A STUDY ON NUMERICAL RANGE OF OPERATORS IN A HILBERT SPACE.

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Approved by ....
Signature

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INTRODUCTION

In classical pure mathematics, the objects of considerable interest were quadratic forms; that is the mappings of the form $x \mapsto \langle Tx,x\rangle$, where T is an nxn matrix, say $T = [a_{ij}]$, x is an n-vector and $\langle\ ,\ \rangle$ is the usual inner product.

$$\langle x,y\rangle = x_1\overline{y_1} + \dots + x_n\overline{y_n}$$

of the unitary space $\mathbb{C}^n$ and $x = (x_1, \dots, x_n)$ $y = (y_1, \dots, y_n)$. It is easily shown that in this case the quadratic form $\langle Tx,x\rangle = \sum_{i=1}^{n}\sum_{j=1}^{n} a_{ij}\,\overline{x_i}\,x_j$

Quadratic forms nowadays play secondary role in Hilbert space theory, although the notion of quadratic forms is not really obsolete. The set $\{\langle Tx,x\rangle : \|x\| = 1\}$ where $\|x\|^2 = \sum_{i=1}^{n}\|x_i\|^2$, is called the numerical range of T. It is clear that if the numerical range of a matrix T is known, then the value of the quadratic form $x \to \langle Tx,x\rangle$ is known at all $x \in \mathbb{C}^n$.

Notwithstanding the fact that the numerical range is still a useful tool for the study of finite matrices, the scope of the concept of numerical ranges has been enlarged by its usefulness in the study of operators in infinite dimensional spaces.

In the following pages we present an expository material on numerical ranges of linear operators in a Hilbert space.

In section 1.1 we define the basic concepts of numerical range, numerical radius and other essentials - of operator theory. Subsequently, we obtain elemtary consequences and properties along with a few examples pertinent to our study.

We study the essential properties of the spectrum of a bounded operator on a Hilbert space, as it relates to the numerical range of the same operator, in section 1.2.

In section 1.3 we present some key results of Hildebrandt and J.P. Williams on numerical range of an operator similar (in the obvious sense) to a bounded operator on a Hilbert space.

Some special points of the numerical range of a bounded operator on a Hilbert space, are discussed in section 1.4. We obtain in this section, a sufficient condition under which a bounded compact operator on a Hilbert space has a closed numerical range.

In section 1.5, we give an elementary proof of a proposition about the power inequality of numerical radius of a bounded operator on a Hilbert space.

In the case of an unbounded operator on a Hilbert space it is not true in general to say that the spectrum of such an operator is contained in the closure of the numerical range (this is shown to be true in the case of bounded operators, see section 1.2). However in certain cases, it is possible to extend such operators to other linear operators about which the said result is true. This problem is dealt with in great details in the rest of the study starting at section 1.6. We give seperate treatment to special cases.

A glossary of symbols used throughout the text is given at the end (see table of content). The reader should assume the usual meanings for symbols not included there. Otherwise the meanings of symbols used will be clear from the context.

1.1 Elementary Consequences and properties of Numerical Range

In this Section, and unless otherwise stated in the subsequent sections, the Symbol H will always represent a Complex Hilbert space with the inner product $\langle , \rangle : H \times H \to \mathbb{C}$ and the symbol $\|.\|$ is the norm determined by the inner product. The symbols S,T,V ... will represent elements of B(H), the Banach algebra of all the bounded linear operators on H to H, I represents the identity operator and T^* represents the adjoint of T in B(H). The symbol o is the zero vector in H. For a subset A of $\mathbb{C}$ and for any $\alpha \in \mathbb{C}$ we shall mean by the symbols $A + \alpha$ $(= \alpha + A)$, αA and $(A)^*$ the subsets $\{z + \alpha : z$ is in $A\}$, $\{\alpha z : z$ is in $A\}$ and $\{\bar{z}, : z$ is in $A\}$ respectively, where $\bar{z}$ is the conjugate of the complex number z.

For any T in B(H), the function $\phi : H \times H \to \mathbb{C}$ defined by

$\phi(x,y) = \langle Tx,y \rangle$ for all x,y in H is sesquilinear and $\hat{\phi} : H \to \mathbb{C}$ defined by

$\hat{\phi}(x) = \langle Tx,x \rangle$ for all x in H is the quadratic form associated with T (and ϕ).

Definition 1.1 Let T be in B(H). The **numerical range.** W(T) of T is the subset of $\mathbb{C}$ given by

$$W(T) = \{\langle Tx,x\rangle : x \in H \text{ and } ||x|| = 1\}.$$

The number, $w(T) = \sup\{|\lambda| : \lambda \in W(T)\}$ is called the **numerical radius** of T.

It is clear that if W(T) is known, then so is the range of the quadratic form $\hat{\phi}$ associated with T.

We collect a few useful results in the following

Proposition 1.2. Let T be in B(H). Then

(i) $W(\alpha T + \beta I) = \alpha W(T) + \beta \quad (\alpha, \beta \in \mathbb{C})$

(ii) $W(T^*) = \{\bar{\lambda} : \lambda \in W(T)\}$ $\{$ie. $W(T^*) = (W(T))^*\}$

(iii) $w(T) \leq ||T||$.

(iv) $|\langle Tx,x\rangle| \leq w(T)\,||x||^2$ for all x in H.

(v) $W(U^* T U) = W(T)$ for unitary U in B(H).

(vi) If $T \geq 0$, then $||Tx||^2 \leq w(T)\langle Tx,x\rangle$ for all x in H.

Remark 1.3 An operator T in B(H) is said to be **positive** in symbols $T \geq 0$, if $\langle Tx,x\rangle \geq 0$ for each x in H. A linear operator U in B(H) is **unitary** if $\langle Ux,Uy\rangle = \langle x,y\rangle$ for all x,y in H, and U is surjective. Standard results in operator theory will be assumed in the sequel and the reader may particularly refer to [1]

Proof (of proposition 1.2)

(i) $W(\alpha T+\beta I) = \{\langle(\alpha T + \beta I)x,x\rangle : x$ is in H and $||x|| =1\}$.

$$= \{ \alpha\langle Tx,x\rangle + \beta : x \text{ is in } H \text{ and } ||x|| =1\}$$

$$= \alpha W(T) + \beta .$$

(ii) Since $\langle T^*x,x\rangle = \langle x,Tx\rangle = \overline{\langle Tx,x\rangle}$ for all x in H,

$$W(T^*) = \{\langle T^*x,x\rangle : x \text{ is in } H \text{ and } ||x|| = 1\}$$

$$= \{\overline{\langle Tx,x\rangle} : x \text{ is in } H \text{ and } ||x|| = 1 \}$$

$$= \{ \bar{\lambda} : \lambda \text{ is in } W(T)\}$$

(iii) By the Cauchy - Schwarz inequality,

$$|\langle Tx,x\rangle| \leq ||T||\,||x||^2 .$$

Now if $||x|| = 1$, we obtain $|\langle Tx,x\rangle| \leq ||T||$;

thus if $\lambda \in W(T)$, then $|\lambda| \leq ||T||$.

Thus $W(T)$ is contained in the closed neighbourhood of radius $||T||$ centred at o and hence is a bounded subset of $\mathbb{C}$. By the completeness axioms for $\mathbb{R}$ it follows that $\sup\{|\lambda| : \lambda \in W(T)\}$ exists ; thus $w(T) \leq ||T||$.

(iv) The result holds trivially when $x = o$

suppose $x \neq o$ and put $x' = ||x||^{-1}x$.

Then $||x'|| = 1$ and $\langle Tx,x\rangle = ||x||^2\langle Tx',x'\rangle$.

Then

$$|\langle Tx,x\rangle| \leq ||x||^2 \sup\{\langle Ty,y\}: y \text{ is in } H \text{ and } ||y|| = 1\}.$$

$$= ||x||^2\, w(T).$$

(v) Now

$$\langle U^* TUx, x\rangle = \langle T(Ux), Ux\rangle \text{ for each } x \in H.$$

since U is unitary, the range R(U) of U is H itself and

$$||Ux|| = ||x|| \text{ for all } x \in H.$$

Hence (putting $Ux = y$)

$$\{\langle U^*TUx, x\rangle : x \text{ is in } H \text{ and } ||x|| = 1\}$$

$$= \{\langle Ty, y\rangle : y \in H \text{ and } ||y|| = 1 \}.$$

ie. $W(U^*TU\} = W(T)$.

(vi) Since $T \geq o$, the sesquilinear functional ϕ defined by

$$\phi(x,y) = \langle Tx, y\rangle \text{ for all } x,y \in H.$$

is positive (i.e. $\phi(x,x) \geq o$ for each $x \in H$) and hence (see [1]. p. 372),

$$|\phi(x,y)|^2 \leq \phi(x,x)\, \phi(y,y) \text{ for all } x,y \in H.$$

Thus $|\langle Tx,y\rangle|^2 \leq \langle Tx,x\rangle\langle Ty,y\rangle$ for all $x,y \in H$.

Putting $y = Tx$, we get

$$||Tx||^4 \leq \langle Tx,x\rangle \langle T^2x, Tx\rangle \text{ for all } x \in H ,$$

and using proposition 1.2 (iv), we obtain

$$||Tx||^2 \leq \langle Tx,x\rangle\, w(T). \text{ if } Tx \neq o \quad .$$

The inequality holds trivially when $Tx = o$.

Lemma 1.4 If $T \geq o$ and $\langle Tx,x\rangle = o$

Then $Tx = o$.

Proof: Indeed since $\langle Tx,x\rangle = o$, Proposition 1.2(vi) gives $Tx = o$.

Corollary 1.5 $W(T) \subseteq \mathbb{R}$ if and only if $T^* = T$.

Proof: Suppose $T = T^*$ and $\lambda \in W(T)$.

Then $\lambda = \langle Tx,x\rangle$ for an $x \in H$ with $||x|| = 1$. Now

$$\begin{aligned}\bar{\lambda} = \overline{\langle Tx,x\rangle} &= \langle x,Tx\rangle \\ &= \langle T^*x,x\rangle \\ &= \langle Tx,x\rangle \\ &= \lambda\end{aligned}$$

and this shows that $\lambda \in \mathbb{R}$. Thus $W(T) \subseteq \mathbb{R}$.

Conversely, Suppose $W(T) \subseteq \mathbb{R}$. Then for each $x \in H$ satisfying $||x|| = 1$,

$$\begin{aligned}\langle (T-T^*)x, x\rangle = \langle Tx,x\rangle - \langle T^*x,x\rangle &= \langle Tx,x\rangle - \langle x,Tx\rangle \\ &= \langle Tx,x\rangle - \overline{\langle Tx,x\rangle} \\ &= o \text{ since } \langle Tx,x\rangle \text{ is in } \mathbb{R}.\end{aligned}$$

Thus the Self-adjoint operator $i(T-T^*)$ is positive and so, by lemma 1.4, we have $i(T-T^*)x = o$ for all $x \in H$. with $||x|| = 1$, Consequently $T-T^* = o$ ie $T = T^*$.

Corollary 1.6 $W(T) \subseteq \mathbb{R}^+ = \{x \in \mathbb{R} : x \geq o\}$ if and only if $T \geq o$.

Proof: Follows from corollary 1.5.

We recollect that a subset M of a linear space X is called **convex** if for all x,y in M, the segment $\{\lambda x + (1-\lambda)y : o \leq \lambda \leq 1\}$ is also contained in M.

If M is any subset of a linear space X the **convex hull** of M, represented by conv M, is the smallest convex subset of X containing M and is thus the intersection of all the convex subsets of X that include M since the intersection of any arbitrary family of convex subsets of X is convex. It can be shown that

$$\text{Conv } M = \{ \lambda_1 x_1 + \ldots + \lambda_n x_n : x_1, \ldots, x_n \in M, \; 0 \leq \lambda_j \text{ and } \sum_{j=1}^{n} \lambda_j = 1 \}.$$

It is a non-trivial fact of finite-dimension Euclidean geometry that a convex hull of a compact set is closed and the most useful formulation of this fact is that the convex hull of a compact subset of $\mathbb{C}$ is the intersection of all the closed half - planes that include it. It is easy to prove that the closure of a convex set is convex. Also closed convex subset of the plane is the intersection of all closed discs containing it. The reader may refer to [2] for all these.

A non - trivial property of the numerical range of an operator T in B(H) is the Hausdorff - Toeplitz theorem:

Proposition 1.7 (Hausdorff - Toeplitz). W(T) is a convex subset of $\mathbb{C}$.

Proof: Let $\lambda = \langle Tx, x \rangle$ and $\mu = \langle Ty, y \rangle$, where x and y are vectors of unit norm in H. We need to show that the segment joining λ and μ is contained in W(T). If $\lambda = \mu$, there is nothing to prove. Let $\lambda \neq \mu$. Then there are complex numbers α and β such that

$$\alpha\lambda + \beta = 1 \quad \text{and} \quad \alpha\mu + \beta = o.$$

It is sufficient to prove that the unit interval $[0,1]$ is included in $W(\alpha T + \beta I)$ (note: $W(\alpha T + \beta I) = \alpha W(T) + \beta$ by proposition 1.2(i)). For if t is in $[0,1]$, then

$$\begin{aligned} t = t.1 + (1-t).o &= t[\alpha\langle Tx,x\rangle + \beta] + (1-t)[\alpha\langle Ty,y\rangle + \beta] \\ &= \alpha\{t\langle Tx,x\rangle + (1-t)\langle Ty,y\rangle\} + \beta \\ &= \alpha\{t\lambda + (1-t)\mu\} + \beta \end{aligned}$$

and $t\lambda + (1-t)\mu$ is a point in the segment joining λ and μ. Hence, without loss of generality, we may assume that $\lambda = 1$ and $\mu = o$. Write

$T = P + iQ$ (cartesian decomposition), where P and Q are self-adjoint operators in $B(H)$. Since $\langle Tx,x\rangle = 1$ and $\langle Ty,y\rangle = o$ it follows that $\langle Qx,x\rangle$ and $\langle Qy,y\rangle$ are both zero. If we replace x by αx where $|\alpha| = 1$, then $\langle Tx,x\rangle$ remains unchanged in value for $\langle T(\alpha x), x\rangle$ $= |\alpha|^2 \langle Tx,x\rangle = \langle Tx,x\rangle$, where as $\langle Qx,y\rangle$, becomes $\alpha\langle Qx,y\rangle$. We may now choose α such that $|\alpha| = 1$ and $\langle Qx,y\rangle$ is purely imaginary. Therefore without loss of generality, we may assume that $\langle Qx,y\rangle$ is purely imaginary. Let $h(t) = tx + (1-t)y$ for $o \leq t \leq 1$. Then $h(t) \neq o$; in fact the set $\{x,y\}$ is linearly independent. This is a consequence of $\langle Tx,x\rangle \neq \langle Ty,y\rangle$. For if x,y were linearly dependent then, since they are unit vectors, either one can be written as a linear multiple of the other, say $x = \theta y$. Then $|\theta| = 1$, and Consequently $\langle Tx,x\rangle = \langle Ty,y\rangle$!! a contradiction, since

$$\begin{aligned}\langle Q\, h(t), h(t)\rangle &= \langle Q(tx+(1-t)y),\ tx+(1-t)y\rangle \\ &= t^2 \langle Qx,x\rangle + t(1-t)\{\langle Qx,y\rangle + \overline{\langle Qx,y\rangle}\} \\ &\quad +(1-t)^2 \langle Qy,y\rangle\end{aligned}$$

the relations

$\langle Qx,x\rangle = \langle Qy,y\rangle = o$ and $\mathrm{Re}\,\langle Qx,y\rangle = o$ give $\langle Q\ h(t),\ h(t)\rangle = o$ for are t in $[o,1]$, and hence that $\langle T\, h(t), h(t)\rangle$ is real for all t in $[o,1]$.

Now the function

$$t \dashrightarrow \frac{\langle Th(t), h(t)\rangle}{||h(t)||} \qquad t \text{ is in } [o,1]$$

is real-valued and continuous on the closed interval $[o,1]$. Its value at o and 1 are o and 1 respectively. Hence the range of the function contains every number in the interval $[o,1]$.

We close this section with a few illustrations of numerical ranges.

Example 1.8 Let H be the two-dimensional Hilbert space $\mathbb{C}^2$ with inner product $\langle z,w\rangle = z_1 \overline{w_1} + z_2 \overline{w_2}$ where $z = (z_1,z_2)$ and $w = (w_1, w_2)$.

(a) Consider the operator T whose matrix with respect to the standard basis $\{(1,o), (o,1)\}$ is

$$\begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}$$

Then W(T) is the closed interval [o,1] For, let $z = (z_1, z_2)$ be in $\mathbb{C}^2$ and $|z_1|^2 + |z_2|^2 = 1$. Then $Tz = (z_1, o)$ and $\langle Tz, z\rangle = \langle (z_1, o), (z_1, z_2)\rangle = |z_1|^2 \leq 1$. When $z_2 = o$, we have $|z_1|^2 = 1$ and $z_1 = o$ gives $|z_1|^2 = o$.

Therefore W(T) is a subset of [o,1] and contains o,1. Since W(T) is convex, we get W(T) = [o,1]

(b) If T has the matrix

$$\begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix}$$

With respect to the standard basis of $\mathbb{C}^2$, then

$W(T) = \{z_1 \bar{z}_2$, where $|z_1|^2 + |z_2|^2 = 1\}$

$= \{ \lambda \in \mathbb{C}$ and $|\lambda| \leq \frac{1}{2}\}$

i.e W(T) is the closed disc with centre o and radius $= \frac{1}{2}$.

1.2. Numerical range and the Spectrum

Some of the most important applications of the numerical range concern the **Spectrum** of T. This is the set Sp(T) of all $\lambda \in \mathbb{C}$ for which $\lambda I - T$ does not have a bounded inverse in B(H). The set of all eigenvalues of T, which is a subset of Sp(T) is called the **point spectrum** of T and represented by Sp(T). The set complement of Sp(T) in $\mathbb{C}$ i.e $\mathbb{C} \setminus$ Sp(T) is called the resolvent of T and we represent it by $\rho(T)$.

It is easily seen that if T is in B(H), then $\lambda \in \rho(T)$ if and only if $\lambda I - T$ has a bounded inverse on the range $R(\lambda I - T)$ of $\lambda I - T$ and $\overline{R(\lambda I-T)} = H$. Of course $\lambda I - T$ has a bounded inverse on its range $R(\lambda I - T)$ if and only if it is bounded below i.e. If and only if $||(\lambda I-T)x|| \geq k||x||$ for all $x \in H$ and for some positive real k. Thus $\lambda \in \rho(T)$ if and only if $\lambda I-T$ is bounded below and

$$\overline{R(\lambda I - T)} = H. \quad (1.1)$$

In our discussions below, we need to characterize two more subsets of Sp(T), namely, the **approximate point spectrum** apSp(T) of T and the **residual spectrum** RSp(T):

A point $\lambda \in \mathbb{C}$ belongs to apSp(T) if and only if there exists a sequence (x_n) of unit vectors of H such that $\lim_{n \to \infty} ||(\lambda I - T)x_n|| = 0$. (This is clearly equivalent to the requirement that for every positive real ε there exists a non-zero vector x such that $||\lambda I - T)x|| < \varepsilon||x||$.)

A point $\lambda \in \mathbb{C}$ belongs to RSp(T) if and only if $R(\lambda I - T)$ is not dense in H and $\lambda I - T$ has bounded or unbounded inverse on $R(\lambda I - T)$. It is clear that $pSp(T) \subseteq Sp(T)$. To see that $apSp(T) \subseteq Sp(T)$ we observe that if $\lambda \notin Sp(T)$, then $(\lambda I - T)^{-1}$ is in B(H) and consequently we have

$$||x|| = ||(\lambda I - T)^{-1}(\lambda I - T)x|| <$$
$$||(\lambda I - T)^{-1}|| \; ||(\lambda I - T)x||$$

for every vector x in H. This implies that

$$||(\lambda I - T)x|| \geq \varepsilon||x||,$$

with $\varepsilon = ||(\lambda I - T)^{-1}||^{-1}$

for every $x \in H$, and hence that $\lambda \notin apSp(T)$.

Lemma 1.9 $pSp(T) \subseteq W(T)$.

Proof: Let $\lambda \in pSp(T)$. Then there exists an x in H such that $||x|| = 1$ and $Tx = \lambda x$. Thus $\langle Tx,x\rangle = \langle \lambda x,x\rangle = \lambda||x||^2 = \lambda$ i.e $\lambda \in W(T)$.

Lemma 1.10 $apSp(T) \subseteq \overline{W(T)}$.

Proof: Let $\lambda \in apSp(T)$. Then there exists a Sequence (x_n) of unit vectors in H such that $\lim_{n \to \infty} ||(\lambda I - T)x|| = 0$. Now

$$|\lambda - \langle Tx_n, x_n \rangle| = |\lambda \langle x_n, x_n \rangle - \langle Tx_n, x_n \rangle|$$

$$= |\langle (\lambda I - T)x_n, x_n \rangle|$$

$$\leq \|(\lambda I - T)x_n\| \dashrightarrow 0 \text{ as } n \dashrightarrow \infty$$

This shows that $\lambda \in \overline{W(T)}$.

Lemma 1.11 $RSp(T) \subseteq W(T)$.

Proof: If λ is in $RSp(T)$, then $\overline{R(\lambda I - T)} \neq H$. i.e $(N(\bar{\lambda} I - T^*))^{\perp} \neq H$. (Where $N(T)$ represents the kernel of the operator T). since $N(\bar{\lambda} I - T^*)$ is a closed linear subspace of H, we have.

$$N(\bar{\lambda} I - T^*) = (N(\bar{\lambda} I - T^*)^{\perp\perp} \neq H^{\perp} = \{0\}.$$

i.e $\bar{\lambda}$ is an eigenvalue of T^*. This is equivalent to the assertion that λ is in $pSp(T)$. But $pSp(T) \subseteq W(T)$ by lemma 1.9. Hence λ is in $W(T)$.

Thus $RSp(T) \subseteq W(T)$. .

Lemma 1.12 λ belongs to $Sp(T)$ if and only if $\bar{\lambda}$ belongs to $Sp(T^*)$.

Proof Suppose $\lambda \notin Sp(T)$, i.e. λ is in $\rho(T)$. Now λ is in $\rho(T)$ if and only if there is an $S \in B(H)$ such that

$$S(\lambda I - T) = I = (\lambda I - T) S.$$

Taking adjoints in the provious line, we get

$$(\bar{\lambda} I - T^*)S^* = I = S^*(\bar{\lambda} I - T^*)$$

i.e $\bar{\lambda}$ is in $\rho(T^*)$, i.e $\bar{\lambda}$ is not in $Sp(T^*)$.

Remark 1.13 Thus $Sp(T^*) = (Sp(T))^*$.

Lemma 1.14 $Sp(T) = (pSp(T^*))^* \cup apSp(T)$

$$= pSp(T) \cup (apSp(T^*))^* \; .$$

Proof. Since

$$pSp(T) = apSp(T) \subseteq Sp(T) \qquad (1.2)$$

We obtain

$$pSp(T^{*}) \subseteq ap\,Sp(T^{*}) \subseteq Sp(T^{*}) \qquad (1.3)$$

Thus (taking "Conjugate Complexes" in the last line)

$$\{\bar{\lambda} : \lambda \text{ is in } pSp(T^{*})\} \subseteq \{\bar{\lambda}: \lambda \text{ is in } apSp(T^{*})\} = Sp(T) \qquad (1.4)$$

Using Lemma 1.12.

From (1.2) and (1.4) we obtain

$$apSp(T) \cup (pSp\,(T^{*}))^{*} \subseteq sp(T) \qquad (1.5)$$

and

$$(apSp(T^{*}))^{*} \cup (pSp(T)) \subseteq Sp(T) \qquad (1.6)$$

We must now establish the reverse set inclusion in (1.5) and (1.6).

Let λ be in $Sp(T)$, i.e λ does not belong to $\rho(T)$. By the remark at (1.1) it follows that

either $\overline{R(\lambda I - T)} \neq H$ or $\lambda I - T$ is not bounded below. If $\overline{R(\lambda I - T)} \neq H$, then from the well - known relation $\overline{R(\lambda I - T)}^{\perp} = N(T^{*} - \bar{\lambda} I)$.

It follows that $N(T^{*} - \bar{\lambda} I) \neq \{ o \}$, so $\bar{\lambda} \in pSp(T^{*})$ i.e $\lambda \in pSp(T)$.

On the other hand, if $\lambda I - T$ is not bounded below it follows that $\lambda \in apSp(T)$. Thus

$$Sp(T) \subseteq (pSp(T^{*}))^{*} \cup (apSp(T)) \qquad (1.7)$$

(1.7) and (1.5) give

$$Sp(T) = (pSp(T^*))^* \cup (apSp(T) \qquad (1.8)$$

Replacing T by T^* in (1.8) and using $(T^*)^* = T$ we get

$$Sp(T^*) = (pSp(T))^* \cup (apsp(T^*)) \qquad (1.9)$$

But $Sp(T^*) = (Sp(T))^*$ (Lemma 1.12)

Taking complex conjugates of the subsets in (1.9). We get $Sp(T) = pSp(T) \cup (apSp(T^*))^*$.

Proposition 1.15: $Sp(T) \subseteq \overline{W(T)}$

Proof: Let $\lambda \in Sp(T)$. Then by lemma 1.14

$\bar{\lambda} \in pSp(T^*)$ or $\lambda \in apSp(T)$.

If $\lambda \in apSp(T)$, then there is a sequence (x_n) of unit vectors in H such that $||(\lambda I - T)x_n|| \to 0$ as $n \to \infty$.

Thus

$$\begin{aligned} |\langle Tx_n, x_n\rangle - \lambda| &= |Tx_n, x_n\rangle - \lambda\langle x_n, x_n\rangle| \\ &= |\langle Tx_n - \lambda x_n, x_n\rangle| \\ &\leq ||(T-\lambda I)x_n|| \, ||x_n|| \\ &\to 0 \text{ as } n \to \infty \end{aligned}$$

Since $||x_n|| = 1$ for all $n \in \mathbb{N}$, it follows that $\langle Tx_n, x_n\rangle \in W(T)$; thus it follows that $\lambda \in \overline{W(T)}$.

If $\bar{\lambda} \in Sp(T^*)$, then since $pSp(T^*) \subseteq W(T^*)$ (see lemma 1.9) and $W(T^*) = (W(T))^*$ (the proposition 1.2 (ii.), we get $\lambda \in W(T)$. Consequently $\lambda \in \overline{W(T)}$, the closure of W(T) Hence we obtain

$$Sp(T) \subseteq \overline{W(T)}$$

Proposition 1.16: Conv Sp(T) $\subseteq$ $\overline{W(T)}$.

Proof: Since W(T) is a convex subset of $\mathbb{C}$, so is $\overline{W(T)}$. By proposition 1.15, sp(T) $\subseteq$ $\overline{W(T)}$. Hence conv Sp(T) $\subseteq$ $\overline{W(T)}$

We note that the closure of the numerical range can be very much larger that Sp(T). In this connection, see example 1.8 (b) where we saw that when

T: $\mathbb{C}^2 \dashrightarrow \mathbb{C}^2$ had the matrix

$$\begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix}$$

the numerical range $W(T) = \{\lambda \in \mathbb{C} : |\lambda| < \frac{1}{2}\}$

$$= N(0, \tfrac{1}{2});$$

however, it is easily seen that Sp(T) = { 0 } in this case. Among hypernormal operators such extreme examples do not exist; for them the closure of the numerical range is as small as the universal properties of spectra and numerical ranges permit.

We recall that an operator T in B(H) is called **hypernormal** if $||T^{*}x|| \leq ||Tx||$ for each x in H. It is easily seen that T is hypernormal if and only if $T\,T^{*} \leq T^{*}T$. Obviously normal operators in B(H) are hypernormal. The **spectral radius** r(T) of an operator T in B(H) is the real number $r(T) = \sup \{ |\lambda| : \lambda$

is in Sp(T)} and it is a well-known fact that $\lim_{n \to \infty} || T^n ||^{1/n}$ exists and equals r(T).

Proposition 1.17: If $T \in B(H)$ is hypernormal, then

$$\text{Conv } Sp(T) = \overline{W(T)}.$$

Proof: If T is hypernormal then for all $\lambda \in \mathbb{C}$, it can be verified that $T - \lambda I$ is hypernormal and $r(T) = ||T||$ (see [4], p10 for details). In view of proposition 1.16, we need show that

$$\overline{W(T)} \subseteq \text{Conv } (Sp(T))$$

In view of the characterisation of Convex hulls in terms of closed discs, the desired result can be formulated this way: If a closed disc includes Sp(T), then it includes W(T). Let $\overline{N}(\lambda, k)$ represent the closed disc in the complex plane with centre λ and radius k and suppose $Sp(T) \subset \overline{N}(\lambda, k)$. Then

$$Sp(T - \lambda I) = Sp(T) - \lambda \subseteq \overline{N}(o,k)$$

and so $r(T - \lambda I) < k$ (by definition of spectral radius). But $T - \lambda I$ is hypernormal, and so $r(T - \lambda I) = || T - \lambda I ||$.

Consequently $|| T - \lambda I || \leq k$ and by proposition 1.2 (iii) it follows that

$$W(T - \lambda I) \subseteq \overline{N}(o,k);$$

But this means $W(T) - \lambda \subseteq \overline{N}(o,k)$, i.e. $W(T) \subseteq \overline{N}(\lambda, k)$, which is what we set to show. Thus every closed disc containing Sp(T) contains W(T). Thus

$$W(T) \subseteq \text{Conv } Sp(T)$$

for conv(Sp(T)) is the intersection of all the closed discs containing Sp(T). Since Sp(T) is a compact subset of $\mathbb{C}$, conv Sp(T) is closed (See remarks on Convexity following corollary 1.6). Hence

$$\overline{W(T)} \subseteq \text{Conv } Sp(T)$$

Remark 1.18: The proposition is obviously valid when $T \in B(H)$ is normal operator since normal operators are hypernormal.

1.3 Similarity and Numerical range.

We have seen in Proposition 1.2(V) that for a T in $B(H)$ the numerical range $W(T)$ is a unitary invariant. However it is far from being a similarity invariant. We recollect that operators T_1, T_2 in $B(H)$ are said to be similar to each other if there is an S in $B(H)$ which is invertible, i.e. S^{-1} exists and belongs to $B(H)$ and is such that $S^{-1} T_2 S = T_1$. What we are therefore asserting is that $W(T)$ and $W(S^{-1} T S)$ are different subsets when S is not unitary. In this connection we prove a result due to S. Hildebrandt (1966) and provide a proof due to J.P. Williams [13]. Some basic material is pertinent to our discussions, which we present first.

Let H be a Hilbert space and let

$$\ell^2_+(H) = H \oplus H \oplus \ldots$$

be the Hilbert space of all sequences $x = (x_n)_{n=1}^{\infty}$ of vectors x_n in H such that $||x||^2 = \sum_{n=0}^{\infty} ||x_n||^2 < \infty$. The **unilateral shift operator** U_+ on $\ell^2_+(H)$ is defined by

$$U_+ (x_1, x_2, \ldots) = (0, x_1, x_2, \ldots)$$

The multiplicity of U_+ is the cardinal number $n = \dim H$. It is easily seen that

$$U_+^* (x_1, x_2, \ldots) = (x_2, x_3, \ldots).$$

and that unilateral shift operators are unitary equivalent if and only if they have the same multiplicity. The operator U_+^* is called the **backward shift.**

We cite next a few well known definitions. A closed linear subspace M of H is invariant with respect to an operator T if $Tx \in M$ for each x in M. An operator T is called a **contraction** if $||T|| \leq 1$ and a **strict contraction** if $||T|| < 1$. A <u>part</u> of an operator is a restriction of it to any invariant subspace.

If M is an invariant subspace for the unilateral shift U_+, then $U_+|_M$ is again a unilateral shift. Since $M^\perp$ is invariant for U_+^*, we ask ourselves, what can be said about the operators $U_+^*|_{M^\perp}$? A remarkable answer was due to Rota [8].

Proposition 1.19. Any strict contraction is similar to an operator of the form $U_+^*|_M$.

Proof: Let T be an operator in a Hilbert space H with $||T|| < 1$. Define a map $S: H \dashrightarrow \ell_+^2(H)$ by

$$S\,x = (x,\ Tx,\ T^2x, \cdots).$$

Clearly S is linear, one-to-one and bounded, since

$$||Sx||^2 = \sum_{n=1}^{\infty} ||T^n x||^2 \leq \sum_{n=1}^{\infty} ||T||^{2n}\, ||x||^2$$

$$= (1-||T||^2)^{-1}\, ||x||^2$$

because $\sum_{n\in N} ||T||^{2n}$ is a geometric series of common ratio less than one which convages to $(1 - ||T||^2)^{-1}$

obviously. Since $||Sx||^2 \geq ||x||^2$ for all x in H, it follows that S is bounded from below and consequently its range R(S) is a closed linear subspace of $l_+^2(H)$. (see, Halmos [6], p 37) For each x in H, we have

$$\begin{aligned} S\,T\,x &= (T\,x,\ T^2 x,\ \ldots.) \\ &= U_+^* (x,\ Tx,\ T^2 x, \ldots) \\ &= U_+^* (Sx) \end{aligned}$$

Thus $S\,T = U_+^* S$. This shows that if $y \in R(S)$ (i.e $y = Sx$ for some x in H) then $U_+^*(y) = STx = S(Tx)$ i.e $U_+^*(y)$ is also in R(S), hence R(S) is invariant with respect to U_+^*. Since S is bounded from below, its inverse S^{-1} exists on R(S) and is bounded i.e $S^{-1} \in B(R(S), H)$ (= normed algebra of all bounded linear transformation on R(S) into H). From $S\,T = U_+^* S$, we get $S\,T = U_+^*|_{R(S)} S$

$$T = S^{-1} U_+^*|_{R(S)} S.$$

which shows that T and $U_+^*|_{R(S)}$ are similar

Proposition 1.20: (Rota [8]) Any operator with spectral radius less than 1 is similar to a part of the backward shift.

Proof: We observe that the hypothesis $||T|| < 1$ was used in proposition 1.18 only to ensure that S is bounded. However, for the latter it is obviously sufficient to have the convergence of the series

$$\sum_{n\in\mathbb{N}} \| T^{n} \|^{2}$$

This Convergence is guaranteed if $\overline{\lim_{n\to\infty}} \| T^{n} \|^{\frac{2}{n}} < 1$. Since for any bounded operator T in $B(H)$,

$$\lim_{n\to\infty} \| T^{n} \|^{\frac{1}{n}}$$ always exists and equals

the spectral radius $r(T)$ (see [1], p 322) it follows that if $r(T)$ is less than 1, then

$\overline{\lim_{n\to\infty}} \| T^{n} \|^{\frac{2}{n}} < 1$. This completes the proof of this proposition.

Corollary 1.21: For any T in $B(H)$,

$$r(T) = \inf \| S^{-1} T S \|,$$

the infimum being taken over all invertible operators S.

Proof: Since $r(T) \leq \|T\|$ for any $T \in B(H)$, for each real $\varepsilon > 0$, the operator $(r(T) + \varepsilon)^{-1} T$ has spectral radius less than one and proposition 1.19 then shows that $(r(T) + \varepsilon)^{-1} T$ is similar to a part of U_{+}^{*}

Hence we can write

$$(r(T) + \varepsilon)^{-1} S^{-1} TS = U_{+}^{*} |_{D(S)} \qquad (1.10)$$

Where D S is a closed linear subspace of $\ell_{+}^{2}(H)$ and $S: D(S) \dashrightarrow H$ is onto, one-to-one and bounded. (See proposition 1.19). Considering norms of both sides of (1.10), we obtain

$$(r(T) + \varepsilon)^{-1} \| S^{-1} T S \| \leq \| U_{+}^{*} |_{D(S)} \| < 1,$$

i.e. $\| S^{-1} T S \| \leq r(T) + \varepsilon$.

Since $\varepsilon > 0$ is arbitrary, we get

$$\| S^{-1} T S \| \leq r(T). \tag{1.11}$$

Obviously, if we consider all bounded operators S from closed subspaces of $\ell_{+}^{2}(H)$ onto H that have bounded inverse, i.e are invertible, then from (1.11) we obtain

$$\inf \{\| S^{-1} TS\| : \text{S invertible}\} \leq r(T) \quad (1.12)$$

The reverse inequality is seen thus: Since for any such invertible S

$$r(T) = r(S^{-1} T S) \leq \| S^{-1} T S \| \quad \text{we get}$$

$$r(T) \leq \inf \{\| S^{-1} T S \| : \text{S invertible}\}.$$

Proposition 1.22: (S Hildebrandt). For any operator T in B(H),

$$\text{Conv } (Sp(T) = \bigcap \{\overline{W}(S^{-1} T S): \text{ S invertible}\}$$

where $\overline{W}$ denotes the closure of the numerical range.

Proof: (due to J.P. Williams [13]).

For any operator A in B(H), we know from proposition 1.15 that $\overline{W(A)}$ is convex and includes Sp(A) Then

$$Sp(T) = Sp(S^{-1} T S) \subseteq \overline{W}(S^{-1} T S).$$

and it easily follows that

$$\text{Conv } Sp(T) \subseteq \bigcap\{ \overline{W}(S^{-1} T S) : \text{ S invertible}\}.$$

The proof of the reverse inclusion is based on the simple fact that any closed convex subset of the plane is the intersection of all open discs containing it. Let D be an open disc containing conv Sp(T) with centre λ and radius r. Then the spectrum of $\frac{1}{r}(T - \lambda I)$ lies in the open unit disc D(o,1), that is $\frac{1}{r}(T - \lambda I)$ is a strict contraction. Hence by corollary 1.20 there is an invertible operator S (from a closed linear subspace D(S) of $\ell^2(H)$ onto H) such that

$$\| \tfrac{1}{r} S^{-1}(T - \lambda I) \| < 1.$$

Thus it follows that

$$\overline{W}(\tfrac{1}{r} S^{-1}(T - \lambda I) S)$$

is contained in the open unit disc D(o,1) or equivalently, using proposition 1.2 (1) $\overline{W}(S^{-1} T S) \subseteq D$

.

J.P. williams ([13]) strengthened the above result by proving that for each Convex set Z containing Sp(T), there exists an invertible operators S on a closed linear subspace of $\ell^2_1(H)$ onto H for which $\overline{W}(S^{-1} T S) \subseteq Z$. We now give an exposition of the proof of this result with relevant preliminaries beginning with the notion of dilations and compressions and some important results concerning the same.

If M is a closed linear subspace of a Hilbert space H and P is the orthogonal projection on H onto M. Then each operator $T \in B(H)$ induces in a natural way an operator S in $B(M)$ defined by

$$Sx = P\,Tx \text{ for all } x \text{ in } M.$$

The relation between S and T can also be expressed by

$$SP = PTP \quad\text{or}\quad S = PT\,|_M$$

Under these conditions S is said to be a **compression** of T to M and T is called a **dilation** of S to H. Recall that $(PT\,|_M)^* = PT^*\,|_M$. This geometric definition of compression and dilation is to be contrasted with the customary concept of restriction and extension. If it happens that M is invariant under T, then it is not necessary to project Tx back to M (it is already there), and, in that case, S is a restriction of T to M and T is an extension of S to H. Restriction - extension is a special case of compression - dilation, the special case in which the operator in the larger space leaves the smaller space invariant !

The concept of operator matrices helps our discussion. We restrict the dicussions with regard to this to the case when the Hilbert space H is expressed as the direct sum $M \oplus M^{\perp}$, where M is a closed linear subspace of H. For Convenience, put $M_1 = M$ and $M_2 = M^{\perp}$. Let T be in $B(H)$. For any x in H, we have the unique decomposition, $x = x_1 + x_2$ where x_1 is in M_1 and

x_2 is in M_2 . Now $Tx = Tx_1 + Tx_2$, and since Tx_1, Tx_2 are in H, we can again write $Tx_i = x_{i1} + x_{i2}$ where x_{ij} is in M_j $(i,j = 1,2)$. Define maps $T_{ij} : M_j \dashrightarrow M_i$ by

$$T_{ij}(x_j) = x_{ij} \qquad (i,j = 1,2)$$

It is a routine verification to see that T_{ij}'s are linear and bounded, namely, $T_{ij} \epsilon B(M_j, M_i)$ $(i = 1,2)$. Thus corresponding to each T in B(H), there is a matrix $[T_{ij}]$ whose entry in row i and column j is the projection onto the i component of the restriction of T to M_j

This correspondence from operators to matrices (induced by a fixed direct decomposition) has all the desirable and right algebraic properties. If T = o, then $T_{ij} = o$ for all i,j; if T = I, then $T_{ij} = o$ when $i \neq j$ and $T_{ii} = I$, the identity on M_i . Moreover, the linear operations on operator matrices are the obvious ones. The matrix of T^* is the adjoint transpose of the matrix of T, that is, the matrix of T^* has the entry T_{ji}^* in row i and column j. The multiplication T S of operators T, S corresponds to the matrix product defined by

$$(T\,S)_{ij} = \sum_k T_{ik} S_{kj}$$

(There being no convergence trouble here, but there may be commutativity trouble and hence the order of the factors must be watched with care). The following situation is the one that occurs most often: A Hilbert space H is given, and the role of what was H in the above discussion is played now by the external direct sum $H \oplus H$, and operators on the direct sum are expressed as two-by-two matrices whose entries are operators on H.

Lemma 1.23. Let T be in B(H), M be a closed linear subspace of H and S be in B(M). Then a necessary and sufficient condition that T be a dilation of S is that the matrix of T with respect to the decomposition $H = M \oplus M$ have the form

$$\begin{bmatrix} S & X \\ Y & Z \end{bmatrix}$$

(Note $T_{11} = S$)

Proof: Suppose S in B(M) is a dilation of T. Then

$$S = PT|_M$$

Where P is the orthogonal projection on H onto M. We stick to the notations employed earlier while representing T by 2 x 2 operator matrix, $[T_{ij}]$ with respect to the decomposition $H = M_1 \oplus M_2$ (where $M_1 = M$ and $M_2 = M^{\perp}$). Let $x \in M$. If $x_1 + x_2$ is the decomposition of x in $M \oplus M^{\perp}$. Then $x_1 = x$ and $x_2 = o$. Writing $x_{11} + x_{12}$ and $x_{21} + x_{22}$ as decompositions of Tx_1 and Tx_2 respectively in $M \oplus M^{\perp}$, we at once obtain

$$x_{21} = o = x_{22} \text{ and } T_{11}x_1 = x_{11} .$$

The latter implies $T_{11} x = x_{11}$ Since $Tx = x_{11} + x_{12}$ is the decomposition of Tx in $M \oplus M^{\perp}$, we have

$$P T x = x_{11} \quad .$$

Thus

$$T_{11} x = P Tx \quad \text{or} \quad T_{11} x = P T|_M x.$$

As this is true for any x M, we get T

$T_{11} = P T|$ i.e $T_{11} = S$.

Conversely, Suppose $T_{11} = S$, since $T_{11} = PT|_M$. it follows that $S = P T|_M$ i.e. T is a dilation of S

It is obvious that any compression of a unitary operator is a contraction. Conversely, any contraction has unitary dilation:

Proposition 1.24: Let T be in $B(H)$

(a) If $||T|| \leq 1$. then T has a unitary dilation.

(b) If $o \leq T \leq I$, then T has a dilation which is a projection.

Proof: (a). Firstly we note the following statements are equivalent;

(i) T is a contraction

(ii) $T^* T \leq I$

(iii) $T T^* \leq I$

(iv) T^* is a contraction.

{indeed (i) $\Leftrightarrow$ (ii) and (i) $\Leftrightarrow$ (iv) since $|| T^* || = || T ||$ and (iv) $\Leftrightarrow$ (iii)}.

Let $S = \sqrt{I - T T^*}$ and $R = \sqrt{I - T^* T}$, where the positive square root are meant; T being a contraction. The desired dilation V can be defined on $H \oplus H$ by

$$V = \begin{bmatrix} T & S \\ R & -T^* \end{bmatrix}$$

That V is a dilation of T is clear from Lemma 1.22. We will show that V is unitary. Since

$$V^* = \begin{bmatrix} T^* & R \\ S & -T \end{bmatrix}$$

We get through direct computation

$$V^* V = \begin{bmatrix} T^* T + R & T^* S - R T^* \\ S T - TR & S^2 + TT^* \end{bmatrix}$$

$$V V^* = \begin{bmatrix} TT^* + S^2 & TR - ST \\ R T^* - T^* S & R^2 + T^* T \end{bmatrix}$$

Since $S^2 = I - TT$ and $R^2 = I - T^* T$, we have $TT^* + S^2 = I = T^* T + R^2$. If we now show that $ST = TR$, then taking adjoints we obtain $T^* S = RT^*$ whence it would follow that $V^* V = \begin{bmatrix} I & o \\ o & I \end{bmatrix} = V V^*$.

i.e. V is unitary on $H \oplus H$. Now

$$T R^2 = T(I - T^* T) = T - TT^* T = (I - TT^*) T = S^2 T,$$

and by induction it follows that

$$TR^{2n} = S^{2n} T \text{ for } n=o,\ 1,\ 2\ \ldots.$$

This implies that

$$T\,p\,(R^2) = p(S^2)T.$$

for all polynomials p. Since the polynomials in x^2 are uniformly dense in $C[o,1]$, it follows that $TR = ST$ (We consider the function f defined on $[o,1]$ by $f(x) = x$, and obtain $T\ f(R) = f(S)\ T$).

(b). Given T with $o \leq T \leq I$, we see that $T(I-T) > o$, let R be the positive square root $\sqrt{T\,(I - T)}$ and let

$$V = \begin{bmatrix} T & R \\ R & I-T \end{bmatrix}$$

be an operator on $H \oplus H$. Clearly $V = V^*$ and $V^2 = V$. Thus V is an orthogonal projection on $H \oplus H$. That V is a dilation of T is obvious from Lemma 1.22 .

Remark 1.25 Part (b) of proposition 1.23 asserts that the compressions of orthogonal projections may be identified as the class of all positive contractions.

Now if T is a dilation of S, it is not necessarily true that T^2 is a dilation of S^2, i.e if $S = PT|_M$, it need not be true that $S^2 = PT^2|_M$ (M is a closed subspace of H). In particular if T is a unitary dilation of $S = PT|_M$, it need not be true that $S^2 = PT^2|_M$

The following example substantiates, this statement.

The least unitary looking contraction is o, but it has a unitary dilation too. For instance

$$\begin{bmatrix} o & I \\ I & o \end{bmatrix}$$

(see proof of proposition 1.23 in this connection) is a unitary dilation of o. The square of this dilation is

$$\begin{bmatrix} I & o \\ o & I \end{bmatrix}$$

which is not a dilation of the square of o (by Lemma 1.22) We have a general definition:

Let H be a Hilbert space and M be a closed subpace of H. An operator T in B(H) is said to be a **power dilation** (Sometimes called a **strong dilation**) of S in B(M) if

$$S^n = PT^n|_M \quad \text{for all integers } n > o,$$

P being the orthogonal projection on H onto M.

It is a well-known result stated in the next

Proposition 1.26: Every Contraction has a unitary power dilation.

Proof The following proof is due to J.J. Schaffer [9] Though it is not the most revealing proof, it is certainly the shortest and serves our purpose (A little knowledge of infinite operator matrices is necessary, and the discussion already provided above can easily be adopted to this situation without much ado).

Given a Hilbert space H, let K be the dirrect sum of countably infinitely many copies of H, indexed by all integers (positive, negative, zero); then each operator on K is an infinite operator matrix, and in particular, the projection P from K to H is given by

$$P = \begin{bmatrix} \ddots & & & & \iddots \\ & 0 & 0 & 0 & \\ & 0 & (I) & 0 & \\ & 0 & 0 & 0 & \\ \iddots & & & & \ddots \end{bmatrix}$$

(the parentheses indicate the entry in position $\langle 0,0\rangle$)
Given a contraction T on H, put

$$V = \begin{bmatrix} \ddots & & & & & & & & \iddots \\ & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \\ & I & 0 & 0 & 0 & 0 & 0 & 0 & \\ & 0 & I & 0 & 0 & 0 & 0 & 0 & \\ & 0 & 0 & S & (T) & 0 & 0 & 0 & \\ & 0 & 0 & -T^* & R & 0 & 0 & 0 & \\ & 0 & 0 & 0 & 0 & I & 0 & 0 & \\ & 0 & 0 & 0 & 0 & 0 & I & 0 & \\ \iddots & & & & & & & & \ddots \end{bmatrix}$$

Where R and S are as in the proof of proposition 1.23
Since V is triangular, its powers are triangular, and the

diagonal entries of the powers are the corresponding powers of the diagonal entries of V. This makes it obvious that V is a power dilation of T. The proof that V is unitary is an obvious computation (which uses the results of proof of proposition 1.23)

A closed subset X of the complex plane is called a **spectral set** for an operator T in B(H) if it contains Sp(T), and if for any rational function f with poles lying outside of X,

$$||f(T)|| \leq \text{Sup}\ \{|f(z)| : z \text{ is in } X\}.$$

(Note: f is a bounded rational function on X).

Proposition 1.27: The closed unit disc $\bar{D}(o,1)$ is a spectral set for any contraction T.

Proof: Let f be holomorphic in a region containing $\bar{D}(o,1)$ and let V be a strong unitary dilation of T. If $f(x) = \sum_{n=0}^{\infty} a_n z^n$, then $\sum_{n=0}^{\infty} a_n T^n$ converges in norm to a bounded operator f(T) and

$$f(T) = \sum_{n=0}^{\infty} a_n (PV^n|_H) = P\, f(V)|_H$$

Now from the spectral mapping theorem

$$Sp(f(V)) = f(Sp(T)) \subseteq \{ f(z): z \text{ is in } \bar{D}(o,1)\},$$

and since the normality of f(V) (note: V is unitary $\Rightarrow$ V is normal; since f is holomorphic, f(V) is normal. These verifications are routine), implies that $||f(V)|| = r(f(V))$, the spectral radius of f(V). Thus it follows that

$$||f(T)|| \leq ||f(V)|| \leq \sup \{|f(z)| : z \text{ is in } \bar{D}(o,1)\}$$

We now reach the goal, namely, to prove J.P. William's strengthening of the result in proposition 1.21.

Proposition 1.28: (J.P. Williams [13]) if Z is a closed convex set containing Sp(T) in its interior then there is an invertible operator S such that $W(S^{-1}TS) \subseteq Z$.

Proof: The following result will be needed. If C is a convex spectral set of T, then $\overline{W}(T) \subseteq C$. It suffices to prove this for C a closed half plane and by translation and rotation it can be assumed that $C = \{z: \text{Re } z > o\}$. It must be shown that $\text{Re}\langle Tx,x\rangle \geq o$ for all x in H. Since $|(1-z)(1+z)^{-1}| \leq 1$ for all z in C, it follows from the definition of spectral set that

$$|| (I-T)(1+T)^{-1} || \leq 1 \qquad (1.13)$$

This is equivalent to

$$|| (1-T)x|| < || (I+T)x|| \text{ for all } x \text{ in } H \qquad (1.14)$$

Indeed (1.13) implies that for all y in H, $||(I-T)(I+T)^{-1}y|| \leq ||y||$. Putting $y = (1+T)x$, We get (1.14) . (Note that $(1+T)^{-1}$ exists, since -1 is not in C) Conversely (1.14) implies (1.13).

Squaring and expanding both sides of (1.14) gives

$$\text{Re } \langle Tx,x\rangle \geq o.$$

as required. Thus $\overline{W}(T) \subseteq C = \{z: \text{Re } z \geq o \}$.

Now let V be the interior of Z, let f be a conformal map of the open unit disc D(o,1) onto V, and g be the inverse map. Then D(o,1) contains the spectrum of g(T). (note: $|g(z)| \leq 1$ for all z in v) and so by corollary 1.21 there is an invertible operator S (from a closed linear subspace D(S) of $\ell^2_+(H)$ onto H) such that

$$\| S^{-1} g(T)S \| = r < 1.$$

Let $\overline{D}_1(o,r)$ be the closed disc of radius r. By proposition 1.27, $\overline{D}_1(o,r)$ is a spectral set $g(S^{-1}TS) = S^{-1}g(T)S$. But f is a uniform limit of polynomials on $\overline{D}_1(o,r)$ from which it follows readily that $f(\overline{D}_1(o,r)$ is a spectral set for $f(g(S^{-1}TS))$

$= S^{-1}TS$. Since Z contains $f(\overline{D}_1(o,r))$ it is also a spectral set and since it is convex the assertion made in the begining of the proof, gives $\overline{W}(S^{-1}TS) \subseteq Z$

1.4 Closure of Numerical range; and some special points of W(T)

If dim $H < \infty$ and $T: H \dashrightarrow H$ is a linear operator, we know that T is bounded. Moreover the subset

$\partial D = \{ x \text{ in } H : \|x\| = 1 \}$ is compact. Hence the map $f: \partial D \dashrightarrow \mathbb{C}$ defined by $f(x) = \langle Tx,x \rangle$ for all $x \in \partial D$ is continuous on ∂D. Indeed, if x is any given element of ∂D and $\varepsilon > o$ is given, then for any $y \in \partial D$ satisfying $\|y - x\| < \varepsilon$ we have

$$
\begin{aligned}
|f(y)-f(x)| &= |\langle Ty,y\rangle-\langle Tx,x\rangle| < |\langle Ty,y\rangle-\langle Tx,y\rangle| + |\langle Tx,y\rangle+\langle Tx,x\rangle| \\
&< ||T||\ ||y-x||\ ||y|| + ||T||\ ||x||\ ||y-x|| \\
&< 2||T||\varepsilon \ ; \quad (||y|| = 1 = ||x||)
\end{aligned}
$$

Thus f is continuous (with norm topology in H and the usual topology in $\mathbb{C}$). Hence the range of f, namely, the numerical range W(T) is a compact subset of $\mathbb{C}$ and hence W(T) is closed.

However the numerical range of an operator T in B(H) in the case of an infinite dimensional H is not always closed, not even if the operator is compact. We see first a

Lemma 1.29: If T is in B(H) and $\lambda \in \mathbb{C}$ is a complex number such that $|\lambda| = ||T||$ and $\lambda \in W(T)$, then λ is an eigenvalue of T.

Proof:- If λ is in W(T), then $\lambda = \langle Tx,x\rangle$ for some $x \in H$ with $||x|| = 1$; then

$$||T|| = |\lambda| = |\langle Tx,x\rangle| \leq ||Tx||\ ||x|| \leq ||T||\ ||x||^2 = ||T||.$$

so that equality holds everywhere. We know that in the Cauchy-Schwarz inequality.

$$|\langle x,y\rangle| \leq ||x||\,||y||.$$

the equality holds if and only if {x,y} is linearly dependent (over $\mathbb{C}$).

Hence, in the above case we have $Tx = \lambda_0 x$ for some $\lambda_0 \in \mathbb{C}$, and this in turn implies that

$$\lambda_0 = \lambda_0\langle x,x\rangle = \langle \lambda_0 x,x\rangle = \langle Tx,x\rangle = \lambda_0$$

So that λ_0 is in Sp(T)

It follows from Lemma 1.29 that if λ is in $\overline{W(T)}$ Such that $|\lambda| = ||T||$ and λ is **not** an eigenvalue of T (and in particular, if T has no eigenvalues), then λ does **not** belong to W(T). In view of this statement, we can construct examples of operators whose numerical range is not closed.

Example 1.30: We have seen that if T is in B(H), then

$$pSp(T) \subseteq W(T) \quad \text{(Lemma 1.9)}.$$

For a normal T in B(H), we know that

$$||T|| = \text{Sup}\,\{|\langle Tx,x\rangle| : x \text{ is in } H \text{ and } ||x|| = 1\} =$$
$$\text{Sup}\,\{|\lambda| : \lambda \in W(T)\}$$

(See [1], p.382) Hence, there always exists a $\lambda \in \overline{W(T)}$ such that $|\lambda| = ||T||$. It follows that if a normal operator has sufficiently many eigenvalues to approximate its norm, but does not have one whose modulus is as large as the norm, then its numerical range will not be closed. As an instance of this, we can take a diagonal operator such that the modulus of the diagonal terms does not attain its supremum.

Example 1.31 Consider T: $\ell^2(\mathbb{C}) \dashrightarrow \ell^2(\mathbb{C})$ whose matrix with respect to the standard orthonormal basis is a diagonal with diagonal $\{1, \frac{1}{2}, \frac{1}{3}, \dots\}$, i.e.

$$Tx = (x_1, \frac{x_2}{2}, \frac{x_3}{3}, \dots) \text{ if } x = (x_1, x_2, x_3, \dots) \in \ell^2(\mathbb{C}).$$

Clearly T is one-to-one, positive and o is not in W(T). In fact W(T) = (o,1]. Now T is compact. This shows that the numerical range may fail to be closed even for compact operators.

Example 1.31 Consider the unilateral shift operator on $\ell^2(\mathbb{C})$ i.e. $U_+(x_1, x_2, \ldots) = (o, x_1, x_2, \ldots)$ for all $x = (x_1, x_2, x_3, \ldots) \in \ell^2(\mathbb{C})$. Then $U_+^*(x_1, x_2, x_3, \ldots) = (x_2, x_3, \ldots)$. It can be seen that every λ in the open unit disc D(o,1) is an eigenvalue of U_+^*. Indeed, let $U_+^* x = \lambda x$ hold for a $\lambda \in \mathbb{C}$ and $x \neq 0$. Putting $x = (x_1, x_2, x_3, \ldots)$, we have $x_{n+1} = \lambda x_n$ for all $n \in \mathbb{N}$. Take $x_1 \neq o$ Then $x = x_1(1, \lambda, \lambda^2, \lambda^3, \ldots)$ If x must be in $\ell^2(\mathbb{C})$, we must have $\sum_{n=0}^{\infty} |\lambda|^{2n} < \infty$, which holds if and only if $|\lambda| < 1$. Also since $x_1 \neq o$, $x \neq o$. Thus it follows that for each $\lambda \in \mathbb{C}$ satisfying $|\lambda| < 1$, there is a non-zero x in $\ell^2(\mathbb{C})$ such that $U_+^*(x) = \lambda x$. Thus the open unit disc D(o,1) is contained in $W(U_+^*)$. Since $W(U_+^*) = (W(U_+))^*$. It follows that $W(U_+)$ is contained in D(o,1). But U_+ has no eigenvalues. Hence by lemma 1.29 and the fact that $\|U_+\| = 1$, it follows that $W(U_+)$ cannot contain any number of modulus 1. Thus $W(U_+)$ is the open disc D(o,1); and hence not closed.

Now the number o plays a special role with respect to the spectrum of a compact operator. That it does play an equally special role with respect to numerical range was shown by G.De Barra, J.R. Giles and B. Sims in 1972. Their result is stated as the next

Proposition 1.32: If T in $B(H)$ is compact and o is in $W(T)$, then $W(T)$ is closed.

Proof: We note that if o is in $W(T)$, then $\langle Tx,x\rangle$ is in $W(T)$ for every x in $D(o,1)$ (and not only for the unit vectors, indeed if $||x|| = 1$, $o \leq t \leq 1$, then

$\langle T(tx), tx\rangle = t^2\langle Tx,x\rangle = t^2\langle Tx,x\rangle + (1-t^2).\ o$ is in $W(T)$ by convexity of $W(T)$

However, the argument in the previous paragraph does not need compactness of T; It goes through for every operator.

We next show that in the presence of compactness of T the quadratic form $x \longmapsto \langle Tx,x\rangle$ is weakly continuous on bounded set. Indeed if (x_η) is a bounded net weakly convergent to x, then

$$|\langle Tx_\eta, x_\eta\rangle - \langle Tx,x\rangle| \leq |\langle Tx_\eta, x_\eta\rangle - \langle Tx, x_\eta\rangle| + |\langle Tx, x_\eta\rangle - \langle Tx,x\rangle|;$$

the first summand tends to o (by compactness of T). Since $x_\eta \xrightarrow{w} x$ implies $Tx_\eta \xrightarrow{s} Tx$, and the second summand tends to o because $x_\eta \longrightarrow x$ weakly.

If both the hypotheses are satisfied (o is in W(T) and T is compact) then

$W(T) = \{ \langle Tx,x\rangle : ||x|| < 1\}$, so that W(T) is the image of a weakly compact set $\overline{D}(o,1)$ (See Halmos, [6]) under a mapping that is weakly continuous on the set (See preceeding paragraph). Hence W(T) is compact and hence closed •

Lemma 1.29 is the simplest result of the kind where we show that certain points of the numerical range are eigenvalues. A more interesting result of this kind was proved by W.F. Donoghue (1957).

Given a complex set K of $\mathbb{C}$, a point $\lambda \in K$ is called a **corner** of K if K is contained in an angle with vertex at λ and magnitude less that π radians.

Proposition 1.33 Let λ be a corner of $\overline{W(T)}$, then $\lambda \in Sp(T)$. If also λ is in W(T) then λ is an eigenvalue of T.

Proof: By proposition 1.2(i), we may suppose that λ =o and that there exists $\delta > o$ such that

$$\text{Re}W(e^{i\theta} T) \subseteq \mathbb{R}^+ \quad (-\delta \leq \theta \leq \delta) \qquad (1.15)$$

Where Re $W(e^{i\theta} T)$ denotes the set of real parts of the elements of $W(e^{i\theta} T)$. Let $S = T+T^*$

Since $\langle Sx,x\rangle = 2 \text{ Re } \langle Tx,x\rangle$, relation (1.15) gives $W(S) \subseteq \mathbb{R}^+$, and so $S \geq o$. Since o is in $\overline{W(T)}$, there exists vectors x_n in H with $||x_n|| = 1$ such that $\lim_{n \to \infty} \langle Tx_n, x_n \rangle = o$. Therefore $\lim_{n \to \infty} \langle Sx_n, x_n \rangle = o$. By proposition 1.2 (Vii), we have $||Sx_n||^2 \leq w(S) \langle Sx_n, x_n \rangle$ and so $s - \lim_{n \to \infty} Sx_n = o$, i.e. $s - \lim_{n \to \infty} (T+T^*) x_n = o$.

By (1.15), we may replace T by $e^{i\theta} T$ with $-\delta \leq \theta \leq \delta$ and so

$$s - \lim_{n \to \infty} (e^{i\theta} T + e^{-i\theta} T^*)x_n = o \text{ for all such}$$

Therefore $s - \lim_{n \to \infty} Tx_n = o$, i.e. $s - \lim_{n \to \infty} (T - o.I)x_n = o$.

This shows that

$$o \text{ is in } apSp(T) \subseteq Sp(T).$$

If also o is in W(T), the sequence (x_n) is replaced by a vector x with $||x|| = 1$ and $\langle Tx, x \rangle = o$. So we obtain $Sx = o$ and finally $Tx = o$ as required.

Let S be a subset of a linear space X. A point z in S is called an **extreme point** of S if the conditions,

x,y in S and $z = rx + (1 - r)y$ for some $o < r < 1$

together imply that $x = y = z$. We represent the set of all extreme points of S by Ext (S); geometrically, z belongs to Ext(S) if and only if z does not lie on any open segment whose end points are in S. Thus if S is a triangle in $\mathbb{R}^2$ with vertices x,y,z, then Ext(S) $= \{x,y,z\}$. If $\bar{D}$ is the closed unit disc in $\mathbb{R}^2$ with p-norm, then for p = 1, Ext($\bar{D}$)

consists of the four points $(1,o)$, $(o,1)$, $(-1,o)$ and $(o,-1)$; for $p = \infty$, Ext $(\bar{D})$ is the set

$$\{(1,1), (1,-1), (-1,1), (-1,-1)\};$$

while if $p = 2$, then Ext (D) is the unit circle which is the boundary of $\bar{D}$.

It is clear that the corners of a convex subset K of $\mathbb{C}$ are extreme points of K.

We refer to example 1.8 (b) where we saw that the operator

$$T = \begin{bmatrix} o & o \\ 1 & o \end{bmatrix}$$ on the Hilbert

space $\mathbb{C}^2$ had

$$W(T) = \{ z \in \mathbb{C} : | z | < \frac{1}{2} \}., \ Sp(T) = \{o\}$$

Thus Ext $(W(T) = \{ z \in \mathbb{C} : | z | = \frac{1}{2} \}$ and we see that extrem points of W(T) do not belong to Sp(T)

However, for certain classes of operators, the extreme points of W(T) are eigenvalues. In this connection, consider the class of bounded self-adjoint operators.

Proposition 1.34 If T in B(H) is self-adjoint, then

Ext $(W(T)) \subset$ pSp (T).

Proof: Let λ be in Ext(W(T)). Then λ is a corner of W(T).

By proposition 1.2(i), we may suppose that $\lambda = o$ then it follows from the proof of the second part of proposition 1.33 that

$T^* = T$, o is in Ext (w(T)), $\langle Tx,x\rangle = o \Rightarrow Tx = o$ (1.16)

i.e. $x \in p\,Sp(T)$.

A deeper result of this kind was proved by C.H. Meng (1957), namely, that

$$\text{Ext } (W(T)) \subset p\,Sp(T)$$

for normal T in B(H). Then J.G. Stampfli [11]. obtained the same conclusion for the wider class of hypernormal operators.

Proposition 1.35: Let $T \in B(H)$ be hypernormal. Then Ext(W(T) $\subset$ pSp(T).

Proof: The proposition follows from (1.16) above which was a special case of proposition 1.33

We prove first that for all $T \in B(H)$, we have

$Im(W(T) \subset \mathbb{R}^+$, $\langle Tx,x\rangle = o \Rightarrow T^*x = Tx$ (1.17)

Where Im (W(T)) = { Im λ : λ is in W(T)} i.e. the set of imaginary parts of the elements of W(T). For all y in H, we have,

$$\langle\frac{1}{i}(T-T^*)y,y\rangle = \frac{1}{i}(\langle Ty,y\rangle - \langle y,Ty\rangle) = 2\ Im\ \langle Ty,y\rangle.$$

Given that Im W(T) $\subseteq \mathbb{R}^+$ and $\langle Tx,x\rangle = o$, this shows that

$\frac{1}{i}(T - T^*) > o$ and that $\langle \frac{1}{i}(T - T^*)x, x\rangle = o$.

Thus Lemma 1.4 gives $\frac{1}{i}(T - T^*)x = o$ and hence (1.17) is proved.

Suppose now that T is hypernormal. Then

$$Tx = T^*x \Rightarrow T T^*x = T^*Tx \qquad (1.18)$$

Indeed, $\langle (T^*T - T T^*)x, x\rangle = \langle Tx\ Tx\rangle - \langle T^*x, T^*x\rangle = o$

But $T^*T - T T^* \geq o$, since T is hypernormal, and so Lemma 1.4 implies (1.18).

Let $N = \{ x$ in H: $T^*x = Tx\}$: We claim that N is a reducing subspace for T and the restriction $T|_N$ is self-adjoint. For let x be in N; then by (1.18). We get $T T^* x = T^*Tx$. But also

$$T(T^*x) = T(Tx) \quad \text{and} \quad T^*(T^*x) = T^*(Tx)$$

Since $T^*x = Tx$. Therefore $T^*(Tx) = T(Tx)$

Which shows that Tx is in N; and $T^*(T^*x) = T(T^*x)$, giving $T^*x \in N$. Thus N is an invariant subspace for T and T^*, this means that N reduces T. It is now clear from the definition of N that $T|_N$ is self adjoint.

Since $\alpha I + \beta T$ is hypernormal for all complex numbers α, β, we may assume that o is in Ext(W(T)), that $\text{Im } W(T) \subseteq i\mathbb{R}^+$ and that $\langle Tx, x\rangle = o$ for some vector x with $||x|| = 1$, and complete the proof by showing that $Tx = o$. By (1.17), x is in N. Let $S = T|_N$; then S is self-adjoint, o is in Ext $(W(T))$ (Since $W(S) \subseteq W(T)$) and $\langle Sx,x\rangle = o$.

Thus by (1.16) above, $Sx = o$, i.e. $Tx = o$

Thus $x \in p\,Sp(T)$.

1.5 Numerical radius, power inequality

In Section 1.1 we introduced the term numerical radius w(T) of a T in B(H): $w(T) = \sup\{|\lambda| ; \lambda \in W(T)\}$. It is easily seen that the function $w: B(H) \dashrightarrow \mathbb{R}$ satisfies the following properties:

(i) $w(T) \geq o$ (ii) $T = o$ implies $w(T) = o$ (iii) $w(\alpha T) = |\alpha|\, w(T)$ for each scalar $\alpha \in \mathbb{C}$ and (iv) $w(S+T) \leq w(S) + w(T)$.

the last inequality being a simple consequence of the inclusion $W(S + T) \subseteq W(S) + W(T)$.

If we prove that $w(T) = o$ implies $T = o$, then it would follow that w is a norm on B(H).

Lemma 1.36 For any T in B(H),

$$\frac{1}{2}||T|| \leq w(T) \leq ||T|| \qquad (1.19)$$

The constant $\frac{1}{2}$ on the left hand side is the best possible.

Proof: If $\phi : H\times H \dashrightarrow \mathbb{C}$ is a sesquilinear functional and $\hat{\phi} : H \dashrightarrow \mathbb{C}$ is the associated quadradic from i.e. $\hat{\phi}(x) = \phi(x,x)$ for all x in H, then we know that ϕ is bounded if and only if $\hat{\phi}$ is bounded and

$$||\hat{\phi}|| < ||\phi|| < 2\,||\phi|| \qquad (1.20)$$

[we recollect that $||\phi|| = \mathrm{Sup}\,\{|\phi(x,y)|$: x,y are in H and $||x|| = ||y|| = 1\}$; $||\hat{\phi}|| = \mathrm{Sup}\,\{1\ \phi(x)1$: x is in H and $||x|| = 1\}$]

(See Halmos [6], p. 37).

Now set $\phi(x,y) = \langle Tx, y \rangle$. It is then known that ϕ is bounded and $\|\phi\| = \|T\|$. The relations (1.20) then yield

$$w(T) < \|T\| < 2\,w(T).$$

from which (1.19) follows easily.

Consider the operator $T = \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix}$ of example 1.8 (b). In that case we have $w(T) = \frac{1}{2}$, whereas $\|T\| = 1$. This proves the second assertion.

If $T \in B(H)$ is self-adjoint, then we know that $\phi : H \times H \to \mathbb{C}$ defined above by $\phi(x,y) = \langle Tx, y \rangle$ is symmetric. For a symmetric ϕ we have $\|\phi\| = \|\hat{\phi}\|$ (see Halmos, [6]. p.37). This shows that $w(T) = \|T\|$.

From the part of the inequality

$$\frac{1}{2}\,\|T\| \leq w(T)$$

in (1.19) we get

$$w(T) = 0 \text{ implies } T = 0.$$

and thus we conclude that the function $w : B(H) \to \mathbb{R}$ is indeed a norm and result (1.19) then shows that the norm is equivalent to the ordinary operator norm in the sense that each is bounded by a constant multiple of the other.

Since $Sp(T) \subseteq \overline{W(T)}$, it is readily seen that

$$r(T) \leq w(T) \quad \text{for all } T \text{ in } B(H).$$

Nothing like the reverse of this inequality could be true. (See for instance, Example 1.8(b) where $r(T) = o$ and $w(T) = \frac{1}{2}$!).

Proposition 1.37: Let T be in B(H).

(a) If $w(I-T) < 1$, then T is invertible, i.e. $T^{-1} \in B(H)$.

(b) If $w(T) = ||T||$, then $r(T) = ||T||$.

Proof: (a) If T is not invertible, i.e. T-o.I is not invertible, then o is in Sp(T); so by the spectral mapping theorem 1 is in Sp(I - T); thus $1 \leq r(I-T) \leq w(I - T)$.

(b) Assume, with no loss of generality, that $||T|| = 1$ (For this, multiply by a suitable constant). The hypothesis $w(T) = ||T||$ then guarantees the existence of a sequence (x_n) of unit vectors in H such that

$|\langle Tx_n , x_n \rangle| \longrightarrow 1$ as $n \longrightarrow \infty$. By proposition 1.2 (i), we may assume, without loss of generality, that $\langle Tx_n , x_n \rangle \longrightarrow 1$ as $n \longrightarrow \infty$ (multiply T by a suitable constant of modulus 1). Since

$|\langle Tx_n , x_n \rangle| \leq ||Tx_n|| \leq 1$ and $\langle Tx_n , x_n \rangle \longrightarrow 1$ as $n \longrightarrow \infty$, it follows that $||Tx_n|| \longrightarrow 1$ as $n \longrightarrow \infty$

This implies that

$||Tx_n - x_n||^2 = ||Tx_n||^2 - 2 \operatorname{Re} \langle Tx_n , x_n \rangle + 1 \longrightarrow o$ as $n \longrightarrow \infty$, so that 1 is in ap. Sp (T);

but ap Sp (T) $\subseteq$ Sp (T); therefore r(T) must be equal to 1.

The norm w has many other pleasant properties: for instance, $w(T^*) = w(T)$ (See proposition 1.2 iii)) $w(T^*T) = ||T||^2$, and w is unitarily invariant in the sense that $w(U^* T U) = w(T)$ whenever U is unitary (see proposition 1.2 (v)). On the other hand the relations between the numerical range and the multiplicative properties of operators are less smooth. Thus, for instance, w is certainly not multiplicative i.e. $w(ST)$ is not always equal to $w(S)\, w(T)$ even if $ST=TS$. This is illustrated in the following example with commutative normal operators.

Example 1.38: Let

$$S = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}, \qquad T = \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}$$

then $w(S) = w(T) = 1$ and $w(ST) = 0$.

The next best thing would be for w to be submultiplicative $(w(S\,T) < w(S)\, w(T))$, but that is also false,

Example 1.39: Let

$$S = \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix}, \qquad T = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}$$

Then $w(S) = w(T) = \frac{1}{2}$ and $w(S\,T) = 1$

Since $w(ST) \leq ||ST|| \leq ||S||\, ||T||$, it follows that for normal operators, w is submultiplicative [because if S and T are normal, then $||S|| = w(S)$ and

$||T|| = w(T)$] and for operators in general $w(ST) \leq 4w(S)w(T)$ [because $||S|| \leq 2\, w(S)$ and $||T|| \leq 2\, w(T)$. by lemma 1.36]

It then follows from example 1.39 that the constant 4 is the best possible here. Commutativity sometimes helps; here it does not. Examples of Commutative operators S and T for which $w(S\,T) > w(S)\,w(T)$ are a little harder to come by, but they exist. For instance, we have

Example 1.40: Let

$$S = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{pmatrix}$$

and $T = S^2$. It is easy to see that $w(S^2) = w(S^3) = \frac{1}{2}$.

The value of $w(S)$ is slightly harder to compute, but it is not needed; the almost obvious relation $w(S) < 1$ will do. Indeed,

$$w(S\,T) = w(S^3) = \frac{1}{2} > w(S)\,\frac{1}{2} = w(S)\,w(T).$$

The above example shows that it is not even true that

$$w(T^{n+m}) \leq w(T^n)\,w(T^m)$$

for T in B(H). Consequently, there was some suprise when C. Berger (see [4]) proved the following power inequality.

Proposition 1.14 For T in B(H)

$$w(T^n) \leq (w(T))^n \quad , n = 1,2,3. \qquad (1.21)$$

Proof:

The following proof is adopted from Pearcy [7]. First we see that (1.21) above is equivalent to

$$\text{If } w(T) \leq 1, \text{ then } w(T^n) \leq 1 \qquad (1.22)$$

That (1.21) implies (1.22) is obvious. Conversely, suppose (1.22) holds. If $w(T) = 0$, then $T = 0$ and everything is trivial. If $w(T) \neq 0$, then $S = \frac{1}{w(T)} T$; clearly $w(S) \leq 1$ Then, by (1.22), $w(S^n) \leq 1$, i.e. $w(T^n) \leq (w(T))^n$.

Next we see that

$w(T) \leq 1$ If and only if $\operatorname{Re}(I - zT) \geq 0$ for $|z| \leq 1$, i.e. iff $\operatorname{Re} \langle (I - zT)x, x \rangle \geq 0$ whenever $|z| < 1$ (1.23).

To see this, let $w(T) \leq 1$, i.e. $|\langle Tx,x \rangle| < 1$ for x in H and $||x|| = 1$. Then $\operatorname{Re} \langle I - zT)x,x \rangle = \operatorname{Re}(1 - z \langle Tx,x \rangle)$ $= 1 - \operatorname{Re} z\langle Tx,x \rangle$. Now if $\langle Tx,x \rangle \leq 1$ and and $|z| \leq 1$, then

$|\operatorname{Re} z\langle Tx,x \rangle| \leq |z\langle Tx,x \rangle| = |z||\langle Tx,x \rangle| < |z|$. Hence $1 - \operatorname{Re} z\langle Tx,x \rangle \geq 1 - |z| > 0$. i.e. $\operatorname{Re} \langle (I - zT)x, x \rangle \geq 0$. Conversely, suppose $\operatorname{Re} \langle (I - zT)x,x \rangle > 0$ when $|z| < 1$ and x is in H.

Thus $\operatorname{Re}(||x||^2 - z\langle Tx,x \rangle) \geq 0$ when $|z| < 1$ and x is in H.

Then this is true, in particular if $z\langle Tx,x\rangle = t|\langle Tx,x\rangle|$, $o \le t \le 1$; since, therefore

$$||x||^2 - t\,|\langle Tx,x\rangle| = \mathrm{Re}\,(||x||^2 - t\,|\langle Tx,x\rangle|) \ge o.$$

It follows (let $t \rightarrow 1^-$) that $|\langle Tx,x\rangle| < ||x||^2$.

This shows that

$\mathrm{Sup}\,\{|\langle Tx,x\rangle|: x$ is in H and $||x|| = 1\} \le 1$ or $w(T) < 1$.

Now if $w(T) \le 1$, then $r(T) \le 1$

(This follows since for any T in $B(H)$ $r(T) \le w(T)$) Also $(I - zT)$ is invertible whenever $r(zT) < 1$, i.e. Whenever $|z|\cdot r(T) < 1$, i.e. Whenever $|z| < 1$. An invertible operator has a positive real part if and only if its inverse has positive real part. (If S is invertible then $\langle S^{-1}x,x\rangle = \langle S^{-1}x, S\,S^{-1}x\rangle$

$$= \overline{\langle S\,(S^{-1}x),\ S^{-1}x\rangle});$$

Hence it follows that

$$w(T) \le 1 \text{ if and only if } \mathrm{Re}(1 - z\,T)^{-1} \ge o$$

for $|z| < 1$.

Next we observe that if n is a positive integer and u is a primitive n-th root of unity (i.e. n is the smallest positive inter such that $u^n = 1$), then,

$$\frac{1}{1 - z^n} = \frac{1}{n}\sum_{k=o}^{n=1} \frac{1}{1 - u^k z}$$

for all z other than powers of u. This identity is, in fact, the partial fraction expression of the left side. For a direct verification, multiply through by $1 - z^n$, and observe that the right side becomes a polynomial of degree n-1 at most that is invariant under each of the n substitutions

$$z \dashrightarrow u^k z \quad (k = o, 1, \dots, n-1).$$

and is therefore constant, and then evaluate the constant by setting z equal to o. (This constant is clearly seen to be 1).

The identity of the previous paragraph implies that if $w(T) < 1$, then

$$(1 - z^n T^n)^{-1} = \frac{1}{n} \sum_{k=o}^{n-1} (1 - u^k z T)^{-1}$$

Whenever $|z| < 1$. Since each summand on the right hand side has positive real part (because $w(u^k T) \leq 1$), it follows that the left hand side has positive real part, and that implies that $w(T^n) \leq 1$. In our argument above, to prove an identity between operators by substitution into an identity between rational functions we make use of the functional calculus for rational functions. Explicitly: If Q_1 and Q_2 are rational functions whose poles are not in Sp(T), so that $Q_1(T)$ and $Q_2(T)$ make sense, then the same is true of each polynomial p in Q_1

and Q_2. If $\phi(\lambda) = p(Q_1(\lambda), Q_2(\lambda))$ then $\phi(T) = p(Q_1(T), Q_2(T))$.

1.6 Unbounded operators in a Hilbert space

When we try to extend the concept of numerical range to unbounded operators, we must proceed cautiously. First, we can define $\langle Tx,x\rangle$ for x in $D(T)$, the domain of T. Hence $W(T) = \{\langle Tx,x\rangle : x \text{ is in } D(T) \text{ and } ||x|| = 1\}$. Thus $W(T)$ if there exists a x in $D(T)$ such that $||x|| = 1$ and $\langle Tx,x\rangle = \lambda$. We are aware that for an operator T from H to H with domain as a linear subspace $D(T)$, the adjoint T^* can be defined in a unique way if and only if $D(T)$ is dense in H and then

$\langle Tx,y\rangle = \langle x, T^*y\rangle$ for each x in $D(T)$ and y in $D(T^*)$, where

$D(T^*) = \{$ y in H for which there exists a $y^* \in H$ such that $\langle Tx,y\rangle = \langle x,y^*\rangle$ for each x in $D(T)$ $\}$.

It is well-known that T^* is closed and that $N(T^*) = R(T)^{\perp}$. Moreover for any α in $\mathbb{C}$, $(\alpha T)^* = \bar{\alpha}\, T^*$ if $\alpha \neq 0$ and $(T + \lambda I)^* = T^* + \bar{\lambda} I$.

Suppose T is a closed linear operator in H with domain $D(T)$ dense in H. Then $\lambda \in Sp(T)$ implies that either $T - \lambda I$ is not bounded from below or $R\,\overline{(T - \lambda I)} \neq H$. If $T - \lambda I$ is not bounded from below, then either

there is an x in $D(T)$ with $||x|| = 1$ such that

$$Tx = \lambda x \tag{1.24}$$

or

We have a sequence (x_n) of elements of $D(T)$ such that $||x_n|| = 1$ and $(T - \lambda I)\, x_n \xrightarrow{s} o$ as $n \to \infty$ (1.25)

In case $\overline{R(T - \lambda I)} \neq H$, then $N(T^* - \bar{\lambda} I) \neq \{o\}$ i.e. there is an $x \in D(T^*)$ such that

$$||x|| = 1 \text{ and } T^*x = \bar{\lambda} x$$

In case of (1.24) we have $\langle Tx,x\rangle = \lambda$ i.e. λ is in $W(T)$ since x is in $D(T)$ and $||x|| = 1$.

In case of (1.25) we have

$$\langle Tx_n , x_n\rangle \to \lambda \text{ as } n \to \infty$$

Since x_n is in $D(T)$ and $||x_n|| = 1$ for all $n \in \mathbb{N}$, We get $\lambda \in \overline{W(T)}$ in this case. However (1.26) implies that $\langle T^*x,x\rangle = \bar{\lambda}$ for an x in $D(T^*)$ with $||x|| = 1$. This does not imply that x is in $D(T)$ (of course if x is in $D(T)$ also, then $\langle Tx,x\rangle = \lambda$, in which case we have $\lambda \in W(T)$).

Thus if $T - \lambda I$ is not bounded from below, then λ is in $\overline{W(T)}$. Hence $\lambda \notin \overline{W(T)}$ implies that $T - \lambda I$ is bounded from below, i.e. $T - \lambda I$ is one - to - one and $R(T - \lambda I)$ is closed in H. We have thus proved

Proposition 1.42: If T is closed, densely defined operator on H and $\lambda \notin \overline{W(T)}$ then $T - \lambda I$ is one-to-one and $R(T - \lambda I)$ is closed in H.

In particular, it may very well happen that $Sp(T)$ is not contained in $\overline{W(T)}$. As we see from proposition 1.42, the fault would be in $R(T - \lambda I)$ not being large enough. The question thus arises as to whether or not we can enlarge D(T) in such a way to make $Sp(T) \subseteq \overline{W(T)}$. We see that this, indeed, can be done under very general circumstances.

In the sequel, as usual H is a complex Hilbert space. Let $D(\phi)$ be a linear subspace of H and $\phi : D(\phi) \times D(\phi) \dashrightarrow \mathbb{C}$ be a sesquilinear functional or form. We write $\hat{\phi}(x)$ in place of $\phi(x,x)$ where x is in $D(\phi)$ and call $D(\phi)$ as the domain of ϕ.

Definition 1.43: The numerical range $W(\phi)$ of the sesquilinear form $\phi : D(\phi) \times D(\phi) \dashrightarrow \mathbb{C}$, is the set of scalars

$$\{ \hat{\phi}(x) : x \text{ is in } D(\phi),\ ||x|| = 1 \}.$$

With any densely defined sesquilinear form $\phi(x,y)$ we associate a linear transformation T on H as follows: Let x be in D^*, if x is in $D(\phi)$ and there is a constant c such that

$$|\phi(x,y)| < c\, ||y|| \quad \text{for all } y \text{ in } D(\phi)\ , \tag{1.27}$$

define $\phi_x(y) = \overline{\phi(x,y)}$. Then by (1.27), for each x in D^*, ϕ_x is a bounded linear functional on $D(\phi)$. Since $\overline{D(\phi)} = H$, ϕ_x can be extended with preservation of norm to a unique element of H^*. Call this extension $\hat{\phi}_x$. (Thus $\hat{\phi}_x |_{D(\phi)} = \phi_x$ and

$\| \hat{\phi}_2 \| = \| \phi_2 \|$). By the Riesz Representation Theorem, there exists (unique) x' in H such that

$$\hat{\phi}_2(y) = \langle y, x' \rangle \text{ for all } y \text{ in } H \tag{1.28}$$

i.e. $\overline{\phi(x,y)} = \langle y, x' \rangle$ for all y in $D(\phi)$.

Hence $\phi(x,y) = \langle x', y \rangle$ for x in D^* and y in $D(\phi)$.(1.29)

The uniqueness of x' in (1.29) (for a given x in D^*) follows from the denseness of $D(\phi)$. Indeed if $\phi(x,y) = \langle x'', y \rangle$ also, then $\langle x' - x'', y \rangle = o$; since $D(\phi)$ is dense in H, we get $x'' = x'$. Define a map $T: D^* \dashrightarrow H$ by $Tx = x'$. Then T is linear. Indeed, if x_1, x_2 are in D^* and α_1, α_2 are in $\mathbb{C}$, then $\phi(x_1, y) = \langle Tx_1 \; y \rangle$, $\phi(x_2 \; y) = \langle Tx_2, y \rangle$ for y in $D(\phi)$. We must first show that $\alpha_1 x_1 + \alpha_2 x_2$ is in D^*. This follows since $\alpha_1 x_1 + \alpha_2 x_2$ is in $D(\phi)$ and by (1.27) for constants c_1 and c_2 we have

$$|\phi(x_1, y)| \leq c_1 \|y\| \quad \text{and} \quad |\phi(x_2, y)| \leq c_2 \|y\|$$

for all y in $D(\phi)$.

Thus by sesquilinearity of ϕ,

$$\begin{aligned} |\phi(\alpha_1 x_1 + \alpha_2 x_2, y)| &= |\alpha_1 \phi(x_1, y) + \alpha_2 \phi(x_2, y)| \\ &\leq |\alpha_1| \, |\phi(x_1, y)| + |\alpha_2| \, |\phi(x_2, y)|. \\ &\leq (|\alpha_1| + |\alpha_2|) \, c \|y\| \text{ for all } y \text{ in } D(\phi). \end{aligned}$$

Where $c = \max \{c_1 \; c_2\}$, which shows that

$\alpha_1 x_1 + \alpha_2 x_2$ is in D^* i.e. D^* is a linear subspace of H. Consequently

$$\phi(\alpha_1 x_1 + \alpha_2 x_2, y) = \langle T(\alpha_1 x_1 + \alpha_2 x_2), y\rangle$$

But the left side is $\alpha_1\langle Tx_1, y\rangle + \alpha_2\langle Tx_2, y\rangle$ i.e. $\langle \alpha_1 Tx_1 + \alpha_2 Tx_2, y\rangle$; hence

$\langle T(\alpha_1 x_1 + \alpha_2 x_2), y\rangle = \langle \alpha_1 Tx_1 + \alpha_2 Tx_2, y\rangle$ for all y in $D(\phi)$. Since $D(\phi)$ is dense in H, we get

$T(\alpha_1 x_1 + \alpha_2 x_2) = \alpha_1 Tx_1 + \alpha_2 Tx_2$ for all x_1, x_2 in D^* and $\alpha_1, \alpha_2 \in \mathbb{C}$ in and linearity of T follows. Thus $D(T) = D^*$.

We shall call the operator T as the operator **associated** with the sesquiliner form $\phi(x,y)$.

Next we prove the following

Proposition 1.44: Let $\phi: H \times H \dashrightarrow \mathbb{C}$ be a densely defined sesquilinear form with associated operator T. Then

(a) If λ is not in $\overline{W(\phi)}$, then $T - \lambda I$ is one-to-one and

$$\|x\| < c\, \|(T - \lambda I)x\|, \quad x \in D(T) \qquad (1.29)$$

(b) If λ is not in $\overline{W(\phi)}$ and T is closed, then $R(T - \lambda I)$ is closed in H.

Proof:

(a) Since λ is not in $\overline{W(\phi)}$, there is a $\delta > 0$ such that

$$|\hat{\phi}(x) - \lambda| \geq \delta \text{ for all } x \text{ in } D(\phi) \text{ with } \|x\| = 1 \qquad (1.30)$$

Thus

$$|\hat{\phi}(x) - \lambda \|x\|^2| \geq \quad \|x\|^2 \text{ for } x \text{ in } D(\phi) \qquad (1.31)$$

Now if x is in D(T) and $(T - \lambda I)x = z$, then

$$\phi(x,y) - \lambda\langle x,y\rangle = \langle Tx,y\rangle - \lambda\langle x,y\rangle$$
$$= \langle z,y\rangle \text{ for all } y \text{ in } D(\phi) \qquad (1.32)$$

In particular

$$\hat{\phi}(x) - \lambda||x||^2 = \langle z,x\rangle; \text{ So}$$
$$|\hat{\phi}(x) - \lambda||x||^2| \leq ||z||\,||x|| \qquad (1.33)$$

Combining this with (1.31) we get

$$||x|| < \frac{||z||}{\delta}$$

which is clearly (1.29). Now (1.29) implies that $T - \lambda I$ is one-to-one. Hence we have proved (a).

(b): To prove (b) we note from (a) that since λ is not in $W(\phi)$, so $T - \lambda I$ is one-to-one and bounded from below. Since T is closed, so is $T - \lambda I$ and hence $R(T - \lambda I)$ is closed in H. (We know the result: If X is a Banach space and T is a closed linear operator in X (with domain D(T)) and T is bounded from below, then R(T) is closed in X).

We recollect that a sesquilinear form $\phi(x,y)$ is symmetric if $\phi(x,y) = \overline{\phi(y, x)}$ for all x,y in $D(\phi)$. An important property of symmetric forms is given by

Lemma 1.45: Let $\phi(x,y)$ and $\psi(x,y)$ be symmetric sesquilinear forms satisfying

$$|\hat{\phi}(x)| \leq M\hat{\psi}(x) \text{ for all } x \in D(\phi) \cap D(\psi) \quad (1.34)$$

(M a constant)

Then

$$|\phi(x, y)|^2 \leq M^2 \hat{\psi}(x)\,\hat{\psi}(y).$$

Proof: Assume first that $\phi(x,y)$ is real. Then

$$\hat{\phi}(x \pm y) = \hat{\phi}(x) \pm 2\,\phi(x,y) + \hat{\phi}(y). \quad (1.36)$$

and hence $4\,\phi(x,y) = \hat{\phi}(x+y) - \phi(x-y)$. Thus

$$4|\phi(x,y)| < M[\psi(x+y) + \psi(x-y]$$

$$= 2M[\hat{\psi}(x) - \hat{\psi}(y)]$$

by (1.34) and (1.36). Replacing x by x and y by y/α, α real and $\neq o$, we get

$$2|\phi(x, y)| \leq M[\alpha^2\hat{\psi}(X) + \frac{\hat{\psi}(y)}{\alpha^2}]. \quad (1.37)$$

If $\hat{\psi}(x) = o$, we let $\alpha \dashrightarrow \infty$, showing that $\phi(x,y) = o$. In this case (1.35) holds trivially. If $\hat{\psi}(y) = o$, we let $\alpha \dashrightarrow o$. In this case as well, $\phi(x,y) = o$ and (1.35) holds. If neither vanishes, set $\alpha^4 = \frac{\hat{\psi}(y)}{\psi(x)}$

This gives (1.35). If $\phi(x,y)$ is not real, then $\phi(x,y) = e^{i\theta}|\phi(x,y)|$. Hence $\phi(e^{i\theta}x,y)$ is real.

Applying (1.35) to this case we have

$$|\phi(e^{-i\theta}x,y)|^2 < M^2\hat{\psi}(e^{i\theta}x)\,\hat{\psi}(y)$$

This implies (1.35) for x and y .

Corollary 1.46: If $\psi(x,y)$ is symmetric but $\phi(x,y)$ is not and (1.34) holds, then

$$|\phi(x,y)|^2 < 4M\ \hat{\psi}(x)\ \hat{\psi}(y), \quad x,\ y \text{ in } D(\phi) \cap D(\psi) \tag{1.38}$$

Proof: Set $\phi_1(x,y) = \frac{1}{2}[\phi(x,y) + \overline{\phi(x,\ y)}]$,

$$\phi_2(x,y) = \frac{1}{2i}[\phi(x,y) - \phi(x,y)] \tag{1.40}$$

Then ϕ_1 and ϕ_2 are symmetric sesquilinear forms, and

$$\phi(x,y) = \phi_1(x,y) + i\ \phi_2(x,y) \tag{1.41}$$

(The forms ϕ_1 and ϕ_2 are known as the real and imaginary parts of ϕ, respectively. Note that in general, they are not real valued). Now by (1.34)

$$|\hat{\phi}_j(x)| \leq M\ \hat{\psi}(x),\ j = 1,2, \quad x \text{ in } D(\phi) \cap D(\psi).$$

Hence (see proof of Lemma 1.45)

$$|\phi_j(x,y)| \leq M(\hat{\psi}(x))^{1/2}\ (\hat{\psi}(y))^{1/2}.$$

i.e.

$$\phi(x,y) \leq 2\ M(\hat{\psi}(x))^{1/2}\ (\hat{\psi}(y))^{1/2}$$

Which implies (1.38).

Corollary 1.47 If $\psi(x,y)$ is symmetric sesquilinear form such that $\hat{\psi}(x) \geq 0$ for all x in $D(\psi)$, then

$$|\psi(x,y)|^2 \leq \hat{\psi}(x)\ \hat{\psi}(y) \text{ for all } x,\ y \text{ in } D(\psi) \tag{1.42}$$

and

$$(\hat{\psi}(x+y))^{1/2} \le (\hat{\psi}(x))^{1/2} + (\hat{\psi}(y))^{1/2}, \quad x, y \in D(\psi) \qquad (1.43)$$

Proof: We have $|\hat{\psi}(x)| = \hat{\psi}(x)$ for all $x \in D(\psi)$. Setting $\phi(x,y) = \psi(x,y)$ in Lemma 1.45, we get (1.42) at once.

Lastly,

$$\begin{aligned} \hat{\psi}(x+y) &= \hat{\psi}(x) + \psi(x,y) + \hat{\psi}(y) + \psi(y,x) \\ &\le \hat{\psi}(x) + 2\,(\hat{\psi}(x))^{1/2}\,(\hat{\psi}(y))^{1/2} + \hat{\psi}(y) \\ &= [(\hat{\psi}(x))^{1/2} + (\hat{\psi}(y))^{1/2}]^2, \text{ by } (1.42). \end{aligned}$$

This proves (1.43).

In passing we note the following criteria for recorgnizing a symmetric sesquilinear form. (We are assuming a complex Hilbert space).

Proposition 1.48: The following statements are equivalent for a sesquilinear form.

(i) $\phi(x,y)$ is symmetric

(ii) $\mathrm{Im}\,\hat{\phi}(x) = 0$ for all x in $D(\phi)$.

(iii) $\mathrm{Re}\,\phi(x,y) = \mathrm{Re}\,\phi(y,x)$ for all x, y in $D(\phi)$.

Proof: Trivially (i) implies (ii) To show that (ii) implies (iii) note that

$$\hat{\phi}(ix+y) = \hat{\phi}(x) + i\,\phi(x,y) - i\,\phi(y,x) + \hat{\phi}(y).$$

Taking the imaginary parts of both sides and using (ii) we get (iii). To prove that (iii) implies (i), observe that by (iii).

$$\begin{aligned} \operatorname{Im} \phi(x,y) &= \operatorname{Im} (-i)\, \phi(ix,y) \\ &= - \operatorname{Re} \phi(ix,y) \\ &= - \operatorname{Re} \phi(y,ix) \\ &= - \operatorname{Re} (-i)\, \phi(y,x) \\ &= - \operatorname{Im} \phi(y,x). \end{aligned}$$

This, together with (iii) gives (i) .

Proposition 1.49: The numerical range of a sesquilinear form is a convex set in the plane.

Proof: The proof is identical to that of Proposition 1.7

Lemma 1.50 A convex set in the plane which is not the whole plane is contained ina half-plane.

Proof: We give an elementary proof. Let V be a convex set in the plane which is not the whole plane. Then $\overline{V}$ cannot be the whole plane either. (This result is intuitively obvious and can also be easily proved).

Let P be a point not in $\overline{V}$. If V is empty, there is nothing to prove. Otherwise, some ray from P intersects $\overline{V}$. The first part of $\overline{V}$ encountered by the ray is boundary point of V.

Let U be the collection of all rays from P which intersect $\overline{V}$ at a point not equal to P, and let M be the set of all points on these rays. Clearly M is convex set. For if S_1 and S_2 are points in M, then the rays PS_1, and PS_2 contain points T_1 and T_2 of $\overline{V}$ respectively. Since $\overline{V}$ is convex, the segment $T_1 T_2$ is in $\overline{V}$ and hence all the rays between PS_1 and PS_2 are in M. This includes the points on the segment $S_1 S_2$. Now a convex set of points consisting of rays from P is merely an angle θ with vertex at P. Clearly, $\theta < \pi$ for otherwise one would have a line segment lying outside M containing points of M. If one extends either of the rays forming the sides of M, one obtains a half-plane free of $\overline{V}$.

Lemma 1.51: If V is a closed convex set in the plane which is not the whole plane, a half plane, a strip or a line, then V is contained in an angle of the form

$$|\arg(z - z_0) - \theta_0| \le \theta < \frac{\pi}{2} \qquad (1.44)$$

(Here z_0 is the vertex of the angle θ_0 = amp z_0 and θ is a constant.)

Proof: We saw in the proof of lemma 1.50, that there is a half plane U containing V such that the boundary line L of U contains a point P of V. The line L is called a Support Line for V. Suppose $L \subseteq V$. If V is not a half-plane or a line, then the interior of U contains a boundary point Q of V (see proof of Lemma 1.50). A support line L_1 through Q cannot intersect L for then we would have points of V on both sides of L_1 (namely, those on L). Hence L_1 must be parallel to L. Moreover, every point in the strip between L and L_1 is on the segment connecting Q and a point on L. This would mean that V is a strip, contrary to assumption. Hence L must contain a point R not in V. Consequently, the ray emitted from R away from P must be free of V.

For simplicity, assume that this ray is the positive real axis, R is the origin, P is the point $-c$, $c > o$, and U is the upper half-plane. We can make the assertion that there is a $\delta > o$ such that V is contained in the angle $\delta \leq \theta \leq \pi$. This would give us exactly what we want. If this were not true, then, for each $\varepsilon > o$, there would be a point z in V such that

$$o < c \, \mathrm{Im}\, z < \varepsilon \, \mathrm{Re}\, z .$$

The segment connecting P and z must be in V. Moreover, the distance from this segment to the origin is less than

$$d = \frac{c \, \mathrm{Im} z}{c + \mathrm{Re} z} < \frac{\varepsilon \, \mathrm{Re}\, z}{c + \mathrm{Re} z} < \varepsilon .$$

(The quantity d is just the distance from the origin to the point where the segment intersects the imaginary axis. Use similar triangles). Since this is true for any $\varepsilon > o$, we see that there is a sequence of points of V converging to the origin, i.e. to R. Since V is closed we must have R in V. But we took R not to be in V. This contradiction proves the Lemma.

Some simple consequences of proposition 1.49 and Lemma 1.51 are

Corollary 1.52: Let ϕ be sesquilinear functional with domain $D(\phi)$. If $\overline{W(\phi)}$ is not the whole plane, a half-plane, a strip or a line, then there are constants r, k, k_c such that $|r| = 1$, $k > 0$, k_c is real and

$$|\hat{\phi}(x)| \leq k[\mathrm{Re}\, r\, \phi(x) + k_c \|x\|^2],$$

$$x \text{ is in } D(\phi) \qquad (1.45)$$

Proof: By proposition 1.49 $W(\phi)$ is convex set; hence, so is $\overline{W(\phi)}$. By lemma 1.51, $W(\phi)$ must satisfy (1.44) for some z_0, θ_0, θ. Now (1.44) is equivalent to

$$|\mathrm{Im}\{e^{-i\theta_0}[\hat{\phi}(x) - z_0]\}| \leq$$

$$\tan\theta\, \mathrm{Re}\{e^{-i\theta_0}[\hat{\phi}(x) - z_0]\} \qquad (1.46)$$

Set $r = e^{-i\theta_0}$. Inequality (1.46) implies

$$|\mathrm{Im}\, r\hat{\phi}(x)| \leq \tan\theta\,[\mathrm{Re}\, r\, \hat{\phi}(x) + k_c],$$

$$\|x\| = 1,\ x \in D(\phi),$$

where

$$k_c = \frac{|\mathrm{Re}\, rz_0| + |\mathrm{Im}\, rz_0|}{\tan\theta}$$

This implies (1.45) with $k = 1 + \tan\theta$

Proposition 1.53 Let $\phi(x,y)$ be a sesquilinear form such that $W(\phi)$ is not the whole plane, a half-plane, a strip or a line. Then there is a symmetric sesquilinear form $\psi(x,y)$ with $D(\phi) = D(\psi)$ such that there is a constant C satisfying

$$C^{-1} | \hat{\phi}(x) | \leq \hat{\psi}(x) \leq | \hat{\phi}(x) | + C||x||^2 ,$$
$$x \text{ in } D(\phi) \qquad (1.47)$$

In particular, if (x_n) is a sequence of points of $D(\phi)$, x in $D(\phi)$, $x_n \xrightarrow{s} x$ and $\phi(x_n - x) \to o$, then

$$\phi(x_n,y) \to \phi(x,y) \text{ for all } y \text{ in } D(\phi) \qquad (1.48)$$

Proof: By corollary 1.52, there are constants r,k, k_o such that (1.45) holds. Let $\psi_1(x,y)$ be the real part of the sesquilinear functional $r\phi(x,y)$ (see (1.40) in the proof of corollary 1.46) and set

$$\psi(x,y) = \psi_1(x,y) + k_o \langle x,y \rangle \qquad (1.49)$$

Then by (1.45)

$$| \hat{\phi}(x) | \leq k \hat{\psi}(x) \text{ for all } x \text{ in } D(\phi)$$

Moreover by (1.49)

$$\hat{\psi}(x) = \mathrm{Re}\, r\, \hat{\phi}(x) + k_o ||x||^2 \leq | \hat{\phi}(x) + |k_o| \; ||x||^2$$

Thus (1.47) holds. To prove (1.48) note that by corollary 1.46

$$\begin{aligned} | \phi(x_n,y) - \phi(x,y)|^2 &= | \phi(x_n - x,y)|^2 \\ &\leq 4C^2\hat{\psi}(x_n - x)\, \psi(y) \\ &\leq 4C^2 \hat{\psi}(y)[| \phi(x_n - x)| \\ &+ C||x_n - x||^2] \\ &\to o \text{ as } n \to \infty . \end{aligned}$$

1.6.1 Closed Forms:

Definition 1.54: A sesquilinear form $\phi(x,y)$ will be called closed if the conditions

$x_n \in D(\phi)$; $x_n \overset{s}{\dashrightarrow} x$ in H, $\phi(x_n - x_m) \dashrightarrow o$ as $m, n \dashrightarrow \infty$. imply that

$x \in D(\phi)$ and $\phi(x_n - x) \dashrightarrow o$ as $n \dashrightarrow \infty$

Proposition 1.55: Let $\phi(x,y)$ be a closed sesquilinear form such that $W(\phi)$ is not a plane half-plane, a strip or a line and such that o is not in $\overline{W(\phi)}$. Then for each linear functional F on $D(\phi)$ satisfying

$$| F(y) |^2 \leq C \, | \hat{\phi}(y) | \quad \text{for all } y \text{ in } D(\phi) \qquad (1.50)$$

there are unique elements z, x in $D(\phi)$ such that

$$F(y) = \phi(y,z) \quad \text{for all } y \text{ in } D(\phi) \qquad (1.51)$$

and

$$F(y) = \overline{\phi(x,y)} \text{ for all } y \text{ in } D(\phi) \qquad (1.52)$$

Proof: Since $\phi(x,y)$ satisfies the hypotheses of proposition 1.53, we know that there is a symmetric sesquilinear form $\psi(x,y)$ such that $D(\psi) = D(\phi)$ and (1.47) holds. Moreover, since o is not in $\overline{W(\phi)}$, there is a $\delta > o$ such that

$$| \hat{\phi}(x) | \geq \delta > o \text{ for all } x \text{ in } D(\phi),$$
$$||x|| = 1 \qquad (1.53)$$

Hence

$$| \hat{\phi}(x) | \geq \delta ||x||^2 \text{ for all } x \in D(\phi) \qquad (1.54)$$

Thus, there are constrants M_1 , M_2 , M_3 such that

$$||x||^2 \leq M_1 \hat{\psi}(x) < M_2 \, | \hat{\phi}(x) | \leq M_3 \hat{\psi}(x)$$

for all x in $D(\phi)$ (1.55)

Let X be the vector space $D(\phi)$ equipped with the scalar product $\psi(x,y)$. We can show that X is a Hilbert space. The only property that needs verification is completeness.

Let (x_n) be strongly fundamental i.e. suppose $\psi(x_n - x_m) \to o$ as m, n --> ∞ . By (1.55) (x_n) is strongly fundamental in H and $\phi(x_n - x_m) \to o$ as m, n --> ∞ . Since H is complete, there is an element x in H such that $x_n \xrightarrow{S} x$ as n --> ∞ . Since $\phi(x,y)$ is a closed sesquilinear form, we know that x is in $D(\phi)$ and $\phi(x_n - x) \to o$ as n --> ∞ . Thus $\psi(x_n - x) \to o$ by (1.55) and X is complete

Now for each $z \in X$, the map G defined by

$Gz = \phi(y,z)$ for all y in X (1.56)

is a linear functional on X. It is bounded by (1.55) and corollary 1.46. Hence by Riesz - Representation theorem, there is an element Sz in X such that

$Gy = \psi(y,Sz)$ for all y in X. (1.57)

clearly S is a linear mapping of X into itself. It is bounded since $(\hat{\psi}(Sz))^2 = (\phi(Sz,z))^2$

$$\leq 4 M_3^2 \hat{\psi}(Sz) \hat{\psi}(z)$$

Whence

$$\hat{\psi}(Sz) \leq 4\ M_3{}^2\hat{\psi}(z) \qquad (1.58)$$

It is also one-to-one and has closed range, since $M_1{}^2(\hat{\psi}(z))^2 \leq M_2{}^2 \mid \hat{\phi}(z) \mid^2 = M_2{}^2 \mid \psi(z,Sz) \mid^2 \leq M_2{}^2\hat{\psi}(z) \cdot \hat{\psi}(Sz)$, by (1.56), (1.57) and (1.58), this gives

$$\hat{\psi}(z) \leq \frac{M_2{}^2\ \psi(Sz)}{M_1{}^2} \qquad (1.59)$$

We can show that R(S) = X. To see this, let u be any element of X orthogonal to R(S), i.e. satisfying

$\psi(u,Sz) = o$ for all z in X, Then by (1.56) and (1.57)

$\phi(u,z) = o$ for all z in X.

In particular, this holds for z = u, showing $\hat{\phi}(u) = o$, which implies u = o by (1.55). Now let F be any functional on $D(\phi)$ satisfying (1.50). Then F is a bounded linear functional on X by (1.55). Hence by Riesz-Representation theorem, there is an element z in X such that

$Fy = \psi(y,z)$ for all y in X

Since S is one-to-one and onto, there is a $u \in X$ such that Su = z

Hence

$F(y) = \psi(y,z) = \psi(y,Su) = \phi(y,u)$ for all y in X.

Which proves (1.51). The proof of (1.52) is almost identical

The importance of closed sesquilinear forms may be seen from the next

Proposition 1.56 Let $\phi(x,y)$ be densely defined closed sesquilinear form with associated operator T. If $\overline{W(\phi)}$ is not the whole plane, a half plane, a strip or a line, then T is closed and

$$Sp(T) \subseteq \overline{W(\phi)} = \overline{W(T)} \qquad (1.60)$$

Proof: To see that T is closed, suppose (x_n) is a sequence of elements of D(T) and $x_n \xrightarrow{s} x$, $Tx_n \xrightarrow{s} z$ in H. Then

$$|\hat{\phi}(x_n - x_m)| = |\langle T(x_n - x_m), x_n - x_m\rangle|$$
$$< ||Tx_n - Tx_m|| \; ||x_n - x_m|| \to o,$$
$$\text{as } m, n \to \infty.$$

Since ϕ is closed, this implies x is in $D(\phi)$ and that $\phi(x_n - x) \to o$. Thus by proposition 1.53,

$$\phi(x_n, y) \to \phi(x,y) \text{ as } n \to \infty, \; y \text{ in } D(\phi) \qquad (1.61)$$

Since $\phi(x_n, y) = \langle Tx_n, y\rangle$ for all y in $D(\phi)$, we have in the limit

$$\phi(x,y) = \langle z,y\rangle \text{ for all } y \text{ in } D(\phi) \qquad (1.62)$$

showing that x is in D(T) and Tx = z. Thus T is a closed operator. To show that $Sp(T) \subseteq \overline{W(\phi)}$, let λ be any scalar not in $\overline{W(\phi)}$. Then by proposition 1.44 (a), $T- \lambda I$ is one-to-one and for all x in D(T) we have $||x|| < C||(T - \lambda I)x||$ for a constant C and $|\hat{\phi}(x) - \lambda||x||^2| \geq \delta||x||^2$. Set $\phi_\lambda(x,y) = \phi(x,y) - \lambda\langle x,y\rangle$. Then ϕ_λ satisfies the hypotheses of proposition 1.55. If z is any element of H, then $\langle y,z\rangle$ is a linear functional on $D(\phi_\lambda) = D(\phi)$ and $|\langle y, z\rangle|^2 \leq ||y||^2\ ||z||^2 \leq C|\ \phi_\lambda(y)|$ by relation (1.31).

Hence by proposition 1.55, there is an x in H such that

$$\phi_\lambda(x,y) = \langle z,y\rangle \text{ for all } y \text{ in } D(\phi) \qquad (1.63)$$

This shows that $x \in D(T)$ and $(T - \lambda I)x = z$. Since z was any element of H, we see that $R(T - \lambda I) = H$, and consequently, λ is in $Sp(T)$.

Lastly in order to prove $\overline{W(\phi)} = \overline{W(T)}$, we shall show that

$$W(T) \subseteq W(\phi)' \subseteq \overline{W(T)} \qquad (1.64)$$

The first inclusion is obvious since x is in D(T) implies $\langle Tx,x\rangle = \hat{\phi}(x)$. To prove the second, we must show that for each x in $D(\phi)$, there is a sequence (x_n) of elements of D(T) such that $\hat{\phi}(x_n) \dashrightarrow \hat{\phi}(x)$. Let $\psi(x,y)$ be a symmetric sesquilinear form satisfying (1.47). Then by corollary 1.46

$$\begin{aligned} |\ \hat{\phi}(y) - \hat{\phi}(x)\ | &\leq |\ \phi(y-x,y)| + |\ \phi(x,y-x)| \\ &\leq 2\ C[(\hat{\psi}(y))^{1/2} + (\hat{\psi}(x))^{1/2}]\ (\hat{\psi}(y-x))^{1/2} \end{aligned}$$

Thus it suffices to show that for each x in $D(\phi)$ there is a sequence (x_n) of elements of D(T) such that $\dot{\psi}(x_n - x) \dashrightarrow 0$.

Now consider $D(\phi)$ as a vector space with scalar product $\psi(x,y) + \langle x,y\rangle$. This makes $D(\phi)$ into a normed linear space with norm $[\hat{\psi}(x) + ||x||^2]^{\frac{1}{2}}$. (Actually, X is a Hilbert space). We want to show that D(T) is dense in X. If it were not, then there would be an element u in X with positive distance from D(T). Then by the Hahn-Banach theorem, there would be a bounded linear functional $F \neq o$ on X which annihilates D(T). Let λ be any scaler in $\overline{W(\phi)}$; and define $\phi_\lambda(x,y)$ as above. Then ϕ_λ satisfies the hypotheses of proposition 1.55. and by relation (1.31) and (1.47)

$$|F(y)|^2 \leq k[\hat{\psi}(y) + ||y||^2] \leq k' \, |\phi_\lambda(y)|,$$

for all y in X.

Thus, by proposition 1.55, there is a u in X such that

$$F(y) = \phi_\lambda(y,u) \text{ for all } y \text{ in } X \qquad (1.65)$$

Since F annihilates D(T), we have

$$\phi_\lambda(y,u) = o \quad \text{for all } y \text{ in } D(T).$$

This is equivalent to

$$\langle (T - \lambda I)\, y,\, u\rangle = o \text{ for all } y \text{ in } D(T).$$

However we have just shown that $R(T - \lambda I) = H$, so that there is a y in D(T) such that $(T - \lambda I)\, y = u$. This shows that u = o, which, by (1.35) implies that F = o, providing a contradiction. Hence D(T) is dense in X and the proof is over

1.6.2 Closed extensions:

Let T be a linear operator on a Hilbert space H. We saw earlier that we may not have $Sp(T) \subseteq \overline{W(T)}$. In this subsection, we shall concern ourselves with the question of when T can be extended to an operator having this property.

It follows immediately from proposition 1.49 that W(T) is convex.

Definition 1.57: A sesquilinear form $\psi(x,y)$ is called an **extension** of a sesquilinear from $\phi(x,y)$ if $D(\psi)$ contains $D(\phi)$ and $\psi(x,y) = \phi(x,y)$ for each, x,y in $D(\phi)$. A set U will be called **dense** in $D(\phi)$ if for each z in $D(\phi)$ and each $\varepsilon > o$ there is a x in U such that

$$\phi(z - x) < \varepsilon \quad \text{and} \quad ||z - x|| < \varepsilon$$

Proposition 1.58: Let $\phi(x,y)$ be a densely defined sesquilinear form such that $\overline{W(\phi)}$ is not the whole plane a half-plane, a strip or a line. Suppose that x_n is in $D(\phi)$, $x_n \xrightarrow{s} o$, $\phi(x_n - x_m) \longrightarrow o$ imply $\hat{\phi}(x_n) \longrightarrow o$. Then $\phi(x,y)$ has a closed extension $\phi_e(x,y)$ such that $D(\phi)$ is dense in $D(\phi_e)$ and $W(\phi) \subseteq W(\phi_e) \subseteq \overline{W(\phi)}$

Proof: Define $\phi_e(x,y)$ as follows: x is in $D(\phi_e)$ If there is a sequence (x_n) of elements of $D(\phi)$ such that $\phi(x_n - x_m) \longrightarrow o$ and $x_n \xrightarrow{s} x$ in H. If (x_n) is such that a sequence for x and (y_n) is such a sequence for y, then define

$$\phi_e(x,y) = \lim_{n \longrightarrow \infty} \phi(x_n, y_n). \qquad (1.66)$$

This limit exists. To see this note that

$$\phi(x_n, y_n) - \phi(x_m, y_m) = \phi(x_n, y_n - y_m) + (x_n - x_m, y_m).$$

Now, by proposition 1.53, there is a symmetric sesquilinear form satisfying (1.47). Hence by corollary 1.46,

$$| \phi(x_n, y_n) - \phi(x_m, y_m)| \le 2C[\hat{\psi}(x_n)^{1/2} (\hat{\psi}(y - y))^{1/2} + (\hat{\psi}(x_n - x_m))^{1/2} (\hat{\psi}(y_m))^{1/2}]$$

This converges to zero by (1.47). Moreover the limit in (1.66) is unique (i.e. it does not depend on the particular sequence chosen). For let (x'_n) and (y'_n) be other sequences for x and y, respectively. Set $x''_n = x'_n - x_n$, $y''_n = y'_n - y_n$,

Then

$$(\hat{\psi}(x''_n - x''_m))^{1/2} \le (\hat{\psi}(x'_n - x'_m))^{1/2} + (\hat{\psi}(x_n - x_m))^{1/2}$$

$\to 0$ as m, n $\to \infty$ by proposition 1.53,

Thus $\hat{\phi}(x''_n - x''_m) \to 0$ and similarly $\hat{\phi}(y''_n - y''_m) \to 0$. Since $x''_n \xrightarrow{s} 0$, $y''_n \xrightarrow{s} 0$ in H, we may conclude, by hypotheses that

$$\hat{\phi}(x''_n) \to 0, \quad \hat{\phi}(y''_n) \to 0 \text{ as } n \to \infty ,$$

which implies

$$\hat{\psi}(x''_n) \to 0, \quad \hat{\psi}(y''_n) \to 0 \text{ as } n \to \infty .$$

Hence

$$| \phi(x'_n, y'_n) - \phi(x_n, y_n)| \le | \phi(x'_n, y''_n) | + | \phi(x''_n, y_n) | \le 2C[(\hat{\psi}(x'_n))^{1/2} (\hat{\psi}(y''_n))^{1/2} + (\hat{\psi}(x''_n))^{1/2} \cdot (\hat{\psi}(y_n))^{1/2}] \to 0$$

From the way $\phi_e(x,y)$ was defined, it is obvious that

$$W(\phi) \subseteq W(\phi_e) \subseteq \overline{W(\phi)} \qquad (1.67)$$

To show that $D(\phi)$ is dense in $D(\phi_e)$, we see that if x is in $D(\phi_e)$, there is a sequence (x_n) of elements of $D(\phi)$ such that $\hat{\phi}(x_n - x_m) \longrightarrow o$ while $x_n \xrightarrow{s} x$ in H, and $\hat{\phi}(x_n) \longrightarrow \phi_e(x)$. In particular, for each n

$$\hat{\phi}(x_n - x_m) \longrightarrow \hat{\phi}_e(x_n - x) \text{ as } m \longrightarrow \infty$$

Now let $\mathcal{E} > o$ be given and take N so large that $|\hat{\phi}(x_n - x_m)| < \mathcal{E}$ for m, $n > N$. Letting $m \longrightarrow \infty$, we obtain

$$|\hat{\phi}_e(x_n - x)| \leq \mathcal{E} \text{ for } n > N.$$

Which shows that

$$\hat{\phi}_e(x_n - x) \longrightarrow o \text{ as } n \longrightarrow \infty \qquad (1.68)$$

This shows that $D(\phi)$ is dense in $D(\phi_e)$

It only remains to show that ϕ_e is closed. To do this, we note that $\overline{W(\phi_e)}$ is not one of the sets mentioned in proposition 1.53. Thus there is a sysmmetric sesquilinear form $\psi_e(x,y)$ satisfying

$$\frac{1}{C}\,|\hat{\phi}_e(x)| \leq \hat{\psi}_e(x) \leq |\hat{\phi}_e(x)| + C||x||^2 \text{ for all}$$

$$x \text{ in } D(\phi_e) \qquad (1.69)$$

Now suppose (x_n) is a sequence of elements of $D(\phi_e)$, $\hat{\phi}_e(x_n - x_m) \longrightarrow o$ and $x_n \longrightarrow x$ in H. Then by (1.69) $\hat{\psi}_e(x_n - x_m) \longrightarrow o$ as m, $n \longrightarrow \infty$

Now by the density of $D(\phi)$ in $D(\phi_e)$, for each n there is a y_n in $D(\phi)$ such that

$$|\hat{\phi}_e(x_n - y_n)| < \frac{1}{n^2}, \qquad ||x_n - y_n|| < \frac{1}{n}$$

Thus by (1.69),

$$\hat{\psi}_e(x_n - y_n) < \frac{1+C}{n^2} \tag{1.70}$$

Since ψ_e is a symmetric form

$(\hat{\psi}_e(y_n - y_m))^{\frac{1}{2}} \leq (\hat{\psi}_e(y_n - x_n))^{1/2} + (\hat{\psi}_e(x_n - x_m))^{1/2} +$
$(\hat{\psi}_e(x_m - y_m))^{\frac{1}{2}} \to 0$ as $m, n \to \infty$

Hence

$$\hat{\phi}(y_n - y_m) = \hat{\phi}_e(y_n - y_m) \to 0 \text{ as } m, n \to \infty$$

Since

$$||y_n - x|| \leq ||y_n - x_n|| + ||x_n - x|| \to 0 \quad n \to \infty$$

We see that $x \in D(\phi_e)$ and $\hat{\phi}_e(y_n - x) \to 0$ as $n \to \infty$ by (1.68). Hence $\hat{\psi}_e(y_n - x) \to 0$ as $n \to \infty$, which implies by (1.70)

$$\hat{\psi}_e(x_n - x) \to 0 \quad \text{as} \quad n \to \infty .$$

Which in turn implies

$$\hat{\phi}_e(x_n - x) \to 0 \text{ as } n \to \infty$$

which implies that ϕ_e is closed, and the proof is complete

Proposition 1.59: Let T be a densely defined linear operator in H such that $\overline{W(T)}$ is not the whole plane, a half-plane, a strip or a line. Then T has a closed extension $\hat{T}$ such that

$$\sigma(\hat{T}) \subseteq \overline{W(T)} \subseteq \overline{W(\hat{T})} \qquad (1.71)$$

Proof: Let $\phi(x,y)$ be the sesquilinear form defined by

$$\phi(x,y) = \langle Tx,y\rangle \quad x,y \text{ in } D(T) \qquad (1.72)$$

with $D(\phi) = D(T)$. Then $W(\phi) = W(T)$. By proposition 1.53, there is a symmetric sesquilinear form $\psi(x,y)$ satisfying (1.47). Now suppose (x_n) is a sequence of points of $D(\phi)$, $x_n \overset{s}{\dashrightarrow} o$ and $\hat{\phi}(x_n - x_m) \dashrightarrow o$. Since

$$\hat{\phi}(x_n) = \phi(x_n, x_n - x_m) + \langle Tx_n, x_m\rangle$$

we have, by corollary 1.46

$$|\hat{\phi}(x_n)| \leq 2C(\hat{\psi}(x_n))^{1/2}(\psi(x_n - x_m))^{1/2} + ||Tx_n||\cdot||x_m|| \qquad (1.73)$$

Now by (1.47)

$$\hat{\psi}(x_n - x_m) \leq |\hat{\phi}(x_n - x_m)| + C||x_n - x_m||^2 \dashrightarrow o \text{ as } m,n \dashrightarrow \infty$$

Thus, there is a constant K such that

$$\hat{\psi}(x_n) \leq K^2, \qquad n = 1, 2, \ldots \qquad (1.74)$$

Now let $\varepsilon > o$ be given. Take N large enough so that

$$\hat{\psi}(x_n - x_m) < \frac{\varepsilon^2}{4C^2K^2} \quad \text{for } m,n > N.$$

Thus $|\hat{\phi}(x_n)| < \varepsilon + ||Tx_n|| \, || x_m ||$, m, n > N.

Letting $m \dashrightarrow \infty$, we obtain $|\hat{\phi}(x_n)| \leq \varepsilon$ when n > N.

This means that $\hat{\phi}(x_n) \dashrightarrow o$ as $n \dashrightarrow \infty$. Thus $\phi(x,y)$ satisfy the hypotheses of proposition 1.58. Therefore, we conclude that $\phi(x,y)$ has a closed extension $\phi_e(x,y)$ with $D(\phi)$ dense in $D(\phi_e)$. Let $\hat{T}$ be the operator associated with $\phi_e(x,y)$. Then by proposition 1.56 $\hat{T}$ is closed and

$$Sp(\hat{T}) \subseteq \overline{W(\phi_e)} = \overline{W(T)}.$$

But by proposition 1.58

$$\overline{W(\phi_e)} = \overline{W(\phi)} = \overline{W(T)}$$

It remains to verify that $\hat{T}$ extends T. Suppose x is in D(T). Then

$$\phi(x,y) = \langle Tx,y \rangle \quad , x \text{ in } \quad D(\phi) = D(T) \tag{1.75}$$

Since $\phi_e(x,y)$ is an extension of $\phi(x,y)$

$$\phi_e(x,y) = \langle Tx,y \rangle \quad \text{for} \quad y \quad \text{in} \quad D(\phi) \tag{1.76}$$

Now we have that (1.76) holds for all y in $D(\phi_e)$. This follows from the fact that $D(\phi)$ is dense in $D(\phi_e)$. Thus, if y is in $D(\phi_e)$, there is a sequence (y_n) of elements of $D(\phi)$ such that $\hat{\phi}_e(y_n - y) \dashrightarrow 0$ and $||y_n - y|| \dashrightarrow 0$. Now $\overline{W(\phi_e)}$ is not the whole plane, a half-plane, a strip, or a line. Thus we may apply proposition 1.53 to conclude that

$$\phi_e(x,y_n) \dashrightarrow \phi_e(x,y)$$

Since $\phi_e(x,y_n) = \langle Tx,y_n \rangle$, we may have in the limit that (1.47) holds. Thus,

$$x \in D(\hat{T}) \text{ and } \hat{T}x = Tx.$$

Thus we have provided sufficient condition that a linear operator T on H have a closed extension $\hat{T}$ satisfying $Sp(\hat{T}) \subseteq \overline{W(\hat{T})}$.

1.6.3 Closable Operators:

If we are only interested in determining whether or not T has a closed extension, then the conditions in proposition 1.59 can be weakened. (We shall see this shortly).

Definition 1.60: Let T be a linear transformation from a normed linear space X to a normed linear space Y and D(T) be its domain. It is called **closable** if the conditions

x_n is in D(T) for all n in $\mathbb{N}$, $x_n \xrightarrow{s} o$ in X and $Tx_n \xrightarrow{s} y$

imply

$$y = o.$$

Clearly, every closed linear transformation is closable.

Lemma 1.61. If $\phi(x,y)$ is a sesquilinear form such that $W(\phi)$ is not the whole plane, then there are constants r, k_o with $|r| = 1$ such that

$$\operatorname{Re}[r\hat{\phi}(x) + k_o\,||x||^2] \geq o \text{ for all } x \text{ in } D(\phi). \quad (1.77)$$

Proof: We know that $W(\phi)$ is convex. Since it is not the whole plane, it is contained in a half-plane (by Lemma 1.50). But every half-plane is of the form

$$\operatorname{Re}[rz + k_o.] \geq o, \quad |r| = 1$$

Thus

$$\operatorname{Re}[r\hat{\phi}(x) + k_o] \geq o, \quad x \text{ in } D(\phi), ||x|| = 1,$$

which implies (1.77).

Proposition 1.62 Let $\phi(x,y)$ be a densely defined sesquilinear form such that $W(\phi)$ is not the whole plane. Let T be the operator associated with $\phi(x,y)$. If $D(T)$ is dense in H then T is closable.

Proof:- By Lemma 1.61, there are constants r, k_o such that (1.77) holds. Set

$$\psi(x,y) = r\phi(x,y) + k_o\langle x,y\rangle$$

and $S = rT + k_oI$. Then S is the operator associated with $\psi(x,y)$. Moreover, T is closable if and only if S is closable. So suppose (x_n) is a sequence of elements of $D(T)$, $x_n \xrightarrow{S} o$ and $Sx_n \xrightarrow{S} y$. Then, for $\alpha > o$ and z in $D(T)$,

$$\begin{aligned}\hat{\psi}(x_n - \alpha z) &= \hat{\psi}(x_n) - \alpha\psi(x_n,z) - \alpha\psi(z,x) + \alpha^2\hat{\psi}(z)\\ &= \langle Sx_n, x_n\rangle - \alpha\langle Sx_n, z\rangle - \alpha\langle Sz, x_n\rangle + \alpha^2\hat{\psi}(z)\\ &\longrightarrow -\alpha\langle y,z\rangle + \alpha^2\hat{\psi}(z).\end{aligned}$$

Hence $\mathrm{Re}\,[-\langle y,z\rangle + \alpha\hat{\psi}(z)] \geq o$ for $\alpha > o$ and all $z \in D(T)$. Letting $\alpha \to o$, we see that

$$\mathrm{Re}\,\langle y,z\rangle \leq o \text{ for all } z \text{ in } D(T) \tag{1.78}$$

Since $D(T)$ is dense in H, there is a sequence (y_n) of elements of $D(T)$ such that $y_n \xrightarrow{s} y$ in H. Since $\mathrm{Re}\,\langle y,y\rangle \leq o$, we have, in the limit, $||y||^2 \leq o$, which shows that $y = o$. Hence T is closable.

Proposition 1.63: If T is densely defined linear operator on H such that W(T) is not the whole complex plane, then T has a closed extension

Proof: Set

$$\phi(x,y) = \langle Tx,y\rangle \text{ for all } x,y \text{ in } D(T).$$

then $\phi(x,y)$ is a sesquilinear form with $D(\phi) = D(T)$ and $W(\phi) = W(T)$. Moreover, T is the operator associated with $\phi(x,y)$. Thus $\phi(x,y)$ satisfies all hypotheses of proposition 1.62. Hence T is closable. The result then follows from

Proposition 1.64: A linear transformation T from a normed linear space X to a normed linear space Y has a closed extension if and only if it is closable.

Proof: For sake of completeness, we prove this result. Suppose T has a closed extension $\hat{T}$ and let (x_n) be a sequence of elements of D(T) such that $x_n \xrightarrow{S} o$ in X while $Tx_n \xrightarrow{S} y$ in Y. Since $\hat{T}$ is an extension of T, x_n is in $D(\hat{T})$ and $\hat{T}x_n \xrightarrow{S} y$. Since $\hat{T}$ is closed, We have $\hat{T}o = y$ Showing that $y = o$. Hence T is closable.

Conversely; assume that T is closable. Define the operator $\overline{T}$ as follows:

An element x in X is in $D(\overline{T})$ if there is a sequence (x_n) of elements of D(T) such that $x_n \xrightarrow{S} x$ in X and Tx_n

converges strongly in Y to some element y. Define $\bar{T}x$ to be y. This definition does not depend on the choice of a particular sequence (x_n). For if (z_n) were another sequence of elements of D(T), $z_n \xrightarrow{s} x$ in X and $Tz_n \longrightarrow w$ in Y, then $x_n - z_n \xrightarrow{s} 0$ and $T(x_n - z_n) \xrightarrow{s} y-w$, since T is closable, we obtain y=w. Clearly, $\bar{T}$ is a linear extension of T and is closed. Indeed, let (x_n) be a sequence of elements of $D(\bar{T})$ and suppose $x_n \xrightarrow{s} x$ and $\bar{T}x_n \xrightarrow{s} y$. Then for each n in $\mathbb{N}$, there is a sequence $(w_{n,k})_{k=1}^{\infty}$ of elements of D(T) such that

$$w_{n,k} \xrightarrow{s} x_n \text{ and } T w_{n,k} \xrightarrow{s} \bar{T}x_n \quad \text{as } k \longrightarrow \infty .$$

In particular, one can find a z_n in D(T) such that

$$|| z_n - x_n || < \frac{1}{n} , ||Tz_n - \bar{T}x_n|| < \frac{1}{n}.$$

Therefore,

$$||z_n - x|| \leq || z_n - x_n || + ||x_n - x|| \longrightarrow 0,$$

and

$$||Tz_n - y|| \leq ||Tz_n - \bar{T}x_n || + ||\bar{T}x_n - y|| \xrightarrow{s} 0$$

This shows that x is in $D(\bar{T})$ and $\bar{T}x = y$. Hence $\bar{T}$ is closed.

Remark 1.65:

The reader will recollect that the operator $\bar{T}$ constructed in the proof of proposition 1.64 is called the closure of T. It is the "smallest" closed extension of T (as may be easily seen)

1.7 Dissipative Operators:

We refer to proposition 1.59 where it was shown that if T is a linear operator on H such that $\overline{D(T)}$ is dense and W(T) is not the whole plane, a half-plane, a strip or a line, then T has a closed extension $\hat{T}$ satisfying

$$\sigma(\hat{T}) \subseteq \overline{W(T)} = \overline{W(\hat{T})} . \tag{1.79}$$

Now there arises a question which was deferred deliberately until this point: What happens when $\overline{W(T)}$ is one of these sets? We now discuss some of these cases.

Suppose $\overline{W(T)}$ is the whole plane. Then (1.79) is vaccuously true for all extensions of T. On the other hand, the existence of a closed extension of T depends on whether or not T is closable. (Proposition 1.64). Hence this case is not interesting.

In all other cases including that discussed in proposition 1.59, W(T) is contained in a half-plane (Lemma 1.50), Thus, there are constants r , k such that $|r| = 1$ and

$$\operatorname{Re} r \langle Tx, x\rangle - k||x||^2 \leq o \text{ for all } x \text{ in } D(T).$$

Set

$$S = r T - k \tag{1.80}$$

Then D(S) = D(T) and

$$\operatorname{Re} \langle Sx,x\rangle \leq o \text{ for all } x \text{ in } D(S) \tag{1.81}$$

An operator S satisfying (1.81) is called **dissipative.**

Proposition 1.66: Let S be a dissipative operator on H with D (S) dense in H. Then S has a closed dissipative extension $\hat{S}$ such that $Sp(\hat{S})$ is contained in the half-plane $\mathrm{Re}\,\lambda \leq 0$.

Proof: By (1.81), we have $\mathrm{Re}\,\langle (I-S)x,x\rangle \geq ||x||^2$ for all x in D(S). This shows that (I-S) is one-to-one. Hence it has an inverse $(I-S)^{-1}$ defined on R(I-S).

Define

$$T = (I+S)(I-S)^{-1}$$

Where D (T) = R(I-S). We have that

$$||Tx|| \leq ||x|| \text{ for all } x \text{ in } D(T) \qquad (1.82)$$

Infact, if

$$y = (I-S)^{-1}\, x \qquad (1.83)$$

Then

$$Tx = (I+S)y \qquad (1.84)$$

Hence $||Tx||^2 = ||y||^2 + ||Sy||^2 + 2\,\mathrm{Re}\,\langle Sy,y\rangle$

$$\leq ||y||^2 + ||Sy||^2 - 2\,\mathrm{Re}\langle Sy,y\rangle = ||(I-S)y||^2 = ||x||^2 ,$$

by (1.81) and (1.83). We can now easily find an extension $\hat{T}$ of T in B(H) such that

$$||\hat{T}|| \leq 1. \qquad (1.85).$$

(Note that operators in B(H) are defined on all of H).

To do this, first extend T to $\overline{D(T)}$, by the usual procedure (extension by continuity). Call this extension

$\bar{T}$. Now let x be in H. By the projection Theorem, $x = z+y$ where z is in $\overline{D(T)}$ and $y \perp D(T)$.

Define $\hat{T}x = \bar{T}z$. Then $||\hat{T}x|| = ||\bar{T}z|| \leq ||z|| \leq ||x||$.

This shows that $\hat{T}$ is in B(H) and (1.85) holds.

Now by (1.83) and (1.84) (1.86)

$$x = y - Sy, \quad Tx = y+Sy$$

or

$$2y = x + Tx, \; 2Sy = Tx-x \tag{1.87}$$

The first equation in (1.87) together with (1.83) shows that (I+T) is one-to-one and that its range is D(S). Hence, We have

$$S = (T - I)(I + T)^{-1} \tag{1.88}$$

A candidate for the extension $\hat{S}$ is

$$\hat{S} = (\hat{T} - I)\,(I + \hat{T})^{-1} \tag{1.89}$$

with $D(\hat{S}) = R(I + \hat{T})$. In order that (1.89) make sense, we must check that $(I+\hat{T})$ is one -to-one. To see this, let x be any element of H such that

$$(1 + \hat{T})x = o \tag{1.90}$$

Let y be any element of H and let α be a positive real number. Set

$$z = (I + \hat{T})\,y. \quad \text{Then by (1.85)}$$

$$||z - y + \alpha x||^2 \leq ||\, y- \alpha x||^2 \tag{1.91}$$

Expanding (1.91) out, we get

$$||z||^2 - 2\,\mathrm{Re}\langle z,y\rangle + 2\alpha\,\mathrm{Re}\langle z,x\rangle \leq o.$$

Divide by α and let $\alpha \dashrightarrow \infty$ This gives

$$\mathrm{Re}\langle z,x\rangle \leq o, \quad z \text{ in } R(I+\hat{T}) \tag{1.92}$$

Since $R(I + \hat{T}) \supseteq R(I + T) = D(S)$, we see that $R(I+\hat{T})$ is dense in H. Hence, there is a sequence (z_n) of elements in $R(I+\hat{T})$ such that $z_n \xrightarrow{S} x \in H$. Thus (1.92) implies $\mathrm{Re}||x||^2 \leq o$, which shows that $x = o$. Thus the operator $\hat{S}$ given by (1.89) is well defined. It is clearly an extension of S. we claim $\hat{S}$ is closed. For suppose (x_n) is a sequence of elements in $D(\hat{S}) = R(I+\hat{T})$ such that

$$x_n \xrightarrow{S} x, \quad \hat{S}x_n \dashrightarrow h \text{ in } H \text{ as } n \dashrightarrow \infty .$$

Since x_n is in $R(I+\hat{T})$, there is a w_n in H such that $x_n = (1+\hat{T})w_n$. By (1.89) we have $\hat{S}x_n = (\hat{T} - I)w_n$. Hence $2w_n = x_n - \hat{S}x_n \dashrightarrow x - h$ as $n \dashrightarrow \infty$. Since $\hat{T}$ is in B(H), this implies

$$2x_n = 2(I + \hat{T})w_n \xrightarrow{S} (I + \hat{T})(x - h)$$

$$2 \hat{S}x_n = 2 (\hat{T}-I)w_n \xrightarrow{S} (\hat{T}-I)(x - h).$$

from which we conclude

$$2x = (I + \hat{T})(x - h), \quad 2h = (\hat{T} - I)(x - h)$$

In particular, we see that x is in $R(I + \hat{T}) = D(\hat{S})$ and

$$\hat{S}x = (\hat{T} - I)(I +\hat{T})^{-1} x = \frac{1}{2}(\hat{T} - I)(x - h) = h .$$

Hence $\hat{S}$ is a closed operator

Next we show that $\hat{S}$ is dissipative. This follows easily, since

$\langle \hat{S}x,x\rangle = \langle(\hat{T}-I)w,(I+\hat{T})w\rangle = ||\hat{T}w||^2 - \langle w,\hat{T}w\rangle + \langle\hat{T}w,w\rangle - ||w||^2$,
where $w = (I+\hat{T})^{-1}x$. Hence

$$\mathrm{Re}\langle\hat{S}x,x\rangle = ||\hat{T}w||^2 - ||w||^2 \leq o \qquad (1.93)$$

by (1.85).

Finally, we must verify that $\lambda \in \rho(\hat{S})$ for $\mathrm{Re}\,\lambda > o$.
Since $\hat{S}$ is dissipative, we have

$\mathrm{Re}\,\langle(\hat{S} - \lambda I)x,x\rangle \leq -\,\mathrm{Re}\,\lambda\,||x||^2$, or

$\mathrm{Re}\,\lambda\,||x||^2 \leq -\,\mathrm{Re}\langle(\hat{S} - \lambda I)x,x\rangle \leq ||(\hat{S}-\lambda I)x||\;||x||$,

Showing that $\hat{S} - \lambda I$ is one-to-one for $\mathrm{Re}\,\lambda > o$. Now

$$(\hat{S} - \lambda I) = [(1-\lambda)\,\hat{T} - (I+\lambda)\,]\,(I+\hat{T})^{-1}\;.$$

Thus one can solve

$$(\hat{S} - \lambda I)\,x = z \qquad (1.94)$$

if and only if one can solve

$$[(1-\lambda)\,T - (1+\lambda)]w = z \qquad (1.95)$$

Not thate (1.95) can be solved for all z in H when $\mathrm{Re}\,\lambda > o$. This is obvious for $\lambda = 1$. If $\lambda \neq 1$, all we need to note is that $\left|\frac{1+\lambda}{1-\lambda}\right| > 1$ for $\mathrm{Re}\,\lambda > o$. Since $||\hat{T}|| \leq 1$, this shows that $\frac{1+\lambda}{1-\lambda}$ is in $\rho(\hat{T})$.
(if $|\lambda| > ||T||$ then $\lambda \in \rho(T)$. Hence (1.95) can be solved for all z in H .

In particular, if $\overline{W(T)}$ is a half-plane, then proposition 1.66 gives a closed extension $\hat{T}$ of T satisfying (1.79). In fact, all we need to do is define

S by (1.80) for appropriate r, k . Then extend S to $\hat{S}$ by proposition 1.66. The extension $\hat{T}$ is thus defined by $\hat{T} = \frac{\hat{S}+kI}{r}$. Hence proposition 1.66 gives

Proposition 1.67 Let T be a densely defined operator on H such that $\overline{W(T)}$ is a half-plane. Then T has a closed extension $\hat{T}$ satisfying (1.79).

If W(T) is a line or a strip then things are more complicated. We can use proposition 1.66 to obtain a closed extension having one of the **adjacent** half-planes in its resolvent set. But it will not be true, in general, that this extension will have the other adjacent half-plane in its resolvent set as well.

1.7.1 The case of a line or a strip: We first prove the following

Lemma 1.68: Let T be a closed linear operator on a Banach space X. If λ is a boundary point of $\rho(T)$ and (λ_n) is a sequence of points in $\rho(T)$ converging to λ, then $||(T - \lambda_n I)^{-1}|| \to +\infty$.

Proof:

If the Lemma 1.68 were not true, there would be a sequence, (λ_n) of points in $\rho(T)$ such that $\lambda_n \to \lambda$ as $n \to \infty$ while

$$||(T - \lambda_n I)^{-1}|| \leq C \qquad (1.96)$$

Since $(T - \lambda_n I)^{-1} - (T - \lambda_m I)^{-1} = (T - \lambda_n)^{-1}(\lambda_n - \lambda_m)(T - \lambda_m)^{-1}$

We have

$$||(T - \lambda_n I)^{-1} - (T - \lambda_m I)^{-1}|| \leq C^2 |\lambda_m - \lambda_n| \to 0 \text{ as } m,n \to \infty.$$

Thus, $(T - \lambda_n I)^{-1}$ converges to an operator S in B(X) as $n \to \infty$. Moreover, if x is any element in X, Then

$$y_n = (T - \lambda_n I)^{-1}x \xrightarrow{s} Sx \text{ as } n \to \infty .$$

But

$$Ty_n = (T - \lambda_n)y_n + \lambda_n y_n \xrightarrow{s} x + \lambda Sx \text{ as } n \to \infty .$$

Since T is closed, Sx is in D(T) and $TSx = x + \lambda Sx$, Whence

$$(T - \lambda I)Sx = x \quad \text{for all } x \text{ in } X \tag{1.97}$$

Similarly if x is in D(T), then

$$(T - \lambda_n I)^{-1}(T - \lambda I)x \xrightarrow{s} S(T - \lambda I)x \text{ as } n \to \infty$$

But $(T - \lambda_n I)^{-1}(T - \lambda I)x = x - (\lambda - \lambda_n)(T - \lambda_n I)^{-1}x \xrightarrow{s} x$

as $n \to \infty$

Hence $S(T - \lambda I)x = x$ for all x in D(T) $\quad$ (1.98)

This shows that λ is in $\rho(T)$, contrary to assumption.

Proposition 1.69: Let T be a closed linear operator on a Banach space X. If λ is a boundary point of $\rho(T)$, then either $N(T - \lambda I) = \{o\}$ or $R(T - \lambda I)$ is not closd in X.

Proof: If the proposition were not true, there would be a constant C such that

$$||x|| \leq C\, ||(T - \lambda I)\, x|| \quad \text{for all } x \text{ in } D(T) \tag{1.99}$$

Since λ is a boundary point of $\rho(T)$, there is a sequence (λ_n) of points of $\rho(T)$ converging to λ. Set

$$S_n = \frac{(T - \lambda_n I)^{-1}}{||(T - \lambda_n I)^{-1}||}$$

Then $||S_n|| = 1$ In particular, for each n in $\mathbb{N}$ there is an element x_n in X such that

$$||x_n|| = 1, \qquad ||S_n x_n|| > \frac{1}{2} \tag{1.100}$$

Now $(T - \lambda I)S_n = (T - \lambda_n I)S_n + (\lambda_n - \lambda)S_n$. Hence

$$||(T - \lambda I)S_n|| \leq ||(T - \lambda_n I)^{-1}|| + |\lambda_n - \lambda|.$$

By Lemma 1.68, this tend to o as n --> ∞. In particular, the norm of $(T - \lambda I)S_n$ can be made less than $\frac{1}{3C}$ for sufficiently large n. But by (1.99) we have

$$\frac{1}{2} < ||S_n x_n|| \leq C||(T - \lambda I)S_n x_n|| < \frac{1}{3}$$

for large n. This contradiction shows that (1.99) does not hold and the proof is complete.

We outline briefly what one can do in case $\overline{W(T)}$ is a line or a strip. Of course, we can consider a line as a strip of thickness o. So suppose that $\overline{W(T)}$ is a strip of thickness a-1 where $a \geq 1$. As before, we can find an operator S of the form

$$S = r\,T - k\,I \tag{1.101}$$

such that $\overline{W(S)}$ is the strip $1-a \leq \mathrm{Re}\, z \leq o$. Thus

$$1-a \leq \mathrm{Re}\langle Sx, x\rangle \leq o, \text{ for all}$$
$$x \text{ in } D(S),\ ||x|| = 1 \tag{1.102}$$

In particular, S is dissipative.

Now suppose S has a closed extension $\hat{S}$ such that $\overline{W(\hat{S})}$ is the same strip and $\rho(\hat{S})$ contains two complementary half-planes. Set

$$\hat{T} = (aI + \hat{S})\,(a - \hat{S})^{-1} \qquad (1.103)$$

Let x be any element of H and set

$$y = (aI - \hat{S})^{-1}x \qquad (1.104)$$

Then

$$\hat{T}x = (aI + \hat{S})\,y \qquad (1.105)$$

Hence

$$||\hat{T}x||^2 - ||x||^2 = 4a\langle \hat{S}y, y\rangle \qquad (1.106)$$

Showing that

$$4a(1-a)||y||^2 \leq ||\hat{T}x||^2 - ||x||^2 \leq o \qquad (1.107)$$

By (1.104) and (1.105) we have

$$(\hat{T} + I)x = 2ay, \quad (\hat{T} - I)x = 2\hat{S}y \qquad (1.108)$$

Thus (1.107) becomes

$$\frac{1-a}{a}\,||(\hat{T} + I)x||^2 \leq ||\hat{T}x||^2 - ||x||^2 \leq o \qquad (1.109)$$

or

$$||\hat{T}x||^2 \leq ||x||^2 \leq (2a - 1)\,||\hat{T}x||^2 + 2(a - 1)\,\mathrm{Re}\,\langle \hat{T}x, x\rangle \qquad (1.110)$$

In short $\hat{T}$ is an extension of

$$T = (aI + S)\,(aI - S)^{-1}, \quad D(T) = R(aI - S) \qquad (1.111)$$

Which satisfies (1.110) and such that $R(\hat{T}) = H$.

Conversely, if we can find an extension $\hat{T}$ of T, then we can show that

$$\hat{S} = a(\hat{T} - I)(\hat{T} + I)^{-1} \tag{1.112}$$

is closed extension of S with $\overline{W(\hat{S})} = \overline{W(S)} \supseteq Sp(\hat{S})$. Infact we have, by the reasoning of section 1.6, that $\hat{S}$ is closed and dissipative while $\rho(\hat{S})$ contains the half-plane $\mathrm{Re}\,\lambda > o$. Moreover, (1.110) and (1.106) imply that $\overline{W(\hat{T})}$ is the strip $1 - a \leq \mathrm{Re}\,\lambda \leq o$.

However, we also want the half-plane $\mathrm{Re}\,\lambda < 1-a$ to be in $\rho(\hat{S})$. We know that $\hat{S} - \lambda I$ is one-to-one and that $R(\hat{S} - \lambda I)$ is closed for such points (see proposition 1.44). It suffices to show that the half-plane contains one point of $\rho(\hat{S})$. We can then apply proposition 1.69. If the half-plane $R\,\lambda < 1 - a$ contains a point of $\rho(\hat{S})$, then the entire half-plane must be $\rho(\hat{S})$. For otherwise, it would contain a boundary point λ of $\rho(\hat{S})$. But this would imply by proposition 1.69 that either $N(\hat{S} - \lambda I) \neq \{o\}$ or $R(\hat{S} - \lambda I)$ is not closed, contradicting the conclusion reached above.

To complete the argument, we show that the point $\lambda = -a$ is, indeed, in $\rho(\hat{S})$. For

$$aI + \hat{S} = \hat{T}(aI - \hat{S}) \tag{1.113}$$

and since $R(\hat{T}) = R(aI - \hat{S}) = H$, it follows that $R(aI + \hat{S}) = H$. This coupled with the fact that

$N(aI + \hat{S}) = \{o\}$, shows that $-a$ is in $\rho(\hat{S})$.

So we must try to find an extension $\hat{T}$ of T satisfying (1.110) and such that $R(\hat{T}) = H$. Let us consider first the case $a = 1$ (i.e. the case of a line). Now by (1.110).

$$||\hat{T}x|| = ||x||, \ x \text{ in } H \tag{1.114}$$

We want T to have a unitary extension $\hat{T}$. By (1.111), the operator T is an isometry of $R(I - S)$ onto $R(I + S)$. By continuity **we can extend** it to be an isometry $\overline{T}$ of $\overline{R(I - S)}$ onto $\overline{R(I + S)}$. Thus, to determine $\hat{T}$, we need only define it on $R(I + S)^{\perp}$. We can see that $\hat{T}$ would have to map $R(I - S)^{\perp}$ into $R(I + S)^{\perp}$. This follows from the general property of isometries on Hibert spaces:

$$\langle \hat{T}x, \hat{T}y \rangle = \langle x,y \rangle.$$

Moreover, $\hat{T}$ must map onto $R(I + S)^{\perp}$, for otherwise we would not have $R(\hat{T}) = H$. Thus we have

Proposition 1.70 Let S be densely defined operator on H such that $W(S)$ is the line $\text{Re}\,\lambda = o$. Then a necessary and sufficient condition that S have a closed extension $\hat{S}$ such that

$$Sp(\hat{S}) \subseteq W(\hat{S}) = W(S) \tag{1.115}$$

is that there exists an isometry from $R(I - S)^{\perp}$ onto $R(I - S)^{\perp}$. In particular, this is true if they both have the same dimension or if they are both seperable and infinite demensional.

The last statement follows from the fact that $R(I + S)^{\perp}$ and $R(I - S)^{\perp}$ have complete orthonormal sequences (ϕ_k) and (ψ_k) respectively. Moreover, these sequences are either both infinite or have thesame finite number of elements. In either case, we can define $\hat{T}$ by

$$\hat{T}\phi_k = \psi_k \quad , \quad k = 1, 2, \ldots. \tag{1.116}$$

This gives the desired isometry.

If $a \neq 1$ (i.e. in the case of a strip), the situation is not so simple. It is necessary for $\hat{T}$ to map $R(aI + S)^{\perp}$ onto a closed subspace M such that

$$H = \overline{R(aI + S)} \oplus M$$

in such a manner that (1.110) holds. The next proposition just gives a sufficient condition:

Proposition 1.71: Let S be a densely defined linear operator on H such that $\overline{W(S)}$ is the strip

$$1 - a \leq \operatorname{Re} z \leq o, \quad a > 1.$$

If $\overline{R(aI - S)} = \overline{R(aI + S)}$, then S has a closed extension $\hat{S}$ satisfying

$$\mathrm{Sp}(\hat{S}) \subseteq \overline{W(\hat{S})} = \overline{W(S)} \tag{1.117}$$

Proof: On $R(aI - S)^{\perp} = \overline{R(aI + S)}^{\perp}$ we define $\hat{T}$ to be $-I$. Then $\hat{T}$ is isometric on this set. Thus (1.109) (and hence (1.110) holds for x in $\overline{R(aI - S)}$ and for x in $R(aI - S)^{\perp}$. For any x in H, $x = x_1 + x_2$ Where x_1 is in $\overline{R(aI - S)}$ and x_2 is in $R(aI - S)^{\perp}$.

Thus

$$\frac{1 - a}{a}||(\hat{T} + I)x||^2 = \frac{1 - a}{a}||(\bar{T} + I)x_1||$$

$$\leq ||\bar{T}x_1||^2 - ||x_1||^2 \leq o$$

But

$$||\bar{T}x_1||^2 - ||x_1||^2 = ||\bar{T}x_1 - x_2||^2 - ||x_1 + x_2||^2$$

$$= ||\hat{T}x||^2 - ||x||^2$$

Hence (1.109) holds and the proof is complete

1.7.2 Self-Adjoint Extensions

If T is a self-adjoint operator then $T = T^*$, however for T^* to be defined we know that D(T) must be dense in H. Moreover T must be closed since T^* is. Also we note that $W(T) \subseteq \mathbb{R}$. Thus W(T) is either a whole real axis or an interval. We have that

$$Sp(T) \subseteq \overline{W(T)} \tag{1.118}$$

when T is self-adjoint. For by proposition 1.42 $N(T - \lambda I) = \{o\}$ and $R(T - \lambda I)$ is closed when $\lambda \notin \overline{W(T)}$. Moreover, if y is orthogonal to $R(T - \lambda I)$, then

$$\langle y, (T - \lambda I)x\rangle = o \text{ for all x in } D(T).$$

Showing that y is in $D(T^*) = D(T)$ and $\langle (T - \bar{\lambda} I)y, x\rangle = o$ for all x in D(T). Since D(T) is dense, we must have

$(T - \bar{\lambda} I)y = o$. Now if $\lambda \notin \overline{W(T)}$, then $\bar{\lambda} \notin \overline{W(T)}$ as well. For if λ is not real, then neither of them is in $\overline{W(T)}$. If λ is real, then $\bar{\lambda} = \lambda$. By what we have just said, $y = o$. This shows that $R(T - \lambda I)$ is dense as well as closed. Thus, $\lambda \in \rho(T - \quad I)$ and the assertion is proved.

Now suppose T is a densely defined linear operator on H. Then $W(T)$ is a subset of $\mathbb{R}$ if and only if T is Symmetric, i.e. if

$$\langle Tx, y \rangle = \langle x, Ty \rangle \text{ for all } x, y \text{ in } D(T) \qquad (1.119)$$

If T is symmetric and (1.118) holds, then we can show that T is self-adjoint. To see this, suppose y is in $D(T^*)$. Let λ be any nonreal point. Then λ and $\bar{\lambda}$ are both in $\rho(T)$, since neither of them is in $\overline{W(T)}$. Hence, there is a $z \in D(T)$ such that

$$(T - \bar{\lambda} I)z = (T^* - \bar{\lambda} I)y.$$

Thus

$$\langle y, (T - \lambda I)x \rangle = \langle (T^* - \bar{\lambda})y, x \rangle = \langle (T - \bar{\lambda} I)z, x \rangle$$

$$= \langle z, (T - \lambda I)x \rangle \quad \text{for all } x \text{ in } D(T).$$

or

$$\langle y-z, (T - \lambda I)x \rangle = o \text{ for all } x \text{ in } D(T).$$

Since $\lambda \in \rho(T)$, this can happen only if $y = z$. Thus, we can see that y is in $D(T)$ and $Ty = T^*y$.

We now answer the following question: Let T be a densely defined linear operator that is symmetric. Does it have a self-adjoint extension?

To examine the question, suppose first that W(T) is not the whole real axis. Then by proposition 1.59, T has a closed extension $\hat{T}$ satisfying

$$Sp(\hat{T}) \subseteq \overline{W(\hat{T})} = \overline{W(T)} \tag{1.120}$$

In particular, $\hat{T}$ is symmetric and satisfies (1.118). Hence $\hat{T}$ is self-adjoint.

On the other hand, if W(T) is the whole real axis, then T has a closed extension $\hat{T}$ satisfying (1.120) if and only if there is an isometry of $R(iI + T)^{\perp}$ onto $R(iI - T)^{\perp}$ (just put $S = iT$ in proposition 1.67). Since, in this case, an extension violating (1.120) could not be self-adjoint, the condition is both necessary and sufficient for T to have a self-adjoint, extension. The problem is, therefore, solved.

GLOSSARY OF SYMBOLS

$\langle , \rangle$	Inner product function
H	Complex Hilbert space
T^*	Adjoint of the operator T
B(T)	Collection of all bounded operators on H
Sp(T)	Spectrum of T
pSp(T)	Point spectrum of T
ap Sp(T)	Approximate point spectrum of T
W(T)	Numerical range of T
R(T)	Range of T (as a map)
N(T)	Kernel of T
D(T)	Domain of T
T^*	Orthogonal conjugate of T
$\rho(T)$	Resolvent set of T
r(T)	Spectral radius of T
w(T)	Numerical radius of T
Conv M	Convex hull of the set M
U_+	Unilateral shift operator
Sup A	Supremum of the set A
Inf B	Infimum of the set B
Rez	Real part of the complex number z
ImZ	Imaginary part of the complex number Z
∂D	Boundary of the D
Ext (S)	Set of all extreme points of S
$X \perp Y$	X is orthogonally perpendicular to Y

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