

**A STUDY OF MHD FLUID FLOW BOUNDED BY TWO PARALLEL VERTICAL  
PLATES IN A POROUS MEDIA WITH HEAT TRANSFER**

**BY**

**KARIUKI JOHN KING'ORI (B.Sc.Ed)**

**I56/CE/28014/2019**

**A PROJECT SUBMITTED IN PARTIAL FULFILMENT OF THE REQUIREMENTS  
FOR THE AWARD OF THE DEGREE OF MASTER OF SCIENCE (APPLIED  
MATHEMATICS) IN THE SCHOOL OF PURE AND APPLIED SCIENCES OF  
KENYATTA UNIVERSITY**

**MAY, 2025**

## Table of Contents

DECLARATION .....	i
DEDICATION .....	ii
ACKNOWLEDGEMENT .....	iii
ABSTRACT.....	iv
NOMENCLATURE .....	v
List of Acronyms .....	ii
List of Figures .....	iii
CHAPTER ONE .....	1
BACKGROUND INFORMATION .....	1
1.1 Introduction.....	1
1.2 Background.....	1
1.3 Definitions.....	2
1.4 Statement of the Problem.....	2
1.5 Justification .....	3
1.6 Objectives .....	3
1.6.1 General Objectives.....	3
1.6.2 Specific Objectives .....	3
1.7 Significance of the study.....	4
CHAPTER TWO .....	5
LITERATURE REVIEW .....	5
2.1 Introduction.....	5
2.2 Literature Review.....	5
CHAPTER THREE .....	12
GENERAL EQUATIONS .....	12
3.1 Introduction.....	12
3.2 Governing Equations.....	12
3.2.1 Continuity Equation .....	12
3.2.2 Momentum equation .....	12
3.2.3 Energy Equation.....	12
3.2.4 Maxwell's Equations .....	13
3.2.5 Darcy's equation.....	13
CHAPTER FOUR.....	14

MATHEMATICAL FORMULATION .....	14
4.1    Introduction.....	14
4.2    Flow Model.....	14
4.3    Assumptions.....	14
4.4    Mathematical Formulation.....	15
4.4.1    Continuity Equation.....	15
4.4.2    Momentum Equation.....	16
4.4.3    Energy Equation.....	20
4.5    Boundary Conditions .....	22
4.6    Non Dimensionalization .....	22
4.6.1    Nondimensionalizing the Momentum Equation .....	23
4.6.2    Nondimensionalizing the Energy Equation .....	24
4.6.3    Non-dimensional Numbers .....	25
4.6.4    Nondimensionalizing the Boundary Conditions .....	27
CHAPTER 5 .....	29
NUMERICAL SOLUTION.....	29
Introduction.....	29
Mesh Definition .....	29
CHAPTER FIVE .....	33
RESULTS AND DISCUSSION .....	33
5.1    Introduction.....	33
5.2    Results and Discussion.....	33
CHAPTER SIX.....	40
CONCLUSION AND RECOMMENDATION.....	40
6.1    Introduction.....	40
6.2    Conclusion .....	40
6.3    Recommendations.....	40
REFERENCES .....	41

**DECLARATION**

I declare that this is my original work and has not been presented to any university in part or whole for the award of a degree.

**John King’ori Kariuki**

Signature.....Date.....

The candidate did the research project under my supervision.

**Dr. Maurine Maraka Wafula (PhD)**

Signature.....Date.....

School of Science and Technology (SST)

United States International University

The candidate did the research project under my supervision.

**Dr. Lawrence Njau (PhD)**

Signature.....Date.....

Department of Mathematics and Actuarial Science

Kenyatta University

**DEDICATION**

To my parents, Patrick Kariuki and Salome Simba, and my siblings, Justin Muthii, Joyce Wangui, and Lucy Wakio. Thank you for all your support.

## ACKNOWLEDGEMENT

I acknowledge the Almighty God for enabling me to pursue my studies this far. I pray He allows me to reach my imaginable heights.

Sincere gratitude to Dr. Maurine Wafula (Ph.D.) for your encouragement and timely feedback; you made the research project an interesting academic aspect. Thanks to Dr. Lawrence Njau (PhD) for his input and support throughout the research project.

I am sincerely grateful to my class teacher and mentor, Mr. Ngubiru, for inspiring me to pursue a Master's degree immediately after my bachelor's. Special thanks to my Mathematics teacher, Mrs. Mwai, for inspiring me to learn mathematics and join university to pursue higher education. And to my history teacher, Mr. Thoronjo (late), for wise counsel and encouragement. May God grant you eternal rest.

My special gratitude to my parents and siblings; you have always been there for me. To Uncle Sam, thank you for providing me with books during my secondary education that helped me revise. You promised to provide all the books I needed as long as you had access. Your inspiration is far-reaching. Thank you.

My colleagues Francis Momanyi, Stephen Chege, and Ali Kulow, the journey has been calm because of your great assistance.

**ABSTRACT**

The study investigated a Newtonian Magneto-hydrodynamic fluid flow bounded by two parallel vertical plates in a porous media with heat transfer. The fluid was considered to be flowing uniformly in the  $x$ -direction. The parallel vertical plates are impermeable and a transverse magnetic field is applied perpendicular to the plates in the positive  $y$ -direction. The plates are heated and kept at constant temperature  $T_w$  and the distance between the two plates was varied. The fluid and the porous matrix are approximated to have the same temperature,  $T_f$ . The effect of varying Darcy number, Hartmann number, Prandtl number, and Reynolds number on velocity and temperature profiles was discussed. The coupled non-linear PDE governing the fluid flow were non-dimensionalized to obtain a dimensionless equation. The resulting equation was discretized using the finite difference method to obtain non-linear algebraic equations which were solved using MATLAB. The obtained results were presented in graphs and then discussed. It was observed that velocity profile decreased when Hartmann number or Reynolds number was increased. On the other hand, velocity profile increased after increasing Prandtl number or Darcy number. It was also observed that temperature profile decreased when Hartmann number or Prandtl number was increased. On the other hand, temperature profile increased when Reynolds number or Darcy number was increased. These results have applications in aerodynamic heating and motor vehicle cooling.

**NOMENCLATURE**

$\rho$  – fluid density

B- Varying Magnetic field

Gr – Thermal Grashof Number

Gc – Mass Grashof Number

Sc – Schmidt Number

$E_c$  – Eckert Number

Ha– Hartmann Number

M- Stuart Number

Re – Reynold's number

Da- Darcy's Number

Pr – Prandtl Number

$\nabla$ . – gradient operator

List of Acronyms

MHD – Magnetohydrodynamics

PDE – Partial Differential Equation

**List of Figures**

Figure 4.1 .....	14
Figure 4.2 .....	29
Figure 5.1 .....	33
Figure 5.2 .....	34
Figure 5.3 .....	35
Figure 5.4 .....	36
Figure 5.5 .....	37
Figure 5.6 .....	38
Figure 5.8 .....	39

## CHAPTER ONE

### BACKGROUND INFORMATION

#### 1.1 Introduction

The chapter focuses on background information, definitions, statement of the problem, and justification for the research problem. Further, it presents objectives and a discussion of the significance of the study.

#### 1.2 Background

When an MHD fluid flows, the induced current experiences some forces due to the magnetic field and in return, the current induces a magnetic field that affects the original field. MHD study for fluid in porous media contributes immensely to physiological laws and engineering. Darcy's law explains how a fluid flows in a porous medium and states that the rate of fluid discharge is proportional to the gradient in the hydraulic head and hydraulic conductivity.

In engineering fields, MHD is applicable in MHD power generators, petroleum engineering, industrial cooling systems, etc. Therefore, continuing with the MHD study is imperative to improve applications in physics, chemistry, engineering, metallurgy, heat exchangers, accelerators, etc. In the above applications, cooling effects help achieve the desired characteristics. The flow field characteristics depend on the orientation and magnitude of the magnetic field. For instance, Hayat et al. (2016) noted that a change in magnetic field strength influences the nature and concentration of the fluid particles and consequently impacts heat transfer. The fluid flow phenomenon past a permeable material is conceptualized as a natural mechanism like absorbing water and minerals in plants and the flow of nutrients in the human body. Industrial processes such as pulp drying, detergent manufacturing, diffusion, capillarity, and

gas management in fuel cells rely on fluid dynamics porous material phenomena. Ahmad et al. (2018) observed that MHD knowledge is applicabel in manufacturing and technological processes, such as, wire drawing, paper and glass fiber production, industrial polymer, and metal production.

### 1.3 Definitions

- i. A porous medium is a material with interconnected voids and passages that allow fluid to penetrate and flow.
- ii. Isotropic material is one whose permittivity and permeability is uniform in all directions.
- iii. Viscosity refers to a measure of fluid's resistance to deformation when subjected to a given force.
- iv. Buoyancy forces cause fluid flow due to a change in density arising from a difference in temperature gradients.
- v. Temperature is the degree of hotness or coldness. It influences fluid flow since a rise in temperature produces a corresponding decrease in density.
- vi. Heat transfer is the flow of heat energy through a physical mechanism either conduction, convection, or radiation.
- vii. Magnetohydrodynamics (MHD) involves the study of fluids which conduct electricity under magnetic field influence.

### 1.4 Statement of the Problem

The effect of varying the distance between two parallel vertical plates for an MHD fluid flow with heat transfer presents a research gap. In this study, we investigated a Newtonian MHD fluid flow bounded by two impermeable vertical plates in a porous media. The plates are heated and kept at constant temperature  $T_w$  and the distance between the two plates is varied. The fluid and the porous matrix are approximated to have the same temperature,  $T_f$ . The fluid flows uniformly

in x-direction between the two vertical plates. A transverse magnetic field is applied perpendicular to the plates in the positive y-direction. The study considered a steady and incompressible fluid. The study examined the effect of varying the vertical plate distances on velocity and temperature profiles.

## **1.5 Justification**

Heat transfer is either into the system or out of the system. The process of adding or removing heat has far reaching implications on fluid viscosity and heat conductivity. MHD heat transfer through a porous medium is applicable in aerodynamic heating and cooling automobiles. Furthermore, the study of MHD flow has applications in glass manufacturing, furnaces, and machine propulsion. Alam et al. (2021) observes that these processes depend on thermally radiating flows. Even though there is extensive research on MHD fluid focusing on heat transfer, mass transfer, and hall effect. There is a gap on the effect of varying the distance between parallel vertical plates for an MHD fluid with heat transfer. This study addresses this gap theoretically and discusses the findings graphically.

## **1.6 Objectives**

### **1.6.1 General Objectives**

To study the effect of varying the distance between the vertical parallel plates on velocity and temperature profiles for an MHD fluid flow in a permeable media with heat transfer.

### **1.6.2 Specific Objectives**

- i. To develop a mathematical model for MHD fluid flow bounded by two parallel vertical plates in a permeable media with heat transfer
- ii. To determine the velocity and temperature profile for an electrically conducting fluid bounded by two parallel vertical plates in a permeable media with heat transfer.

- iii. To investigate the effects of Reynolds number, Hartmann number, Prandtl number, and Darcy number on velocity and temperature profiles for this flow.

### **1.7 Significance of the study**

The study of MHD fluid flow has significant applications in the industries, biomedical fields, technology, and manufacturing, especially in separation processes and chemical operations. Furthermore, the pumping of molten metal and MHD stirring devices help intensify heat transfer when preparing alloys to average the temperature in the melt volume. This study is significant since it focuses on analyzing the effect of varying fluid properties on an MHD fluid flow. Varying the parallel plate distance helps determine the nature of flow and temperature fields. The fields arise from varying Reynolds, Hartmann, Darcy, and Prandtl number. These findings have significant application in engineering and manufacturing.

## CHAPTER TWO

### LITERATURE REVIEW

#### 2.1 Introduction

The chapter presents a review of literature in the study area and also presents other relevant works of literature that inform the formulation of the research problem.

#### 2.2 Literature Review

Reddy et. Al (2023) studied heat generation and absorption on MHD fluid with heat transfer. The Newtonian fluid was flowing through a permeable stretching cylinder. The study considered 2D flow whose resulting governing equations were highly non-linear and were solved using Keller-Box technique which is a highly accurate implicit finite difference method. It was observed that increasing the intensity of magnetic field dampened the velocity due to generation of drag forces. The Lorentz forces lead to resistance to fluid flow thus fluid velocity decreases. Furthermore, it was observed that an increase in Prandtl number leads to a decrease in the thermal boundary layer and consequently, fluid temperature decreases.

Kocić et al. (2023) studied a micro-polar fluid through a permeable material. The electrically conducting fluid studied helps to understand flow of fluids such as blood in human beings and animals. It can also help to describe flows such as colloidal fluids and in chemical suspensions. The study considered the flow in a constant pressure gradient bounded by two parallel plates set at different constant temperatures. It was observed that the applied external magnetic field tends to reduce flow velocity due to generation of drag forces. This is because the generated Lorentz force acts in a direction opposite to the fluid flow and thus retards fluid movement. The velocity field was observed to be uniform along the channel created by the vertical plates.

Onyinkwa and Chepkwony (2023) studied Mass and heat transfer of an electrically conducting fluid over a semi-infinite inclined permeable plate. The fluid flow was modeled in the positive x-direction and a transverse magnetic field on the permeable plate. The resulting first order differential equations were solved using an implicit central difference approximation method. It was observed that increasing the value of magnetic field strength causes a decrease in fluid velocity and an increase in its temperature. It was also observed that high values of porosity led to an increase in fluid temperature. This implies that at high values of porosity, fluid velocity is increased and thus it enhances convective heat transfer.

Lavanya (2019) investigated unsteady two-dimensional MHD flow via a permeable material, considering it in a revolving parallel plate channel. The flow parameters, heat transfer rate, velocity, shear stress, mass transfer rate, and volume flow rate were analyzed non-dimensionally. The researcher observed that increasing the Hartmann number reduces velocity and enhances permeability parameters ( $K$ ). Furthermore, increasing  $Gr$ ,  $Gc$ , and slip parameters enhanced resultant velocities. Notably, an increase in thermal radiation produces a corresponding reduction in heat transfer rate and temperature at any specific point. However, thermal radiation has no significant effect on velocity.

Lavanya (2017) examined an MHD transient fluid flow passing through two vertical walls in the presence of radiation to investigate the effect on mass and heat transfer rates. The researcher adopts a perturbation technique to determine PDEs governing velocity and temperature fields. Lavanya assumed a fluctuating temperature on one plate and graphically represented the flow parameters. Also, he developed equations governing skin friction on the parallel plates.

Appidi et al. (2021) examined MHD free convection flow subjected to a heat source and chemical reaction past a vertical permeable plate. Using the finite element method, the researchers

presented the problem using the non-direct fractional differential method and solved the resulting equations. The researcher concludes that the temperature increases with the increasing heat source parameter. On the other hand, the fluid flow temperature decreases on increasing radiation parameters and Pr. Finally, an increasing velocity field leads to a corresponding Gr and permeability parameter increase.

Sehra et al. (2021) postulated an incompressible Newtonian viscous fluid problem with exponential heating. The scholars formulated three governing equations and corresponding IBCs, which were solved using Laplace transform methods. The researchers also investigated the flow parameters and concluded that fluid motion increases as the Grashof number increases, which agrees with Appidi et al. (2021) and Lavanya (2019). Furthermore, higher values of non-dimensional heat source and Prandtl number led to decelerating temperature profiles. Rubbab et al. (2013) observed that neglecting the heat source parameter leads to a high temperature.

Sehra et al. (2021) observed fluid parameters such as velocity increase without a non-dimensional heat source and chemical reaction parameter. Conversely, a decrease in velocity was observed due to a higher Prandtl number, magnetic parameter, non-dimensional heat source, chemical reaction, and Schmidt number. Rubbab et al., (2013) observed that Pr and Gr have a significant effect on fluid motion and that fluid velocity increases as a function of Gr when cooling the plate and decreases as a function of Gr when heating the plate.

Sakthikala & Lavanya (2020) studied a non-Newtonian MHD fluid over a permeable material to investigate the fluid velocity, temperature, and concentration. The scholars established that fluid velocity varies directly to K, M, and inversely to viscoelastic parameters for a heated plate. Further, the temperature is proportional to M, while concentration varies inversely to Sc with respect to time. Abel et al. (2012) studied an MHD flow in the presence of thermal radiation and

observed that the elastic parameter and magnetic field suppress the velocity for upper convected Maxwell (UCM) fluid and enhance the temperature field. Furthermore, fluid temperature above the sheet is proportional to  $Ec$  and varies inversely to the radiation parameter.

Ngesa (2018) studied an MHD fluid flow past an erect permeable plate and observed that the effects of convection currents' on the plates have no significant effect on the heat transfer rate. Furthermore, a strong magnetic field affects the heat boundary layer due to the Hall effect. Sehra et al. (2021) observed that the fluid velocity for an MHD flow past an upright plate with heat-mass transfer varies inversely with  $Pr$ ,  $Sc$ , and magnetic parameters. Conversely, fluid flow is proportional to large values of  $Gr$ . Finally, the researchers observed that the temperature profile varies inversely to increase in  $Pr$  and heat source term.

Rubbab et al., (2013) studied fluid flow near vertical plates for unsteady convection and incompressible fluid is due to shear stress and plate heating. The governing equations were solved using Laplacian transform techniques using shear stress boundary conditions. Flow parameters, i.e., temperature and velocity, were developed using Boussinesq approximations. The researchers aimed to provide exact solutions for the said flow near an infinite plate with exponential heating. First, the researchers had remarkable findings that indicated that fluid velocity is a sum of mechanical and thermal components, i.e.,  $u(y,t) = u_m(y,t) + u_t(y,t)$ . Secondly, and in agreement with Appidi et al. (2021) and Sehra et al. (2021), the fluid temperature is inversely proportional to the Prandtl number. Rubbab et al. (2013) further concluded that fluid temperature increases with time. Furthermore, heating the plates leads to higher  $Pr$  values or small positive values of  $Gr$ , and the fluid velocity decreases. A reverse effect is observed on cooling as fluid velocity increases as a function of  $Gr$ .

Ngesa (2018) studied MHD fluid through a semi-infinite erect permeable plate in an inclined magnetic field. The findings indicate that the angle of inclination does not affect temperature profiles. On the other hand, a change in the angle produces a corresponding change in primary and secondary velocity profiles near the plates. Furthermore, treating the flow parameters unchanging, i.e.,  $Gr$ , heat source parameter, and suction velocity, increasing inclination angle led to a corresponding reduction in heat transfer rate and skin friction. Ngesa (2018) and Rubbab et al. (2013) noted that  $Sc$  as well as  $Gr$  affect the concentration, temperature, and velocity profiles.

Abel et al. (2012) studied an incompressible 2D steady MHD to investigate heat transfer dependence on various flow characteristics, e.g., Eckert and Prandtl number, elastic, magnetic, and radiation parameters, leading to remarkable observations. First, raising the magnetic parameter or elastic parameter value reduces velocity since the two parameters above the stretching sheet suppress the velocity field and enhance the temperature profile. Secondly, the Prandtl value decreases the thickness of the heat boundary layer, leading to uniform temperature distribution. Notably, small values of  $Pr$  equal thermal conductivity increase and, consequently, ease heat diffusion from the heated surfaces. Higher  $Ec$  increases fluid temperature above the boundary and is effective for cooling. In particular, the wall temperature is less than the fluid temperature near the wall, and thus, heat is dissipated from the fluid to the sheet. On the other hand, an increase in radiation parameters produces a corresponding decrease in fluid temperature above the boundary; this phenomenon can increase the cooling rate.

Falade et al. (2017) studied an oscillatory MHD flow through a permeable channel. The researchers subjected the fluid to a perpendicular magnetic field and a non-uniform wall temperature. The lower plate was used to carry out investigations on flow parameters since it was set at a low velocity. On solving the governing dimensionless equations, the researcher concludes

that there is a significant effect on flow parameters such as temperature, velocity profile, heat transfer rate, and skin friction. The solutions indicated that increasing suction or injection enhanced flow velocity at the wall and increased the fluid temperature, enhancing skin friction on the plates. Finally, an increase in the injection or suction decreased the heat transfer rate on plates while increasing the walls.

Acharya et al. (2014) studied convective MHD flow past an erect permeable plate. The plate temperature from a heat source varied in space and time. The scholars established that flow characteristics are not affected by porous media and instead, convective current leads to heating and cooling of the plates. Furthermore, heating the plates leads to fluid backflow, Lorentz forces retards velocity profiles, and a cooling current from viscous dissipation accelerates velocity.

Sakthikala & Lavanya (2020) studied a non-newtonian MHD flow in a magnetic field through a permeable material with varying wall temperatures. The governing equations were solved using regular perturbation methods and reduced to boundary value problems. The solutions on flow parameters indicate that velocity increases with increasing hall and permeability parameters. Furthermore, velocity increased with higher values of the viscoelastic parameter. On the other hand, low values of Hartmann's number correspond to velocity decrease. Finally, temperature and concentration decrease, corresponding to a decrease in the Prandtl number and Schmidt number, respectively.

Hayat et al. (2016) investigated the unsteady MHD fluid between two parallel rotating disks with porous media. The problem was formulated in Joule heating, thermal radiation, and stratification, and the governing PDEs were converted to nonlinear differential equations, and the solutions explained the flow parameters. The tangential velocity profile decreased as a function of the Hartmann number. The magnitude of radial and axial velocity depends on the values of

Reynold numbers. The researchers also observed that an increase in thermal stratification parameters and Prandtl values led to fluid temperature decrease, which agrees with Appidi et al. (2021). The scholars observed that fluid flow temperature decreases as an increasing radiation parameter and Prandtl number. Furthermore, Hayat et al. (2016) and Abel et al. (2012) findings agrees that fluid temperature increases with high Eckert number and radiation parameters. Finally, large Prandtl, Reynolds, and Eckert values increase the heat transfer rate.

Ersoy (2012) studied unsteady flow due to oscillations of eccentrically spinning disks and the fluid flow in the opposite direction. The researcher investigated the influence of the oscillations, Reynolds number, and time on the fluid velocity and noted that short times increase velocity. On the other hand, decreasing the oscillation frequency delays the periodic motion. When the disks oscillate in the x-direction, translational velocity is observed along the same axis while negligible in the y-axis. High Reynolds numbers decrease boundary layer thickness. Finally, the researchers observed a phase lag between flow velocity and disk oscillation during periodic motion.

The above studies demonstrate substantial contributions by researchers in studying MHD fluid flow with heat transfer. However, there is no study focusing on varying the distance between vertical plates bounding an MHD fluid flow. We are focusing on the effect of resulting dimensionless numbers from non dimensionalization of the specific governing equations on temperature and velocity profile. This study involves an MHD fluid flow bounded by two impermeable vertical plates.

## CHAPTER THREE

### GENERAL EQUATIONS

#### 3.1 Introduction

The chapter presents the general fluid flow equation, i.e., continuity, momentum, energy, Darcy's, and Maxwell's equations.

#### 3.2 Governing Equations

##### 3.2.1 Continuity Equation

The equation of continuity is expressed mathematically in rectangular coordinates is as

$$\frac{\partial \rho}{\partial t} + \rho \left( \frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} + \frac{\partial w}{\partial z} \right) = 0 \dots \dots \dots (3.1)$$

$\rho$  – fluid density,  $\mathbf{u}, \mathbf{v}, \mathbf{w}$  - velocity component in (x, y, z) rectangular coordinate system.

##### 3.2.2 Momentum equation

The momentum equation is expressed mathematically as

$$\rho \left( \frac{\partial \mathbf{u}}{\partial t} + (\mathbf{V} \cdot \nabla) \mathbf{V} \right) = -\nabla P + \mathbf{F} + \mu \nabla^2 \mathbf{V} \dots \dots \dots (3.2)$$

On LHS, the first term is the temporal acceleration followed by conventional acceleration. On

RHS, the first term is pressure followed by body force, and finally the dynamic viscosity.

##### 3.2.3 Energy Equation

The first principles of thermodynamics imply that: -

$$dQ = dE + dW \dots \dots \dots (3.3)$$

where Q-heat added, E-internal energy, W-Work done

### 3.2.4 Maxwell's Equations

Maxwell's equation describes the link between current density, magnetic and electric field. These equations are:

$$\text{Gauss's law of electricity } \nabla \cdot E = \frac{\rho}{\epsilon_0} \dots \dots \dots (3.4)$$

$$\text{Gauss's law of magnetism } \nabla \cdot B = 0 \dots \dots \dots (3.5)$$

$$\text{Faraday's law of magnetism } \nabla \times E = -\frac{\partial B}{\partial t} \dots \dots \dots (3.6)$$

$$\text{Ampere's rule } \nabla \times B = \mu_0 j + \mu_0 \epsilon_0 \frac{\partial E}{\partial t} \dots \dots \dots (3.7)$$

### 3.2.5 Darcy's equation

The equation is expressed mathematically as

$$\frac{Q}{A} = -\frac{dh}{dl} \dots \dots \dots (3.8)$$

Further, we consider the Darcy model which represents momentum balance for a fluid in a porous medium.

$$V = -\frac{k}{\mu} (\nabla p - \rho g e_g) \dots \dots \dots (3.9)$$

## CHAPTER FOUR

### MATHEMATICAL FORMULATION

#### 4.1 Introduction

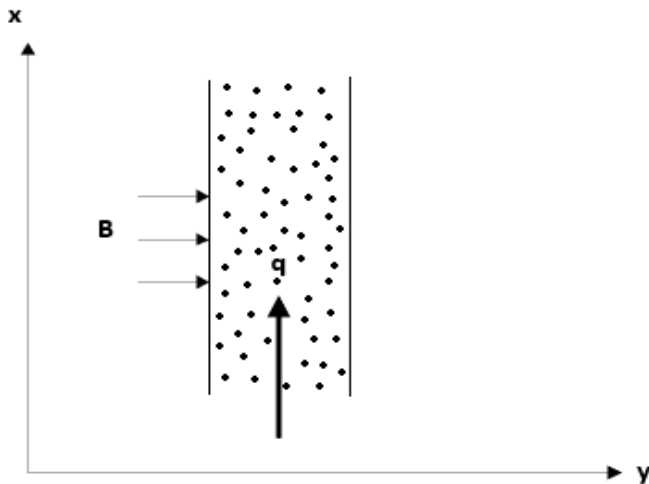
The chapter presents a geometrical model of the flow and mathematical formulation of the governing equations to fit our study.

#### 4.2 Flow Model

Consider a 2-D fluid flow in an isotropic porous media bounded by two vertical plates. The fluid is incompressible, steady, and flows only in the x-direction. The distance between the vertical plates is  $H$ . There is a magnetic field acting on one of the plates in the y-direction. The fluid flow is analysed using the model in Figure 4.1.

**Figure 4.1**

*Flow Model*



#### 4.3 Assumptions

Assumptions made for this study are:

- i) The fluid is incompressible

- ii) No slip condition.
- iii) The flow is steady.
- iv) The plates are maintained at a constant temperature  $T_w$ .
- v) The fluid flows uniformly in x.

**4.4 Mathematical Formulation**

**4.4.1 Continuity Equation**

From equation (3.2), the equation of continuity is written mathematically as

$$\frac{\partial \rho}{\partial t} + \rho \left( \frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} + \frac{\partial w}{\partial z} \right) = 0 \dots\dots\dots (4.1)$$

The flow is steady thus the first term reduces to zero. Furthermore, the fluid is incompressible and therefore equation (4.1) reduces to,

$$\frac{\partial}{\partial x}(u) + \frac{\partial}{\partial y}(v) + \frac{\partial}{\partial z}(w) = 0 \dots\dots\dots (4.2)$$

The flow is in 2 D and thus velocity along z-direction is zero. Equation (4.2) becomes,

$$\frac{\partial}{\partial x}(u) + \frac{\partial}{\partial y}(v) = 0 \dots\dots\dots (4.3)$$

Storey (2018) observed that inlet flow fields are uniform in space with a single velocity. In particular, a uniform flow field in the x-direction implies that the vertical velocity is zero. Since our fluid flows in the x-direction, the y-direction is vertical to the flow field implying that  $v=0$  and thus equation 4.3 becomes

$$\frac{\partial u}{\partial x} = 0 \dots\dots\dots (4.4)$$

Integrating the equation to determine the value of u

$$\int \frac{\partial u}{\partial x} = 0$$

$$u = u_0 \dots\dots\dots (4.3)$$

**4.4.2 Momentum Equation**

From equation (3.2), the Momentum equation is expressed Mathematically as

$$\rho \frac{\partial u}{\partial t} + \rho(V \cdot \nabla)V = -\nabla P + \mu \nabla^2 V + F \dots\dots\dots (4.4)$$

The body forces (F) acting on the fluid are the Lorentz force ( $J \times B$ ) and Buoyancy force ( $\rho g$ ).

Furthermore, we introduce a Darcy term ( $\frac{\mu}{k} V$ ) to model fluid flow in a porous Matrix. The Darcy term accounts for filtration velocity in the porous media.

$$\rho \frac{\partial u}{\partial t} + \rho(V \cdot \nabla)V = -\nabla P + \mu \nabla^2 V - \rho g + J \times B - \frac{\mu}{k} V \dots\dots\dots (4.5)$$

The flow field is in 2D and thus, momentum equation in X and Y direction can be written from equation (4.5) respectively as

$$\begin{aligned} \rho \frac{\partial u}{\partial t} + \rho \left( u \frac{\partial u}{\partial x} + v \frac{\partial u}{\partial y} + w \frac{\partial u}{\partial z} \right) \\ = -\frac{\partial P}{\partial x} + \mu \left( \frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} + \frac{\partial^2 u}{\partial z^2} \right) + \rho g_x + J \times B - \frac{\mu}{k} u \dots\dots\dots (4.6) \end{aligned}$$

$$\begin{aligned} \rho \frac{\partial v}{\partial t} + \rho \left( u \frac{\partial v}{\partial x} + v \frac{\partial v}{\partial y} + w \frac{\partial v}{\partial z} \right) \\ = -\frac{\partial P}{\partial y} + \mu \left( \frac{\partial^2 v}{\partial x^2} + \frac{\partial^2 v}{\partial y^2} + \frac{\partial^2 v}{\partial z^2} \right) + \rho g_y + J \times B - \frac{\mu}{k} v \dots\dots\dots (4.7) \end{aligned}$$

The fluid flow is steady, i.e., it is independent of time making the first term of equation (4.6) and (4.7) reduce to zero.

$$\begin{aligned} \rho \left( u \frac{\partial u}{\partial x} + v \frac{\partial u}{\partial y} + w \frac{\partial u}{\partial z} \right) \\ = -\frac{\partial P}{\partial x} + \mu \left( \frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} + \frac{\partial^2 u}{\partial z^2} \right) + \rho g_x + J \times B - \frac{\mu}{k} u \dots \dots \dots (4.8) \end{aligned}$$

$$\begin{aligned} \rho \left( u \frac{\partial v}{\partial x} + v \frac{\partial v}{\partial y} + w \frac{\partial v}{\partial z} \right) \\ = -\frac{\partial P}{\partial y} + \mu \left( \frac{\partial^2 v}{\partial x^2} + \frac{\partial^2 v}{\partial y^2} + \frac{\partial^2 v}{\partial z^2} \right) + \rho g_y + J \times B - \frac{\mu}{k} v \dots \dots \dots (4.9) \end{aligned}$$

The flow field is 2D and thus velocity component in z- direction is zero. Equation (4.9) and (4.8) becomes

$$\rho \left( u \frac{\partial u}{\partial x} + v \frac{\partial u}{\partial y} \right) = -\frac{\partial P}{\partial x} + \mu \left( \frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} \right) + \rho g_x + J \times B - \frac{\mu}{k} u \dots \dots \dots (4.10)$$

$$\rho \left( u \frac{\partial v}{\partial x} + v \frac{\partial v}{\partial y} \right) = -\frac{\partial P}{\partial y} + \mu \left( \frac{\partial^2 v}{\partial x^2} + \frac{\partial^2 v}{\partial y^2} \right) + \rho g_y + J \times B - \frac{\mu}{k} v \dots \dots \dots (4.11)$$

The flow field is uniform in x-direction implying that the vertical velocity (v)is zero. Therefore, momentum equation (4.11) in Y direction becomes

$$0 = \rho g_y - \frac{\partial p}{\partial y} \dots \dots \dots (4.12)$$

The gravitational force in y direction is negligible, thus we neglect the first term

$$-\frac{\partial p}{\partial y} = 0 \dots \dots \dots (4.13)$$

Integrating equation (4.11) we have that

$$\int \left( -\frac{\partial P}{\partial y} \right) = P(x) \dots \dots \dots (4.14)$$

This implies that pressure gradient is a function of (x) and the negative sign implies that the fluid flows from a region of high pressure to low pressure.

Equation (4.10) is the momentum equation in X-direction. The equation can be simplified further to obtain viscous term. The fluid velocity at different points in x depend on y only. Thus, equation (4.10) becomes

$$\rho \left( u \frac{\partial u}{\partial x} + v \frac{\partial u}{\partial y} \right) = - \frac{\partial P}{\partial x} + \mu \left( \frac{\partial^2 u}{\partial y^2} \right) + \rho g_x + J \times B - \frac{\mu}{k} u \dots \dots \dots (4.15)$$

The velocity field is uniform in the x-direction and therefore velocity (u) in equation (4.15) can be substituted with  $u_0$  obtained in equation (4.3). Furthermore, the value of velocity in the Darcy term is also  $u_0$  since, the porous material is assumed to be isotropic meaning that the flow properties will be the same.

$$\rho \left( u_0 \frac{\partial u}{\partial x} + v \frac{\partial u}{\partial y} \right) = - \frac{\partial P}{\partial x} + \mu \left( \frac{\partial^2 u}{\partial y^2} \right) + \rho g_x + J \times B - \frac{\mu}{k} u_0 \dots \dots \dots (4.16)$$

We need to solve equation (4.16) for pressure gradient near the vertical plates where velocity is minimum  $u \rightarrow 0$  and density  $\rho \rightarrow \rho_\infty$ .

$$0 = \rho_\infty g - \frac{\partial p}{\partial x} \dots \dots \dots (4.17)$$

Equation (4.17) can be solved to obtain

$$-\rho_\infty g + \rho g = -g(\rho_\infty - \rho) \dots \dots \dots (4.18)$$

Substituting equation (4.18) into (4.16),

$$\rho \left( u_0 \frac{\partial u}{\partial x} + v \frac{\partial u}{\partial y} \right) = \mu \frac{\partial^2 u}{\partial y^2} - J \times B - g(\rho_\infty - \rho) - \frac{\mu}{k} u_0 \dots \dots \dots (4.19)$$

Buoyancy term arises from heating leading to a temperature difference which causes density change and consequently, the fluid flows. For this reason, we need to introduce a coefficient of thermal expansion  $\beta$  which is given mathematically as

$$\beta = \frac{1}{v} \left( \frac{\Delta V}{\Delta T} \right) = \frac{1}{v} \left( \frac{\partial v}{\partial \rho} \frac{\partial \rho}{\partial T} \right) \dots \dots \dots (4.20)$$

The RHS of equation (4.20) can further be evaluated

$$v = \frac{1}{\rho} \dots \dots \dots (4.21)$$

and thus

$$\frac{\partial v}{\partial \rho} = -\frac{1}{\rho^2} \dots \dots \dots (4.22)$$

also

$$\frac{\partial \rho}{\partial T} = \frac{\rho_\infty - \rho}{T - T_\infty} \dots \dots \dots (4.23)$$

Substituting equation (4.21), (4.22), and (4.23) into (4.20), we have the coefficient of thermal expansion being given by

$$\beta = \frac{\rho}{1} \left( -\frac{1}{\rho^2} \right) \left( \frac{\rho_\infty - \rho}{T - T_\infty} \right) = -\frac{1}{\rho} \left( \frac{\rho_\infty - \rho}{T - T_\infty} \right) \dots \dots \dots (4.24)$$

Expressing density change in terms of thermal expansion coefficient (4.24) becomes

$$-\beta \rho (T - T_\infty) = \rho_\infty - \rho \dots \dots \dots (4.25)$$

Substituting equation 4.25 into equation 4.19, we get

$$\rho \left( u_0 \frac{\partial u}{\partial x} + v \frac{\partial u}{\partial y} \right) = \mu \frac{\partial^2 u}{\partial y^2} + \mathbf{J} \times \mathbf{B} + g\beta\rho(T - T_\infty) - \frac{\mu}{k} u_0 \dots \dots \dots (4.26)$$

Now working out the Lorentz term,

$$\mathbf{J} \times \mathbf{B} = \delta(\mathbf{V} \times \mathbf{B}) \times \mathbf{B} \dots \dots \dots (4.27)$$

where V is a velocity component given as  $q_1, q_2, q_3$  and the magnetic field B is given as  $(0, B_y, 0)$

$$\delta(\mathbf{V} \times \mathbf{B}) = \delta \begin{pmatrix} i & j & k \\ q_1 & q_2 & q_3 \\ 0 & B_y & 0 \end{pmatrix} = \delta(B_y q_3 i + B_y q_1 k)$$

$$\delta(\mathbf{V} \times \mathbf{B}) \times \mathbf{B} = \delta \begin{pmatrix} i & j & k \\ B_y q_3 & 0 & q_1 B_y \\ 0 & B_y & 0 \end{pmatrix} = \delta(q_1 B_y^2 i - B_y^2 q_3 k)$$

The cross product can be simplified further since the flow is in 2D, thus dropping the term with  $q_3$ , i.e., velocity in the z-direction.

$$\delta u B_y^2 i \dots \dots \dots (4.28)$$

Thus, substituting (4.28) into (4.26) gives

$$\rho \left( u_0 \frac{\partial u}{\partial x} + v \frac{\partial u}{\partial y} \right) = \mu \frac{\partial^2 u}{\partial y^2} + \delta u B_y^2 + g\beta\rho(T - T_\infty) - \frac{\mu}{k} u_0 \dots \dots \dots (4.29)$$

### 4.4.3 Energy Equation

From energy equation (3.3), it is written in general as

$$dQ = DE + Dw \dots \dots \dots (4.30)$$

where Q-heat added, E-internal energy, W-Work done

The general energy equation in fluid flow is expressed as

$$\rho c_p \left( \frac{\Delta T}{\Delta t} \right) = k \nabla^2 T + \phi \dots \dots \dots (4.31)$$

where the first term is the convective heat transfer, conductive heat transfer, and Joule heating.

The first term and the third term can further be expressed as follows: -

$$\frac{\Delta T}{\Delta t} = \frac{\partial T}{\partial t} + (\mathbf{V} \cdot \nabla) T \dots \dots \dots (4.32)$$

$$\phi = \frac{J^2}{\delta} \dots \dots \dots (4.33)$$

where  $J = \delta (E + V \times B)$ , assuming that the electric field is negligible,  $J = \delta (V \times B)$ .

Taking the cross product considering that  $\mathbf{V} = q_1 i + q_2 j + q_3 k$  and magnetic field  $\mathbf{B} = B_x i + B_y j + B_z k$ . Also, note that the velocity is steady and uniform in x, thus  $q_2 = q_3 = 0$  and  $B_x = B_z = 0$ . The terms for cross product simplify to: -  $\mathbf{V} = q_1 i + 0j + 0k$  and  $\mathbf{B} = 0i + B_j + 0k$ .

Therefore,

$$\mathbf{J} = (\mathbf{V} \times \mathbf{B}) = \delta \begin{pmatrix} i & j & k \\ q_1 & 0 & 0 \\ 0 & B_y & 0 \end{pmatrix} = \delta B_y q_1 \mathbf{k} \dots \dots \dots (4.34)$$

From equation (4.27), we obtain  $J = \delta u B_y$  consequently equation 4. 26 becomes,

$$\phi = \frac{J^2}{\delta} = \delta u^2 B_y^2 \dots \dots \dots (4.35)$$

Substituting equation 4.25, and 4.28 into 4.24, we obtain

$$\rho c_p \left( \frac{\partial T}{\partial t} + (\mathbf{V} \cdot \nabla) T \right) = k \nabla^2 T + \delta u^2 B_y^2 \dots \dots \dots (4.36)$$

Furthermore, equation 5.19 can be expressed in Cartesian coordinate as follows,

$$\rho c_p \left( \frac{\partial T}{\partial t} + u \frac{\partial T}{\partial x} + v \frac{\partial T}{\partial y} \right) = k \left( \frac{\partial^2 T}{\partial x^2} + \frac{\partial^2 T}{\partial y^2} + \frac{\partial^2 T}{\partial z^2} \right) + \delta u^2 B_y^2 \dots \dots \dots (4.37)$$

Equation (5.19) can further be simplified since the flow is steady and thermal conductivity is a variable in y, thus

$$u \frac{\partial T}{\partial x} + v \frac{\partial T}{\partial y} = \frac{k}{\rho c_p} \left( \frac{\partial^2 T}{\partial y^2} \right) + \frac{\delta u^2 B_y^2}{\rho c_p} \dots \dots \dots (4.38)$$

Equation 4.38 represents the energy equation in 2-D.

#### 4.5 Boundary Conditions

The initial and boundary conditions used in this study are derived with the help of the model in Fig 4.1. Also, putting into consideration the assumption that at the fluid-plate boundary, velocity is zero. Furthermore, since the two plates are heated and maintained at constant temperature, the temperature difference between the plate and the fluid compels the fluid to move.

$$u = v = 0 \text{ at } y = 0 \text{ and } y = H \dots \dots (4.39)$$

$$T = T_w \text{ at } y = 0 \text{ and } y = H \dots \dots (4.40)$$

$$u = u_0 \text{ at } y = \frac{H}{2} \dots \dots (4.41)$$

$$T = T_f \text{ at } y = \frac{H}{2} \dots \dots (4.42)$$

#### 4.6 Non Dimensionalization

It is a process of reducing a physical problem to its essential variables and removing the units of measurement from those variables. The physical variables are divided by appropriate scaling factors chosen based on the problem being studied. The resultant dimensionless quantities are often called "parameters" and describe the system's behavior under different conditions.

### Non-dimensionalizing Variables

Equations (4.29) and (4.38) will be dimensionalized using the following variables which are carefully chosen to achieve desired dimensionless numbers.

Using the following non-dimensional variables,

$$x^* = \frac{x}{H}, y^* = \frac{y}{H}, T^* = \frac{T - T_f}{T_w - T_f} = \frac{T - T_f}{\Delta T}, B_y^* = B_0, u_0^* = \frac{u_0}{U}, u^* = \frac{u}{U}, v^* = \frac{v}{U} \dots \dots \dots (4.43)$$

$U, V, H,$  and  $\nabla T$  are characteristic velocity, length, pressure, and temperature difference scales.

Rearranging the non-dimensional variables, we have

$$x = x^*H, y = y^*H, T - T_\infty = \Delta T T^* B_y^* = B_0 u_0 = U u_0^* u = U u^* v = v^*U \dots \dots \dots (4.44)$$

#### 4.6.1 Nondimensionalizing the Momentum Equation

We apply the non-dimensional variables (4.44) to equation (4.29)

$$\begin{aligned} \rho \left( u_0 \frac{\partial u}{\partial x} + v \frac{\partial u}{\partial y} \right) &= \mu \frac{\partial^2 u}{\partial y^2} + \delta u B_y^2 + g\beta\rho(T - T_\infty) - \frac{\mu}{k} u_0 \rho \left( U u_0^* \frac{\partial(Uu^*)}{\partial(x^*H)} + U v^* \frac{\partial(Uu^*)}{\partial(y^*H)} \right) \\ &= \mu \frac{\partial}{\partial y} \left( \frac{\partial(Uu^*)}{\partial(y^*H)} \right) + \delta(Uu^*)(B_0)^2 + g\beta\rho\Delta T T^* - \frac{\mu}{k} U u_0^* \dots \dots \dots (4.45) \end{aligned}$$

Pulling the like terms together and carrying out differentiations

$$\rho \left( U \left( \frac{U}{H} \right) \left[ u_0^* \frac{\partial u^*}{\partial x^*} + v^* \frac{\partial u^*}{\partial y^*} \right] \right) = \mu \frac{U}{H} \frac{\partial}{\partial y} \left( \frac{\partial(u^*)}{\partial(y^*)} \right) + \delta U B_0^2 u^* + g\beta\rho\Delta T T^* - \frac{\mu}{k} U u_0^* \dots \dots \dots (4.46)$$

$$\rho \left( U \left( \frac{U}{H} \right) \left[ u_0^* \frac{\partial u^*}{\partial x^*} + v^* \frac{\partial u^*}{\partial y^*} \right] \right) = \mu \frac{U}{H} \frac{1}{H} \left( \frac{\partial^2 u^*}{\partial y^{*2}} \right) + \delta U B_0^2 u^* + g\beta\rho\Delta T T^* - \frac{\mu}{k} U u_0^* \dots \dots \dots (4.47)$$

$$\rho \frac{U^2}{H} \left[ u_0^* \frac{\partial u^*}{\partial x^*} + v^* \frac{\partial u^*}{\partial y^*} \right] = \frac{\mu U}{H^2} \left( \frac{\partial^2 u^*}{\partial y^{*2}} \right) + \delta U B_0^2 u^* + g\beta\rho\Delta T T^* - \frac{\mu}{k} U u_0^* \dots \dots \dots (4.48)$$

Multiplying every term in the equation by  $\frac{H}{\rho U^2}$

$$\begin{aligned} & \frac{H}{\rho U^2} \times \rho \frac{U^2}{H} \left[ u_0^* \frac{\partial u^*}{\partial x^*} + v^* \frac{\partial u^*}{\partial y^*} \right] \\ &= \frac{H}{\rho U^2} \times \frac{\mu U}{H^2} \left( \frac{\partial^2 u^*}{\partial y^{*2}} \right) + \frac{H}{\rho U^2} \times \delta U B_0^2 u^* + \frac{H}{\rho U^2} \times g \beta \rho \Delta T T^* \\ & \quad - \frac{H}{\rho U^2} \\ & \quad \times \frac{\mu}{k} U u_0^* \dots \dots \dots (4.49) \end{aligned}$$

Simplifying equation (4.45)

$$\left[ u_0^* \frac{\partial u^*}{\partial x^*} + v^* \frac{\partial u^*}{\partial y^*} \right] = \frac{\mu}{\rho U H} \left( \frac{\partial^2 u^*}{\partial y^{*2}} \right) + \frac{H \delta B_0^2}{\rho U} u^* + \frac{H g \beta \Delta T}{U^2} T^* - \frac{H \mu}{\rho U k} u_0^* \dots \dots \dots (4.50)$$

Expressing the last term of (4.46) to introduce the dimensionless numbers we obtain

$$\begin{aligned} & u_0^* \frac{\partial u^*}{\partial x^*} + v^* \frac{\partial u^*}{\partial y^*} \\ &= \frac{\mu}{\rho U H} \left( \frac{\partial^2 u^*}{\partial y^{*2}} \right) + \frac{H \delta B_0^2}{\rho U} u^* + \frac{H g \beta \Delta T}{U^2} T^* - \left( \frac{H^2}{k} \right) \left( \frac{\mu}{\rho U H} \right) u_0^* \dots \dots \dots (4.51) \end{aligned}$$

Equation (4.51) is the nondimensionalized momentum equation.

### 4.6.2 Nondimensionalizing the Energy Equation

Applying the non-dimensional variables (4.40) to equation 4.34, we obtain

$$U u^* \frac{\partial (\Delta T T^*)}{\partial (x^* H)} + v^* U \frac{\partial (\Delta T T^*)}{\partial (y^* H)} = \frac{k}{\rho c_p} \left[ \frac{\partial}{\partial y} \left( \frac{\partial (\Delta T T^*)}{\partial (y^* H)} \right) \right] + \frac{\delta (U u^*)^2 B_0^2}{\rho c_p} \dots \dots \dots (4.52)$$

Pulling the like terms together and carrying out differentiations

$$\frac{U\Delta T}{H}u^*\frac{\partial(T^*)}{\partial(x^*)} + \frac{U\Delta T}{H}v^*\frac{\partial(T^*)}{\partial(y^*)} = \frac{k}{\rho c_p} \left[ \frac{\Delta T}{H} \frac{\partial}{\partial y} \left( \frac{\partial T^*}{\partial y^*} \right) \right] + \frac{\delta U^2 B_o^2 u^{*2}}{\rho c_p} \dots\dots\dots (4.53)$$

$$\frac{U\Delta T}{H} \left( u^* \frac{\partial T^*}{\partial x^*} + v^* \frac{\partial T^*}{\partial y^*} \right) = \frac{k}{\rho c_p} \left[ \frac{\Delta T}{H} \frac{1}{H} \frac{\partial^2 T^*}{\partial y^{*2}} \right] + \frac{\delta U^2 B_o^2 u^{*2}}{\rho c_p} \dots\dots\dots (4.54)$$

$$\frac{U\Delta T}{H} \left( u^* \frac{\partial T^*}{\partial x^*} + v^* \frac{\partial T^*}{\partial y^*} \right) = \frac{k\Delta T}{H^2 \rho c_p} \left( \frac{\partial^2 T^*}{\partial y^{*2}} \right) + \frac{\delta U^2 B_o^2 u^{*2}}{\rho c_p} \dots\dots\dots (4.55)$$

Multiply the equation throughout by  $\frac{H}{U\Delta T}$

$$\frac{H}{U\Delta T} \times \frac{U\Delta T}{H} \left( u^* \frac{\partial T^*}{\partial x^*} + v^* \frac{\partial T^*}{\partial y^*} \right) = \frac{H}{U\Delta T} \times \frac{k\Delta T}{H^2 \rho c_p} \left( \frac{\partial^2 T^*}{\partial y^{*2}} \right) + \frac{H}{U\Delta T} \times \frac{\delta U^2 B_o^2 u^{*2}}{\rho c_p} \dots\dots\dots (4.56)$$

Simplifying equation (4.53)

$$u^* \frac{\partial T^*}{\partial x^*} + v^* \frac{\partial T^*}{\partial y^*} = \frac{k}{UH\rho c_p} \left( \frac{\partial^2 T^*}{\partial y^{*2}} \right) + \frac{H\delta UB_o^2 u^{*2}}{\Delta T \rho c_p} \dots\dots\dots (4.57)$$

Expressing (4.54) in a manner we can introduce dimensionless numbers we have

$$u^* \frac{\partial T^*}{\partial x^*} + v^* \frac{\partial T^*}{\partial y^*} = \left( \frac{\mu}{\rho U H} \right) \left( \frac{k}{\mu c_p} \right) \left( \frac{\partial^2 T^*}{\partial y^{*2}} \right) + \left( \frac{U^2}{\Delta T c_p} \right) \left( \frac{H\delta B_o^2}{\rho U} \right) u^{*2} \dots\dots\dots (4.58)$$

Equation (4.58) is the nondimensionalized energy equation.

### 4.6.3 Non-dimensional Numbers

Non-dimensional numbers are mathematical quantities that represent ratios or relationships between physical quantities without any units of measurement.

#### i) Grashof Number (Gr)

It indicates the relationship between the force caused by spatial changes in fluid density due to temperature differences and the restraining force due to the fluid's viscosity. It is given by

$$Gr = \frac{H^3 \rho^2 \alpha g \Delta T}{\mu^2}$$

Where H- length scale,  $\rho$  – fluid density,  $\alpha$  – thermal diffusivity, g - gravitational acceleration

**ii) Prandtl Number (Pr)**

It represents the ratio of fluid viscosity to heat diffusion in a fluid. Fluids with small Pr are good thermal conductors.

$$Pr = \frac{k}{C_p \mu}$$

Where k- thermal conductivity,  $C_p$ - specific heat capacity  $\mu$  – viscosity

**iii) Reynold's Number**

It is a dimensionless parameter which describes the nature of fluid flow under different conditions.

$$Re = \frac{HU\rho}{\mu}$$

**iv) Hartmann Number**

It describes the ratio of electromagnetic and viscous forces in a conductive fluid.

$$Ha = \frac{H\delta B}{\rho U \mu}$$

where B- Magnetic field

**v) Darcy Number**

It describes permeability in a permeable medium.

$$Da = \frac{k}{H^2}$$

**vi) Eckert number**

It is a dimensionless number that describes the relationship between kinetic energy and thermal energy.

$$E_c = \frac{U^2}{\Delta T c_p}$$

**vii) Stuart number**

It is a dimensionless number that estimates the importance of magnetic fields on fluid flow.

$$M = \frac{H \delta B_o^2}{\rho U}$$

Introducing the non-dimensional numbers to equations (4.51) and (4.58), we obtain

$$u_0^* \frac{\partial u^*}{\partial x^*} + v^* \frac{\partial u^*}{\partial y^*} = \frac{1}{Re} \left( \frac{\partial^2 u^*}{\partial y^{*2}} \right) + Ha u^* + Gr T^* - \frac{Da}{Re} u_0^* \dots \dots \dots (4.59)$$

$$u^* \frac{\partial T^*}{\partial x^*} + v^* \frac{\partial T^*}{\partial y^*} = \frac{Pr}{Re} \left( \frac{\partial^2 T^*}{\partial y^{*2}} \right) + E_c M u^{*2} \dots \dots \dots (4.60)$$

**4.6.4 Nondimensionalizing the Boundary Conditions**

The boundary conditions can be nondimensionalized to obtain them in a form that can be used in numerical computations. Using the nondimensional variables (4.40) and applying them to boundary conditions (4.35) (4.36) (4.37) (4.38), we obtain: -

$$\mathbf{u} = \mathbf{v} = \mathbf{0} \text{ at } \mathbf{y} = \mathbf{0} \text{ and } \mathbf{y} = \mathbf{H}$$

$$u^* = \frac{0}{U} = 0, \quad v^* = \frac{0}{U} = 0, \quad y^* = \frac{y}{H} = \frac{0}{H} = 0, \quad y^* = \frac{y}{H} = \frac{H}{H} = 1$$

$$u^* = v^* = 0 \text{ at } y^* = 0 \text{ and } y^* = 1 \dots (4.61)$$

$$T = T_w \text{ at } y = 0 \text{ and } y = H$$

$$T^* = \frac{T_w - T_f}{T_w - T_f} = 1, \quad y^* = \frac{y}{H} = \frac{0}{H} = 0, \quad y^* = \frac{y}{H} = \frac{H}{H} = 1$$

$$T^* = 1 \text{ at } y^* = 0 \text{ and } y^* = 1 \dots \dots (4.62)$$

$$u = u_0 \text{ at } y = \frac{H}{2}$$

$$u^* = \frac{u_0}{U} = \frac{u_0 * U}{U} = u_0^*, \quad u^* = u_0^* \Rightarrow \frac{u^*}{u_0^*} = 1 \quad y^* = \frac{y}{H} = \frac{\left(\frac{H}{2}\right)}{H} = \frac{1}{2}$$

$$u^* = 1 \text{ at } y^* = \frac{1}{2} \dots \dots (4.63)$$

$$T = T_f \text{ at } y = \frac{H}{2}$$

$$T^* = \frac{T_f - T_f}{T_w - T_f} = 0, \quad y^* = \frac{y}{H} = \frac{\left(\frac{H}{2}\right)}{H} = \frac{1}{2}$$

$$T^* = 0 \text{ at } y^* = \frac{1}{2} \dots \dots (4.64)$$

In summary, the dimensionless boundary conditions are

$$u^* = v^* = 0 \text{ at } y^* = 0 \text{ and } y^* = 1 \dots \dots (4.65)$$

$$T^* = 1 \text{ at } y^* = 0 \text{ and } y^* = 1 \dots \dots (4.66)$$

$$u^* = 1 \text{ at } y^* = \frac{1}{2} \dots \dots (4.67)$$

$$T^* = 0 \text{ at } y^* = \frac{1}{2} \dots \dots (4.68)$$

## CHAPTER 5

### NUMERICAL SOLUTION

#### Introduction

The resulting equations after nondimensionalisation are equation (4.59) and (4.60)

$$u_0^* \frac{\partial u^*}{\partial x^*} + v^* \frac{\partial u^*}{\partial y^*} = \frac{1}{Re} \left( \frac{\partial^2 u^*}{\partial y^{*2}} \right) + Ha u^* + Gr T^* - \frac{Da}{Re} u_0^* \dots \dots \dots (5.1)$$

$$u^* \frac{\partial T^*}{\partial x^*} + v^* \frac{\partial T^*}{\partial y^*} = \frac{Pr}{Re} \left( \frac{\partial^2 T^*}{\partial y^{*2}} \right) + E_c M u^{*2} \dots \dots \dots (5.2)$$

These equations are nonlinear partial differential equations which should be solved using the corresponding boundary conditions. We adopt numerical methods to solve the equation since they are difficult to obtain solutions using the exact methods.

We employ the finite difference method to transform equations (5.1) and (5.2) into algebraic equations. The finite difference method is easy to use and unconditionally stable.

#### Mesh Definition

We consider a uniform rectangular mesh which is bounded and the mesh points approach infinity. The rectangular mesh helps determine the relation between partial derivatives and function values at a particular nodal point. The equations to be solved are in 2D and consequently, we develop grids that are uniform in width and height on an X-Y plane.

**Figure 4.1**

*Rectangular Grid*

m-1, n+1	m, n+1	m+1, n+1
m-1, n	m, n	m+1, n
m-1, n-1	m, n-1	m+1, n-1

Let  $\Delta x$  and  $\Delta y$  be the dimensions of the grids on the plane. Furthermore, we let  $m$  and  $n$  refer to  $x$  and  $y$  respectively such that,  $x = m\Delta x$  and  $y = m\Delta y$ . The finite difference approximations for equations (5.1) and (5.2) will be obtained from the expansion of Taylor's series on the dependent variable.

Consider a Taylor's series for a function  $\vartheta(y)$  about an arbitrary value, "a." By definition, the Taylor's series expansion at that point is given by

$$\vartheta(y) = \frac{\vartheta^0(a)}{0!}(y-a)^0 + \frac{\vartheta^1(a)}{1!}(y-a)^1 + \frac{\vartheta^2(a)}{2!}(y-a)^2 + \dots + \varepsilon \quad (5.3)$$

Considering this definition, the Taylor series for a given function about the stated point is given by

$$\vartheta(m+1, n) = \vartheta(m, n) + \vartheta'(m, n)\Delta x + \frac{1}{2}\vartheta''(m, n)(\Delta x)^2 + \dots \quad (5.4)$$

$$\vartheta(m-1, n) = \vartheta(m, n) - \vartheta'(m, n)\Delta x + \frac{1}{2}\vartheta''(m, n)(\Delta x)^2 + \dots \quad (5.5)$$

For our computations, we shall ignore higher order terms of the series since the error is assumed to be small.

Subtracting equation (5.4) and (5.5), we obtain

$$\begin{aligned} \vartheta(m+1, n) - \vartheta(m-1, n) \\ = 2\vartheta'(m, n)\Delta x + \dots \quad (5.6) \end{aligned}$$

Rearranging equation (5.6) to determine the first derivative  $\vartheta'(m, n)$

$$\vartheta'(m, n) = \frac{\vartheta(m+1, n) - \vartheta(m-1, n)}{2\Delta x} + \dots \quad (5.7)$$

Adding equation (5.4) and (5.5), we obtain

$$\vartheta(m + 1, n) + \vartheta(m - 1, n) = 2\vartheta(m, n) + \vartheta''(m, n)(\Delta x)^2 \dots \dots \dots (5.8)$$

Rearranging (5.8) to determine the approximation for the second derivate  $\vartheta''(m, n)$

$$\vartheta''(m, n) = \frac{\vartheta(m + 1, n) + \vartheta(m - 1, n) - 2\vartheta(m, n)}{(\Delta x)^2} \dots \dots \dots (5.9)$$

Similarly, the approximations for the first and second derivatives in y direction will be given by

$$\vartheta'(m, n) = \frac{\vartheta(m, n + 1) - \vartheta(m, n - 1)}{2\Delta y} \dots \dots \dots (5.10)$$

$$\vartheta''(m, n) = \frac{\vartheta(m, n + 1) + \vartheta(m, n - 1) - 2\vartheta(m, n)}{(\Delta y)^2} \dots \dots \dots (5.11)$$

Substituting equations (5.7), (5.10), and (5.11) to equation (5.1)

$$\begin{aligned} u_{m,n}^* \frac{(u_{m+1,n}^* - u_{m-1,n}^*)}{2\Delta x} + v_{m,n}^* \frac{(u_{m,n+1}^* - u_{m,n-1}^*)}{2 \Delta y} \\ = \frac{1}{Re} \frac{(u_{m,n+1}^* + u_{m,n-1}^* - 2u_{m,n}^*)}{(\Delta y)^2} + Ha u_{m,n}^* + Gr T_{m,n}^* - \frac{Da}{Re} u_{0m,n}^* \dots \dots \dots (5.12) \end{aligned}$$

Making  $u_{m+1,n}^*$  the subject, equation 5.12 becomes

$$\begin{aligned} u_{m+1,n}^* = u_{m-1,n}^* \\ + \frac{2\Delta x}{u_{m,n}^*} \left( -v_{m,n}^* \frac{(u_{m,n+1}^* - u_{m,n-1}^*)}{2 \Delta y} + \frac{1}{Re} \frac{(u_{m,n+1}^* + u_{m,n-1}^* - 2u_{m,n}^*)}{(\Delta y)^2} + Ha u_{m,n}^* \right. \\ \left. + Gr T_{m,n}^* - \frac{Da}{Re} u_{0m,n}^* \right) \dots \dots \dots (5.13) \end{aligned}$$

Substituting equations (5.7), (5.10), and (5.11) to equations (5.2)

$$\begin{aligned}
& u_{m,n}^* \frac{(T_{m+1,n}^* - T_{m-1,n}^*)}{2\Delta x} + v_{m,n}^* \frac{(T_{m,n+1}^* - T_{m,n-1}^*)}{2\Delta y} \\
&= \frac{Pr (T_{m,n+1}^* + T_{m,n-1}^* - 2T_{m,n}^*)}{Re (\Delta y)^2} + E_c Mu_{m,n}^{*2} \dots \dots \dots (5.14)
\end{aligned}$$

Making  $T_{m+1,n}^*$  the subject, equation (5.14) becomes

$$\begin{aligned}
T_{m+1,n}^* &= T_{m-1,n}^* \\
&+ \frac{2\Delta x}{u_{m,n}^*} \left( -v_{m,n}^* \frac{(T_{m,n+1}^* - T_{m,n-1}^*)}{2\Delta y} + \frac{Pr (T_{m,n+1}^* + T_{m,n-1}^* - 2T_{m,n}^*)}{Re (\Delta y)^2} \right. \\
&\left. + E_c Mu_{m,n}^{*2} \right) \dots \dots \dots (5.15)
\end{aligned}$$

Applying the boundary conditions obtained in (4.65), (4.66), (4.67), (4.68) to the resulting equations (5.13) and (5.15) using the Matrix Laboratory programming software we obtain solutions. The distances  $\Delta x$  and  $\Delta y$  were set as small increments when writing the MATLAB code.

## CHAPTER FIVE

### RESULTS AND DISCUSSION

#### 5.1 Introduction

The chapter presents graphical solutions for the equations. The graphs represent the effects of Reynolds number, Hartmann number, Prandtl number, and Darcy number on velocity and temperature profiles for a fluid bounded by parallel vertical plates in a porous media.

#### 5.2 Results and Discussion

**Figure 5.1**

*Effect of Hartmann number on velocity profile*

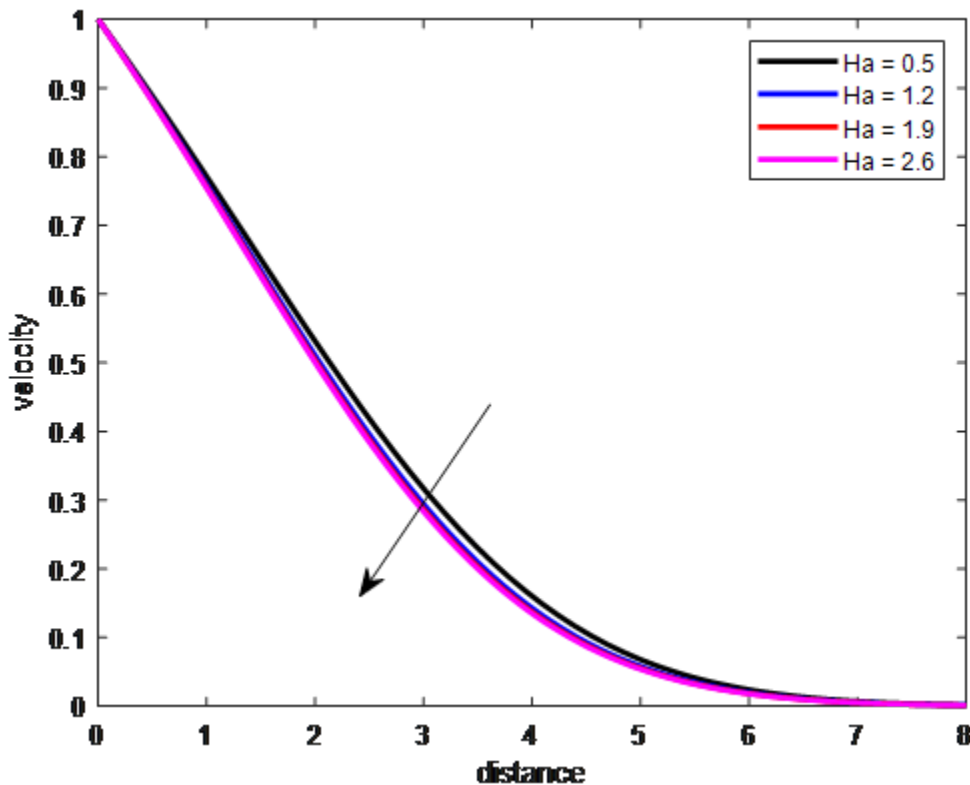


Figure 5.1 shows the effect of varying Hartmann number on velocity profile. The velocity decreases as Hartmann number is increased. An increase in Hartmann number causes magnetic forces to be greater than inertia forces. Consequently, this increases drag causing a reduction in

fluid velocity. High Hartmann numbers play crucial role in stabilizing fluid flow to achieve a laminar flow which is essential in liquid metal cooling. Mohamed et al (2021) observed that in physical terms, high Hartmann numbers generate a counter-force to fluid flow and thus at low Hartmann number, fluid velocity will be high. Momanyi and Wafula (2023) observed that high Hartmann number values tend to decrease fluid movement.

**Figure 5.2**

*Effect of Hartmann Number on the temperature profile*

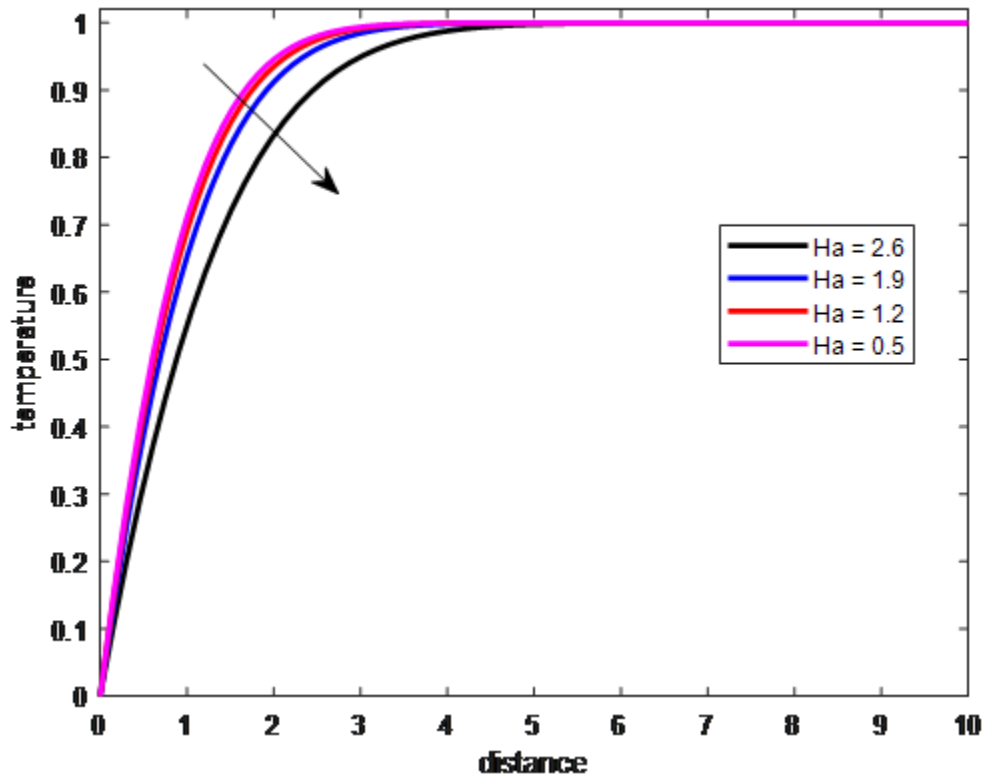


Figure 5.2 shows the effect of varying Hartmann number on temperature profile. Thermal temperature decreases as Hartmann number is increased. Large values of Hartmann number cause magnetic forces to dominate over viscous forces making it difficult for the fluid to carry heat away from the plates since the fluid particles can hardly mix. The damping effect caused by magnetic

forces reduce convective heat transfer altering temperature distribution due to reduced fluid velocity. Noghrehabadi et al. (2014) noted that an increase in Ha value lead to a temperature decrease.

**Figure 5.3**

*Effect of varying Reynolds number on the velocity profile*

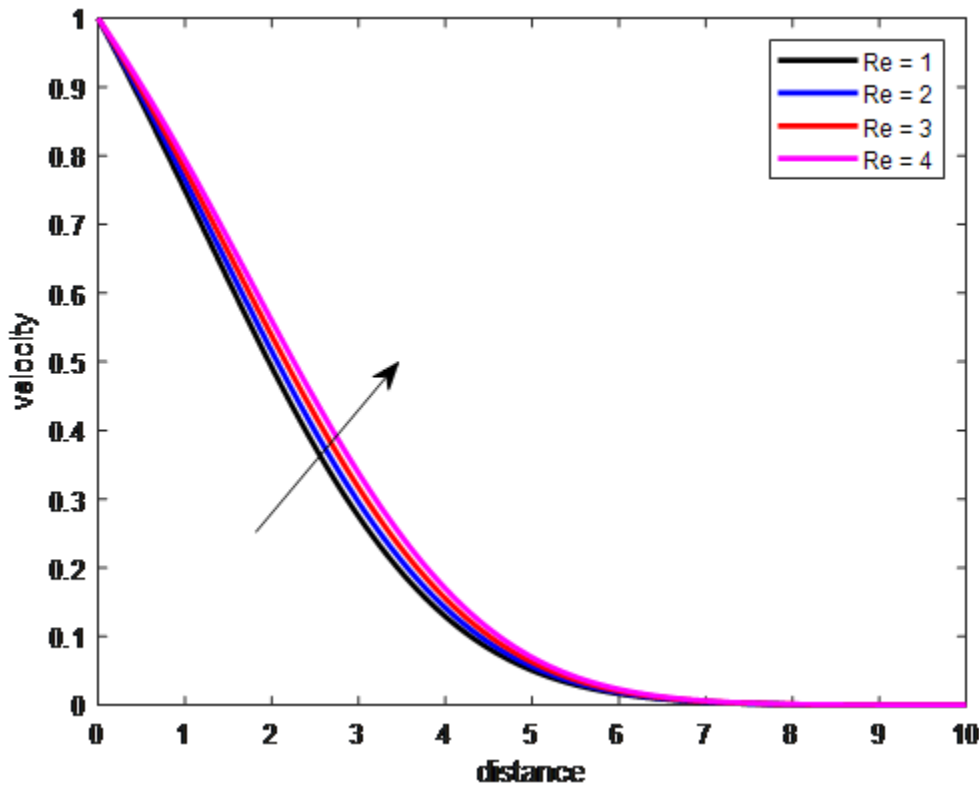


Figure 5.3 shows the effect of varying Reynolds number on velocity profile. The fluid velocity decreases as Reynolds values are increased. The Reynolds number represent the ratio of inertia forces to viscous forces. At small values of Reynolds numbers, the viscous forces increase while inertial forces decrease and consequently fluid velocity decreases. Reynold numbers help in predict flow behaviour and consequently influences designing of pipes, pumps, reactors etc. in

aerospace and chemical engineering. The results agree with Qi et al. (2022) observed that an increase in Reynolds number leads to a decrease in flow velocity.

**Figure 5.4**

*Effect of Reynolds number on the temperature profile.*

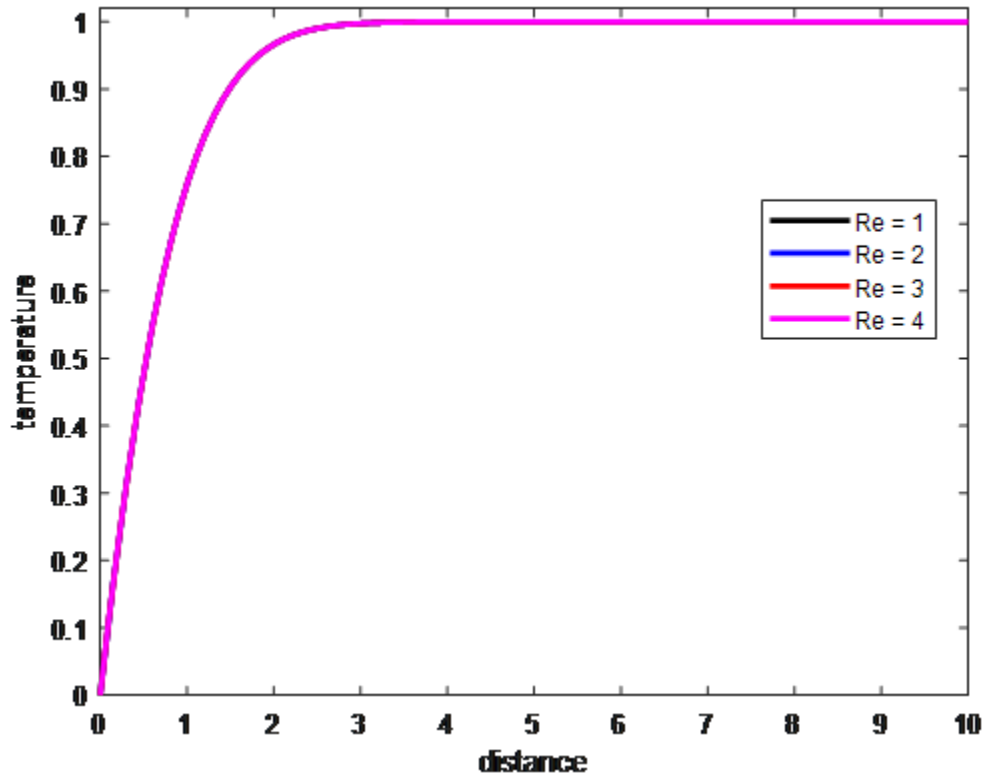


Figure 5.4 shows the effect of varying Reynolds number on temperature profile. An increase in Reynolds number causes a corresponding increase in the temperature of the fluid. Large values of Reynolds number lead to an increase in convection since the fluid velocity increases and thus improves the rate of heat transfer. Reynolds number affects the temperature profile through altering the flow regime and efficiency of heat transfer. This helps in promoting fluid mixing and designing systems that attain desired temperature distributions

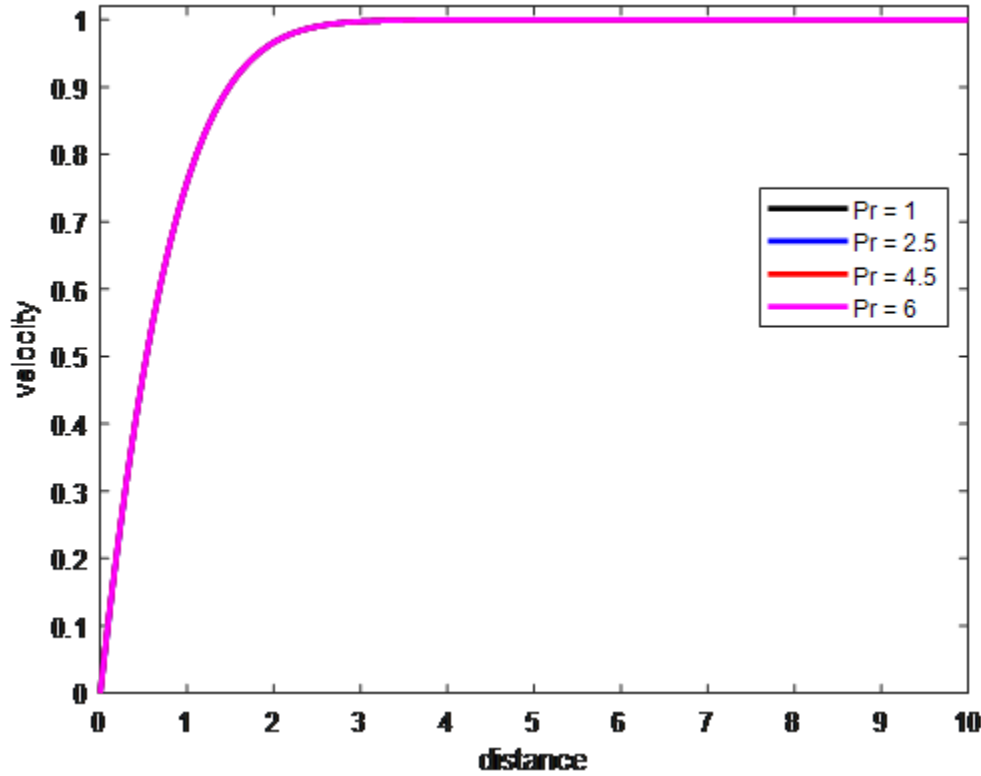
**Figure 5.5***Effect of Prandtl Number on the velocity profile*

Figure 5.5 shows the effect of varying Prandtl number on the velocity profile. An increase in Prandtl number causes a velocity increase. Large values of Prandtl number causes the velocity to become more uniform allowing larger regions for momentum transfer. Prandtl number shows the relationship between momentum and thermal boundary layers. A higher Prandtl number causes a thinner thermal boundary layer compared to the velocity boundary layer. Understanding the effects of Prandtl number on fluid velocity helps in designing cooling systems and chemical reactors. Ndiaye et al. (2020) observed that the Pr number has a weak influence on fluid velocity especially in a porous medium.

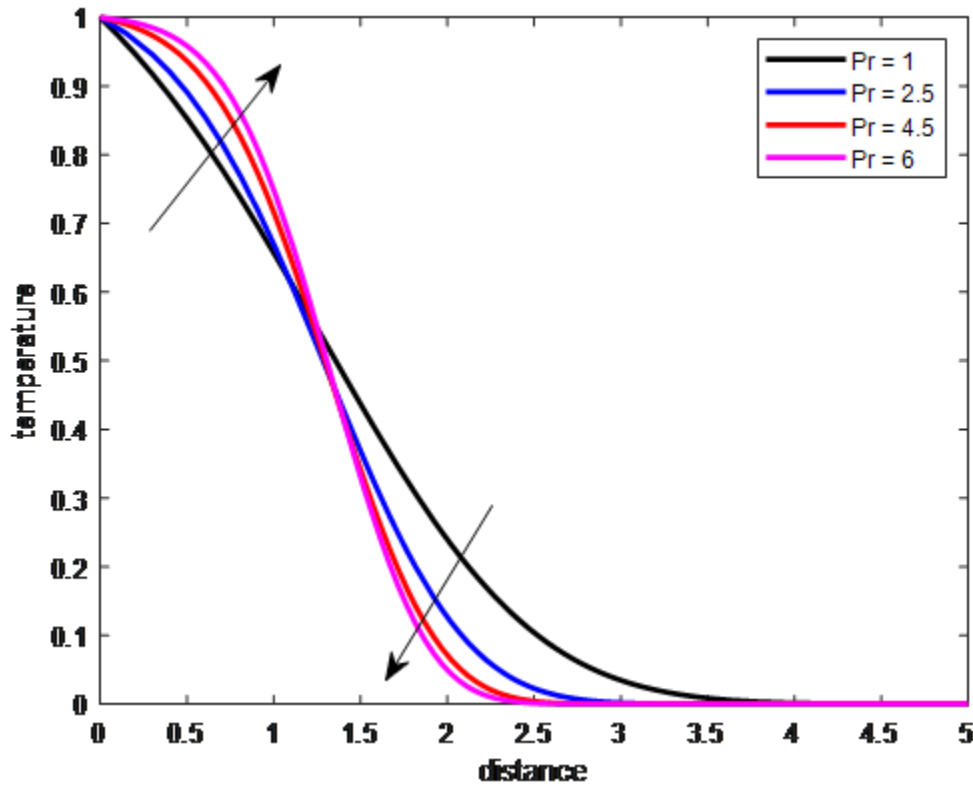
**Figure 5.6***Effect of Prandtl Number on Temperature Profile*

Figure 5.6 shows the effect of varying Prandtl number on temperature profile. An increase in Prandtl number leads to a decrease in temperature. A higher Prandtl number causes a thinner thermal boundary layer compared to the velocity boundary layer. This means that heat transfer is localized close to the walls and thus heat transfer is low due to reduced thermal diffusivity. Prandtl numbers are fluid dependent and thus they influence design and operation of heat exchangers and reactors. Rubbab et al. (2013) observed that fluid temperature drops with a rise in Prandtl number since it lowers thermal boundary layer thickness.

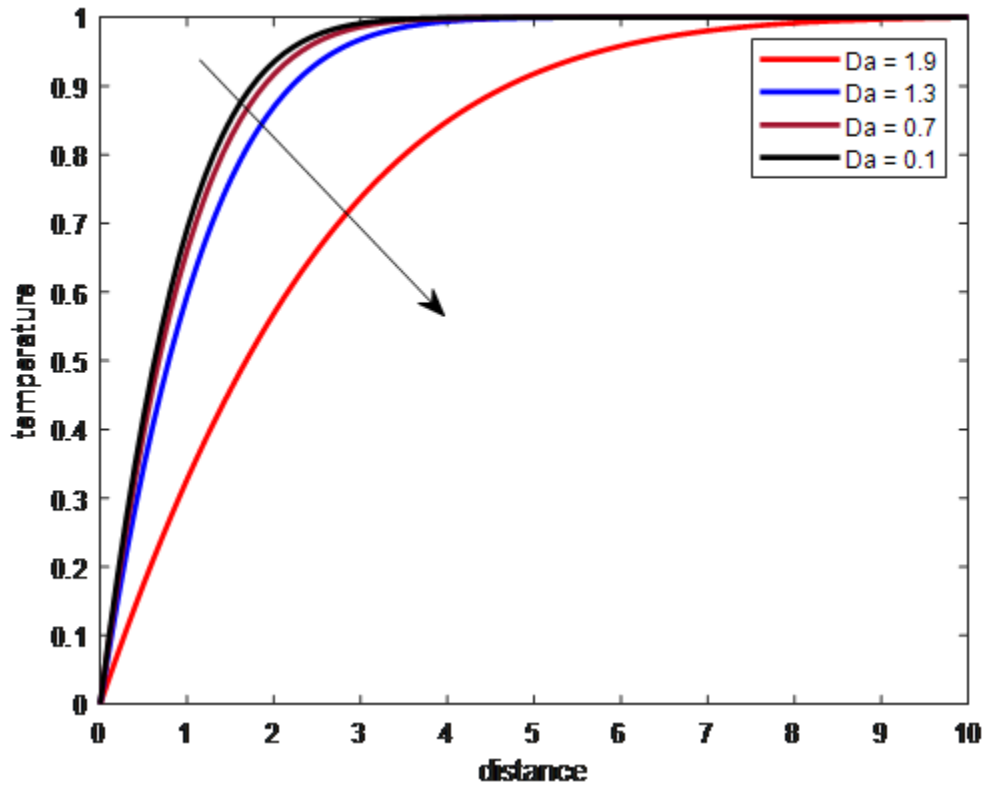
**Figure 5.7***Effect of Darcy Number on Temperature Profile*

Figure 5.8 shows the effect of Darcy number on temperature profile. Heat transfer in a porous media can occur through conduction and convection. An increase in Darcy number leads to an increase in fluid temperature due to enhanced convection. Higher values of Darcy number ensure a thin thermal boundary layer which improves heat transfer rates through convection. Understanding the effect of Darcy number on temperature profile is crucial in designing geothermal energy systems and chemical reactors to optimize performance and enhance efficient heat transfer.

## CHAPTER SIX

### CONCLUSION AND RECOMMENDATION

#### 6.1 Introduction

The chapter present the conclusions and recommendations drawn from the research.

#### 6.2 Conclusion

From this study, we conclude that fluid velocity decreases with increase in Reynolds number since the inertia forces are greater than viscous forces and fluid velocity increases on raising the value of Prandtl number. On the other hand, fluid temperature decreases with increasing values of Prandtl number and Hartmann number. Increasing the value of Darcy number leads to an increase on temperature.

#### 6.3 Recommendations

Following this study, we recommend a study of a compressible MHD fluid bounded by vertical plates bounded in a porous media. The study may consider the case of variable distances between the plates.

## REFERENCES

- Abel, M. S., Tawade, J. V., & Shinde, J. N. (2012). The Effects of MHD Flow and Heat Transfer on the UCM Fluid over a Stretching Surface in Thermal Radiation. *Advances in Mathematical Physics*, 1-21.
- Acharya, A. K., Dash, G. C., & Mishra, S. R. (2014). Free Convective Fluctuating MHD Flow through Porous Media Past a Vertical Porous Plate with Variable Temperature and Heat Source. *Physics Research International*, 1-8.
- Ahmad, S., Yousaf, M., Khan, A., & Zaman, G. (2018). Magnetohydrodynamic fluid flow and heat transfer over a shrinking sheet under the influence of thermal slip. *Heliyon*, 4(10), e00828.
- Appidi, L., Malga, B. S., Matta, S., & Kumar, P. P. (2021). Effects of Thermal Radiation on Temperature and Concentration on MHD Free Convection Flow past a Vertical Porous Plate in the Presence of Chemical Reaction and Heat Source. *Indian Journal of Science and Technology*, 14(17), 1354-1363.
- Atangana, A. (2018). Principle of Groundwater Flow. In Fractional Operators with Constant and Variable Order with Application to Geo-Hydrology (pp. 15-27). *Academic Press*. <https://doi.org/10.1016/C2015-0-05711-2>
- Ersoy, H. V. (2012). Unsteady Flow Produced by Oscillations of Eccentric Rotating Disks. *Mathematical Problems in Engineering*, 1-14.
- Falade, J. A., Ukaegbu, J. C., Egere, A. C., & Adesanya, S. O. (2017). MHD oscillatory flow through a porous channel saturated with porous medium. *Alexandria Engineering Journal*, 56(1), 147-152.
- Hasanuzzaman, M., Afroj, R., Sharin, S., & Miyara, A. (2022). Thermal Radiation Effect on Unsteady Convection and Mass Transport over a Stretching Sheet in a Saturated Porous Medium with Chemical Reaction. *Research Square*, 1-21. doi:10.21203/rs.3.rs-1254356/v1
- Hayat, T., Qayyum, S., Imtiaz, M., & Alsaedi, A. (2016). MHD Flow and Heat Transfer between Coaxial Rotating Stretchable Disks in a Thermally Stratified Medium. *PLoS One*, 11(5), 1-23.
- Kocić, M., Stamenković, Ž., Petrović, J., & Bogdanović-Jovanović, J. (2023). Control of MHD Flow and Heat Transfer of a Micropolar Fluid through Porous Media in a Horizontal Channel. *Fluids*, 93. doi:10.3390/fluids8030093
- Lavanya, B. (2017). Effect of radiation on free convection heat and mass transfer flow through a porous medium in a vertical channel with heat absorption/generation and chemical reaction. *AIP Conference Proceedings*, 1859(1), 020023.
- Lavanya, B. (2019). MHD Rotating Flow through a Porous Medium with Heat and Mass Transfer. *Journal of Advanced Research in Fluid Mechanics and Thermal Sciences*, 54(2), 221-231.

- Mohamed, R. A., Ahmed, S. E., Aly, A. M., Abo-Dahab, S. M., & Soliman, M. S. (2021). MHD Three-Dimensional Flow of Couple Stress Nanofluids over a Stretching Sheet through a Porous Medium in Presence of Heat Generation/Absorption and Nonlinear Thermal Radiation. *Challenges in Nano and Micro Scale Science and Technology*, 135-150. doi:10.22111/CNMST.2021.37182.1203
- Momanyi, F. O., & Wafula, M. M. (2023). MHD in a Porous Medium across Parallel Plates with Mass Transfer in a Rotating System. *The University Journal*, 225-234.
- Ndiaye, P., Ndiaye, M., Mbow, C., & Ndiaye, G. (2020). Influence of Reynolds and Prandtl Numbers on Thin Film Condensation in Forced Convection in a Canal Covered with a Porous Material. *International Journal of Engineering Applications*, 178-187. doi:10.15866/irea.v8i5.18678
- Ngesa, J. O. (2018). Heat and Mass Transfer past a Semi-Infinite Vertical Porous Plate in MHD Flows in a Turbulent Boundary Layer. *Proceeding of the 1st Annual International Conference* (pp. 1-38). Machakos, Kenya: Machakos University.
- Noghrehabadi, A., Ghalambaz, M., Izadpanahi, E., & Pourrajab, R. (2014). Effect of magnetic field on the boundary layer flow, heat, and mass transfer of nanofluids over a stretching cylinder. *Journal of Heat and Mass Transfer Research* 1, 9-16.
- Onyinkwa, L. M., & Chepkwony, I. (2023). Heat and Mass Transfer in MHD Flow about an Inclined Porous Plate. *Journal of Engineering Research and Reports*, 106-115. doi: 10.9734/JERR/2023/v25i4906
- Poddar, S., Islam, M. M., Ferdouse, J., & Alam, M. M. (2021). Characteristical analysis of MHD heat-mass transfer dissipative and radiating fluid flow with magnetic field induction and suction. *SN Applied Sciences*, 3(470).
- Qi, R., Chen, F., Xu, L., Yu, J., & Chen, X. (2022). Effects of Reynolds Number on the Overall Characteristics of Flow and Heat Transfer in the Long Micro-Tube with Dimples. *Energy Processes*, 2696. doi:10.3390/pr10122696
- Reddy, D., Goud, S., Nisar, K. S., Alshahrani, B., Mahmoud, M., & Park, C. (2023). Heat absorption/generation effect on MHD heat transfer fluid flow along a stretching cylinder with a porous medium. *Alexandria Engineering Journal*, 659-666. doi:10.1016/j.aej.2022.08.049
- Rubbab, Q., Vieru, D., Fetecau, C., & Fetecau, C. (2013). Natural Convection Flow near a Vertical Plate that Applies a Shear Stress to a Viscous Fluid. *PLOS ONE*, 8(11), 1-7.
- Sakthikala, R., & Lavanya, V. (2020). MHD Oscillatory Flow of Non-Newtonian Fluid through Porous Medium in the Presence of Radiation and Chemical Diffusion with Hall Effects. *International Journal on Emerging Technologies*, 11(2), 1093-1099.
- Sehra, Haq, S. U., Shah, S. I., Nisar, K. S., Jan, S. U., & Khan, I. (2021). Convection heat mass transfer and MHD flow over a vertical plate with chemical reaction, arbitrary shear stress, and exponential heating. *Scientific Reports*, 11(4265).

- Storey, B., (2018). Fluid dynamics and heat transfer: An introduction to the fundamentals. Olin College.
- Yu, S., Tang, T., Li, J., & Yu, P. (2020). Effect of Prandtl Number on Mixed Convective Heat Transfer from a Porous Cylinder in the Steady Flow Regime. *Entropy*, 184. doi:10.3390/e22020184