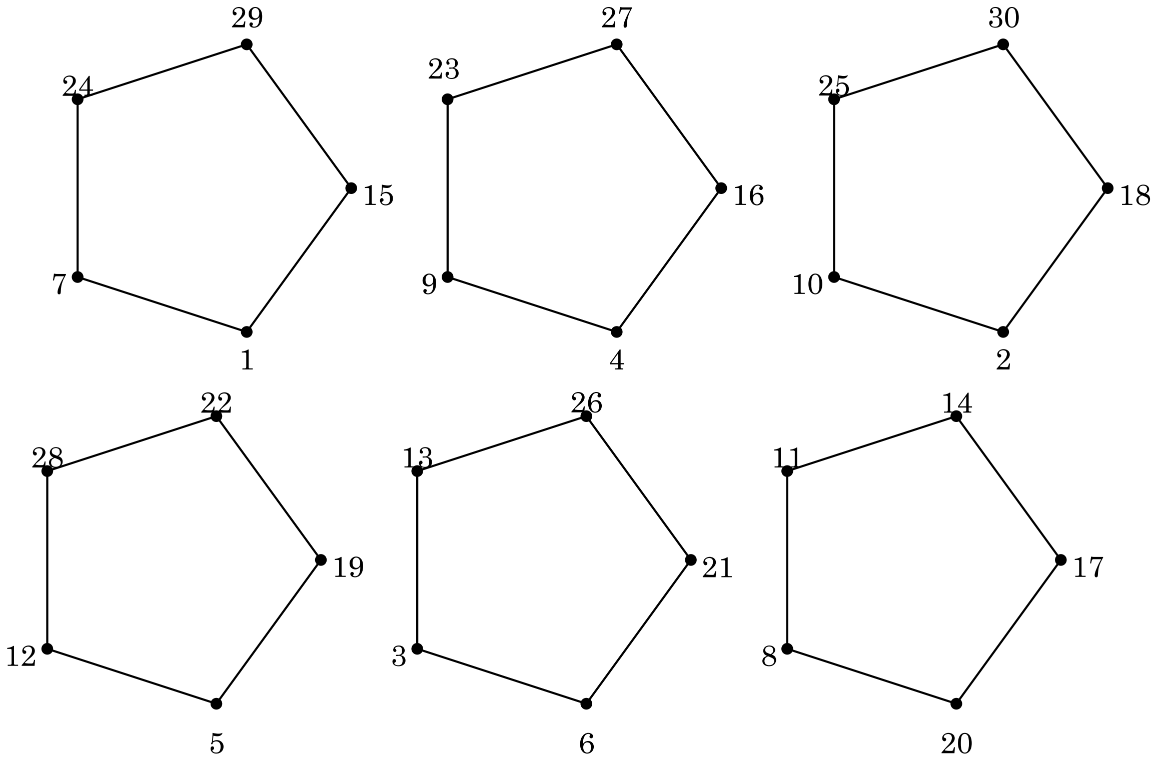


Figure 4.4.6 Graph Γ_5

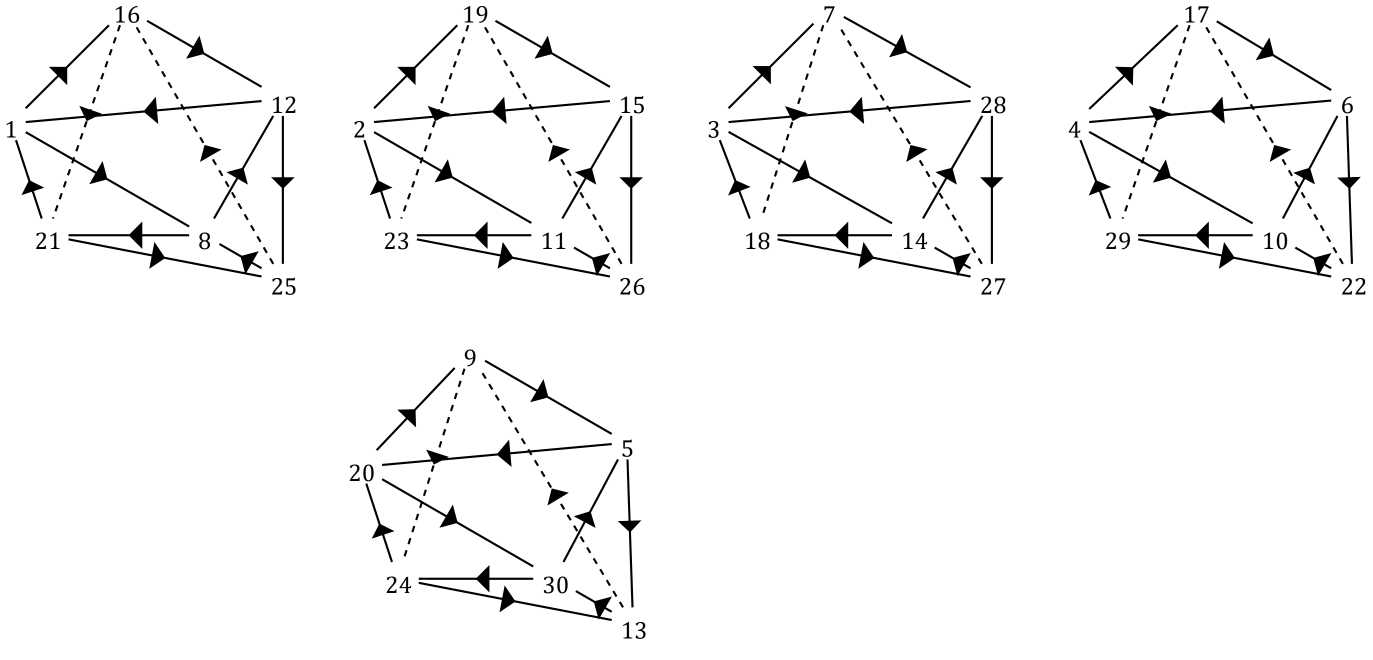


Figure 4.4.7: Graph Γ_6

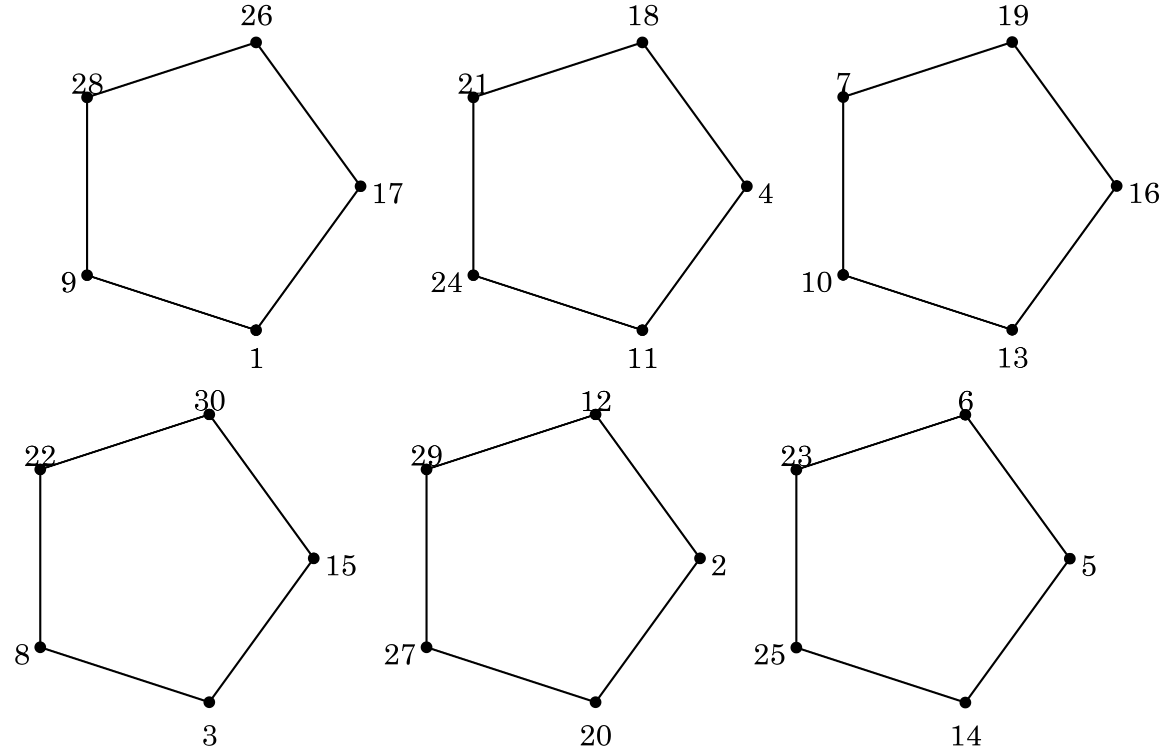


Figure 4.4.8: Graph Γ_7

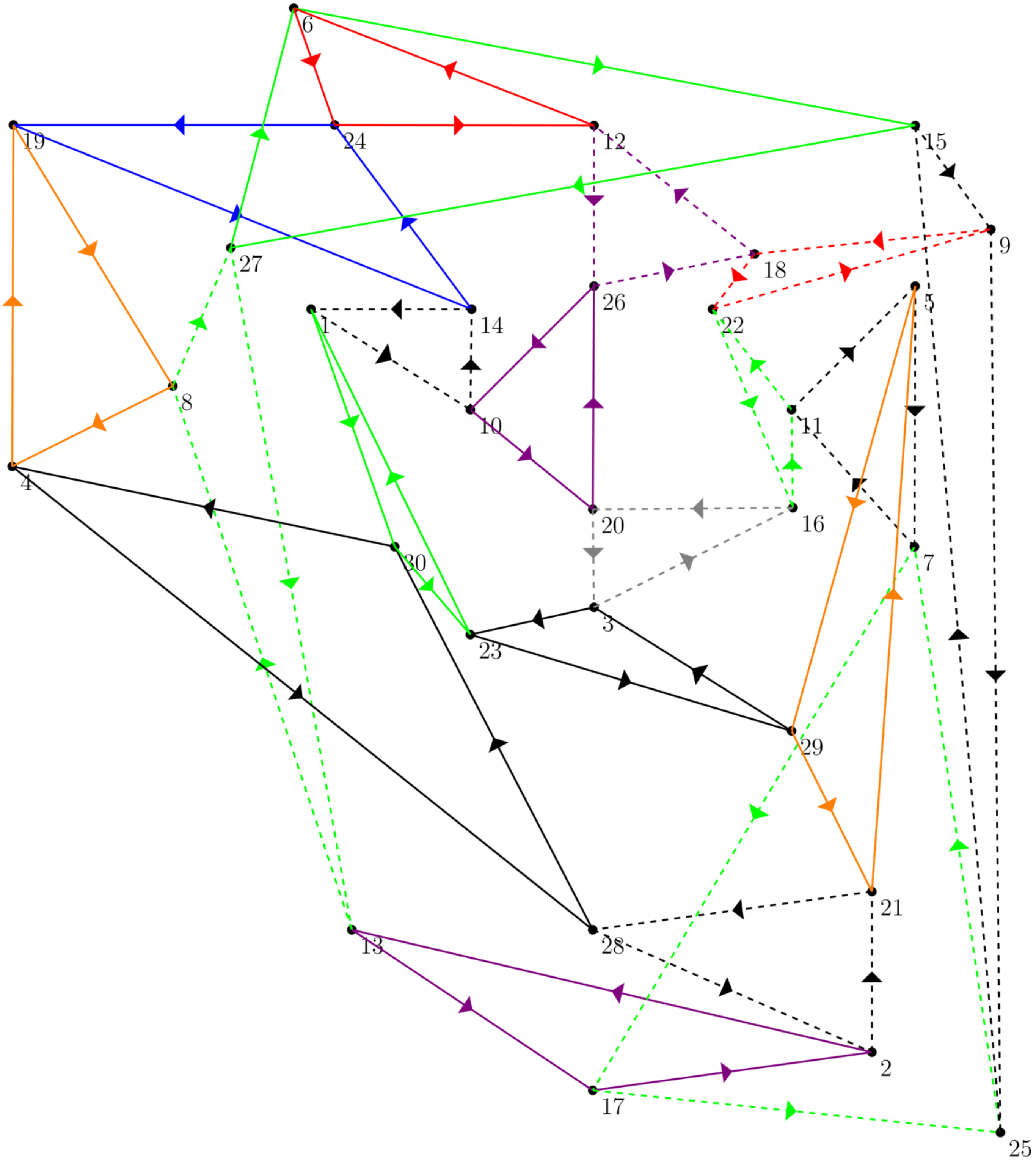


Figure 4.4.9: Graph Γ_8

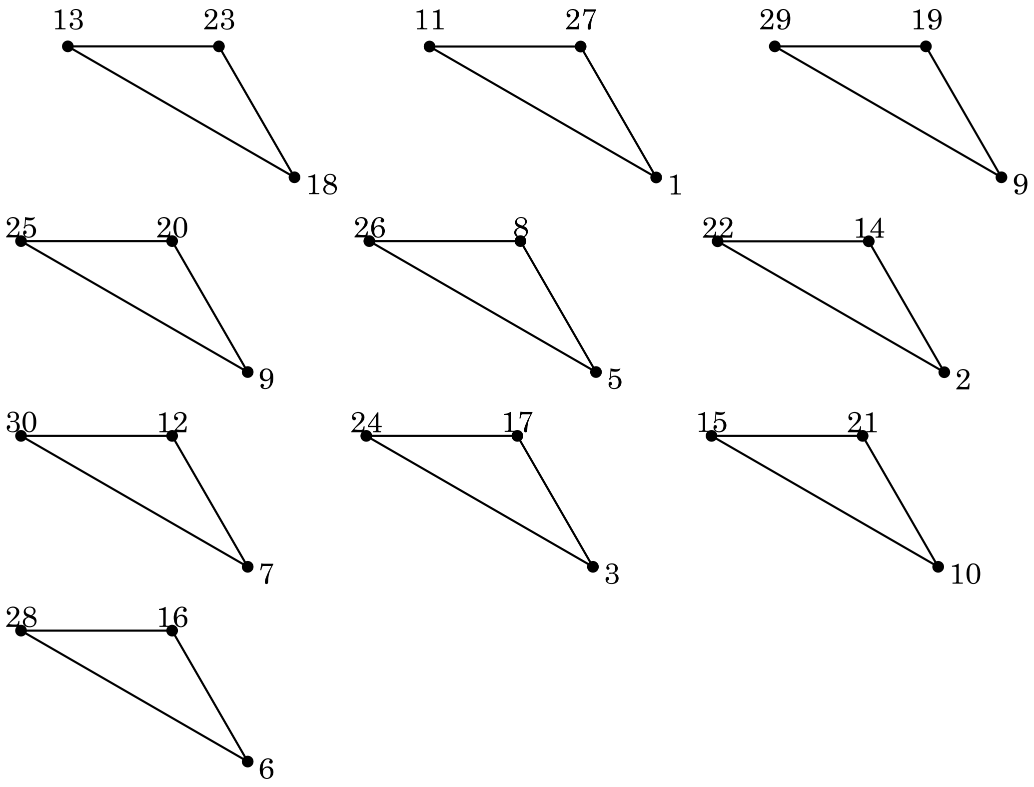


Figure 4.4.10: Graph Γ_9

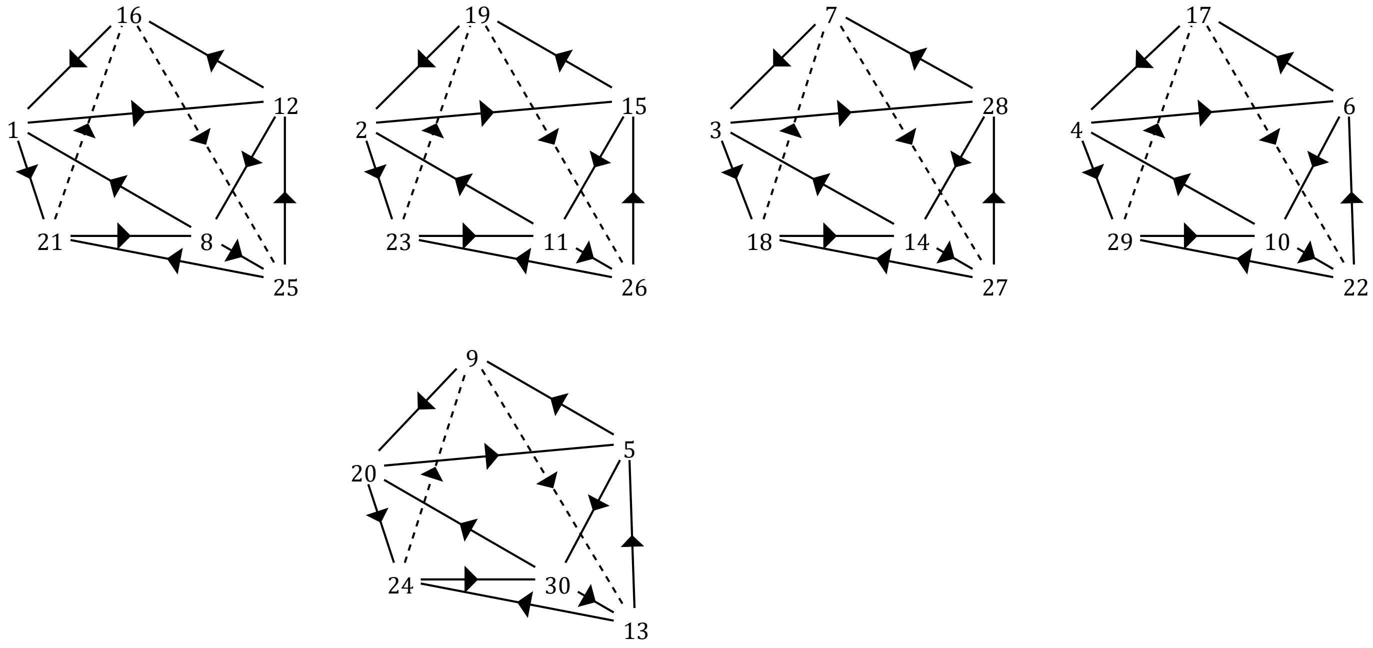


Figure 4.4.11: Graph Γ_{10}

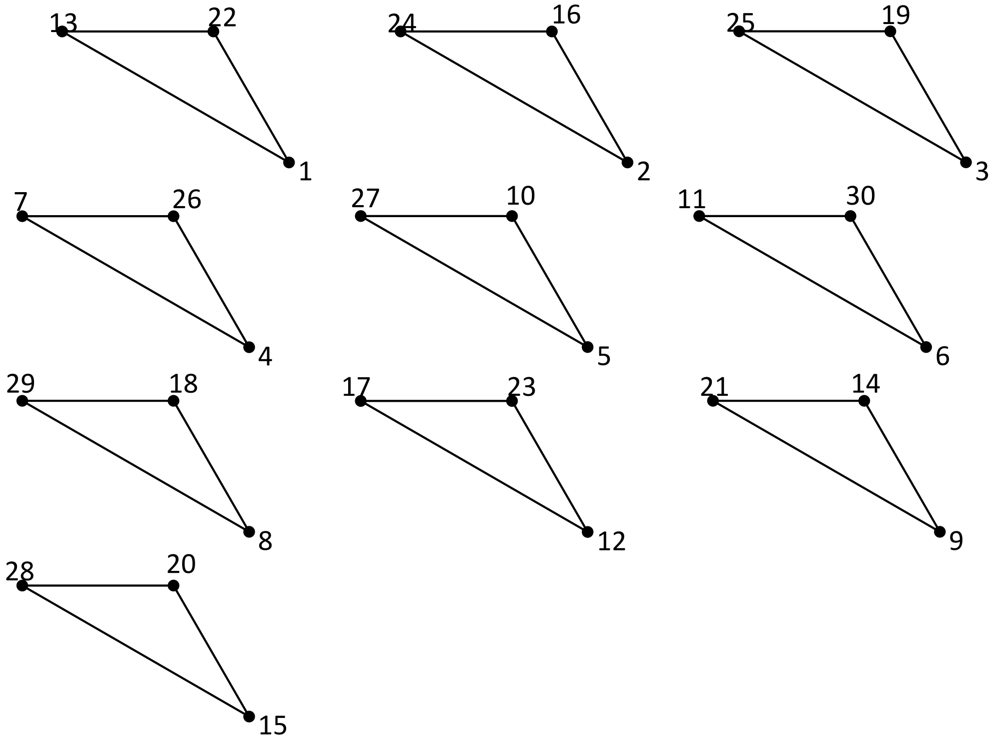


Figure 4.4.12: Graph Γ_{11}

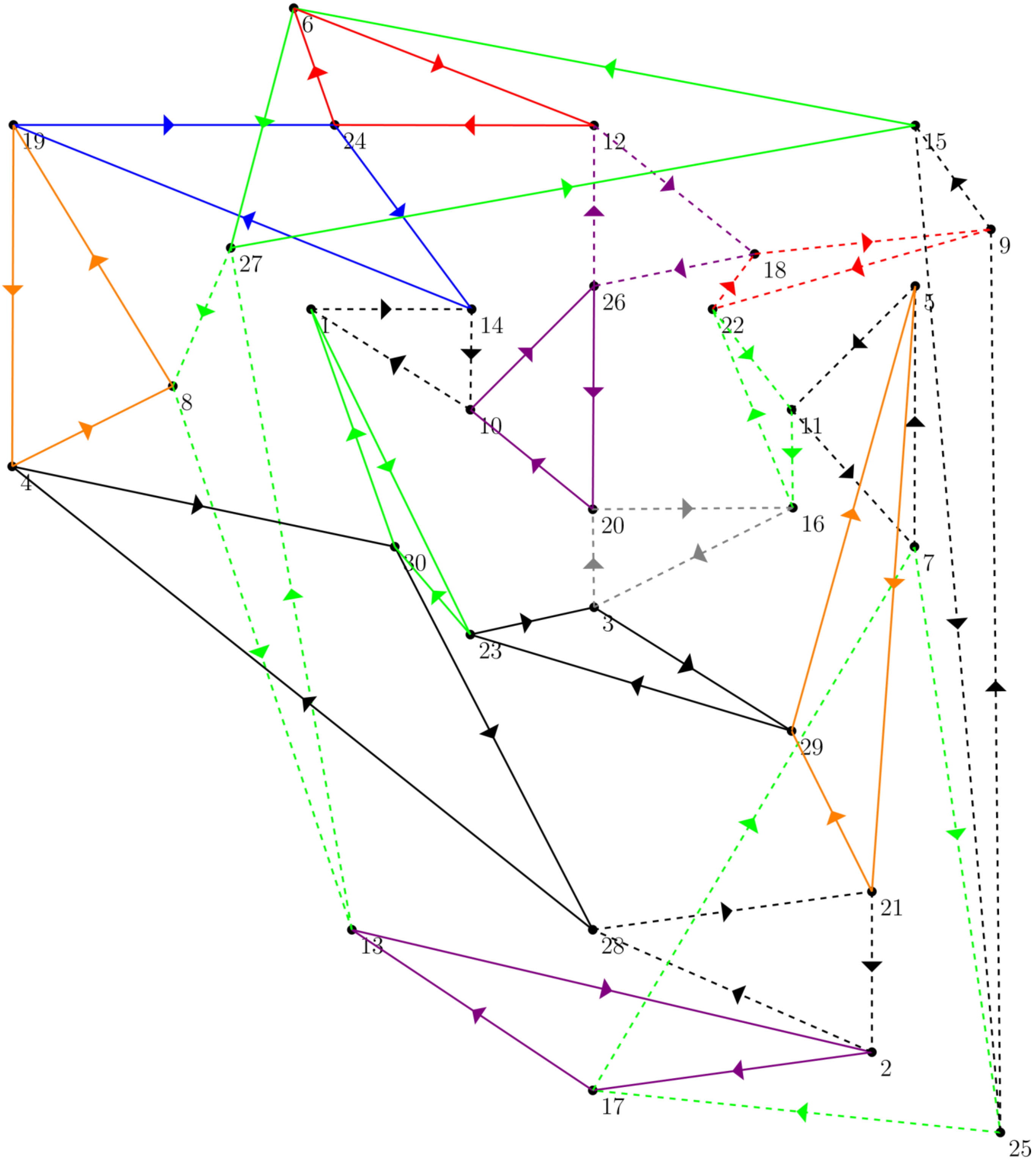


Figure 4.4.13: Graph Γ_{12}

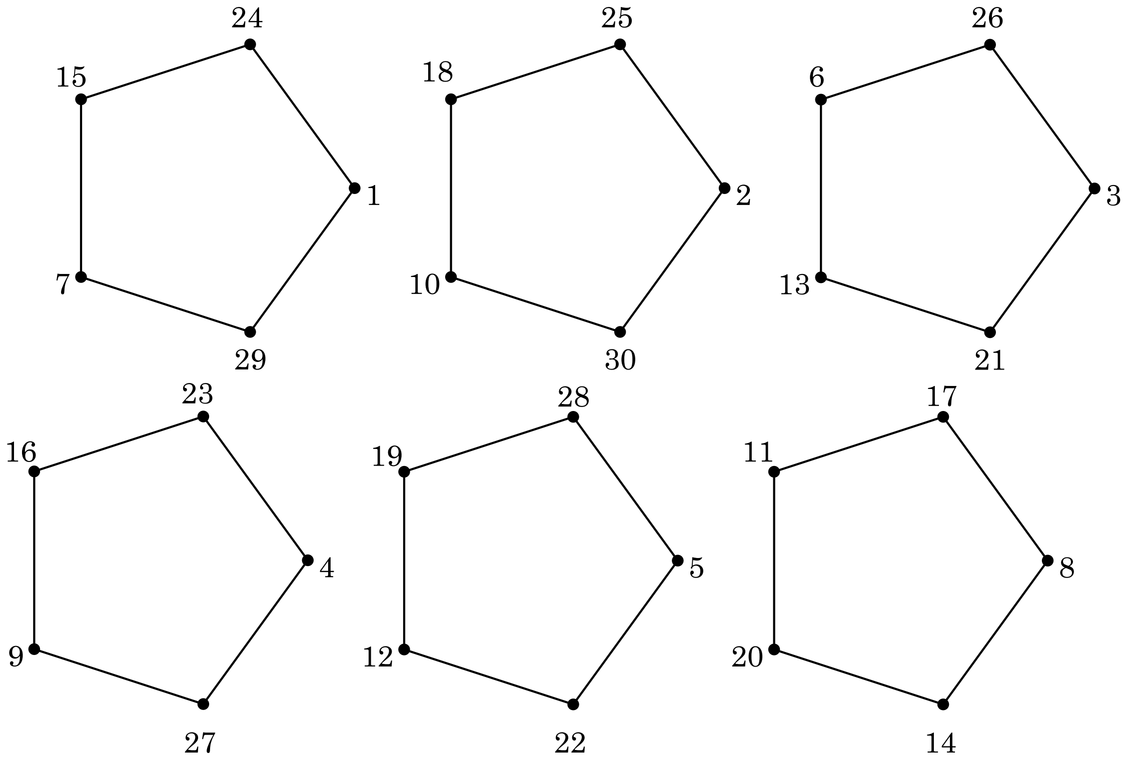


Figure 4.4.14: Graph Γ_{13}

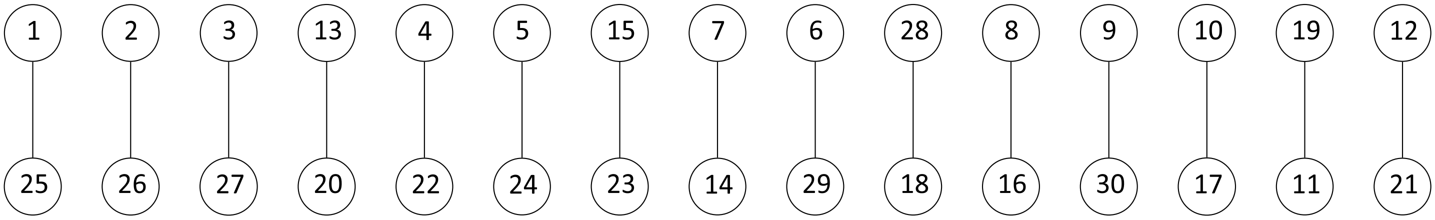


Figure 4.4.15: Γ_{14}

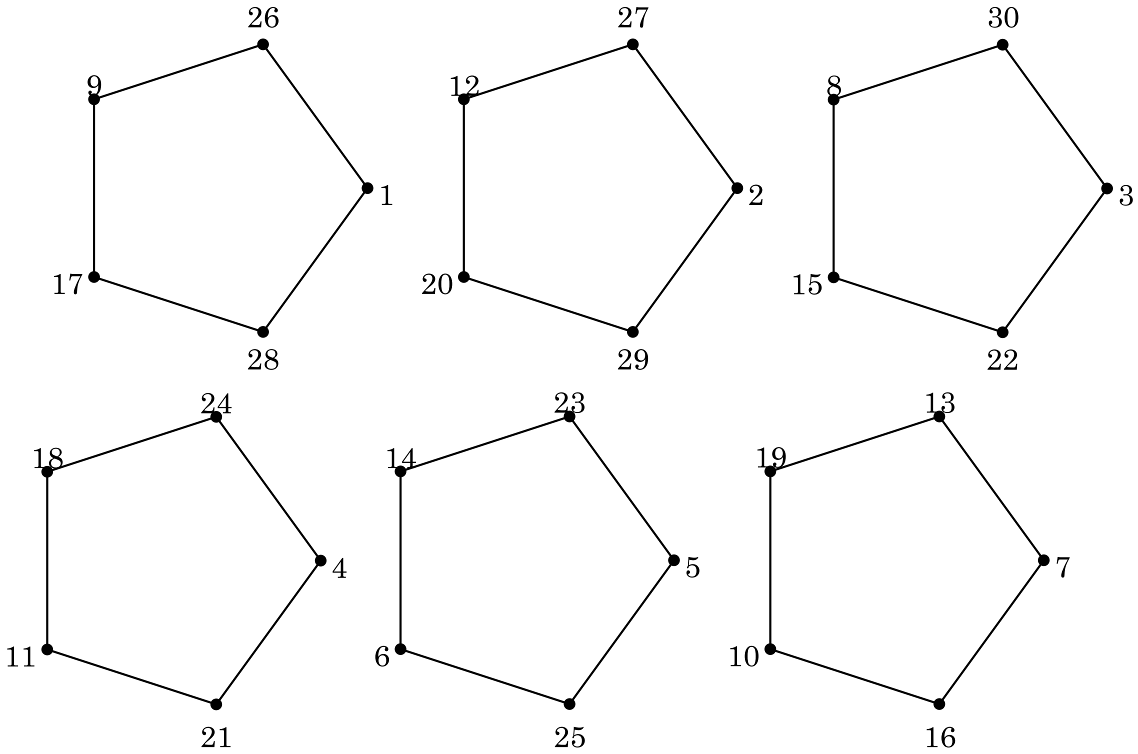


Figure 4.4.16: Graph Γ_{15}

Appendix V: Suborbitals and suborbital graphs of an icosahedron

Using **Definition 1.5.27** we compute the following suborbitals,

X×X = { (1,1), (1,2), (1,3), (1,4), (1,5), (1,6), (1,7), (1,8), (1,9), (1,10), (1,11), (1,12), (1,13), (1,14), (1,15), (1,16), (1,17), (1,18), (1,19), (1,20), (1,21), (1,22), (1,23), (1,24), (1,25), (1,26), (1,27), (1,28), (1,29), (1,30), (2,1), (2,2), (2,3), (2,4), (2,5), (2,6), (2,7), (2,8), (2,9), (2,10), (2,11), (2,12), (2,13), (2,14), (2,15), (2,16), (2,17), (2,18), (2,19), (2,20), (2,21), (2,22), (2,23), (2,24), (2,25), (2,26), (2,27), (2,28), (2,29), (2,30), (3,1), (3,2), (3,3), (3,4), (3,5), (3,6), (3,7), (3,8), (3,9), (3,10), (3,11), (3,12), (3,13), (3,14), (3,15), (3,16), (3,17), (3,18), (3,19), (3,20), (3,21), (3,22), (3,23), (3,24), (3,25), (3,26), (3,27), (3,28), (3,29), (3,30), (4,1), (4,2), (4,3), (4,4), (4,5), (4,6), (4,7), (4,8), (4,9), (4,10), (4,11), (4,12), (4,13), (4,14), (4,15), (4,16), (4,17), (4,18), (4,19), (4,20), (4,21), (4,22), (4,23), (4,24), (4,25), (4,26), (4,27), (4,28), (4,29), (4,30), (5,1), (5,2), (5,3), (5,4), (5,5), (5,6), (5,7), (5,8), (5,9), (5,10), (5,11), (5,12), (5,13), (5,14), (5,15), (5,16), (5,17), (5,18), (5,19), (5,20), (5,21), (5,22), (5,23), (5,24), (5,25), (5,26), (5,27), (5,28), (5,29), (5,30), (6,1), (6,2), (6,3), (6,4), (6,5), (6,6), (6,7), (6,8), (6,9), (6,10), (6,11), (6,12), (6,13), (6,14), (6,15), (6,16), (6,17), (6,18), (6,19), (6,20), (6,21), (6,22), (6,23), (6,24), (6,25), (6,26), (6,27), (6,28), (6,29), (6,30), (7,1), (7,2), (7,3), (7,4), (7,5), (7,6), (7,7), (7,8), (7,9), (7,10), (7,11), (7,12), (7,13), (7,14), (7,15), (7,16), (7,17), (7,18), (7,19), (7,20), (7,21), (7,22), (7,23), (7,24), (7,25), (7,26), (7,27), (7,28), (7,29), (7,30), (8,1), (8,2), (8,3), (8,4), (8,5), (8,6), (8,7), (8,8), (8,9), (8,10), (8,11), (8,12), (8,13), (8,14), (8,15), (8,16), (8,17), (8,18), (8,19), (8,20), (8,21), (8,22), (8,23), (8,24), (8,25), (8,26), (8,27), (8,28), (8,29), (8,30), (9,1), (9,2), (9,3), (9,4), (9,5), (9,6), (9,7), (9,8), (9,9), (9,10), (9,11), (9,12), (9,13), (9,14), (9,15), (9,16), (9,17), (9,18), (9,19), (9,20), (9,21), (9,22), (9,23), (9,24), (9,25), (9,26), (9,27), (9,28), (9,29), (9,30), (10,1), (10,2), (10,3), (10,4), (10,5), (10,6), (10,7), (10,8), (10,9), (10,10), (10,11), (10,12), (10,13), (10,14), (10,15), (10,16), (10,17), (10,18), (10,19), (10,20), (10,21), (10,22), (10,23), (10,24), (10,25), (10,26), (10,27), (10,28), (10,29), (10,30), (11,1), (11,2), (11,3), (11,4), (11,5), (11,6), (11,7), (11,8), (11,9), (11,10), (11,11), (11,12), (11,13), (11,14), (11,15), (11,16), (11,17), (11,18), (11,19), (11,20), (11,21), (11,22), (11,23), (11,24), (11,25), (11,26), (11,27), (11,28), (11,29), (11,30), (12,1), (12,2), (12,3), (12,4), (12,5), (12,6), (12,7), (12,8), (12,9), (12,10), (12,11), (12,12), (12,13), (12,14), (12,15), (12,16), (12,17), (12,18), (12,19), (12,20), (12,21), (12,22), (12,23), (12,24), (12,25), (12,26), (12,27), (12,28), (12,29), (12,30), (13,1), (13,2), (13,3), (13,4), (13,5), (13,6), (13,7), (13,8), (13,9), (13,10), (13,11), (13,12), (13,13), (13,14), (13,15), (13,16), (13,17), (13,18), (13,19), (13,20), (13,21), (13,22), (13,23), (13,24), (13,25), (13,26), (13,27), (13,28), (13,29), (13,30), (14,1), (14,2), (14,3), (14,4), (14,5), (14,6), (14,7), (14,8), (14,9), (14,10), (14,11), (14,12), (14,13), (14,14), (14,15), (14,16), (14,17), (14,18), (14,19), (14,20), (14,21), (14,22), (14,23), (14,24), (14,25), (14,26), (14,27), (14,28), (14,29), (14,30), (15,1), (15,2), (15,3), (15,4), (15,5), (15,6), (15,7), (15,8), (15,9), (15,10), (15,11), (15,12), (15,13), (15,14), (15,15), (15,16), (15,17), (15,18), (15,19), (15,20), (15,21), (15,22), (15,23), (15,24), (15,25), (15,26), (15,27), (15,28), (15,29), (15,30), (16,1), (16,2), (16,3), (16,4), (16,5), (16,6), (16,7), (16,8), (16,9), (16,10), (16,11), (16,12), (16,13), (16,14), (16,15), (16,16), (16,17), (16,18), (16,19), (16,20), (16,21), (16,22), (16,23), (16,24), (16,25), (16,26), (16,27), (16,28), (16,29), (16,30), (17,1), (17,2), (17,3), (17,4), (17,5), (17,6), (17,7), (17,8), (17,9), (17,10), (17,11), (17,12), (17,13), (17,14), (17,15), (17,16), (17,17), (17,18), (17,19), (17,20), (17,21), (17,22), (17,23), (17,24), (17,25), (17,26), (17,27), (17,28), (17,29), (17,30), (18,1), (18,2), (18,3), (18,4), (18,5), (18,6), (18,7), (18,8), (18,9), (18,10), (18,11), (18,12), (18,13), (18,14), (18,15), (18,16), (18,17), (18,18), (18,19), (18,20), (18,21), (18,22), (18,23), (18,24), (18,25), (18,26), (18,27), (18,28), (18,29), (18,30), (19,1), (19,2), (19,3), (19,4), (19,5), (19,6), (19,7), (19,8), (19,9), (19,10), (19,11), (19,12), (19,13), (19,14), (19,15), (19,16), (19,17), (19,18), (19,19), (19,20), (19,21), (19,22), (19,23), (19,24), (19,25), (19,26), (19,27), (19,28), (19,29), (19,30), (20,1), (20,2), (20,3), (20,4), (20,5), (20,6), (20,7), (20,8), (20,9), (20,10), (20,11), (20,12), (20,13), (20,14), (20,15), (20,16), (20,17), (20,18), (20,19), (20,20), (20,21), (20,22), (20,23), (20,24), (20,25), (20,26), (20,27), (20,28), (20,29), (20,30), (21,1), (21,2), (21,3), 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Orb$_G$ (1, 1) = O_0 = {(1, 1), (2, 2), (3, 3), (4, 4), (5, 5), (6, 6), (7,7), (8,8), (9,9), (10,10), (11,11), (12,12), (13, 13), (14, 14), (15, 15), (16, 16), (17, 17), (18, 18), (19, 19), (20, 20), (21, 21), (22,22), (23,23), (24,24), (25,25), (26, 26), (27,27), (28,28), (29,29), (30,30)}

Orb$_G$ (1, 2) = O_1 = {(1,2), (3,1), (12,3), (27,16), (7,6), (26,27), (22,30), (28,6), (21,22), (7,9), (12,10), (14,13), (15,16), (12,21), (20,17), (20,19), (27,28), (8,22), (19,13), (18,20), (6,29), (9,2), (1,15), (5,4), (28,26), (29,28), (23,24), (25,29), (24,25), (30,8), (22,12), (21,19), (25,23), (13,20), (17,18), (26,24), (4,3), (6,5), (8,7), (10,9), (15,14), (16,17), (4,27), (3,5), (2,10), (10,11), (11,14),

(14,1), (5,7), (29,30), (23,21), (13,11), (11,12), (19,23), (18,26), (16,4), (17,15), (24,18), (30,25), (9,8)}

Orb_G (1, 3) = O_2 = {(1,3), (3,2), (2,1), (27,4), (7,5), (26,28), (22,8), (28,29), (21,12), (7,8), (12,11), (14,11), (15,17), (12,22), (20,18), (20,13), (27,26), (8,30), (19,20), (18,17), (6,28), (9,10), (1,14), (5,3), (28,27), (29,6), (23,25), (25,30), (24,23), (30,22), (22,21), (21,23), (25,24), (13,19), (17,20), (26,18), (4,5), (6,7), (8,9), (10,2), (15,1), (16,15), (4,16), (3,4), (2,9), (10,12), (11,13), (14,15), (5,6), (29,25), (23,19), (13,14), (11,10), (19,21), (13,24), (16,27), (17,16), (24,26), (30,29), (9,7)}

Orb_G (1, 4) = O_3 = {(1,4), (3,9), (2,14), (27,5), (7,3), (26,29). (22,9), (28,25), (21,11), (7,30), (12,13), (14,10), (15,20), (12,8), (20,24), (20,14), (27,18), (8,29), (19,18), (18,16), (6,27), (9,12), (1,11), (5,2), (28,4), (29,7), (23,30), (25,22), (24,19), (30,21), (22,23), (21,25), (25,26), (13,21), (17,13), (26,17), (4,6), (6,8), (8,10), (10,1), (15,3), (16,1), (4,15), (3,16), (2,7), (10,22), (11,19), (14,17), (5,28), (29,24), (23,20), (13,15), (11,2), (19,12), (18,23), (16,26), (17,27), (24,28), (30,6), (9,5)}

Orb_G (2, 4) = O_4 = {(2,4), (1,9), (3,14), (16,5), (6,3), (27,29), (30,9), (6,25), (22,11), (9,30), (10,13), (13,10), (16,20), (21,8), (17,24), (19,14), (28,18), (22,29), (13,18), (20,16), (29,27), (2,12), (15,11), (4,2), (26,4), (28,7), (24,30), (29,22), (25,19), (8,21), (12,23), (19,25), (23,26), (20,21), (18,13), (24,17), (3,6), (5,8), (7,10), (9,1), (14,3), (17,1), (27,15), (5,16), (10,7), (11,22), (14,19), (1,17), (7,28), (30,24), (21,20), (11,15), (12,2), (23,12), (26,23), (4,26), (15,27), (18,28), (25,6), (8,5)}

Orb_G (4, 1) = O_5 = {(4,1), (9,3), (14,2), (5,27), (3,7), (29,26), (9,22), (25,28), (11,21), (30,7), (13,12), (10,14), (20,15), (8,12), (24,20), (14,20), (18,27), (29,8), (18,19), (16,28), (27,6), (12,9), (11,1), (2,5), (4,28), (7,29), (30,23), (22,25), (19,24), (21,30), (23,22), (25,21), (26,25), (21,13), (13,17), (17,26), (6,4), (8,6), (10,8), (1,10), (3,15), (1,16), (15,4), (16,3), (7,2), (22,10), (19,11), (17,14), (28,5), (24,29), (20,23), (15,13), (2,11), (12,19), (23,18), (26,16), (27,17), (28,24), (6,30), (5,9)}

Orb$_G$ (1, 5) = O$_6$ = {(1,5), (3,10), (2,15), (27,3), (7,4), (26,6), (22,7), (28,30), (21,10), (7,22), (12,14), (14,12), (15,18), (12,30), (20,26), (20,11), (27,24), (8,25), (19,17), (18,15), (6,26), (9,11), (1,13), (5,1), (28,16), (29,5), (23,29), (25,8), (24,21), (30,12), (22,19), (21,24), (25,18), (13,23), (17,19), (26,20), (4,7), (6,9), (8,2), (10,3), (15,2), (16,14), (4,17), (3,27), (2,8), (10,21), (11,20), (14,16), (5,29), (29,23), (23,13), (13,1), (11,9), (19,22), (18,25), (16,28), (17,4), (24,27), (30,28), (9,6)}

Orb$_G$ (1, 6) = O$_7$ ={(1,6), (3,12), (2,17), (27,2), (7,16), (26,7), (22,5), (28,22), (21,2), (7,21), (12,15), (14,22), (15,24), (12,29), (20,28), (20,10), (27,23), (8,24), (19,16), (18,1), (6,18), (9,13), (1,19), (5,14), (28,15), (29,3), (23,6), (25,9), (24,12), (30,11), (22,20), (21,26), (25,17), (13,25), (17,21), (26,13), (4,8), (6,10), (8,1), (10,4), (15,9), (16,11), (4,20), (3,26), (2,30), (10,23), (11,18), (14,27), (5,25), (29,19), (23,14), (13,3), (11,7), (19,8), (18,30), (16,29), (17,5), (24,4), (30,27), (9,28)}

Orb$_G$ (6, 1) = O$_8$ ={(6,1), (12,3), (17,2), (2,27), (16,7), (7,26), (5,22), (22,28), (2,21), (21,7), (15,12), (22,14), (24,15), (29,12), (28,20), (10,20), (23,27), (8,24,8), (16,19), (1,18), (18,6), (13,9), (19,1), (14,5), (15,28), (3,29), (6,23), (9,25), (12,24), (11,30), (20,22), (26,21), (17,25), (25,13), (21,17), (13,26), (8,4), (10,6), (1,8), (4,10), (9,15), (11,16), (20,4), (26,3), (30,2), (23,10), (18,11), (27,14), (25,5), (19,29), (14,23), (3,13), (7,11), (8,19), (30,18), (29,16), (5,17), (4,24), (27,30), (28,9)}

$\mathrm{Orb}_G\,(2,6) = O_9$ = {(2,6), (1,12), (3,17), (16,2), (6,16), (27,7), (30,5), (6,22), (22,2), (9,21), (10,15), (13,22), (16,24), (21,29), (17,28), (19,10), (28,23), (22,24), (13,16), (20,1), (29,18), (2,13), (15,19), (4,14), (26,15), (28,3), (24,6), (29,9), (25,12), (8,11), (12,20), (19,26), (23,17), (20,25), (18,21), (24,13), (3,8), (5,10), (7,1), (9,4), (14,9), (17,11), (27,20), (5,26), (10,30), (11,23), (14,18), (1,27), (7,25), (30,19), (21,14), (11,3), (12,7), (23,8), (26,30), (4,29), (15,5), (18,4), (25,27), (8,28)}

$\mathrm{Orb}_G\,(6, 2) = O_{10}$ = {(6,2), (12,1), (17,3), (2,16), (16,6), (7,27), (5,30), (22,6), (2,22), (21,9), (15,10), (22,13), (24,16), (29,21), (28,17), (10,19), (23,28), (24,22), (16,13), (1,20), (18,29), (13,2), (19,15), (14,4), (15,26), (3,28), (6,24), (9,29), (12,25), (11,8), (20,12), (26,19), (17,23), (25,20), (21,18), (13,24), (8,3), (10,5), (1,7), (4,9), (9,14), (11,17), (20,27), (26,5), (30,10), (23,11), (18,14), (27,1), (25,7), (19,30), (14,21), (3,11), (7,12), (8,23), (30,26), (29,4), (5,15), (4,18), (27,25), (28,8)}

$\mathrm{Orb}_G\,(1, 21) = O_{11}$ = {(1,21), (3,18), (2,29), (27,19), (7,23), (26,14), (22,18), (28,1), (21,28), (7,15), (12,6), (14,26), (15,7), (12,17), (20,2), (20,29), (27,9), (8,14), (19,9), (18,22), (6,12), (9,27), (1,28), (5,24), (28,21), (29,20), (23,15), (25,16), (24,5), (30,4), (22,3), (21,1), (25,10), (13,4), (17,6), (26,8), (4,13), (6,17), (8,26), (10,25), (15,23), (16,25), (4,30), (3,22), (2,20), (10,16), (11,5), (14,8), (5,11), (29,2), (23,7), (13,30), (11,24), (19,27), (18,3), (16,10), (17,12), (24,11), (30,13), (9,19)}

$\mathrm{Orb}_G\,(3, 21) = O_{12}$ = {(3,21), (2,18), (1,29), (4,19), (5,23), (28,14), (8,18), (29,1), (12,28), (8,15), (11,6), (11,26), (17,7), (22,17), (18,2), (13,29), (26,9), (30,14), (20,9), (17,22), (28,12), (10,27), (14,28), (3,24), (27,21), (6,20), (25,15), (30,16), (23,5), (22,4), (21,3), (23,1), (24,10), (19,4), (20,6), (18,8), (5,13), (7,17), (9,26), (2,25), (1,23), (15,25), (16,30), (4,22), (9,20), (12,16), (13,5),

(15,8), (6,11), (25,2), (19,7), (14,30), (10,24), (21,27), (24,3), (27,10), (16,12), (26,11), (29,13), (7,19)}

$\text{Orb}_G(4, 21) = O_{13}$ = {(4,21), (9,18), (14,29), (5,19), (3,23), (29,14), (9,18), (25,1), (11,28), (30,15), (13,6), (10,26), (20,7), (8,17), (24,2), (14,29), (18,9), (29,14), (18,9), (16,22), (27,12), (12,27), (11,28), (2,24), (4,21), (7,20), (30,15), (22,16), (19,5), (21,4), (23,3), (25,1), (26,10), (21,4), (13,6), (17,8), (6,13), (8,17), (10,26), (1,25), (15,30), (16,22), (7,20), (22,16), (19,5), (17,8), (28,11), (6,13), (5,19)}

$\text{Orb}_G(5, 21) = O_{14}$ = {(5,21), (10,18), (15,29), (3,19), (4,23), (6,14), (7,18), (30,1), (10,28), (22,15), (14,6), (12,26), (18,7), (30,17), (26,2), (11,29), (24,9), (25,14), (17,9), (15,22), (26,12), (11,27), (13,28), (1,24), (16,21), (5,20), (29,15), (8,16), (21,5), (12,4), (19,3), (24,1), (18,10), (23,4), (19,6), (20,8), (7,13), (9,17), (2,26), (3,25), (2,23), (14,25), (17,30), (27,22), (8,20), (21,16), (20,5), (16,8), (29,11), (23,2), (13,7), (1,30), (9,24), (22,27), (25,3), (28,10), (4,12), (27,11), (28,13), (6,19)}

$\text{Orb}_G(6, 21) = O_{15}$ = {(6,21), (12,18), (17,29), (2,19), (16,23), (7,14), (5,18), (22,1), (2,28), (21,15), (15,6), (22,26), (24,7), (29,17), (28,2), (10,29), (23,9), (24,14), (16,9), (1,22), (18,12), (13,27), (19,28), (14,24), (15,21), (3,20), (6,15), (9,16), (12,5), (11,4), (20,3), (26,1), (17,10), (25,4), (21,6), (13,8), (8,13), (10,17), (1,26), (4,25), (9,23), (11,25), (20,30), (26,22), (30,20), (23,16), (18,5), (27,8), (25,11), (19,2), (14,7), (3,30), (7,24), (8,27), (30,3), (29,10), (5,12), (4,11), (27,13), (28,19)}

Next we construct the corresponding non-trivial suborbital graphs,

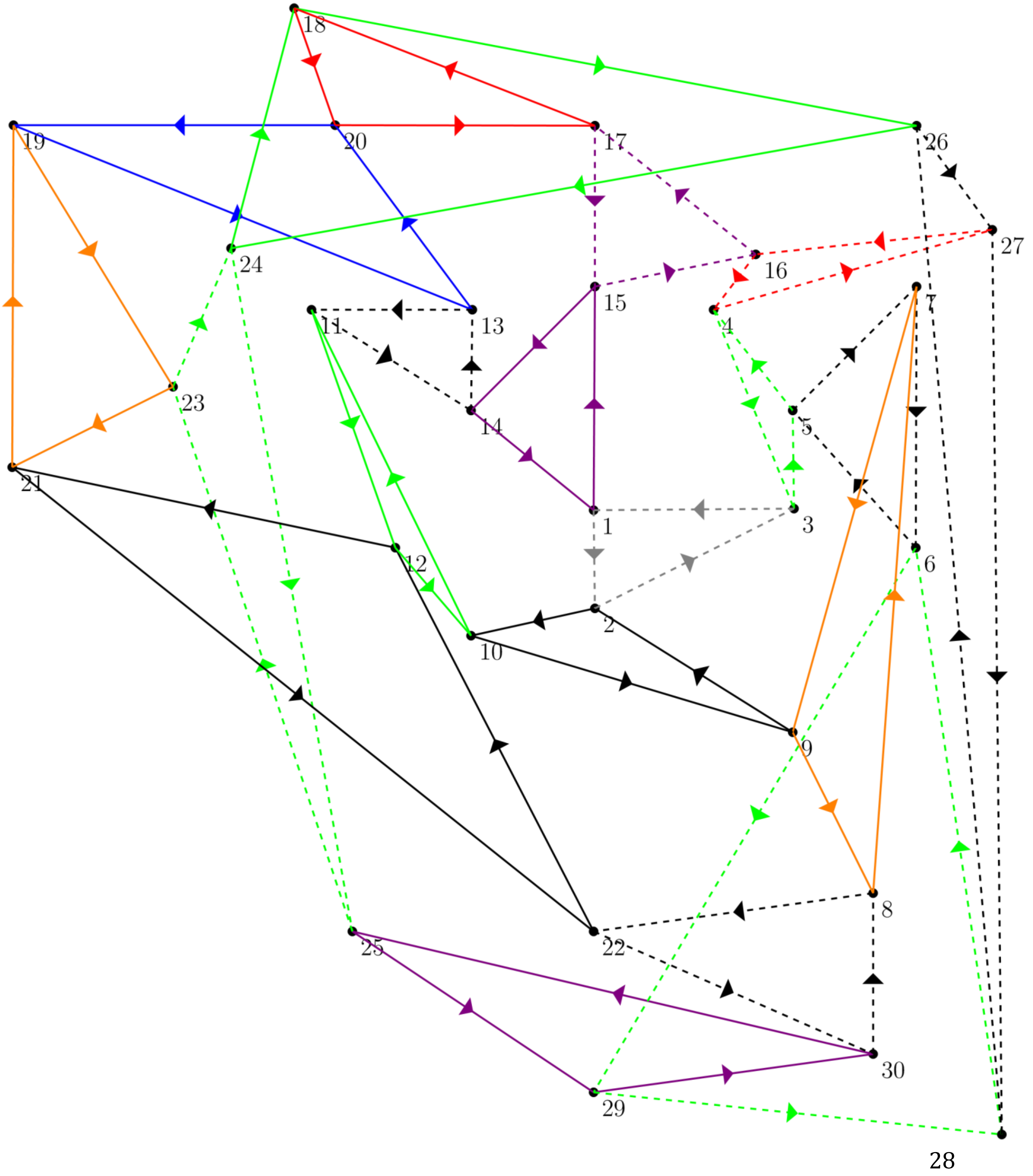

Figure 4.5.2: Graph Γ_1

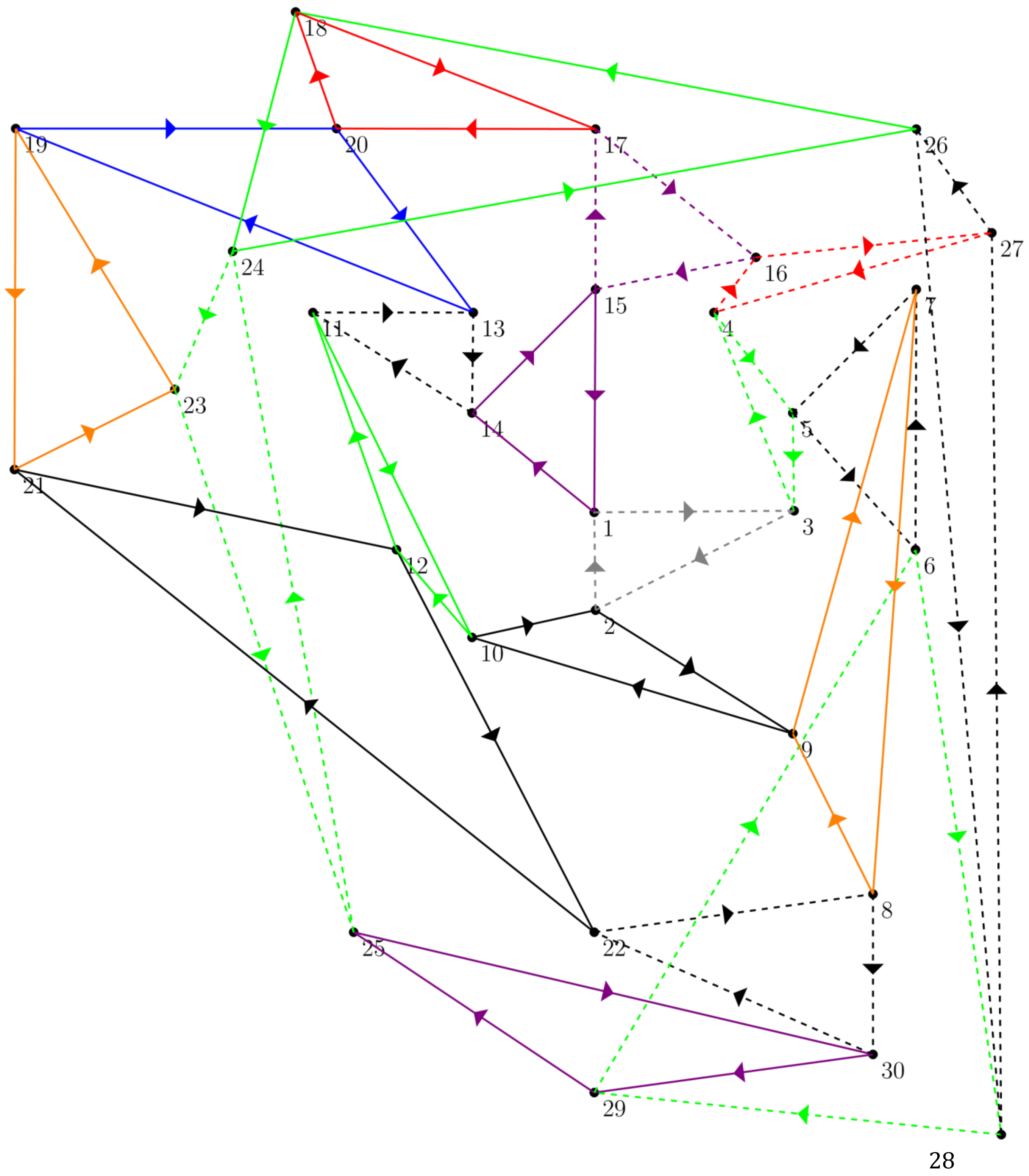

Figure 4.5.3: Graph Γ₂

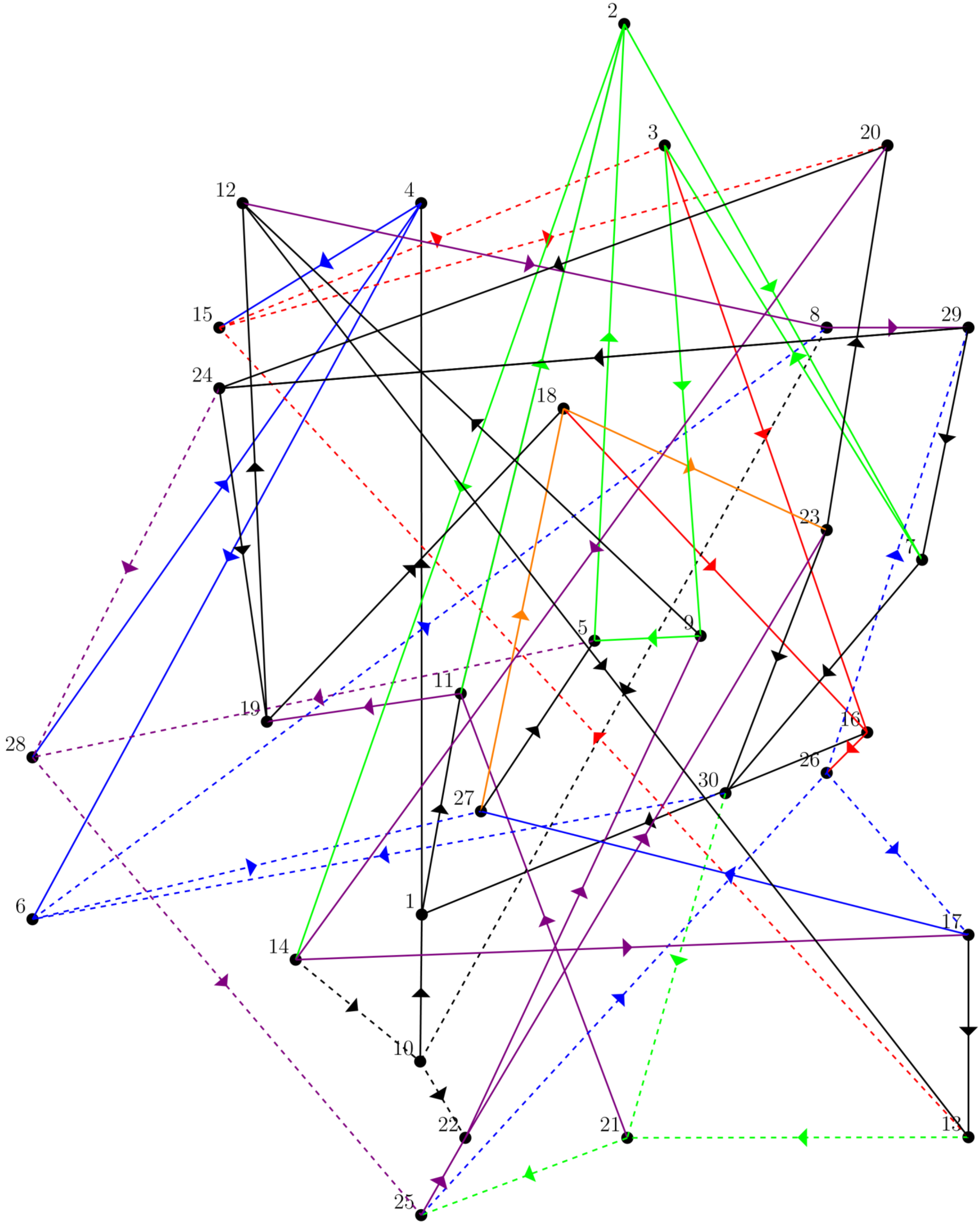

Figure 4.5.4: Graph Γ_3

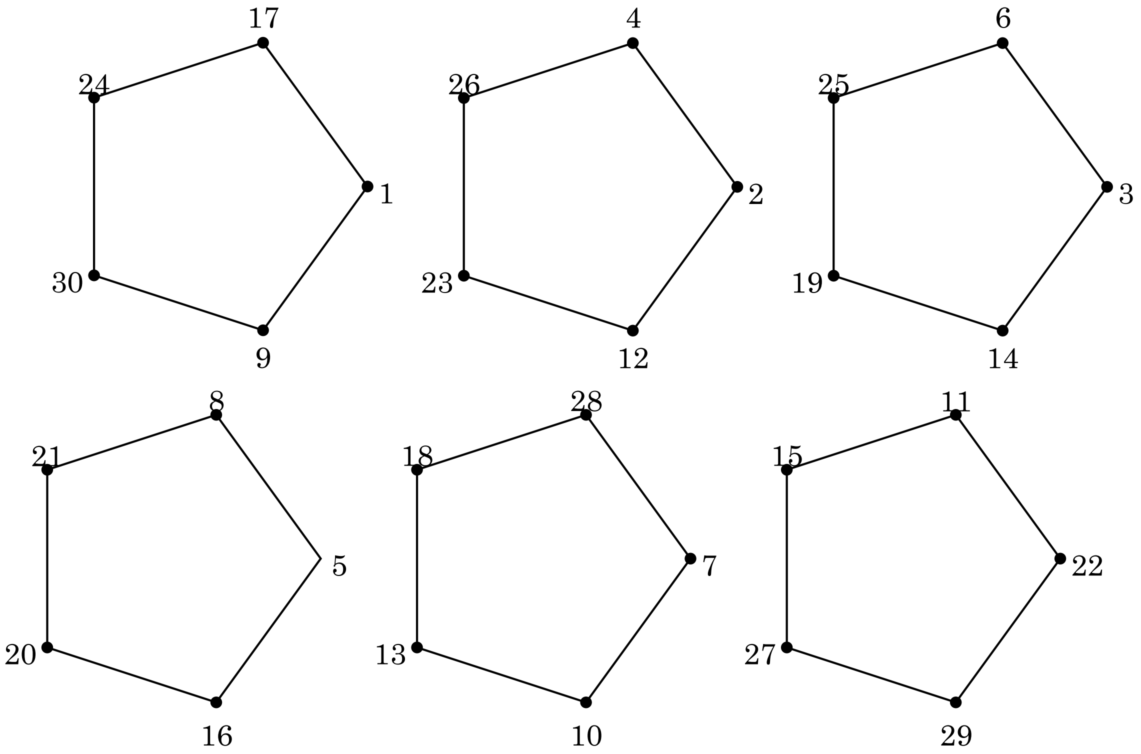

Figure 4.5.5: Graph Γ_4

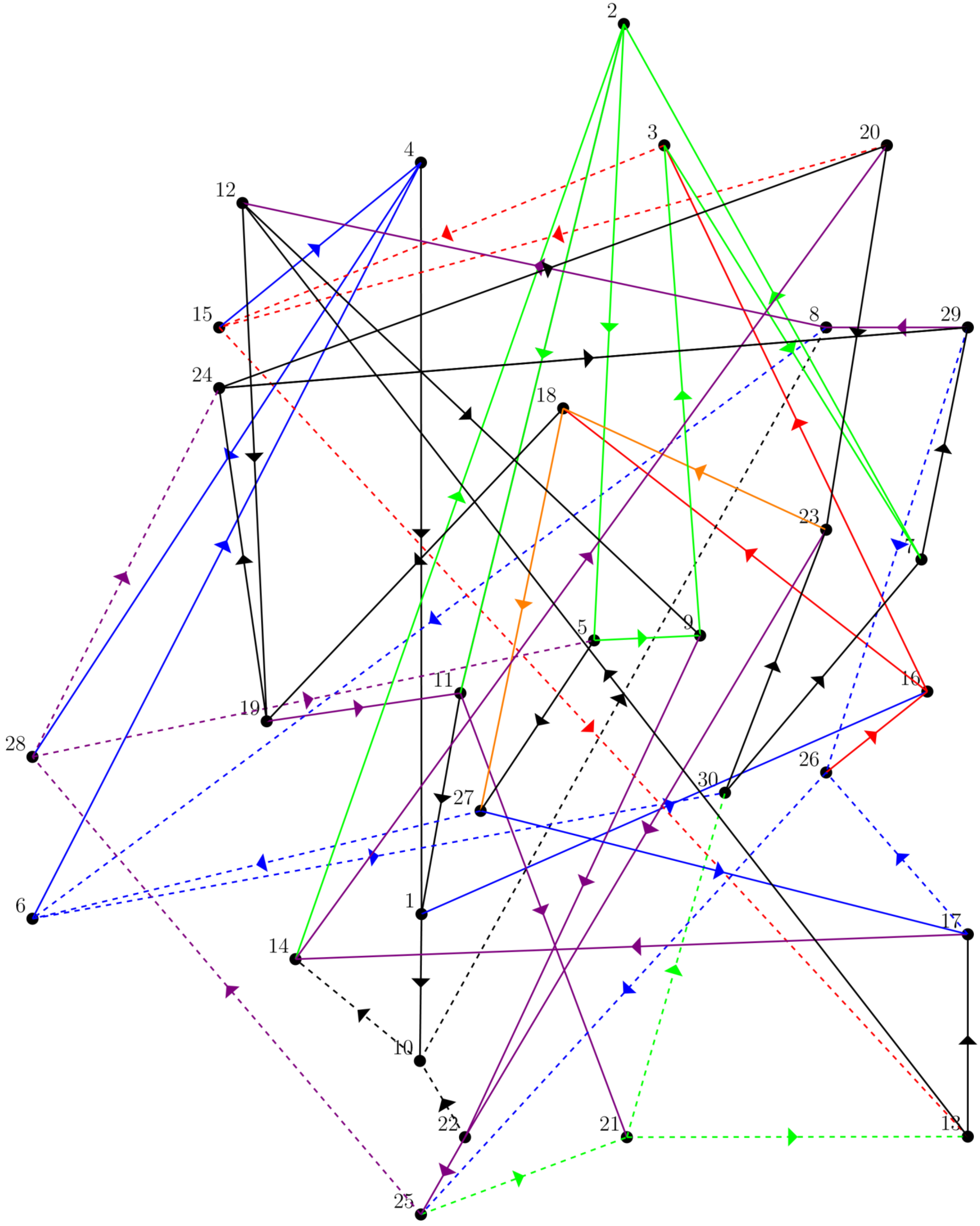

Figure 4.5.6: Graph Γ_5

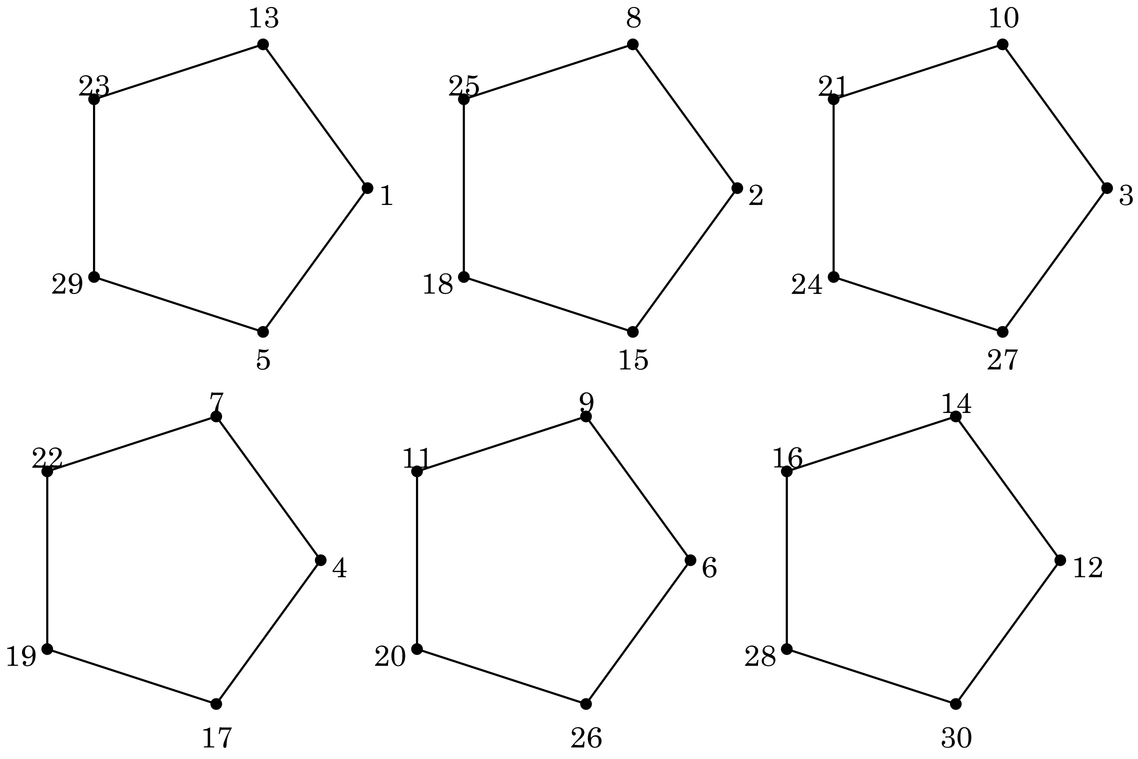

Figure 4.5.7: Graph Γ_6

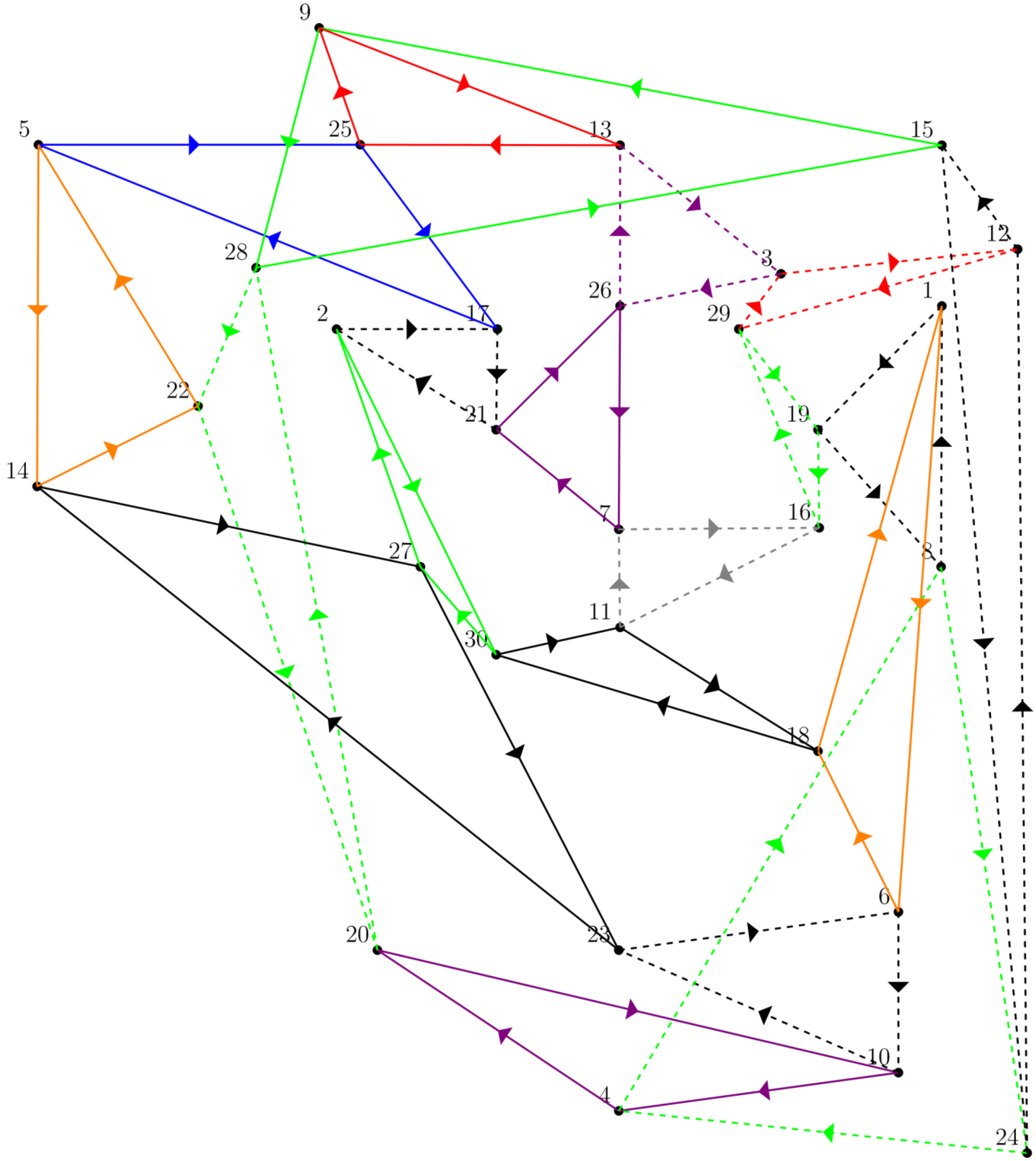

Figure 4.5.8: Graph Γ_7

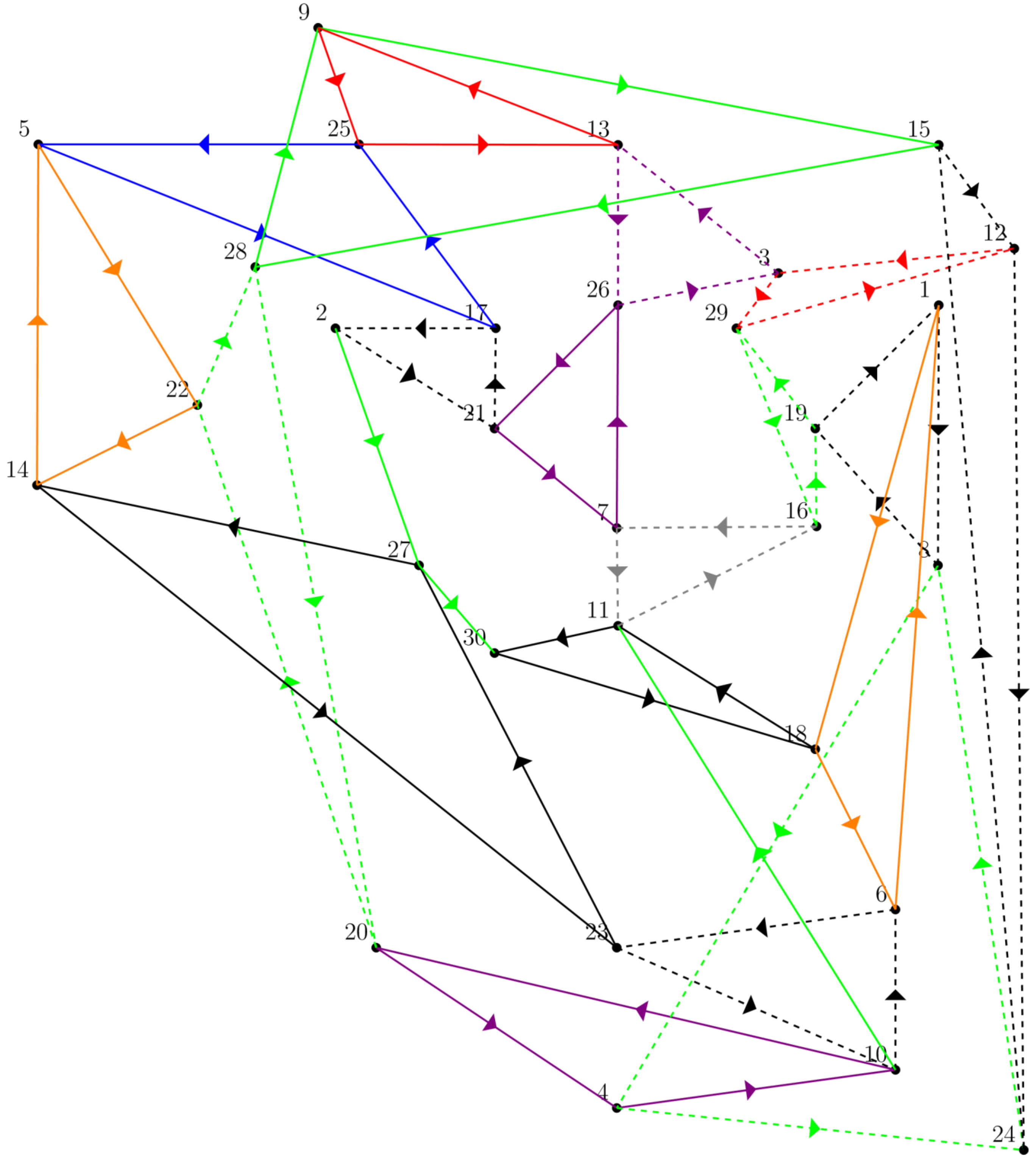

Figure 4.5.9: Graph Γ_8

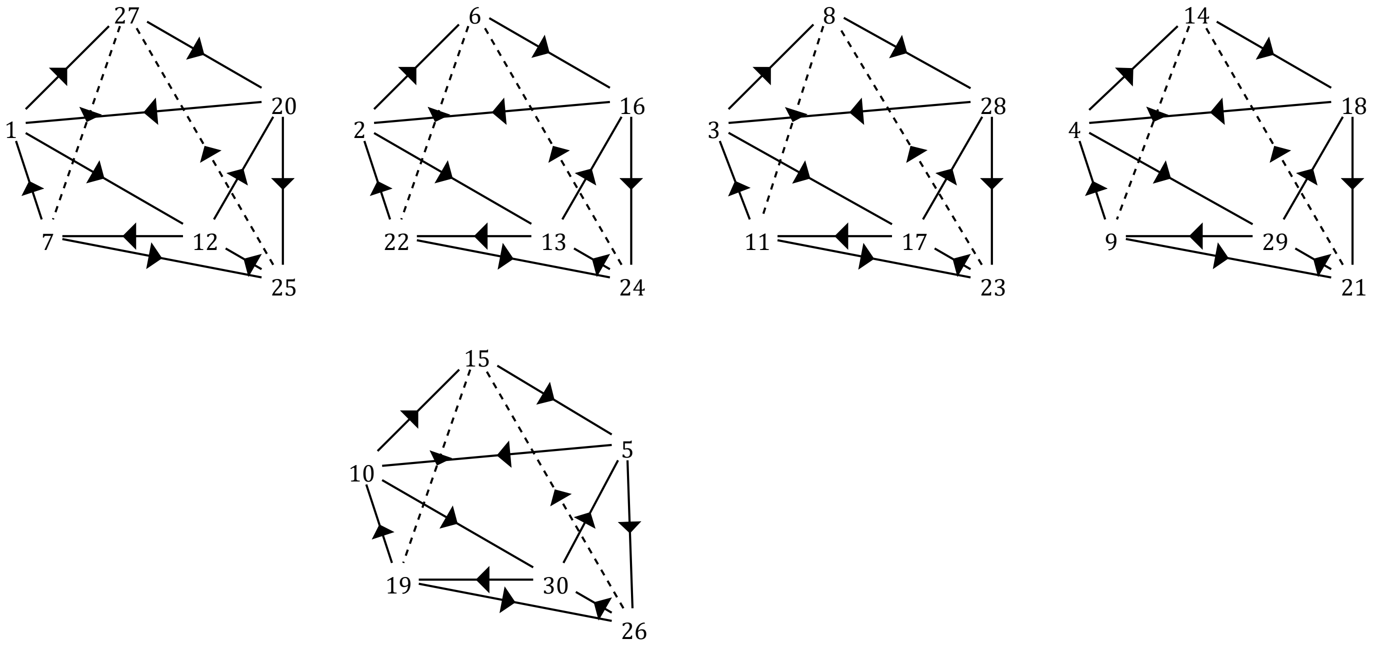

Figure 4.5.10: Graph Γ_9

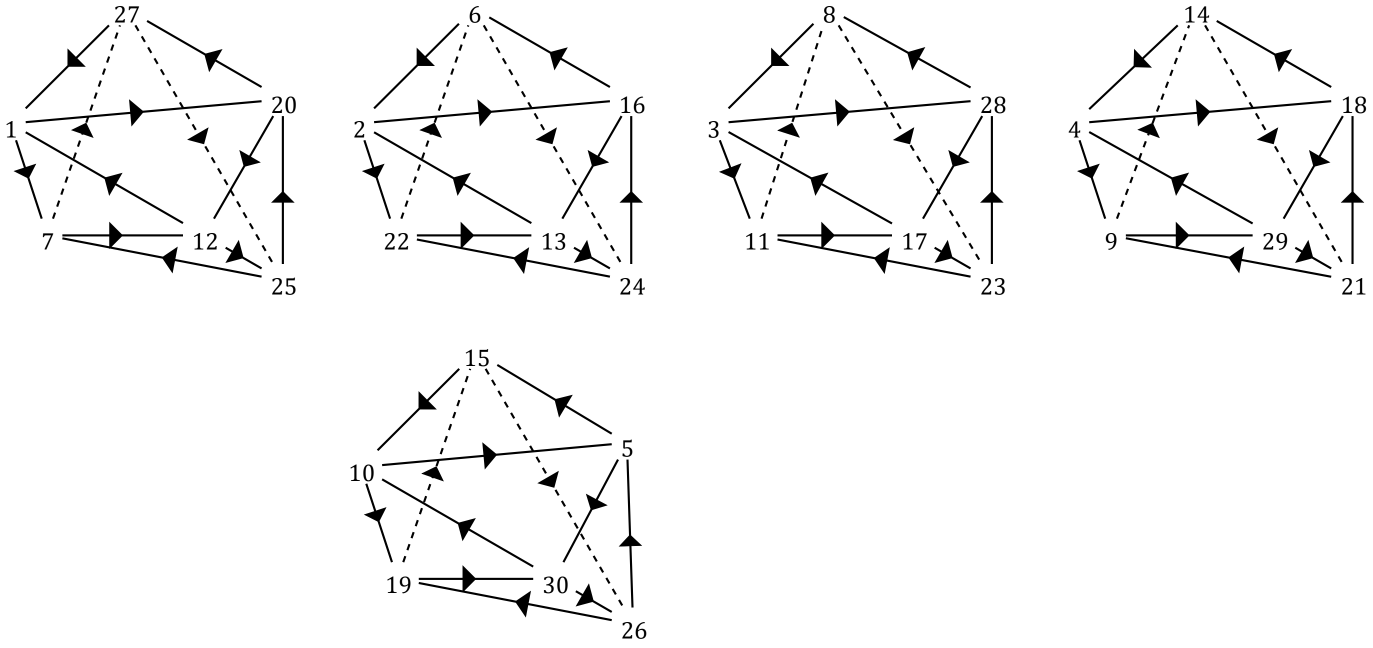

Figure 4.5.11: Graph Γ_{10}

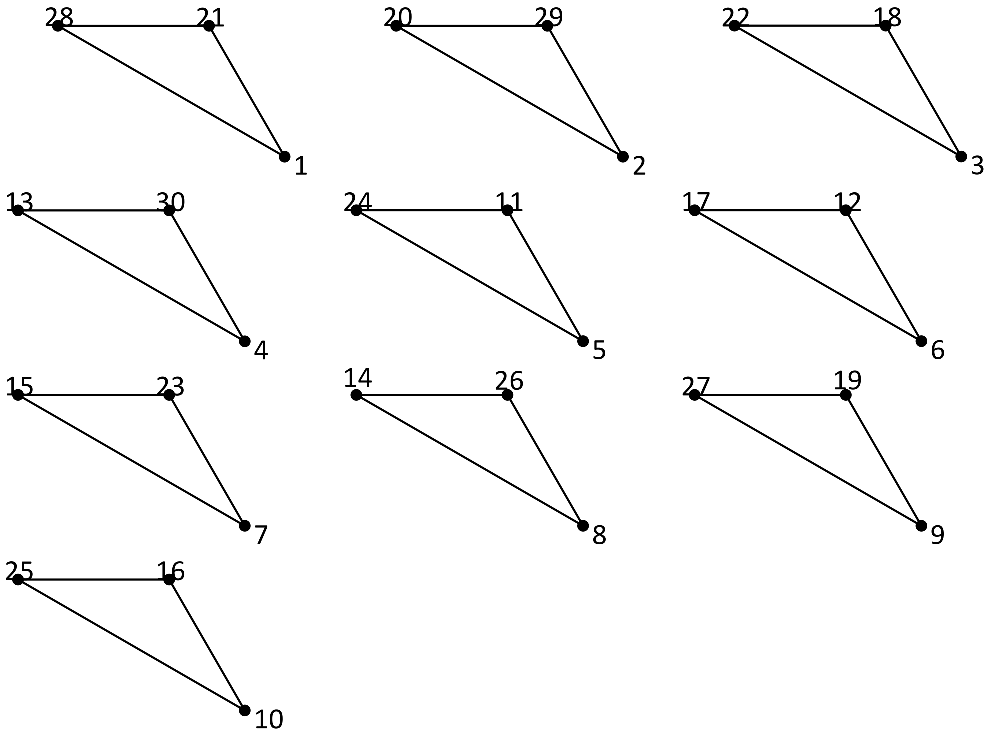

Figure 4.5.12: Graph Γ_{11}

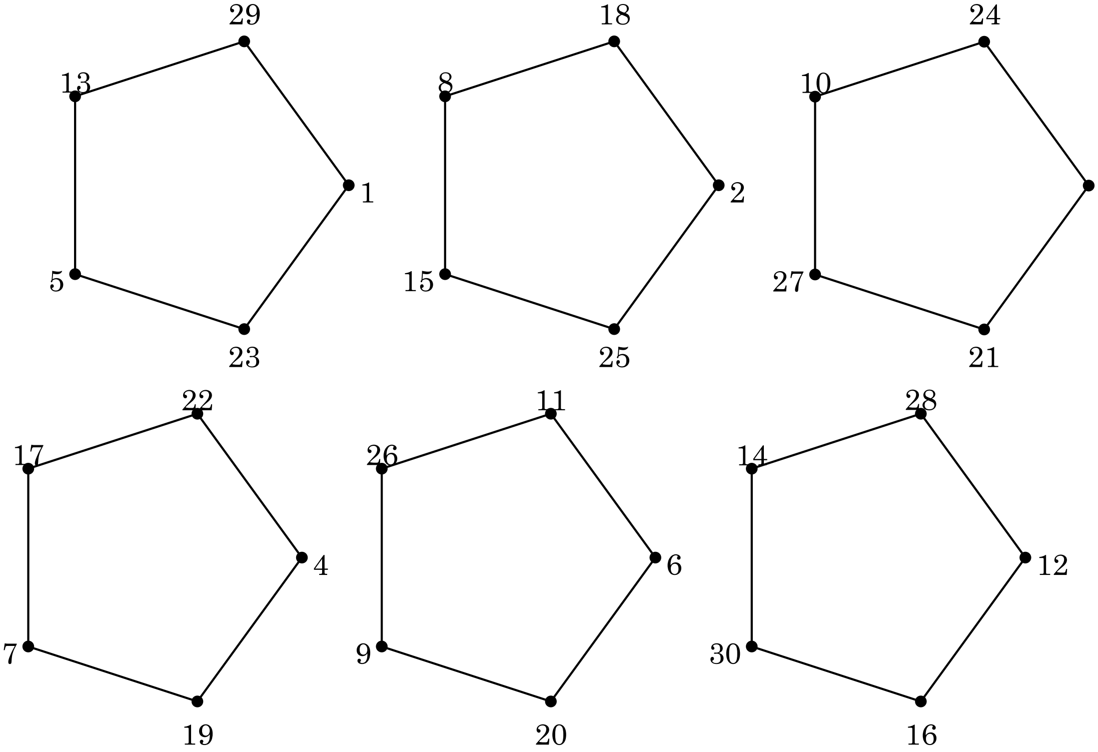

Figure 4.5.13: Graph Γ_{12}

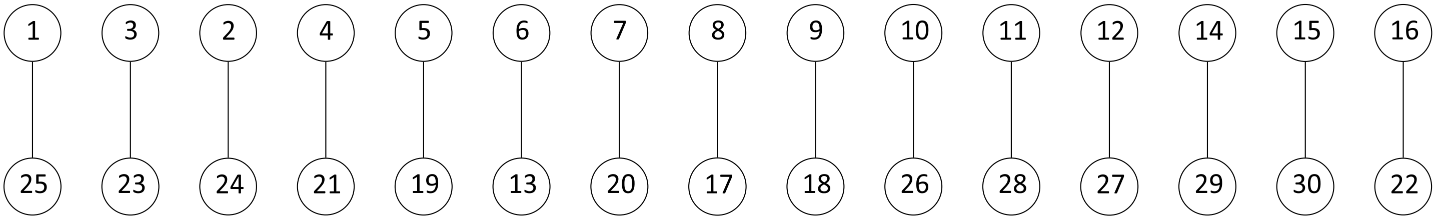

Figure 4.5.14: Graph Γ_{13}

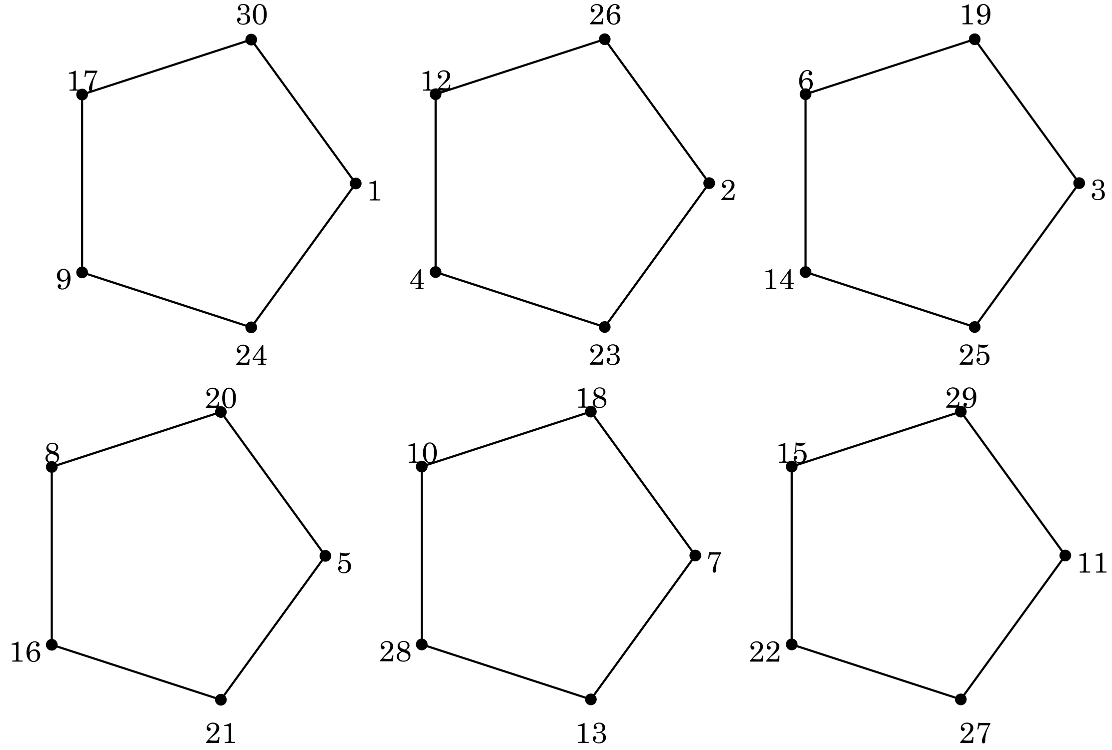

Figure 4.5.15: Graph Γ_{14}

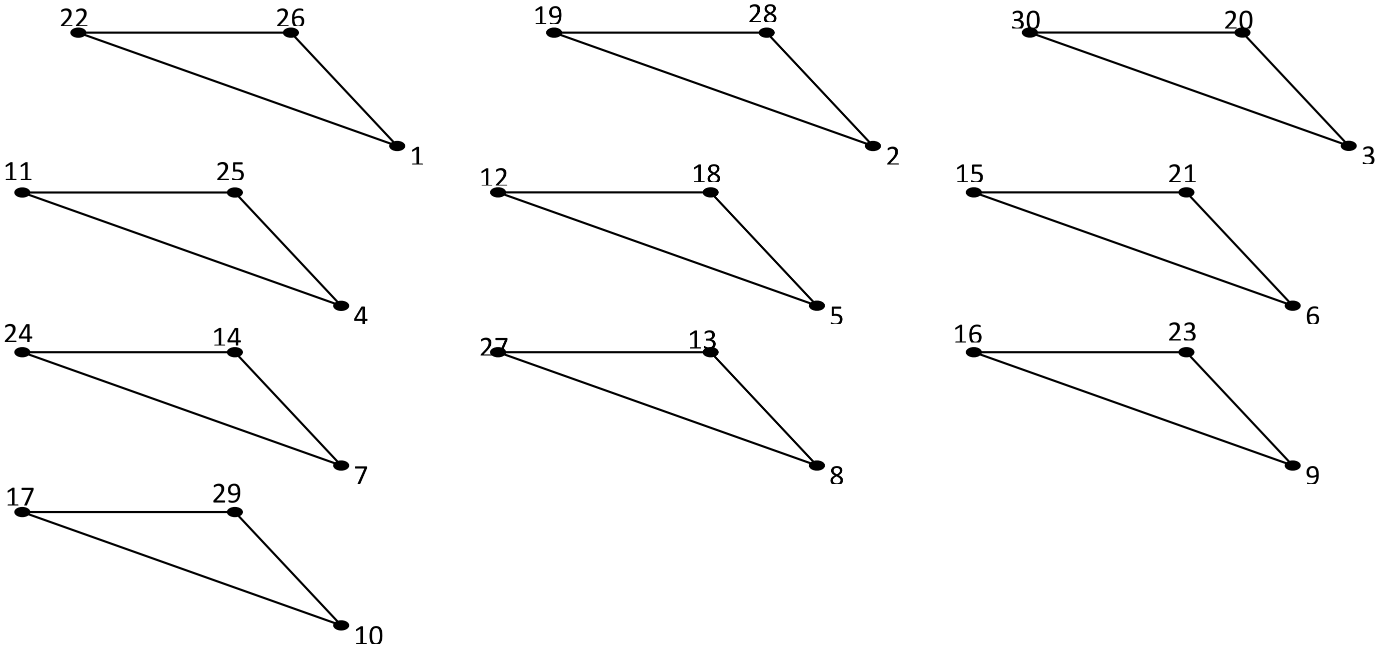

Figure 4.5.16: Graph Γ_{15}