STUDY OF PROJECTIVE CURVATURE TENSOR, $W^{i}_{jkh}(x,\dot{x})$ IN BI-RECURRENT FINSLER SPACE, $2R - F_n$

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DECLARATION

This thesis is my original work and has not been presented for a degree or other award in any other university.

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DEDICATION

To my father Melitus Opondo, my mother Patricia Godia and

my late grandparents: Oudo, Alego, Owuama, Aduol and Adik

for instilling in me the value of education.

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LIST OF SYMBOLS

G^i_{jk}	- Berwald connection
H^i_{jkh}	- Berwald curvature tensor
$2R - F_n$	- Bi-recurrent Finsler space
C_{ijk}	- Cartan's tensor
S^i_{hjk}	- Cartan's first curvature tensor
P^i_{hjk}	- Cartan's second curvature tensor
R^i_{hjk}	- Cartan's third curvature tensor
Γ^i_{jk}	- Chern-Rund coefficient
$\gamma_{ijk},\ \gamma^i_{jk}$	- Christoffel symbols
:	- Covariant derivative with respect to Berwald's connection
I	- Covariant derivative with respect to Cartan's connection
;	- Covariant derivative with respect to Riemannian connection
$\partial_i = \frac{\partial}{\partial x^i}$	- Derivative
$\dot{\partial}_i = \frac{\partial}{\partial y^i}$	- Derivative
L	- Finsler metric
F	- Finsler space
G^i_{hjk}	- hv-curvature tensor
R^i_{jk}	- torsion tensor field for R
δ^i_j	- Kronecker delta
L_{ijk}	- Landberg coefficients
L_V	- Lie-derivative
g_{ij}	- Metric tensor
g^{ij}	- Inverse or conjugate of g_{ij}

N^i	- Normal vector
H_α	- Normal curvature tensor
F_n-n	- dimensional Finsler space
N_j^i	- Non-linear connection
W_j^i	- Projective deviation tensor
$\bar{F}_n$	- Recurrent Finsler space
α	- Riemannian metric
H_{kh}	- Ricci tensor
A_{lm}	- Symmetric recurrence tensor field
T_j^i	- Mixed tensor field
P_{ijk}	- Torsion tensor field for P
W_{jkh}^i	- Weyl projective curvature tensor
X_n	- n- dimensional space
x^i	- Co-ordinate system
$\bar{x}^i$	- Transformed co-ordinate system
$\dot{x}$	- Direction vector
ds^2	- Arc length
$\lvert g_{ij} \rvert$	- Determinant of matrix g_{ij}
$\sum_i^N$	- Summation
T_n	- Tangent space
V^i	- Contravariant vector field
ξ^k	- Variables
$\widetilde{K}_{jkh}^i$	- Relative curvature tensor
H_k^i	- Berwald deviation tensor

H_{jk}^{i}	- Berwald torsion tensor
$T_{[\ ]}$	- Successive differentiation
$T^{i_1 \dots i_r}$	- Contravariant tensor of rank r
$T_{j_1 \dots j_s}$	- Covariant tensor of rank s
$T_{j_1 \dots j_s}^{i_1 \dots i_r}$	- Mixed tensor of order (r, s)
D_j^i	- Deflection tensor
Γ_{jk}^{*i}	- Connection coefficient
$T\dot{x}^h$	- Transvection by $\dot{x}^h$
δt	- Infinitesimal point constant
K_l	- Recurrence vector field
$\alpha(x)$	- Non zero scalar
$V_{(j)}^i$	- Contra condition for curvature symmetry
λ	- Constant not necessarily non zero
$\Psi_{j_1 \dots j_s}^{i_1 \dots i_r}$	- Decomposition tensor of order (r, s)
P^i, X^i	- Contravariant decomposition vector
X_j	- Covariant decomposition vector
$T_{(l)}$	- Covariant differentiation of tensor T
N^r	- Rank r in an N- dimensional space
$F(x^i, \dot{x}^i)$	- Metric function

ABSTRACT

The Weyl (W) projective curvature tensor, $W^i_{jkh}(x,\dot{x})$ has properties that have a wide range of applications in various fields but are still not easy to understand in a fundamental way since $W^i_{jkh}(x,\dot{x})$ is a function of both position and direction. In this study three selected properties of the projective curvature tensor, $W^i_{jkh}(x,\dot{x})$ namely inheritance symmetry, collineation symmetry and decomposition are investigated in bi-recurrent Finsler space for purposes of new applications. Many authors have studied inheritance, collineation and decomposition properties for H, K, N, R and U curvature tensors in recurrent Finsler spaces and in this thesis we have extended these studies to W curvature tensor in bi-recurrent Finsler space which is still relatively under explored. The Finsler space is viewed as regular metric space and the three properties are described from the modern geometry view point using the relevant geometric and symbolic computation tools such as Lie derivatives, transvection, commutation and covariant differentiation in the sense of Berwald. The W-Curvature inheritance is defined by a Lie derivative (L_v) proportional to the projective curvature tensor $W^i_{jkh}(x,\dot{x})$ while the W-Curvature collineation is defined by vanishing Lie derivative (L_v) of $W^i_{jkh}(x,\dot{x})$. The results of the study show that every motion admitted in a bi-recurrent Finsler space is also a W-curvature inheritance if the space is isotropic otherwise it is a W-curvature collineation. The contra field and concurrent field are considered as special cases. The study reveals that both fields do not admit a W-curvature inheritance however they both admit a W- curvature collineation when the vector field (V^i) of the infinitesimal transformation is orthogonal to the recurrence vector (K_l). The geometrical properties of both inheritance and collineation symmetries have physical significance which make them useful in spacetime applications. The decomposition property of the projective curvature tensor $W^i_{jkh}(x,\dot{x})$ is also investigated for specified decomposition tensors denoted by $\Psi(x,\dot{x})$ using different symbolic tensor computation algorithms. The study has established that tensor decomposition for the projective curvature tensor $W^i_{jkh}(x,\dot{x})$ is not unique but specific conditions discussed in this study introduce uniqueness into the decomposition algorithm. The study has also established that in both recurrent and bi-recurrent Finsler spaces the decomposition tensors have some properties similar to those of the original tensor and therefore decomposition can be used to compress tensors for further applications. The tensorial computation algorithms are presented in form of step by step equations and the principal results obtained have helped to identify some hidden components of the projective curvature tensor $W^i_{jkh}(x,\dot{x})$. The results of the study are summarized in form of theorems which have already been verified and can be used in various fields for theoretical investigations and practical applications of tensors.

CHAPTER ONE

INTRODUCTION

This is an introductory chapter in which the elementary concepts of Finsler geometry upon which the geometrical development of this thesis depends have been examined. The chapter is divided into six sections. In section 1.1, known definitions, tensor notations and some preliminary results which were needed in the research have been given. Basic concepts such as Finsler spaces and connections have been discussed. Finsler geometry has many connections such as Berwald, Cartan, Chern, Rund, Civita and others. This research was mainly concerned with Berwald connection. In section 1.2, Berwald curvature tensor and other tensors in Finsler space have been defined since preliminary results on mathematical analysis of these tensors formed the background of the study. In section 1.3, the problem statement and justification have been given and in section 1.4 the main objective and the specific objectives have been stated. In section 1.5 the significance of the study to relevant areas have been presented. Finally in section 1.6 the general layout of the thesis highlighting the contents of each chapter has been given.

1.1 Mathematical spaces

Definition 1.1.1: Space

A space is defined by a set of rectangular coordinates system or axes. It is the boundless three dimensional extent in which objects and events have relative position and direction. If time is included it becomes a four dimensional continuum known as space time (Eisenhart, 1925). In modern mathematics spaces are defined as sets with some added structure and are described as different types of manifolds. Common types of spaces include Euclidean, Riemannian, Finsler, Banach, Hilbert, Topological spaces and others (Rund, 1959).

Definition 1.1.2: n-dimensional space

A real coordinate space which is described by rectangular coordinates is called a Euclidean space. In three dimensional rectangular space, the Cartesian coordinates of a point are *(x,y,z)* which can be conveniently written as (x^1, x^2, x^3). The fact that one cannot visualize points in space of dimension higher than three does not mean they do not exist. A space in which the co-ordinates are given by $(x^1, x^2, x^3 \dots, x^n)$ is called an *n*-dimensional space and is denoted by X_n (Eisenhart, 1925).

Definition 1.1.3: Differentiable manifold

A continuous space of points is called a manifold. In an n-dimensional manifold, to any point P of the manifold there corresponds co-ordinates $(x^1, x^2, x^3 \dots, x^n)$ which are in fact part of R^n. When the derivatives of fields on the manifold can be defined, the manifold is said to be differentiable. A manifold is called isotropic if its geometry remains the same regardless of direction.

Definition 1.1.4: Coordinate transformation

Let $(\bar{x}^1, \bar{x}^2, \bar{x}^3 \dots \bar{x}^n)$ be coordinates of the same point in Y-coordinate system and let $\bar{x}^1, \bar{x}^2, \bar{x}^3 \dots \bar{x}^n$ be independent single valued functions of $x^1, x^2, x^3 \dots x^n$ so that

$$\bar{x}^i = \bar{x}^i(x^1, x^2, x^3 \dots x^n), \quad i = 1,2, \dots n \qquad (1.1.1)$$

Solving these equations and expressing x^i as functions of $\bar{x}^1, \bar{x}^2, \bar{x}^3 \ldots \bar{x}^n$ so that

$$x^i = x^i(\bar{x}^1, \bar{x}^2, \bar{x}^3 \ldots \bar{x}^n), \quad i = 1,2,. . ., n \tag{1.1.2}$$

the equations (1.1.1) and (1.1.2) are said to be a transformation of coordinates from one coordinate system to another. The equation (1.1.1) indicates that the co-ordinates x^i of a point P of X_n are expressed in the new co-ordinate system by the new variable $\bar{x}^i$ (Eisenhart, 1925).

Definition 1.1.5: Metric of a space

A non-negative function $g(x^i, x^j)$describing the distance between each neighbouring points or each pair of elements of a set is called a metric. A set possessing a metric is called a metric space. A real fundamental form is taken as the basis of the metric of the space (Eisenhart, 1925).

$$\varphi = g_{ij} dx^i dx^j, i, j = 1,2, \ldots, n \tag{1.1.3}$$

where the g's are functions of the x's such that the determinant of the coefficient matrix is subject to the condition

$$g = \begin{vmatrix} g_{11} \; g_{12} g_{13} \cdots \; g_{1n} \\ g_{21} \; g_{22} g_{23} \cdots \; g_{2n} \\ \cdots \cdots \cdots \cdots \cdots \cdots \cdots . \\ g_{n1} \; g_{n2} g_{n3} \cdots \; g_{nn} \end{vmatrix} \neq 0 \tag{1.1.4}$$

In this thesis equation (1.1.4) has conventionally been given as

$$g = |g_{ij}| \neq 0 \tag{1.1.5}$$

Definition 1.1.6: Curve and arc length

A curve C of X_n is a set of points whose co-ordinates can be expressed as functions of a single parameter t. Thus the equations

$$x^i = x^i(t) \tag{1.1.6}$$

define a curve of X_n.

The derivatives of the functions x^i with respect to the parameter t, are given by

$$\dot{x}^i = \frac{dx^i}{dt} \tag{1.1.7}$$

and is called the tangent vector to C (Rund, 1959).

The length ds of the arc between the points whose coordinates are x^i and $(x^i + dx^i)$ given by

$$ds^2 = g_{ij}\, dx^i dx^j, \qquad i, j = 1,2 \dots \dots n \tag{1.1.8}$$

If S is the arc length of the curve between the points P_1 and P_2 on the curve which correspond to the two points t_1 and t_2 of the parameter t, then

$$S = \int_{t_1}^{t_2} (g_{ij} \frac{dx^i}{dt}\frac{dx^j}{dt})^{1/2} dt \tag{1.1.9}$$

If $g_{ij}\frac{dx^i}{dt}\frac{dx^j}{dt}$ = 0 along a curve, then S = 0 and such a curve is called a minimal curve or null curve. A curve of minimum arc length joining two given points on or a given point and a given hypersurface is called a geodesic (Rund, 1959).

Definition 1.1.7: Line element

The combination of $(x^i, \frac{dx^i}{dt})$ *or* $(x^i, \dot{x}^i)$ is called a line element of the curve C (Rund, 1959). In this combination x^i is the positional co-ordinates and $\dot{x}^i$ is the directional co-ordinates. If $(x^i, \dot{x}^i)$ and $(x^i, \dot{\bar{x}}^i)$ are any two line elements attached to the same centre x^i then the triangle inequality holds good and

$$F\left(x^i, x^i + \dot{\bar{x}}^i\right) \leq F(\left(x^i, \dot{x}^i\right) + F((x^i, \dot{\bar{x}}^i) \tag{1.1.10}$$

where $\leq$ occurs if and only if there exists a relation of the form $\dot{x}^i = v\dot{\bar{x}}^i$ with $v > 0$.

Definition 1.1.8: Riemannian space

Riemann defined element of length in n-dimensions by means of a quadratic differential

$$ds^2 = eg_{ij}\,dx^i dx^j \qquad (1.1.11)$$

where e is positive or negative so the right hand side of (1.1.8) shall be positive unless it is zero and the *g's* are functions of the *x's* (Eisenhart, 1925). If the function F is of the form

$$F\left(x^i,dx^i\right) = \sqrt{g_{ij}\,(x^k)dx^i dx^j} \qquad (1.1.12)$$

where $g_{ij}(x^k)$ are coefficients that are independent of the $dx^i{'}s$ then the metric defined by (1.1.12) is a metric of a Riemannian space. This space is one that cannot be covered with a set of rectangular coordinates (not Euclidean).

Definition 1.1.9: Finsler space

We consider a function $F(x^i,\dot{x}^i)$ of the line element $(x^i,\dot{x}^i)$ of curves defined in X_n. We assume that the function F is at least of class C^5 in all its *2n* arguments. Suppose that this function also defines the distance between points $P(x^i)$ and $Q(x^i + dx^i)$ of C as

$$ds = F(x^i + dx^i) \qquad (1.1.13)$$

then the space X_n equipped with fundamental function defining the metric (1.1.13), is called a Finsler space (Rund, 1959) provided $F(x^i,\dot{x}^i)$ satisfies the three conditions below:

Condition I: The function $F(x^i,\ \dot{x}^i)$ is positively homogeneous of degree one in the directional co-ordinate $\dot{x}^i$, that is

$$F\left(x^i,k\dot{x}^i\right) = kF(x^i,\dot{x}^i), \quad \text{with } k > 0, \qquad (1.1.14)$$

This condition is necessary and sufficient for the integral $\int_{t_1}^{t_2} F(x^i,\, \dot{x}^i)dt$ to be independent of the parameter t.

By Euler's theorem on homogeneous functions, *condition I* implies

$$\dot{x}^i \dot{\partial}_i F\left(x^i, \dot{x}^i\right) = F(x^i, \dot{x}^i) \tag{1.1.15}$$

and also

$$\partial^2_{ij} F\left(x^i, \dot{x}^i\right)\left(\dot{x}^i\right) = 0 \tag{1.1.16}$$

Condition II: The function $F(x^i, \dot{x}^i)$ is positive unless when all its variables $\dot{x}^i$ vanish simultaneously, that is

$$F\left(x^i, \dot{x}^i\right) > 0 \ with \sum_i (\dot{x}^i)^2 \neq 0 \tag{1.1.17}$$

This condition is included because we assume that distances are naturally positive.

Condition III: The quadratic form

$$F^2{}_{\dot{x}^i \dot{x}^j}\left(x^i, \dot{x}^i\right)\xi^i \xi^j = \partial^2_{ij} F^2(\left(x^i, \dot{x}^i\right)\xi^i \xi^j \tag{1.1.18}$$

is assumed to be positive definite for all variables ξ^i and for any line element $F\left(x^i,\right.$ $\left.\dot{x}^i\right)$. Thus only positive values are assumed by (1.1.18) unless ξ^i vanish simultaneously.

On applying Euler's formula of homogenous function, the *Condition I* implies

$$F_{\dot{x}^i}\left(x^i, \dot{x}^i\right) = F\left(x^i, \dot{x}^i\right) \tag{1.1.19}$$

and

$$F_{\dot{x}^i \dot{x}^j}\left(x^i, \dot{x}^j\right)\dot{x}^i = 0 \tag{1.1.20}$$

By writing $g_{ij}(x, \dot{x}) = \frac{1}{2}\dot{\partial}^2 F^2\left(x^i, \dot{x}^i\right)$, we have from theory of quadratic form and *Condition III* that the determinant

$$\left|g_{ij}\left(x^i, \dot{x}^i\right)\right| = g\left(x^i, \dot{x}^i\right) > 0 \tag{1.1.21}$$

for all elements $(x^i, \dot{x}^i)$.

In this thesis we have denoted an n-dimensional Finsler space by F_n .

Definition 1.1.10: Tangent space

We consider a change of local co-ordinates as represented by transformation equations (1.1.1). Along the curve represented by equation (1.1.6) the new components of the tangent vector are obtained by differentiating the relations

$$\bar{x}^i = \bar{x}^i(x^i(t)) \qquad (1.1.22a)$$

with respect to *t*. This gives

$$d\bar{x}^i = \frac{\partial \bar{x}^i}{\partial x^i} dx^i \qquad (1.1.22b)$$

An *n*-dimensional linear vector space obeying the transformation law given by equation (1.1.22b) is called the tangent space at the point $P(x^i)$ of X_n.

The tangent space at a point $P(x^i)$ of F_n will be denoted by $T_n(P)$ *or* $T_n(x^i)$ (Rund 1959, Vanger and Whitehead, 1932). Since the transformation (1.1.22a) is homogeneous, the tangent space may be regarded as centered affine space with origin at the point P. It is noted that a transformation (1.1.1) in X_n induces a linear transformation (1.1.22a) in each of the tangent spaces. The usual laws of vector algebra apply to the elements of the tangent space $T_n(P)$.

Definition 1.1.11: Indicatrix

Let us consider the metric function $F(x^i, \dot{x}^i)$ defined for all line elements $(x^i, \dot{x}^i)$ in Finsler space F_n and satisfying *Conditions I,II* and *III* in Definition (1.1.9). We also consider a fixed point $P(x^i)$ of X_n and corresponding tangent space $T_n(P)$. The equation

$$F(x^i,\dot{x}^i) = 1 \tag{1.1.23}$$

for which x^i is fixed and $\dot{x}^i$ variable, represents an *(n-1)* dimensional locus in $T_n(P)$ which is a hypersurface of $T_n(P)$. The hypersurface (1.1.23) of $T_n(P)$ is called indicatrix. It plays the role of unit sphere in the geometry of tangent space (Vanger, 1949). We also note that the function $F(x^i,\dot{x}^i)$ of (1.1.23) uniquely defines the metric function of $T_n(P)$. A metric function in F_n induces a metric $T_n(P)$ and conversely a metric of $T_n(P)$ determines locally the metric F_n.

Definition 1.1.12: Minkowskian space

A vector space whose metric function satisfies the *Conditions I, II* and *III* is called Minkowskian space (Berwald 1947, Bluementhal 1953, Rund 1959). The metric of the Finsler space may therefore be regarded as locally Minkowskian just as the metric of the Riemannian space is locally Euclidean. The indicatrix is symmetric about the origin on imposing the condition below:

$$F(x^i,\dot{x}^i) = F(x^i,-\dot{x}^i) \tag{1.1.24}$$

Let us denote the Minskowskian distance between distinct points $\dot{x}^i$ and $\bar{x}^i$ of T_n by $F(x^i,\dot{x}^i - \dot{\bar{x}}^i)$, then

$$F(x^i,\dot{x}^i - \dot{\bar{x}}^i) = F(x^i,\dot{\bar{x}}^i - \dot{x}^i) \tag{1.1.25}$$

Definition 1.1.13: Tensor in n-dimensional space

Mathematically a tensor in an n-dimensional space is a set of N^r functions or quantities which transform according to certain rules under change of co-ordinate systems.

The rank of a tensor, r represents the number of dimensions an array of numbers or functions has to have to represent a tensor.

1.2 Background information on curvature tensors

The theory of curvature describes the essential differences between the underlying manifold and its tangent spaces. This cannot be done in terms of connection coefficients since they do not possess suitable transformation properties that satisfy equation (1.1.1) and therefore it had to be done with the help of a set of tensors which satisfy a large number of identities. The simplest analytical approach to this problem was based on the fact that covariant differentiation is not in general commutative. Berwald (1926) in his scientific work used tensors to set up the basic theory of curvature. He used the connection coefficient $G^i_{jk}(x,\dot{x})$ for the definition of covariant derivatives and commutation formulae for repeated covariant derivatives gave rise to the required curvature tensors such as the Relative curvature tensor, $\tilde{K}^i_{jkh}$ and the Berwald curvature tensor, H^i_{jkh}. These tensors have close affinity to the projective curvature tensor, W^i_{jkh} which is the subject of our research.

1.2.1 Curvature tensors in Finsler space

The property that distinguishes a curved space from a flat space is curvature of the space and this is defined by the curvature tensor.

We consider a vector $X^i(x^k, \xi^k)$ depending on the position and direction vector field $\xi^k = \xi^k(x^k)$. The covariant δ-derivative of X^i at the point x^k in the direction ξ^k is given by

$$X^i_{;h} = \partial_h X^i + \dot{\partial}_i X^i \partial_h \xi^l + \Gamma^{*i}_{rh}(x,\xi) X^r \tag{1.2.1}$$

while

$$X^i_{jhk} = \partial_k (X^i; h) + \dot{\partial}_l (X^i; h) \partial_k \xi^l + \Gamma^{*i}_{jk} X^j_{;h} - \Gamma^{*i}_{jk} X^i_{;j} \tag{1.2.2}$$

Forming the derivative $X^i_{;kh}$ by interchanging h and k and using the tensor difference equation $2T_{[jk]} = T_{jk} - T_{kj}$

$$2X^i_{;[kh]} = \tilde{K}^i_{jkh} X^j \tag{1.2.3}$$

where

$$\tilde{K}^i_{jkh} = 2\partial_{[h} \Gamma^{*i}_{k]j} + 2\dot{\partial}_l \Gamma^{*i}_{j[k} \partial_{h]} \xi^l + 2\Gamma^{*i}_{m[h} \Gamma^{*i}_{k]j} \tag{1.2.4}$$

gives rise to the tensor $\tilde{K}^i_{jkh}(x,\xi)$ which is called Relative curvature tensor (Rund, 1954).

The relative curvature tensor satisfies the following identities

$$\tilde{K}^i_{jkh} = -\tilde{K}^i_{jhk} \tag{1.2.5}$$

and

$$\tilde{K}^i_{[jkh]} = 0 \tag{1.2.6}$$

The Bianchi identity satisfied by the relative curvature tensor is given by

$$\tilde{K}^r_{i[jk;h]} = 0 \tag{1.2.7}$$

Let the vector field ξ^l be stationary at a point under consideration, thus it satisfies the relation

$\xi^i_{;h}(x,\xi) = 0$. It follows that

$$\partial_h \xi^l = -\Gamma^{*l}_{rh}(x,\xi)\xi^r = -\dot{\partial}_h G^l(x,\xi) \qquad (1.2.8)$$

Substituting (1.2.8) into (1.2.1), we obtain Cartan's covariant derivative $X^i_{\mathrm{I}h}$ provided the line element ξ^l is regarded as the element of support $\dot{x}^l$.

Thus the tensor (1.2.4) becomes

$$K^i_{jkh} \overset{\text{def}}{=} 2\partial_{[h}\Gamma^{*i}_{k]j} + 2\dot{\partial}_l \Gamma^{*i}_{j[h}\partial_{k]}G^l + 2\Gamma^{*i}_{m[h}\Gamma^{*m}_{k]j} \qquad (1.2.9)$$

This curvature tensor depends only on the line element, thus only on position and direction.

If we apply Cartan's theory of element of support in which case $\xi^i = \dot{x}^i$ and using the definition (1.2.9), we obtain

$$2X^i_{\mathrm{I}[hk]} = K^i_{jkh}X^j - \dot{\partial}_j X^i K^j_{rkh}\dot{x}^r \qquad (1.2.10)$$

The curvature tensor K^i_{jkh} satisfies the identities

$$K^i_{jkh} = -K^i_{jhk} \qquad (1.2.11)$$

and

$$K^i_{[jkh]} = 0 \qquad (1.2.12)$$

We then write

$$K_{ijhk} = g_{rj}K^r_{ihk} \qquad (1.2.13)$$

In view of (1.2.9), (1.2.13) and $g_{ij\mathrm{I}k} = 0$, we have

$$K_{jikh} = -K_{ijkh} - 2C_{ijl}K^l_{rkh}\dot{x}^r \qquad (1.2.14)$$

This in contrast to Riemannian geometry the curvature tensor in F_n is not skew symmetric with respect to the first two indices.

The curvature tensor K^i_{jkh} satisfies the Bianchi identity

$$K^i_{l[jhIk]} + \dot{\partial}_l \Gamma^{*r}_{i[j} K^l_{\mathrm{IsI}hk]} \dot{x}^s = 0 \qquad (1.2.15)$$

1.2.2 Curvature tensors arising from Berwald's connection

The equation of geodesic deviation has been given in the form below by Rund (1959)

$$\frac{\delta^2 z^i}{\delta u^2} + H^i_k(x, \dot{x}) z^i = 0 \qquad (1.2.16)$$

where the vector z^i is called the variation vector while the tensor $H^i_k(x, \dot{x})$is called the deviation tensor which is defined by,

$$H^i_k(x, \dot{x}) = K^i_{jhk} \dot{x}^j \dot{x}^h \qquad (1.2.17)$$

The deviation tensor can also be written as

$$H^i_k = 2\partial_k G^i - \partial_h \dot{\partial}_k G^i \dot{x}^h + 2G^i_{kl} G^l - \dot{\partial}_l G^i \dot{\partial}_k G^l \qquad (1.2.18)$$

where the function $G^l(x, \dot{x})$ is positively homogeneous of degree two in x^i.

The curvature tensors H^i_{jk} is called torsion tensor and is defined by

$$H^i_{jk} = \frac{2}{3} \dot{\partial}_{[j} H^i_{k]} \qquad (1.2.19)$$

The tensor H^i_{hjk} is called Berwald's curvature tensor field and is defined by

$$H^i_{hjk} = \dot{\partial}_h H^i_{jk} \qquad (1.2.20)$$

In terms of $G^l(x, \dot{x})$ the tensors H^i_{hjk} and H^i_{jk}can be expressed as:

$$H^i_{jk} = 2\partial_{[k} \dot{\partial}_{j]} G^i + 2G^i_{r[k} \dot{\partial}_{j]} G^r \qquad (1.2.21)$$

and

$$H^i_{hjk} = 2\partial_{[k} G^i_{j]h} + 2G^r_{h[j} G^i_{k]r} + 2G^i_{rh[k} \dot{\partial}_{j]} G^r \tag{1.2.22}$$

In the above equations

$$G^i_{hjk} = \dot{\partial}_h G^i_{jk} \tag{1.2.23}$$

which satisfies

$$G^i_{hjk} \dot{x}^h = 0 \tag{1.2.24}$$

The Berwald curvature tensor H^i_{hjk} and the Cartan's curvature tensor K^i_{hjk} are related by the equations below:

$$H^i_{hjk} = K^i_{hjk} + \dot{\partial}_h K^i_{rjk} \dot{x}^r \tag{1.2.25}$$

and

$$H_{ihjk} \dot{x}^i = K_{ihjk} \dot{x}^i \tag{1.2.16a}$$

where

$$H_{ihjk} = g_{lh} H^l_{ijk} \tag{1.2.26b}$$

On contraction of the indices h and k in the tensors $H^h_k, H^h_{jk}, H^h_{ijk}$, we obtain

$$H^h_h = (n-1)H \tag{1.2.27}$$

$$H^h_{ih} = H_i \tag{1.2.28}$$

and

$$H^h_{ijh} = H_{ij} = \dot{\partial}_i H_j \tag{1.2.29}$$

By applying equations (1.2.22) and (1.2.29), we deduce that

$$H^h_{hij} = 2H_{[ji]} \tag{1.2.30}$$

1.3 Problem statement and justification

The study of Finsler spaces which is commonly known as Finsler geometry has continued to grow and develop since its introduction in 1854 and various theories have been developed. With particular reference to the theory of curvature a lot of research has been done with the most recent work being on curvature inheritance symmetry and curvature collineation symmetry and decomposition. However, these studies have concentrated on, *H*- curvature, $\widetilde{K}$- curvature and *N*-curvature tensors and not much work has been done on the mentioned properties for the *W*- projective curvature tensor which is closely related to *H*-curvature tensor. The *W*- projective curvature tensor is not a generalised curvature tensor and therefore it exhibits some geometric properties that are different from those of other generalised tensors. Most of the previous studies and investigations have also concentrated on recurrent Finsler space and hence this study deals with bi-recurrent Finsler space.

1.4 Objectives of the study

Main Objective: To use a Finsler framework and geometrical methods to investigate the inheritance, collineation and decomposition properties of the Weyl projective curvature tensor $W^i_{jkh}(x,\dot{x})$ in bi-recurrent Finsler space.

Specific Objectives

i. To develop properties of W- curvature inheritance in bi- recurrent Finsler space.

ii. To study properties of W- curvature collineation in bi- recurrent Finsler space.

iii. To verify the developed properties for W- curvature inheritance and W- curvature collineation by using symbolic algorithms.

iv. To develop factorization algorithms for tensor decomposition of W-curvature tensor in recurrent and bi-recurrent and study the properties of the decomposition tensor.

1.5 Significance of study

This study has established some relations in the area of W-curvature inheritance, W-curvature collineation and W- curvature decomposition in bi-recurrent Finsler space and by doing so new results have been realised. The results of the study have provided new and valuable information on curvature inheritance, collineation and decomposition for the Weyl projective curvature tensor $W^i_{jkh}(x,\dot{x})$. Tensors in Finsler spaces being mathematical objects which can be used to represent real world systems have geometrical significance and have been used in several applications in Physics, Biology and Computer Science as well as other fields of study. For example a variety of problems from general relativity, thermodynamics, optics, diffusion imaging, ecology, signal processing and machine learning are now being modelled using a tensorial approach. The results of this study can also be used in similar applications. In the decomposition of the W- projective curvature tensor the study has broken down the Weyl projective curvature tensor $W^i_{jkh}(x,\dot{x})$ and expressed it in terms of other curvature tensors which make it easy for application.

1.6 Outline of the Thesis

In Chapter One the basic concepts of Finsler geometry which include curvature and tensors are reviewed. The known results are discussed in form of definitions, equations and Theorems. The problem statement, objectives and significance of the study have also been given in this chapter.

In Chapter Two, Literature on the work related to this research is discussed with the focus on bi-recurrence, inheritance, collineation and decomposition properties of W-projective curvature tensor.

In Chapter Three, symbolic and geometrical computation tools, materials and methods required for tensor computations algorithms are discussed.

The results and discussion of this research study are presented in chapter Four and in chapter Five.

In Chapter Four, the key research findings for the bi-recurrence, inheritance and collineation properties of W- projective curvature tensor are presented. In Chapter Five, four different factorization algorithms have been developed for the decomposition of W- projective curvature tensor. Symbolic algorithms are used in the discussion and results given in form of Theorems.

In Chapter Six, the conclusions, recommendations and suggestions for further research related to this study are given.

The references and some work related to this research which has been published are given at the end of the thesis.

CHAPTER TWO

LITERATURE REVIEW

In this chapter previous studies that have been done on recurrent Finsler spaces with reference to the W- projective curvature tensor $W^i_{jkh}(x, \dot{x})$ and its inheritance, collineation and decomposition properties is presented. The chapter is divided into five sections. In section 2.1 the literature on the historical development of Finsler spaces and the studies in the recent past has been discussed. In section 2.2 the recurrence in Finsler spaces, recurrent and bi-recurrent Finsler spaces are examined. Sections 2.3 and Section 2.4 have paid particular attention to the previous studies on the two symmetry properties studied in this thesis namely curvature inheritance and curvature collineation respectively. Finally in Section 2.5 previous works on the decomposition of curvature tensor fields in Finsler space is presented.

2.1 Development of the geometry of Finsler space

Finsler geometry is a kind of differential geometry and is considered as a generalization of the Riemannian geometry. The fundamental idea of a Finsler space may be traced back to Bernhard Riemann who in 1854 discussed various possibilities by means of which an *n*-dimensional manifold may be endowed with a metric, (Rund, 1959). He introduced the notion of curvature for spaces with a family of inner products and paid particular attention to a metric defined by the positive square root of a positive definite quadratic differential form. The first systematic study of manifolds endowed with such metric came more than 60 years later in 1918 when Paul Finsler studied the variation problem in regular metric spaces, (Finsler, 1918). Finsler also studied the geometry of spaces with generalized metric which were eventually named after him. The study of a Finsler space is called Finsler Geometry and Finsler's thesis of 1918 is regarded as the first in this direction. In his thesis Finsler generalised a number of theorems of classical differential geometry, (Yano, 1957).

Finsler did not make use of the tensor calculus, he was guided in principle by the notations of the calculus of variations and he therefore did not investigate the geometric properties of Finsler spaces broadly. In the middle 1920s methods of tensor calculus were applied to the theory independently but almost simultaneously by Synge (1925), Taylor (1925) and Berwald (1926). It was found that the derivatives of half of the square of $F(x, dx)$ with respect to the differentials, dx served well as components of a metric tensor in analogy with Riemannian geometry. From the differential equations of the geodesic connection coefficients could be derived by means of which a generalization of Levi-Civitas parallel displacement could be defined, (Rund, 1959).

While the corresponding covariant derivatives as introduced by Synge (1925) and Taylor (1925) coincide, the theory of Berwald shows a marked difference. Berwald was the first to introduce the concept of connection in Finsler geometry and defined a connection called Berwald connection, (Berwald, 1926). In this geometry, the lemma of Ricci which in Riemannian geometry implies the vanishing of the covariant derivative of the metric tensor is no longer valid. Nevertheless Berwald continued to develop his theory with particular reference to the theory of curvatureas well as to two dimensional spaces. The significance of his work was enhanced by the advent of the general geometry of paths**,** a generalization of the so-called Non- Riemannian geometry due to

Douglas (1927) and Knebelman (1929). The initial approach of Berwald was to establish a close affinity between these branches of metric and non-metric differential geometry.

The theory took a new and unexpected turn in 1934 when Cartan published his tract on Finsler spaces in which he introduced and defined a connection called Cartan connection, (Cartan, 1934). He showed that it was indeed possible to define connection coefficients and hence a covariant derivative such that the preservation of Ricci's lemma was ensured. On this basis Cartan developed a theory of curvature and particularly all subsequent investigations concerning the geometry of Finsler spaces were dominated by this approach. In 1943, Chern introduced a connection called the Chern connection, Chern (1943) and Bao and Chern (1943). Several mathematicians such as Wagner (1949), Davies (1955), Golab, Hombu and Varga (1955) have studied Finsler geometry along Cartan's approach. They expressed the opinion that the theory had thus attained its final form. However, this was not totally true.

The above mentioned theories make use of a device which basically involves the consideration of a space whose elements are not the points of the underlying manifold but the line elements of the later which form a $(2n-1)$ dimensional variety. This facilitates the introduction of what Cartan calls the "Euclidean connection" which may be derived uniquely from the fundamental metric $F(x, dx)$. The method also depends on the introduction of the so called "element of support", which implies that at each point a previously assigned direction must be given, which then serves as a directional argument in all functions depending on direction as well as position. Thus for instance the length of a vector and the vector obtained from it by an infinitesimal parallel displacement depend on the arbitrary choice of the element of support. It is this device which led to the development of Finsler geometry in terms of direct generalizations of methods of Riemannian geometry.

In 1950 Hano Rund introduced a new process of parallelism from the point of Minkowskian geometry, (Rund, 1950). Cartan had introduced parallelism from the point of locally Euclidean geometry. Later on Davies (1955) and Deicke (1955) showed that the two concepts of parallelism were the same. Several mathematicians such as Barthel (1955), Deicke (1955) and Laugwitz (1965), have studied Finsler spaces on Rund's approach and obtained various results.

In 1959 Rund published his book "The Differential Geometry of Finsler Spaces" (Rund, 1959) which became the key text in the area. Since then Finsler geometry continued to develop and important contributions resulted one after the other. In 1963 Akabar developed the modern theory of Finsler spaces based on the geometry of connections of fibre bundles. The reason for modernization was to establish a global definition of connections in Finsler spaces and to re-examine the Cartan's system of axioms. In 1970 Matsumoto organised a symposium on the models of Finsler spaces. The symposium saw the emergence of a number of mathematicians and physicists such as Sinha and Singh (1971), Pandey (1978) who began to study special Finsler spaces. Singh (1978) studied and published results on some tensors in Finsler space.

The book "Foundations of Finsler geometry and special Finsler spaces"(Matsumoto, 1986) attracted special attention towards special Finsler spaces such as Berwald space, Minkowskian space, C-reducible Finsler space, Finsler space with (α, β)-metric, Randers space, Matsumoto space and recurrent space and bi-recurrent space among others.

2.2 Recurrent and Bi-recurrent Finsler spaces

Finsler spaces have different connections, because of this the recurrence of different curvature tensors have been studied by various mathematicians. The spaces with recurrent curvature were first studied by Ruse in 1949. He considered a three dimensional Riemannian space having the recurrent curvature tensor and he called such a space Riemannian space of recurrent curvature (Ruse, 1949). This idea was extended to n-dimensional Riemannian and non-Riemannian space. Walker (1950) discussed Ruse's Space of recurrent curvature, Worng and Yano (1961) studied projectively flat spaces with recurrent curvature. Worng (1962) discussed linear connections with zero torsion and recurrent curvature. The idea was extended to Finsler spaces for the first time by Moor (1963). Recurrent tensor fields and generalized Finsler spaces of recurrent curvature were studied by Sinha and Singh (1968, 1969). The book "Integral formulae in Riemannian geometry" was published by Yano (1970) and it highlighted symbolic relations involving recurrence. The relativistic significance of curvature tensors was discussed by Pokhariyal and Mishra (1970) and this encouraged more studies in the area. The concept of C^h-recurrent was introduced (Matsumoto, 1971). Certain theorems on the recurrence vector field, the curvature tensor fields and the projective tensor fields in a recurrent Finsler space were obtained by Sinha and Singh (1971). They used these to define recurrent Finsler spaces of second order. Sinha and Singh (1972) studied a special Kawaguchi space of recurrent curvature tensor field of second order. Further work by Sinha and Singh (1973) was dedicated to the study of the properties of the projective tensor and the recurrent tensor fields in recurrent Finsler spaces of second order. In 1973, Mishra published his book "Tensor Analysis with Applications" which became a useful text.

In literature there are several works on recurrence in Finsler spaces done in the mid and late 1970s. In the 1980s a lot of work was done in the area by several mathematicians including Pandey and by Singh. A recurrent Finsler manifold with a cocircular vector field was discussed by Pandey (1980) and on further work he discussed the recurrence vector (Pandey, 1981). Dubbey and Srivastava (1981) studied recurrent Finsler spaces while projective recurrent Finsler manifold I was studied by Pandey and Mishra (1981). Singh (1980, 1981) studied the recurrent Finsler space and curvature field of second order which he extended to bi-recurrent generalised Finsler space (Singh, 1982). The concept of a Lie- recurrence was introduced by Pandey (1982). He concluded that it is an infinitesimal transformation with respect to which the Lie-derivative of curvature tensor is proportional to itself. Pandey also established that Weyl projective curvature tensor is Lie-recurrent with respect to a Lie-recurrence but its converse is not necessarily true. However, an infinitesimal transformation with respect to which Weyl projective curvature tensor as well as Ricci tensor is Lie- recurrent, is necessarily a Lie-recurrence. Results on the third order recurrent Finsler spaces were obtained in (Pandey and Tripathi, 1982) and this was extended to T- recurrent Finsler spaces by (Pandey and Dwivedi, 1987).

During the 1990s the recurrence of different curvature tensors such as *R*, *H*, *K*, and *N* continued to be an area of interest and new concepts were studied and discussed. The recurrence of Cartan's curvature tensor R^i_{jkh}Verma (1991) while Upal (1991) studied the properties of tensors in generalised recurrent Finsler space. Generalised recurrent manifolds were studied by De and Guha (1991). A concurrent vector field on a generalized Finsler space was introduced by Singh (1991) and on further work he discussed a symmetric tensor in the same space (Singh, 1993). Dikshit (1992) studied types of recurrences and introduced a Finsler space whose Berwald curvature tensor H^i_{jkh} satisfies the recurrence property in the sense of Berwald. With increase in applications of recurrence in Finsler spaces, the book on "The theory of sprays and Finsler spaces with applications to physics and biology" was published (Antonelli, Ingarden & Matsumoto, 1993). Gatoto (1996) also discussed various aspects of recurrence in Finsler spaces. The generalized recurrent and concircular recurrent manifolds were introduced and discussed by Muralebhavi and Rathamma (1999).

Singh (1999) investigated affine and projective motion with contra field in Finsler space and extended this to the study of Projective motion in Finsler spaces Singh (2002). Pandey and Pal (2003) studied hypersurface of recurrent Finsler spaces. On further work Singh (2006, 2007, 2011) studied projective motion in recurrent and bi-recurrent Finsler spaces and proved various theorems on the same. The concept of C^h-recurrent Finsler space introduced earlier by Matsumoto has been extended to C^h-recurrent torsion tensor field of second order by Mishra and Lodhi (2008). Singh (2010) investigated projective motion in bi-recurrent Finsler spaces. The recurrence of tensor fields in Finsler spaces has continued to be an active area of research into the current decade where *U, H, K, R,* and *BR*- recurrent spaces have been studied by various mathematicians.

Qasem and Saleem (2010) discussed general Finsler spaces for hv curvature tensor U^i_{jkh} satisfying the bi-recurrent property with respect to Berwald coefficient G^i_{jk} and called it *UBR*- Finsler space. A Finsler space whose Berwald curvature tensor satisfies the generalized recurrence property in the sense of Berwald was introduced by Pandey, et al., (2011). They called it *H*- recurrent Finsler space. Pandey and Kumari (2016) studied and proved certain results for recurrent and projective recurrent Finsler space of dimension greater than two. Qasem and Al-Qashbari (2016) introduced a generalized H^h- recurrent space and studied properties of the space. They also defined a generalised R^h- recurrent space and obtained some identities that are satisfied in such a space. On further research Qasem and Abdallah (2016) defined a generalized *BR*- recurrent space. They obtained the necessary and sufficient conditions for the Berwald curvature tensor and Cartan's fourth curvature tensor to be generalized recurrent. The recurrence of generalized second order in a Finsler space equipped with a non-symmetric connection has been studied by Srivastava and Mishra (2016). They made a study of special bi-recurrent and special generalized bi-recurrent Finsler spaces of first and second kind and obtained significant results. They also studied the recurrence of third order in such a space with reference to the first curvature tensor. The generalized *BK*-recurrent space was introduced by Qasem and Baleedi (2016). This is a space whose Cartan's fourth order satisfies a recurrence relation. They showed that the K- Ricci tensor, the curvature vectors and curvature scalar are non-vanishing in the *BK*-recurrent space.

2.3 Curvature inheritance in Finsler spaces

The concept of Lie-recurrence was introduced by Pandey in 1982. It was defined as an infinitesimal transformation $\bar{x}^i = x^i + \in V^i(x^j)$ in which the Lie-derivative (L_v) of the curvature tensor is proportional to itself. Pandey (1982) also established that the Weyl Projective curvature tensor, W^i_{jkh} is Lie-recurrent with respect to a Lie-recurrence but the converse is not true. Duggal (1992) investigated curvature inheritance symmetry in Riemannian spaces and he also identified various applications of the same to fluid space times. There was not much work done in this area during the last decade of the 20^{th} Century. However the topic became an active area of research in subsequent years. Singh (2003) studied an infinitesimal transformation with respect to which the Lie-derivative of curvature tensor is proportional to itself and called such transformation curvature inheritance. A curvature inheritance can therefore be seen as a Lie-recurrence. On further work Singh (2004) considered a curvature inheritance which is a projective motion and called it projective curvature inheritance. He also studied the cases generalized by contra and concurrent vector fields and obtained several results. Mishra and Yadav (2007) investigated projective curvature in an NP-F_n. Gatoto and Singh (2008) studied $\tilde{K}$- curvature inheritance and projective $\tilde{K}$- curvature inheritance. The concept of *K*-curvature was also studied by Pandey and Pandey (2009) who established theorems on the same and proved that an affine motion is a *K*-curvature inheritance. In their studies Saxena and Pandey (2011) generalized several theorems and concluded that no contra vector field or concurrent vector field can generate Lie-recurrence and projective Lie- recurrence. On further work Saxena and Pandey (2011) studied properties of Weyl projective curvature tensor by considering conformal transformation. In 2012 the *H*- curvature inheritance in bi-recurrent Finsler spaces was defined (Singh, 2012). Mishra and Lodhi (2012) studied the curvature inheritance symmetry in Finsler spaces and Ricci-inheriting symmetry and investigated some results. In an extension of their previous work, Pandey and Saxena (2012) introduced and discussed *N*-curvature inheritance in Finsler space. The properties of curvature inheritance in $R-\oplus$ recurrent were investigated and various theorems established (Gatoto and Singh, 2013). Curvature inheritance for the Weyl curvature tensor field was introduced and called W-curvature inheritance (Singh, 2013). This study was restricted to recurrent Finsler spaces only. The *N*-curvature inheritance in recurrent Finsler spaces was defined and studied by Gatoto (2014) and several results were obtained and special cases discussed. Opondo and Singh (2014) defined and studied the concept of W- curvature inheritance in bi-recurrent Finsler space an obtained new results which have become part of this thesis.

In recent years curvature inheritance symmetry has been studied for physical significance and related applications. Calvaruso and Zaemi (2016) studied some applications of curvature inheritance symmetry in three manifolds with recurrent curvature. Duggal (2017) established the link between curvature inheritance symmetry and Ricci solitons using mathematical models.

2.4 Curvature collineation in Finsler spaces

There are about 25 different types of curvature collineations that have been studied (Zafar, 2014). The literature of such collineations is very large and is still expanding. In this study only collineations related to the Weyl curvature tensor field and collineations in recurrent Finsler spaces have been discussed. The study of collineations took shape when Pande, Kumar and Dubey (1978) investigated the different cases under which a special conformal motion is a projective curvature collineation. *CA*-collineation in a bi-recurrent Finsler manifold was defined and studied by Pandey (1978).

Singh and Singh (1981) have defined *N*-curvature collineations and discussed the existence of *N*-curvature collineations of different types in Finsler spaces. Singh (1984) defined the *H*-curvature collineation of a Finsler space and proved that every homothetic transformation admitted in a Finsler space is *H*-curvature collineation. The projective curvature collineation for Berwald curvature tensor was developed by Singh (1987). He discussed the collineation property for curvature tensors H^i_{jkh} and R^{*i}_{jkh} in Finsler spaces and later extended to Finsler spaces of order *II,* Singh (1987). Gatoto (1996) discussed various aspects of Finsler spaces including projective curvature collineation in Finsler spaces. Singh (1999) established a number of results on projective motion and projective curvature collineations in Finsler spaces. In 2001, Singh developed special concircular projective curvature collineation in recurrent Finsler space and concluded that the scalar function in such a case is homogeneous of degree zero. Kashif (2003) studied curvature collineation of some spacetimes and discussed their physical interpretation. Curvature collineations for the curvature tensor constructed form a Bianchi Type-V metric were studied by Tiwari (2005). The study was concerned with a symmetry property of space time and he also discussed the physical and kinematical properties of the model.

Mishra and Yadav (2007) have considered the transformation which defines a projective motion for normal projective curvature tensor field N^i_{jkh}. Uppal and Singh (2008) have studied the Q- curvature collineation in projective Finsler space. Misra and Lodhi (2010) defined Torse forming N- curvature collineations and studied the corresponding properties. Saxena and Pandey (2011) have studied projective Lie-recurrence for Weyl projective curvature tensor with respect to projective motion. They concluded that this is a form of curvature collineation. The properties of the W-curvature collineation in Finsler spaces for Weyl curvature tensor field were studied by Singh (2013). He considered an infinitesimal transformation and obtained several results for recurrent spaces. Zafar and Musavir (2014) established a relationship between W- curvature tensor and its divergence with that of other curvature tensors. He introduced a symmetry of space-time named W-curvature collineation and obtained the conditions under which the space-times of general relativity may admit such collineations. Ahsan and Ali (2017) studied curvature tensor for spacetime of general relativity. Curvature collineation is a fundamental symmetry property of spacetime general relativity remains an active research area. Most recently Donia and Shenawy (2020) studied W^*- curvature tensor on Relativistic spacetimes and investigated the physical significance of its properties.

2.5 Decomposition of curvature tensor

The decomposition of curvature tensor field in Finsler space was first studied by Takano (1961). This was followed by a study of decomposition of the recurrent curvature tensor fields in an affinely connected space (Takano, 1967). Sinha and Singh (1970) have decomposed different recurrent curvature tensor fields in Finsler spaces. Singh (1973) studied decomposition of recurrent curvature tensor field in special Kawaguchi space and Singh (1975) extended the study to Generalized Finsler space. Gama (1978) has discussed similar decomposition for the recurrent tensor in a space of the submetric class. The decomposition of curvature tensor in Finsler manifold was studied by Pandey (1981). The results previously obtained by Singh were used to extend the decomposition to *2-R* Generalised space and new theorems were developed (Singh, 1982). Uppal (1991) studied decomposition of projective curvature tensor field in recurrent Finsler space and Gatoto (1996) studied decomposition of curvature tensor field in conformal Finsler space.

Curvature decomposition in Conformal Finsler Spaces has been studied by several authors. The decomposability of curvature tensor in second order recurrent Conformal Finsler space has been established and various theorems proved, (Singh and Gatoto, 2001). Hakan and Fatma (2007) studied the recurrent Riemannian space having semi symmetric metric connection and the curvature tensor which decomposed in one form. They proved some theorems concerning such space. Mishra and Lodhi (2008) studied the decomposability of the curvature tensor in recurrent conformal Finsler space.

Tamara and Brett (2009) highlighted some applications of W- curvature tensor decomposition in other fields.

The research on decomposition of different curvature tensor fields has continued to grow due to the several application areas available. Thakur and Mishra (2014) extended their research and decomposed the projective curvature tensor in recurrent Finsler space. They also studied the properties of the decomposition tensor. The curvature tensor fields in conformal Finsler space of order *II* have been decomposed and new results obtained by Gatoto and Singh (2015). The normal projective curvature tensor has also been decomposed and some properties of the decomposition tensor field established by Qasem (2015). Due to increasing use of computer based tensorial algorithms, tensor decomposition techniques are widely used in practical applications. Sidiropoulos and Lieven (2017) discussed the use of tensor decomposition in Signal Processing and Machine Learning. There is also a lot of unpublished work on the use of tensor decomposition in data analysis for COVID-19 related research. In this study we have decomposed the Weyl projective curvature tensor in recurrent and bi-recurrent Finsler spaces and compared the properties of the decomposition tensor with those of the original tensor for application purposes.

CHAPTER THREE

MATERIALS AND METHODS

This chapter consists of the necessary concepts and mathematical tools that have been used in the development of the study. These tools have been presented in form of definitions and results which are stated as Theorems. The methodology used is symbolic and geometrical computations involving curvature tensors. The chapter is divided into seven sections and in each section the relevant concept or computation tool is presented.

3.1 Tensor Analysis

3.1.1 Contravariant and covariant vectors

Definition 3.1.1: Contravariant vector

If N quantities $A^1, A^2. . . A^N$ in a co-ordinate system $(x^1, x^2. . . x^N)$ are related to other quantities $\bar{A}^1, \bar{A}^2. . . \bar{A}^N$ in another coordinate system $(\bar{x}^1, \bar{x}^2. . . \bar{x}^N)$ or in another notation $x^i \rightarrow \bar{x}^i$ by the transformation equations (Rund, 1959)

$$\bar{A}^i = \sum_{i=1}^{N} \frac{\partial \bar{x}^i}{\partial x^j} A^j \quad , \ i = 1,2 \dots N \qquad (3.1.1)$$

by using the Eistein's summation convention, we drop the sigma sign and write

$$\bar{A}^i = \frac{\partial \bar{x}^i}{\partial x^j} A^j \qquad (3.1.2)$$

then, these are called components of a contravariant vector or a contravariant tensor of rank one.

The rule for partial differentiation gives $dx^i = \frac{\partial x^i}{\partial x^j} dx^j$, hence the coordinate differential is a contravariant vector.

Definition 3.1.2: Covariant vector

A vector A_i is said to be a covariant vector if under the coordinate transformation $x^i \rightarrow \bar{x}^i$ the A_is transform according to the law

$$\bar{A}_i = \frac{\partial x^j}{\partial \bar{x}^i} A_j \qquad (3.1.3)$$

Generally, a superscript is used to indicate contravariant components whereas a subscript is used to indicate covariant components except in the notation for coordinates. Any index which is repeated in a given term is called a dummy index.

3.1.2 Contravariant and covariant tensors

Tensors can be viewed as a generalized form of mathematical quantities such as scalars, vectors and vectors of higher rank. One advantage of tensors is that if physical laws are expressed as relations among tensors they will be valid for any choice of coordinates.

Definition 3.1.3: Contravariant tensor

If N^2 quantities A^{qs} in a co-ordinate system $(x^1, x^2 . . . x^N)$ are related to other quantities $\bar{A}^{pr}$ in another coordinate system $(\bar{x}^1, \bar{x}^2 . . . \bar{x}^N)$ or in another notation $x^i \rightarrow \bar{x}^i$ by the transformation equations

$$\bar{A}^{pr} = \sum_{s=1}^{N} \sum_{q=1}^{n} \frac{\partial \bar{x}^p}{\partial x^q} \frac{\partial \bar{x}^r}{\partial x^s} A^{qs} \quad , \; p, r = 1,2 \ldots N \qquad (3.1.4)$$

they are called contravariant components of rank two (Eisenhart, 1925). In convention form we write

$$\bar{A}^{pr} = \frac{\partial \bar{x}^p}{\partial x^q} \frac{\partial \bar{x}^r}{\partial x^s} A^{qs} \qquad (3.1.5)$$

Generally, the set of N^r quantities $T^{i_1 - i_r}$ is said to constitute the components of a contravariant tensor of rank r at a point in an N-dimensional space if under the coordinate transformation $\bar{x}^j = \bar{x}^j(x^i)$ these quantities transform according to the law

$$\bar{T}^{i_1 \dots i_r} = \frac{\partial \bar{x}^{i_1}}{\partial x^{j_1}} \cdots \frac{\partial \bar{x}^{i_r}}{\partial x^{j_r}} T^{j_1 \dots j_r} \qquad (3.1.6)$$

Definition 3.1.4: Covariant tensor

The N^2 quantities A_{qs} are called covariant components of a tensor of rank two if

$$\bar{A}_{pr} = \frac{\partial x^q}{\partial \bar{x}^p} \frac{\partial x^s}{\partial \bar{x}^r} A_{qs} \qquad (3.1.7)$$

Generally the set of N^s quantities is said to constitute the components of a covariant tensor of rank s at a point P in an n-dimensional space if under the co-ordinate transformation $\bar{x}^j = \bar{x}^j(x^i)$ these quantities transform according to the law

$$\bar{T}_{i_1 \dots i_s} = \frac{\partial x^{j_1}}{\partial \bar{x}^{i_1}} \cdots \frac{\partial x^{j_s}}{\partial \bar{x}^{i_s}} T_{j_1 \dots j_s} \qquad (3.1.8)$$

(Eisenhart, 1925).

Definition 3.1.5: Mixed tensor

The N^2 quantities A^q_s are called components of a mixed tensor of rank two if

$$\bar{A}^p_r = \frac{\partial \bar{x}^p}{\partial x^q} \frac{\partial x^s}{\partial \bar{x}^r} A^q_s \qquad (3.1.9)$$

A set of N^r_s quantities $T^{i_1 \dots i_r}_{j_1 \dots j_s}$ is said to constitute the components of a mixed tensor of rank (r+s) or tensor of type (r, s) contravariant of rank r and covariant of rank s at a point p in an N-dimensional space if under the coordinate transformation $\bar{x}^i = \bar{x}^i(x^i)$ these quantities transform according to the law

$$\bar{T}^{i_1 \dots i_r}_{j_1 \dots j_s} = \frac{\partial \bar{x}^{i_1}}{\partial x^{k_1}} \cdots \frac{\partial x^{i_r}}{\partial \bar{x}^{k_r}} \frac{\partial x^{i_1}}{\partial \bar{x}^{j_1}} \cdots \frac{\partial x^{i_s}}{\partial \bar{x}^{j_s}} T^{k_1 \dots k_r}_{i_1 \dots i_s} \qquad (3.1.10)$$

(Rund, 1956).

The tensor field A_{ij}^{qst} is a mixed tensor of rank 5 and the law of transformation is

$$\bar{A}_{ij}^{prm} = \frac{\partial \bar{x}^p}{\partial x^q}\frac{\partial \bar{x}^r}{\partial x^s}\frac{\partial \bar{x}^m}{\partial x^t}\frac{\partial x^k}{\partial \bar{x}^i}\frac{\partial x^l}{\partial \bar{x}^j}A_{kl}^{qst} \qquad (3.1.11)$$

3.1.3 Fundamental operations with tensors

Theorem 3.1.1: The sums and differences of two or more tensors of the same rank and type are also tensors of the same rank and type.

Consider two mixed tensorsA_k^{ij} and B_k^{ij}, by Theorem 3.1.1

$$C_k^{ij} = A_k^{ij} \mp B_k^{ij} \qquad (3.1.12)$$

Theorem 3.1.2: The outer product of two tensors of types (r_1, s_1) and (r_2, s_3) at a point P in x_n is a tensor of type $(r_1 + r_2\ , s_1 + s_2)$

Definition 3.1.6: Tensor contraction

If one contravariant and one covariant index of a tensor of type (r, s) are set equal, the resulting sum is a tensor of rank two less than that of the original tensor thus tensor of type (r-1, s-1). The process is called contraction (Eisenhart, 1925).

Definition 3.1.7: Inner multiplication

The process of outer multiplication of two tensors followed by a contraction is called inner multiplication.

Theorem 3.1.3: An inner product of two tensors of types (r_1, s_1) and (r_2, s_3) at a point P in x_n is a tensor of type $(r_1 + r_2\ - 1, s_1 + s_2 - 1)$ provided that the contraction is over a pair of indices one contravariant and the other covariant.

Let T^{ij} be a contravariant tensor of rank 2 so that its transformation law is given by

$$\bar{T}^{ij} = \frac{\partial \bar{x}^i}{\partial x^r}\frac{\partial \bar{x}^j}{\partial x^s}T^{rs} \tag{3.1.13}$$

Let C_p and F_q be two covariant vectors with transformation laws

$$\bar{C}_p = \frac{\partial x^l}{\partial \bar{x}^p}C_l \ \ ,\bar{F}_p = \frac{\partial x^m}{\partial \bar{x}^q}F_m \tag{3.1.14}$$

Then

$$\bar{T}^{ij}\bar{C}_p\bar{F}_p = \frac{\partial \bar{x}^i}{\partial x^r}\frac{\partial \bar{x}^j}{\partial x^s}\frac{\partial x^l}{\partial \bar{x}^p}\frac{\partial x^m}{\partial \bar{x}^q}T^{rs}C_lF_m \tag{3.1.15}$$

If $Q^{ij}_{pq} = T^{ij}C_pF_q$ then

$$\bar{Q}^{ij}_{pq} = \frac{\partial \bar{x}^i}{\partial x^r}\frac{\partial \bar{x}^j}{\partial x^s}\frac{\partial x^l}{\partial \bar{x}^p}\frac{\partial x^m}{\partial \bar{x}^q}Q^{rs}_{lm} \tag{3.1.16}$$

Contracting over the indices j and p we have

$$\bar{Q}^{ij}_{pq} = \frac{\partial \bar{x}^i}{\partial x^r}\frac{\partial \bar{x}^j}{\partial x^s}\frac{\partial x^l}{\partial \bar{x}^p}\frac{\partial x^m}{\partial \bar{x}^q}Q^{rs}_{lm} = \frac{\partial \bar{x}^i}{\partial x^r}\frac{\partial x^m}{\partial \bar{x}^q}T^{rs}_{lm} \tag{3.1.17}$$

We observe that the contraction has reduced both upper and lower indices by one and such a product is called inner product. The process is known as transvection.

Theorem 3.1.4: Suppose it is not known whether a quantity X is a tensor or not, if an inner product of X with an arbitrary tensor is itself a tensor, then X is also a tensor. This theorem is also known as the quotient law for tensors.

3.1.4 Symmetric and skewsymmetric tensors

Definition 3.1.8: Symmetric tensor

A tensor is called symmetric with respect to two contravariant or two covariant indices if its components remain unaltered upon interchange of indices. If $A_{qs}^{mpr} = A_{qs}^{pmr}$ then the tensor is symmetric in *m* and *p* (Eisenhart, 1925).

Theorem 3.1.5: If a tensor is symmetric with respect to two indices in any coordinate system, it remains symmetric with respect to these indices in any other coordinate system.

If $A_{q}^{mpr} = A_{q}^{pmr}$ then according to tensor transformation law

$$\bar{A}_{q}^{mpr} = \frac{\partial \bar{x}^m}{\partial x^i}\frac{\partial \bar{x}^p}{\partial x^j}\frac{\partial \bar{x}^r}{\partial x^k}\frac{\partial x^s}{\partial \bar{x}^q}A_{s}^{ijk} = \frac{\partial \bar{x}^p}{\partial x^j}\frac{\partial \bar{x}^m}{\partial x^m}\frac{\partial \bar{x}^r}{\partial x^k}\frac{\partial x^s}{\partial \bar{x}^q}A_{s}^{jik} = \bar{A}_{q}^{pmr} \qquad (3.1.18)$$

Thus the given tensor is symmetric with respect to the first two indices in the new coordinate system. Similar results can be proved for covariant indices.

Theorem 3.1.6: A symmetric tensor of rank two in *n*-dimensional space has at most $\frac{n(n+1)}{2}$ independent components.

The total number of components in the array is n^2 out of which *n* diagonal terms will be different and the rest $(n^2 - n)$ will be equal in pairs with the number of pairs as $(\frac{n^2-n}{2})$.

Definition 3.1.9: Skew symmetric

A tensor whose components change in sign but not in magnitude when two covariant or contravariant indices are interchanged is said to be anti-symmetric or skew symmetric with respect to the two indices (Eisenhart, 1925).

If $A_{qs}^{mpr} = -A_{qs}^{pmr}$ then the tensor field A is skew symmetric in *m* and *p*.

Theorem 3.1.7: If a tensor is skew symmetric with respect to two indices in any coordinate system, it remains skew symmetric with respect to these two indices in any other coordinate system.

Let $A_q^{mpr} = -A_q^{pmr}$, then

$$\bar{A}_q^{mpr} = \frac{\partial \bar{x}^m}{\partial x^i}\frac{\partial \bar{x}^p}{\partial x^j}\frac{\partial \bar{x}^r}{\partial x^k}\frac{\partial x^s}{\partial \bar{x}^q}A_s^{ijk} = -\frac{\partial \bar{x}^p}{\partial x^j}\frac{\partial \bar{x}^m}{\partial x^m}\frac{\partial \bar{x}^r}{\partial x^k}\frac{\partial x^s}{\partial \bar{x}^q}A_s^{jik} = -\bar{A}_q^{pmr} \qquad (3.1.19)$$

For an anti-symmetric tensor the components change sign under an odd permutation of its indices and do not change sign under an even permutation of indices. The number of independent components of such a tensor of rank 2 is $\frac{n(n-1)}{2}$.

3.2 The metric tensor

Definition 3.2.1: Metric

A metric is a function of a topological space that gives for any two points in the space a value equal to the distance between them.

In three dimensional Euclidean space the distance between two neighbouring points *(x, y, z)* and *(x+dx, y+dy, z+dz)* is given by

$$(ds)^2 = dx^2 + dy^2 + dz^2 \qquad (3.2.1)$$

In *n*-dimensional space, Riemann defined the distance *ds* between two neighbouring points x^i and $x^i + dx^i$, $(i = 1,2, \dots n)$ by quadratic differential form

$$\begin{aligned}(ds)^2 = {} & g_{11}dx^1dx^1 + g_{12}dx^1dx^2 + \dots + g_{1n}dx^1dx^n \\ & + g_{12}dx^2dx^1 + g_{22}dx^2dx^2 + \dots + g_{2n}dx^2dx^n \\ & + \quad . \quad . \quad . \; + \; . \quad . \quad . \\ & + g_{n1}dx^ndx^1 + g_{n2}dx^ndx^2 + \dots + g_{nn}dx^ndx^n \qquad (3.2.2)\end{aligned}$$

(Eisenhart, 1925).

We express the above metrical relation as

$$(ds)^2 = \sum_{i=1}^{n}\sum_{j=1}^{n} g_{ij}dx^i dx^j \tag{3.2.3}$$

and by Einstein's convention we omit the summation signs and write the sum as

$$(ds)^2 = g_{ij}dx^i dx^j = g_{ij}(x,\dot{x}) \;\; (i,j = 1,2,\ldots n) \tag{3.2.4}$$

where g_{ij} are the functions of the coordinates x^i such that $g = |g_{ij}| \neq 0$ as in the equation (1.1.5).

Let us define a set of quantities

$$g_{ij}(x,\dot{x}) = \frac{1}{2}\dot{\partial}_{ij}^2 F^2(x,\dot{x}) \tag{3.2.5}$$

The quantities $g_{ij}(x,\dot{x})$ which form the components of a covariant tensor of order two are positively homogeneous of degree zero in $\dot{x}^i$ and symmetric with respect to its lower indices. In view of the homogeneity of the function $F(x,\dot{x})$, we have

$$F^2(x,\dot{x}) = g_{ij}(x,\dot{x})\dot{x}^i\dot{x}^j \tag{3.2.6}$$

The quadratic differential form (3.2.3) is called the Riemannian metric or just metric or line element for n-dimensional space. Such an n-dimensional space is called Riemannian space and is denoted by V_n.

The tensor g_{ij} is a covariant symmetric tensor of rank two. In the coordinate transformation $x^i \to \bar{x}^i$ the metric $ds^2 = g_{ij}dx^i dx^j$ transforms to $ds^2 = \bar{g}_{ij}d\bar{x}^i d\bar{x}^j$ and so

$$ds^2 = g_{ij}dx^i dx^j = \bar{g}_{ij}d\bar{x}^i d\bar{x}^j \tag{3.2.7}$$

The inverse of the metric tensor is also very useful and is denoted by the conjugate tensor. The inverse of the covariant metric tensor g_{ij} is the contravariant metric tensor g^{ij} and is defined by

$$g^{ij} = \frac{B_{ij}}{g} \tag{3.2.8}$$

where B_{ij} is the cofactor of g_{ij} and the determinant $g = |g_{ij}| \neq 0$.

Definition 3.2.2: Fundamental tensor

The tensors with covariant components $g_{ij}(x, \dot{x})$ and contravariant components $g^{ij}(x, \dot{x})$ are known as the fundamental tensors or metric tensors of the Finsler space F_n (Matsumoto, 1971).

Since the rank of the matrix $\|g_{ij}(x, \dot{x})\|$ is n, we have the inverse matrix $\|g^{ij}(x, \dot{x})\|$ such that

$$g_{ij}(x, \dot{x}) g^{ij}(x, \dot{x}) = \delta_i^k \tag{3.2.9}$$

where δ_i^k is the Kronecker delta which is a mixed tensor of rank two defined by

$$\delta_i^k = \begin{cases} 1 \ if \ i = k \\ 0 \ if i \neq k \end{cases} \tag{3.2.10}$$

A curvature tensor may be written in terms of the metric g_{ij} and the Kronecker delta δ_i^k by including all the necessary symmetries.

Computing the metric for a general two geometry and then imposing constant curvature gives a set of differential equations. We compute the tensor C_{ijk} given by

$$C_{ijk}(x, \dot{x}) \overset{\text{def}}{=} \frac{1}{2} \dot{\partial}_k g_{ij}(x, \dot{x}) = \frac{1}{4} \dot{\partial}_{ijk}^3 F^2(x, \dot{x}) \tag{3.2.11}$$

which is positively homogeneous of degree-1 and symmetric with respect to the three lower indices.

Applying the homogeneity of $F(x, \dot{x})$, we obtain

$$C_{ijk}(x,\dot{x})\dot{x}^i = C_{ijk}(x,\dot{x})\dot{x}^j = C_{ijk}(x,\dot{x})\dot{x}^k = 0 \qquad (3.2.12)$$

In this thesis the study has been restricted to coordinate systems or flat surfaces with constant metrics. In cases where the metric is variable the finite interval lengths cannot be expressed in terms of finite component differences. However the equation (3.2.4) will still apply.

3.3 Christoffel symbols and Geodesics

Definition 3.3.1: Christoffel symbols

The Christoffel symbols of the first kind have been defined as follows

$$\Upsilon_{ijk} = \frac{1}{2}\left[\partial_k g_{ij} + \partial_i g_{ik} - \partial_j g_{ik}\right], (i, j, k = 1,2 \ldots n), \qquad (3.3.1)$$

where $\partial_k = \frac{\partial}{\partial x^k}$.

Likewise the Christoffel symbols of the second kind have been defined as

$$\dot{\Upsilon}^i_{jk} = g^{ih}(x,y)\dot{\Upsilon}_{ihk}(x,\dot{x}) \qquad (3.3.2)$$

The Christoffel symbols are symmetric in their covariant indices and satisfy the relations

$$\Upsilon^i \partial_k g_{ij}(x,\dot{x}) = \Upsilon_{ijk} + \Upsilon_{jik} \qquad (3.3.3)$$

and

$$\partial_k g^{ij}(x,y) = -g^{hj}(x,y)\Upsilon^i_{hk} - g^{hi}(x,y)\Upsilon^i_{hk} \qquad (3.3.4)$$

where

$$y_i = g_{ij}(x,\dot{x})\dot{x}^j \qquad (3.3.5)$$

Definition 3.3.2: Geodesics

Geodesics are defined as the curves of extremal arc length between any two given points or a given point and a given hypersurface.

In a Riemannian space endowed with a metric tensor $g_{ij}(x^k)$, the geodesics are given by the differential equation of order two

$$\frac{d^2x^i}{ds^2} + \Upsilon^i_{jk}(x, x') \frac{dx^j}{ds} \frac{dx^k}{ds} = 0 \qquad (3.3.6)$$

where arc length s is the parameter defined by the equation (1.1.9) and the Christoffel symbols (Υ^i_{jk}) are functions of position only.

3.4 Covariant differentiation in Finsler space, F_n

The partial differential equations are very important in solving physical systems and they contain partial derivatives. However partial derivatives of tensors do not transform like tensors since they are not co-ordinate invariant. Therefore the transformation law of the tensor is obtained by constructing a new derivative operator that reduces to the usual partial derivative in Cartesian coordinates.

Definition 3.4.1: Covariant derivative

We shall take the Finsler space F_n to be locally Minkowkian space. Hence we define the differential for a vector x^i as

$$\frac{\delta x^i}{\delta t} = \frac{dx^i}{dt} + P^i_{hj} x^h \dot{x}^j \qquad (3.4.1a)$$

where $P^i_{hj}(x, \dot{x})$ is the hv-torsion given by

$$P^i_{hj}(x, \dot{x}) = \Upsilon^i_{hj}(x, \dot{x}) - C^i_{hk}(x, \dot{x}) \Upsilon^k_{ij}(x, \dot{x}) \dot{x}^k \qquad (3.4.2)$$

and forms the components of a covariant vector.

This process of differentiation is called δ-differentiation or covariant differentiation (Rund, 1959).

It results into parallel displacement of vectors. The vector $(x^i + dx^i)$ of $T_n(x^i + dx^i)$ is said to be parallelly displaced vector of x^i of $T_n(P)$ if $dx^i = 0$, that is

$$dx^i = -P^i_{hj}(x, dx)x^h dx^j \tag{3.4.3}$$

The quantity $X^i_{;j}$ defined by

$$X^i_{;j} = \partial_j x^i + P^{*i}_{hj}(x, \dot{x})x^h \tag{3.4.4a}$$

is the partial δ-derivative of x^i with respect to x^j and is a mixed tensor.

The coefficient $P^{*i}_{hj}(x, \dot{x})$ are given by

$$g_{ij} P^*_{jk} = P^*_{jik}(x, \dot{x}) \tag{3.4.5}$$

where

$$P^*_{jik} = \Upsilon_{jki} + [C_{kjh}P^h_{il} + C_{jih}P^h_{kl} - C_{ihk}P^h_{jl}]\dot{x}^i \tag{3.4.6}$$

and it is symmetric in the lower indices.

If the vector x^i depends on the line element (x^k, ξ^k) instead of the position only, then the equations (3.4.1a) and (3.4.4a) take the forms

$$\frac{\delta x^i(x,\xi)}{\delta t} = \partial_j x^i \frac{dx^j}{dt} + \dot{\partial}_j x^i \frac{d\xi^i}{dt} + P^i_{hj} x^h \dot{x}^j \tag{3.4.1b}$$

and

$$x^i_j(x, \xi) = \partial_j x^i + \partial_h x^i \partial_j \xi^h + P^{*i}_{hj} x^h \tag{3.4.4b}$$

respectively.

These types of derivatives follow the same rules of addition and multiplication as that of the ordinary derivatives. The δ-derivative of a scalar is its ordinary derivative.

The three connections $\Upsilon^i_{jk}, P^i_{jk}$ and P^{*i}_{jk} are connected by the relations below:

$$\Upsilon^i_{jk}\dot{x}^j = P^i_{jk}\dot{x}^j$$

$$P^i_{jk}\dot{x}^j = P^{*i}_{jk}\dot{x}^j$$

$$\Upsilon^i_{jk}\dot{x}^j\dot{x}^k = P^{*i}_{jk}\dot{x}^j\dot{x}^k \tag{3.4.7}$$

Definition 3.4.2: Auto parallelism

A curve is called auto parallel if its tangent vector at any point results by successive infinitesimal displacement of type (3.4.3).

In the Finsler space F_n, the geodesics are auto parallel curves. Hence their equations are given by

$$\frac{d^2x^i}{ds^2} + P^{*i}_{jk}(x,x')\dot{x}'^j\dot{x}'^k = 0 \tag{3.4.8}$$

The covariant derivative of the fundamental tensor does not vanish in general and it is given by

$$g_{ij,k}(x,\xi) = 2C_{ijh}(x,x')\xi^h_{,k} \tag{3.4.9a}$$

But if $\xi^i = x'^i = \frac{dx^i}{ds}$ is the tangent vector of the geodesic, then we have

$$g_{ij,k}(x,x') = 2C_{ijh}(x,x')\frac{\delta x'^h}{\delta s} = 0 \tag{3.4.9b}$$

Hence a δ-derivative of the fundamental tensor vanishes along the arc of the geodesic. The δ-derivative and partial derivative (covariant derivative) are related as follows

$$\frac{\delta X^i}{\delta t} = X^i_{;j}(x,\dot{x})\frac{dx^j}{dt} = x^i_{;j}(x,\dot{x})\dot{x}^j \tag{3.4.10}$$

The covariant derivative of the fundamental tensor is non-vanishing and hence the length of the vector x^i does not remain invariant under the parallel displacement as defined by equation (3.4.3). In particular, the length of the vector remains invariant under the parallel displacement if the displacement is taken in its direction.

Generally covariant differentiation involves the use of a differential operator to work with a connection on a Finsler space. In this thesis we have denoted the covariant derivative with $\dot{\partial}_j$ and it acts on the component of a vector X^i as

$$\dot{\partial}_j X^i = \frac{\partial X^i}{\partial x^j} + \Gamma^i_{jk} X^i \qquad (3.4.11)$$

where Γ^i_{jk} are coefficients called connection coefficients.

3.5 Finsler Connections

In an *n*-dimensional Finsler space F_n, any quantities is a function of line element $(x^i, \dot{x}^i)$. If $\mathrm{S}(x^i, \dot{x}^i)$ is a scalar field in F_n the $\frac{\partial S}{\partial x^i}$ are not components of a covariant vector. If we let $N^i_j(x^i, \dot{x}^i)$ be a non-linear connection then we can have the covariant vector field whose components are

$$S_{\mathrm{I}k} = \frac{\delta S}{\delta x^i}, \quad \frac{\delta}{\delta x^i} = \frac{\partial}{\partial x^i} - N^i_j \frac{\partial}{\partial \dot{x}^i}$$

Further, if we have quantities $F^i_{jk}(x^i, \dot{x}^i)$ which obey the transformation rule similar to equation (3.3.3) for Christoffel symbol then the covariant derivative of any tensor field K^i_j of type (1, 1) are defined by

$$K^i_{j\mathrm{I}k} = \frac{\delta K^i_j}{\delta x^k} + K^r_j F^i_{rk} - K^i_r F^r_{jk} \qquad (3.5.1)$$

The partial derivatives of K_j^i with respect to $\dot{x}^i$ give a new tensor field which we shall write as,

$$K_j^i \text{I}_k = \frac{\partial K_j^i}{\partial \dot{x}^k} + K_j^r C_{rk}^i - K_r^i \tag{3.5.2}$$

where $C_{jk}^i(x^i, \dot{x}^i)$ are components of a tensor field of type (1, 2).

Definition 3.5.1: Finsler connection

The collection $(F_{jk}^i, N_j^i, C_{jk}^i)$ is called a Finsler connection denoted by $(F\Gamma)$ and the covariant derivatives given by equations (3.5.1) and (3.5.2) are called h- and v-covariant derivatives of K_j^i respectively.

For any Finsler connection $(F_{jk}^i, N_j^i, C_{jk}^i)$, we have five torsion tensors and three curvature tensors which are as given below:

(h)h-torsion: $T_{jk}^i = F_{jk}^i - F_{kj}^i$ (3.5.3)

(v)v-torsion: $S_{jk}^i = C_{jk}^i - C_{kj}^i$ (3.5.4)

(v)hv-torsion: C_{jk}^i as the vertical connection C_{jk}^i (3.5.5)

v(h)-torsion: $R_{jk}^i = \frac{\delta N_j^i}{\delta x^k} - \frac{\delta N_k^i}{\delta x^j}$ (3.5.6)

(v)hv-torsion: $P_{jk}^i = \frac{\delta F_{jk}^i}{\delta \dot{x}^k} - F_{kj}^i$ (3.5.7)

h-curvature: $R_{hjk}^i = \frac{\delta F_{hj}^i}{\delta x^k} - \frac{\delta N_{hk}^i}{\delta x^k} + F_{hj}^m F_{mk}^i - F_{hk}^m F_{mj}^i + C_{hm}^i R_{jk}^m$ (3.5.8)

hv-curvature: $P_{hjk}^i = \frac{\delta F_{hj}^i}{\delta \dot{x}^k} - C_{hkIj}^i + C_{hm}^i P_{jk}^m$ (3.5.9)

v-curvature: $S_{hjk}^i = \frac{\delta C_{hj}^i}{\delta \dot{x}^k} - d\,\frac{\delta C_{hk}^i}{\delta \dot{x}^j} + C_{hj}^m C_{mk}^i - C_{hk}^m C_{mj}^i$ (3.5.10)

The deflection tensor field of a Finsler connection is denoted by a type (1,1) tensor D_j^i which is given by

$$D_j^i = \dot{x}^k F_{jk}^i - N_j^i \tag{3.5.11}$$

Given a Finsler metric we can determine various Finsler connections. The commonly known examples of connections are Cartan's connection $(C\Gamma)$, Rund's connection $(R\Gamma)$, Berwald's connection $(B\Gamma)$ and Hashinguchi's connection $(H\Gamma)$.

In this thesis we have mainly used Cartan's connection and Berwald's connection which we now define.

Definition 3.5.2: Cartan connection

According to Matsumoto (1966), the Cartan's connection $C\Gamma = (\Gamma_{jk}^{*i}, \Gamma_{ok}^{*i}, C_{jk}^i)$ is determined by the following axioms:

$$i)\ g_{ij\mathrm{I}k} = 0\ ,\ \ (ii) g_{ij}\mathrm{I}_k = 0,\ \ (iii) F_{jk}^i = F_{kj}^i,\ (iv) C_{jk}^i = C_{kj}^i,\ \ (v) D_j^i = 0) \tag{3.5.12}$$

The Cartan's connection is given by

$$(\mathrm{i})\Gamma_{jk}^{*i} = \frac{1}{2} g^{ih}\left(\frac{\delta g_{jh}}{\delta x^k} + \frac{\delta g_{kh}}{\delta x^j} - \frac{\delta g_{jk}}{\delta x^h}\right)$$

$$(\mathrm{ii})\ \Gamma_{ok}^{*i} = G_j^i = \Upsilon_{jk}^i \dot{x}^k - 2C_{km}^i G^m$$

$$(iii) C_{jk}^i = C_{kj}^i = \frac{1}{2} g^{ih} \frac{\delta g_{jh}}{\delta \dot{x}^k} \tag{3.5.13}$$

Cartan connections generalize the idea of affine connections and therefore describe the geometry manifolds that are modelled in spaces which are homogenous.

Deffinition 3.5.3: Rund connection

The Rund's connection is determined from the fundamental metric function $F(x^i, \dot{x}^i)$ by the axioms below

$$i)\ g_{ij|k} = 0\ , \quad (ii) F^i_{jk} = F^i_{kj}, \quad (iii) C^i_{jk} = 0, \ \ (iv) D^i_k = 0 \qquad (3.5.14)$$

The Rund's connection is derived from Cartan's connection by the C-process. The C-process involves removing the torsion tensor C^i_{jk} so that (i) and (ii) in equation (3.5.13) in definition (3.5.2) remain unaltered while (iii) is equal to zero. Hence the Rund's connection of the F_n Finsler space is denoted by $R\Gamma = \left(\Gamma^{*i}_{jk}, \Gamma^{*i}_{ok}, 0\right)$. We note that the h-covariant differentiations with respect to Cartan's connection and Rund's connection coincide with each other.

Definition 3.5.4: Berwald connection

The Berwald's connection is uniquely determined from the metric function by the following axioms

$$i)\ L_{|i} = 0\ , \quad (ii) F^i_{jk} = F^i_{kj}, \ (iii) C^i_{jk} = 0, \ \ (iv) D^i_j = 0, \ (v) P^i_{jk} = \dot{\partial}_k N^i_j - F^i_{kj} = 0 \qquad (3.5.15)$$

The Berwald's connection is thus obtained from the Rund's connection by the P^i – process which involves expelling the torsion tensor P^i_{jk}. The Berwald's connection is denoted by $B\Gamma = \left(\mathrm{G}^i_{jk}, \mathrm{G}^i_j, 0\right)$ where $\mathrm{G}^i_{jk} = \dot{\partial}_k G^i_k$ and $\mathrm{G}^i_k = \dot{\partial}_k G^i$. It is easily expressed in terms of the geodesic coefficients and that makes it the most commonly used in projective Finsler geometry.

Definition 3.5.5: Haashiguchi's connection

The Hashiguchi's connection is obtained from Cartan's connection by the P^i-process and is given by $H\Gamma = (\mathrm{G}^i_{jk}, \mathrm{G}^i_k, \mathrm{C}^i_{jk})$.

3.6 Covariant derivative in the sense of Cartan

The Cartan connections lead to covariant derivatives and other differential operators since Cartan's approach does not depend on co-ordinate systems. We have used a Cartan geometrical approach in this thesis because it is a generalization of the idea of congruence between objects in space and allows for the existence of curvature. Cartan's geometry also relates infinitesimally close an infinitesimal transformation and in the process attaches to points.

We consider the metric tensor $g_{ji}(x,\dot{x})$. The square of the length of an arbitrary covariant vector X^i when transformed from $(x^i,\dot{x}^i)$ to $(x^i + dx, \dot{x}^i + d\dot{x}^i)$ is defined by

$$(X^i)^2 = g_{ij}(x,\dot{x})x^i x^j \qquad (3.6.1)$$

When the element of the support $(x^i,\dot{x}^i)$ undergoes an infinitesimal change, the variation of the vector X^i is represented by a covariant differential

$$DX^i = dX^i + C^i_{kh}(x,\dot{x})X^k d\dot{x}^h + \Gamma^i_{kh}(x,\dot{x})X^k d\dot{x}^h \qquad (3.6.2)$$

where the coefficients C^i_{kh} and Γ^i_{kh} are functions of the element of support.

If the vector X^i is transformed from $(x^i,\dot{x}^i)$ to $(x^i + dx, \dot{x}^i + d\dot{x}^i)$ by parallel displacement the covariant differential Dx^i vanishes, that is

$$dX^i = -C^i_{kh}(x,\dot{x})X^k d\dot{x}^h - \Gamma^i_{kh}(x,\dot{x})X^k d\dot{x}^h \qquad (3.6.3)$$

and this implies that the length of X^i, given by (3.6.1) remains invariant.

The contravariant unit vector l^i, in the direction of the element of support $(x^i, \dot{x}^i)$ is given by

$$l^i = \dot{x}^i F^{-1}(x, \dot{x}), \tag{3.6.4}$$

the covariant unit vector is given by

$$l_i = \partial_i F(x, \dot{x}) = g_{ij}(x, \dot{x}) l^j \tag{3.6.4a}$$

and its covariant differential is given by

$$Dl^i = dl^i + \Gamma^i_{jk} l^j dx^k \tag{3.6.5}$$

On solving the above equations for dX^i, we have

$$d\dot{x}^i = FDl^i + \dot{x}^i \frac{dF(x,\dot{x})}{F} - \Gamma^i_{jk} \dot{x}^j dx^k \tag{3.6.6}$$

In view of (3.5.6) we eliminate $d\dot{x}^l$ from (3.6.2) and we obtain

$$Dx^i = \left(F \partial_j X^i + F C^i_{kj} X^k\right) Dl^j + X^i_{lj} dx^j \tag{3.6.7}$$

where

$$X^i_{\mathrm{I}j} = \partial_j X^i - \dot{\partial}_k X^i \dot{\partial}_j G^k + X^k \Gamma^{*i}_{jk}(x, \dot{x}) \tag{3.6.8}$$

and

$$2G^i(x, \dot{x}) \stackrel{\text{def}}{=} \Upsilon^i_{hk}(x, \dot{x}) \dot{x}^h \dot{x}^k \tag{3.6.9}$$

The vertical bar followed by $\Gamma^{*i}_{jk}(x, \dot{x})$ is the connection coefficient introduced by Cartan (1933) and it is symmetric in its lower indices.

The connection Γ^{*i}_{jk} satisfies the equations

$$\Gamma_{kij} = \Gamma^{*i}_{kij} + C_{kih} \dot{\partial}_j G^h \tag{3.6.10}$$

$$\Gamma_{kij} = \Upsilon_{kij} - C_{jih} \dot{\partial}_k G^h + C_{kih} \dot{\partial}_j G^h \tag{3.6.11}$$

where

$$\Gamma_{ijh} = g_{kj}\Gamma_{ih}^{k} \tag{3.6.12}$$

and

$$\Gamma_{ijh}^{*i} = g_{kj}\Gamma_{ih}^{*k} \tag{3.6.13}$$

The coefficients of Dl^i and $X_{\mathrm{I}j}^{i}$ of (3.6.7) are components of a tensor. The expression $X_{\mathrm{I}j}^{i}dx^j$ represents the variation of X^i if the element of support were transported by parallel displacement from the point x^i to $(x^i + dx^i)$. The values of both Dg_{ij} and $g_{ij\mathrm{I}h}$ vanish identically.

We also have

$$l_{\mathrm{I}h}^{i} = 0 \tag{3.6.14}$$

and

$$F_{\mathrm{I}h}(x,\dot{x}) = 0 \tag{3.6.15}$$

Furthermore the covariant derivative of C_{ij} vanishes and satisfies the relations

$$C_{ijh\mathrm{I}k}x^i = C_{ijh\mathrm{I}k}\dot{x}^j = C_{ijhIk}\dot{x}^h = 0 \tag{3.6.16}$$

We define a new tensor A_{ih}^{k} as

$$A_{ih}^{k}(x,\dot{x}) = FC_{ih}^{k}(x,\dot{x}) \tag{3.6.17a}$$

which gives

$$A_{ijh} = g_{jk}A_{ih}^{k} = \frac{1}{4}\dot{\partial}_{ijh}^{3}F^2 \tag{3.6.17b}$$

The covariant derivative of A_{ijh} is given by

$$A_{ijh\mathrm{I}k} = FC_{ijh\mathrm{I}k} = F(\partial_k C_{ijh} - \dot{\partial}_l C_{ijh}\dot{\partial}_k G^l - C_{ijh}\Gamma_{ik}^{*l} - C_{ilh}\Gamma_{jk}^{*l} - C_{ijl}\Gamma_{kh}^{*l}) \tag{3.6.18}$$

From the above relation it follows that

$$A_{ijh\mathrm{I}k}\dot{x}^i = A_{ijh\mathrm{I}k}\dot{x}^j = A_{ijh\mathrm{I}k}\dot{x}^h = 0 \tag{3.6.19}$$

From equations (3.4.8), (3.4.9), (3.4.9), (3.4.10) and (3.4.11) it is noted that the connection coefficients satisfy the relation

$$\Gamma^i_{kj}\dot{x}^k = \Gamma^{*i}_{kj}\dot{x}^k = P^{*i}_{kj}\dot{x}^k = P^i_{kj}\dot{x}^k = \dot{\partial}_j G^i \tag{3.6.20}$$

The connection coefficient P^{*i}_{kj} of partial δ-derivative and the covariant derivative in the sense of Cartan are the same (Rund, 1959).

3.7 Berwald connection and covariant derivation

Berwald has defined partial covariant derivative in a manner analogous to that of Cartan as given by equation (3.6.8). The only difference is that the connection coefficient Γ^{*i}_{jk} is replaced by G^i_{jk}, where

$$a) G^i_{jk} = \dot{\partial}_j G^i_k, \quad b)\ G^i_k = \Gamma^{*i}_{jk} = \ \dot{\partial}_k G^i, \quad c)\ G^i_{jk} = \dot{\partial}^2_{jk} G^i \tag{3.7.1}$$

The covariant derivative in the sense of Berwald is denoted by

$$X^i_{(j)} = \dot{\partial}_j X^i - \dot{\partial}_j X^i \dot{\partial}_j G^h + G^i_{jh} X^h \tag{3.7.2}$$

In this case, the partial covariant derivatives of $F(x, \dot{x})$ and the unit vector l^i in the direction of the element of support vanish identically. That is $F_{(k)} = l^i_{(k)} = 0$.

The covariant derivative of g_{ij} is given by

$$g_{ij(k)} = 2A_{ijk\mathrm{I}h}\, l^h \tag{3.7.3}$$

The connections Γ^{*i}_{jk} and G^i_{jk} are related by

$$G^i_{jk} = \Gamma^{*i}_{jk} + C^i_{jk\mathrm{I}h}\dot{x}^h \tag{3.7.4}$$

and

$$G^i_{jk}\dot{x}^j = \Gamma^{*i}_{jk}\dot{x}^j \tag{3.7.5}$$

The Finsler spaces for which the G^i_{jk} are independent of the directional argument $\dot{x}^i$ are called "affinely connected spaces". If the metric of the space is such that $C^i_{jk\text{I}h} = 0$, it follows from (3.7.4) that $G^i_{jk} = \Gamma^{*i}_{jk}$. Thus the affinely connected Finsler spaces are characterized by the condition,

$$C^i_{jk\text{I}h} = 0 \tag{3.7.6}$$

and also the equations $\dot{\partial}_l G^i_{jk} = 0$ imply that

$$\dot{\partial}_l \Gamma^{*i}_{jk} = 0 \tag{3.7.7}$$

The Berwald curvature tensor H^i_{jkh} has the torsion tensor H^i_{jk} from (1.2.19) and deviation tensor H^i_j from equation (1.2.18) which arise from the covariant differentiation in the sense of Berwald.

On transvecting the tensors $H^i_j, H^i_{jk}, H^i_{jkh}$ by the directional arguments $\dot{x}^j$ of the line element $(x, \dot{x})$, we obtain

$$H^i_j\dot{x}^j = 0,\ \dot{\partial}_k H^i_j\dot{x}^j = -H^i_k, \qquad \dot{\partial}_r H^r_j\dot{x}^j = -(n-1)H \tag{3.7.8}$$

$$H^i_{jk}\dot{x}^j = H^i_k \tag{3.7.9}$$

and

$$H^i_{jkh}\dot{x}^j = H^i_{kh} \tag{3.7.10}$$

Since the tensor H_i is homogeneous of degree one in the directional argument $\dot{x}^j$, we get

$$H_{ij}\dot{x}^j = H_j \tag{3.7.11}$$

$$H_{ij}\dot{x}^i\dot{x}^j = (n-1)H \tag{3.7.12}$$

and

$$H_i\dot{x}^i = (n-1)H \tag{3.7.13}$$

The tensors H^i_{jk} and H^i_{jkh} are skew-symmetric in any two of the covariant indices and therefore they satisfy the following identities:

$$H^i_{jk} = -H^i_{kj} \tag{3.7.14}$$

$$H^i_{jkh} = -H^i_{jhk} \tag{3.7.15}$$

and

$$H^i_{[jkh]} = 0 \tag{3.7.16}$$

The Bianchi identities for the tensors H^i_j, H^i_{jk} and H^i_{jkh} take the forms,

$$H^r_{k(l)} - H^r_{l(k)} + H^r_{k(i)}\dot{x}^i = 0 \tag{3.7.17}$$

$$H^r_{[ik(l)]} = 0 \tag{3.7.18}$$

$$H^r_{j[ik(l)]} + H^m_{[ik}G^r_{l]mj} = 0 \tag{3.7.19}$$

In the above identities the covariant derivative in the sense of Berwald with respect to x is represented by the index in the round bracket.

The commutation formulae involving the curvature tensor fields H^i_{jk} and H^i_{jkh} are expressed as below (Rund, 1959)

$$2T^i_{j[(h)(k)]} = -\dot{\partial}_r T^i_j H^r_{hk} + T^r_j H^i_{rhk} - T^i_r H^r_{jhk} \tag{3.7.20}$$

$$\dot{\partial}_k(T^i_{j(h)}) - \left(\dot{\partial}_k T^i_j\right)_{(h)} = T^r_j G^i_{rkh} - T^i_r G^r_{jkh} \tag{3.7.21}$$

where T_j^i denotes the components of an arbitrary tensor covariant order one and contravariant order one and $G_{rkh}^i = \dot{\partial}_r G_{hk}^i$ are components of a symmetric tensor.

We use this to obtain commutation formulae involving arbitrary tensors of orders zero, two and four as below:

$$T_{(h)(k)} - T_{(k)(h)} = \frac{\partial T}{\partial \dot{x}^i} H_{hk} \tag{3.7.22}$$

$$T_{ij(h)(k)} - T_{ij(k)(h)} = \frac{\partial T_{ij}}{\partial \dot{x}^r} H_{hk}^r - T_{rj} H_{ihk}^r - T_{ir} H_{jhk}^r \tag{3.7.23}$$

$$T_{jkh(l)(m)}^i - T_{jkh(m)(l)}^i = -\dot{\partial}_r T_{jkh}^i H_{lm}^r + T_{jkh}^r H_{rlm}^i - T_{rkh}^i H_{jlm}^r - T_{jrh}^i H_{klm}^r - T_{jkr}^i H_{hlm}^r \tag{3.7.24}$$

CHAPTER FOUR

RESULTS AND DISCUSSION ON W-CURVATURE INHERITANCE AND W-CURVATURE COLLINEATION IN BI-RECURRENT FINSLER SPACE

In this chapter the definitions, geometric objects and tools of computation developed in the previous chapters have been used to study the concept of curvature inheritance symmetry and the concept of curvature collineation symmetry for the Weyl projective curvature tensor $W^i_{jkh}(x,\dot{x})$ in bi-recurrent Finsler spaces.

The chapter is divided into four sections. In the first Section 4.1, the Weyl projective curvature tensor if defined and some of its algebraic properties in relation to the Berwald curvature tensor $H^i_{jkh}(x,\dot{x})$ are determined. Geometrical computations have been used to describe recurrence and bi-recurrence in Finsler spaces and the Lie derivative for the projective curvature tensor $W^i_{jkh}(x,\dot{x})$ in such spaces determined. Section 4.2 contains the curvature inheritance property while Section 4.3 contains the curvature collineation property.

4.1 Introduction

4.1.1 The Weyl projective curvature tensor field, $W^i_{jkh}(x,\dot{x})$

The main object in this study is the Weyl projective curvature tensor field, $W^i_{jkh}(x,\dot{x})$ which is a mixed tensor of type (1, 3) thus contravariant order one and covariant order 3. In this study the simplified notation W^i_{jkh} has been used to represent $W^i_{jkh}(x,\dot{x})$ especially in symbolic equations.

Definition 4.1.1: W- projective curvature tensor

In an n-dimensional Finsler space F_n , the Weyl projective curvature tensor field denoted as W^i_{jkh} is defined by

$$W^i_{jkh} = H^i_{jkh} + \frac{\delta^i_j}{n+1}(H_{kh} - H_{hk}) + \frac{\dot{x}^i}{n+1}(\dot{\partial}_j H_{kh} - \dot{\partial}_j H_{hk}) + \frac{\delta^i_k}{n^2-1}(nH_{jh} + H_{hj} + \dot{x}^r \partial_j H_{hr}) - \frac{\delta^i_h}{n^2-1}\left(nH_{jk} + H_{kj} + \dot{x}^r \partial_j H_{kr}\right) \quad (4.1.1)$$

where H^i_{jkh} is the Berwald curvature tensor defined by equation 1.2.22 and $\dot{\partial}_j = \frac{\partial}{\partial \dot{x}_j}$ (Rund, 1959).

The tensor W^i_{jkh} is skew-symmetric in the last two indices *k* and *h* and positively homogeneous of degree zero in its directional arguments.

In the same space F_n, the projective tensor field W^i_{jk} is defined by

$$W^i_{jk} = H^i_{jk} + \frac{\dot{x}^i}{n+1}(H_{jk} - H_{kj}) + \frac{\delta^i_j}{n^2-1}(nH_k + \dot{x}^r H_{kr}) - \frac{\delta^i_k}{n^2-1}(nH_j + \dot{x}^r H_{jr}) \quad (4.1.2)$$

and the deviation tensor field W^i_j is define by W^i_j

$$W^i_j = H^i_j - H\delta^i_j - \frac{1}{n+1}(\dot{\partial}_k H^k_j - \dot{\partial}_j H)\dot{x}^i \quad (4.1.3)$$

(Rund, 1959). The torsion tensor H^i_{jk} and the deviation tensor H^i_j are defined by equations (1.2.19) and (1.2.18) respectively.

The projective tensor W^i_{jk} is skew-symmetric in its covariant indices and positively homogeneous of degree one in their directional arguments. The deviation tensor W^i_j is homogeneous of degree two in its directional arguments.

Transvecting each of the tensors W^i_{jkh}, W^i_{jk} and W^i_j successively by $\dot{x}$, we obtain the following identities.

$$W^i_{jkh}\dot{x}^j = W^i_{kh} \tag{4.1.4a}$$

$$W^i_{jkh}\dot{x}^j\dot{x}^k = W^i_h \tag{4.1.4b}$$

$$W^i_{jk}\dot{x}^j = W^i_k \tag{4.1.5a}$$

$$W_{jk}\dot{x}^j = W_k \tag{4.1.5b}$$

$$W = \frac{1}{n-1}W_{jk}\dot{x}^j\dot{x}^k \tag{4.1.5c}$$

and

$$W^i_k\dot{x}^k = 0 \tag{4.1.6}$$

The deviation tensor field W^i_k is homogeneous of degree two in its directional arguments.

Using the skew-symmetric property of Definition 3.9 we note that the projective curvature tensor W^i_{jkh} and the torsion tensor W^i_{jk} satisfy the following identities,

$$W^i_{jkh} = -W^i_{jhk} \tag{4.1.7}$$

$$W^i_{jk} = -W^i_{kj} \tag{4.1.8}$$

and

$$W^i_{[jkh]} = W^i_{jkh} + W^i_{khj} + W^i_{hjk} = 0 \tag{4.1.9}$$

From equation (4.1.3), it follows that

$$\dot{\partial}_h W^i_k\dot{x}^k = -W^i_h \tag{4.1.10}$$

and

$$\dot{\partial}_h W^i_k = 0 \tag{4.1.11}$$

Contracting j and k of the tensor fields W^i_{jkh}, W^i_{kh} and W^i_k, we obtain

$$W^i_{jkh} = 0 \tag{4.1.12}$$

$$W^i_{kh} = 0 \tag{4.1.13}$$

$$W^i_i = 0 \tag{4.1.14}$$

respectively.

The above tensor fields satisfy the relations

$$a)\ W^i_{jkh} = \dot{\partial}_j W^i_{kh}, \quad b) W^i_{kh} = \frac{1}{3}\left(\dot{\partial}_k W^i_h - \dot{\partial}_h W^i_k\right) \tag{4.1.15}$$

$$W^i_h = H^i_h - H\delta^i_h - \frac{x^i}{n+1}(\dot{\partial}_r H^r_h - \dot{\partial}_h H) \tag{4.1.16}$$

where H is a scalar curvature defined by $H = \frac{1}{(n-1)} H^i_i$ and $\dot{\partial}_r = \frac{\partial}{\partial \dot{x}^r}$H is derived from the Berwald curvature tensor H^i_{jkh} by successive transvections as described by equations (3.7.12) and (3.7.13).

We let $T^{i_1 \dots i_r}_{j_1 \dots j_s}$ be the mixed tensor defined by the equation (3.1.10). Then the commutation formulae involving the projective curvature tensor field W^i_{jkh} are as follows

$$T_{(h)(k)} - T_{(k)(h)} = \frac{-\partial T}{\partial \dot{x}^i} W^i_{hk} \tag{4.1.17}$$

$$T^i_{j(h)(k)} - T^i_{j(k)(h)} = \frac{-\partial T}{\partial \dot{x}^r} W^r_{hk} - T^i_r W^r_{jkh} + T^r_j W^i_{rkh} \tag{4.1.18}$$

$$\left(\frac{\partial T^i_j}{\partial \dot{x}^k}\right)_{(h)} - \frac{\partial T^i_{j(h)}}{\partial \dot{x}^k} = T^i_r G^r_{jkh} - T^r_j G^i_{rkh} \tag{4.1.19}$$

$$T^i_{jkh(l)(m)} - T^i_{jkh(m)(l)} = -\dot{\partial}_r T^i_{jkh} W^r_{lm} + T^r_{jkh} W^i_{rlm} - T^i_{rkh} W^r_{jlm} - T^i_{jrh} W^r_{klm} - T^i_{jkr} W^r_{hlm} \tag{4.1.20}$$

where $\frac{\partial T}{\partial \dot{x}^k} = \dot{\partial}_k$ and the index enclosed between round brackets denotes covariant derivative with respect to $\dot{x}$s for the connection parameter G^i_{jkh}.

4.1.2 W- curvature tensor in recurrent and bi- recurrent Finsler spaces

Definition 4.1.2: Recurrence

In a non-flat Finsler space F_n, if there exists a non-zero covariant vector K_l whose components are positively homogeneous functions of degree zero in $\dot{x}^i$, such that the W-curvature tensor field W^i_{jkh} satisfies the relation

$$W^i_{jkh(l)} = K_l W^i_{jkh} \tag{4.1.21}$$

then the space is called a recurrent Finsler space of first order (Sinha and Singh,1971). We denote this space by $R - F_n$.

In equation (4.1.21), $W^i_{jkh} \neq 0$ and $K_l \neq 0$ is called the recurrence vector field. The recurrence vector of a recurrent space $R - F_n$ is independent of the directional arguments provided the scalar curvature W$\neq 0$.

Transvecting (4.1.21) successively by $\dot{x}^j$ and $\dot{x}^k$ and using equations (4.1.4) and (4.1.5), we get

$$W^i_{jkh(l)}\dot{x}^j = K_l W^i_{jkh}\dot{x}^j$$

$$W^i_{kh(l)} = K_l W^i_{kh} \tag{4.1.22}$$

and

$$W^i_{kh(l)}\dot{x}^k = K_l W^i_{kh}\dot{x}^k$$

$$W^i_{h(l)} = K_l W^i_h \tag{4.1.23}$$

The torsion tensor field W^i_{kh} and the deviation tensor W^i_h above are also recurrent.

Contracting the tensor W^i_{jkh} in indices i and h in equation (4.1.21), we obtain

$$W_{jk(l)} = K_l W_{jk} \tag{4.1.24}$$

Contracting the tensor W^i_{jkh} in indices i and k in equation (4.1.22), we get

$$W_{h(l)} = K_l W_h \tag{4.1.25}$$

Transvecting equation (4.1.25) by $\dot{x}^h$ and using (4.1.7)

$$W_{h(l)}\dot{x}^h = K_l W_h \dot{x}^h$$

$$W_{(l)} = K_l W \tag{4.1.26}$$

In a recurrent Finsler space $R - F_n$ if $G^i_{jkh} = 0$ then by cyclic change of indices j, k and h in (4.1.21) and using the identity (4.1.9) the projective curvature tensor W^i_{jkh} satisfies the identity below

$$W^i_{jkh(l)} + W^i_{khj(l)} + W^i_{hjk(l)} = 0 \tag{4.1.27}$$

In a recurrent Finsler space $R - F_n$ the Bianchi identities of the projective tensor field W^i_{jk} and the projective curvature tensor field W^i_{jkh} take the forms

$$K_h W^i_{jk} + K_j W^i_{kh} + K_k W^i_{hj} = 0 \tag{4.1.28}$$

$$K_h W^i_{mkj} + K_j W^i_{mhk} + K_k W^i_{mjh} = \frac{\partial V_m}{\partial \dot{x}^h} H^i_{jk} + \frac{\partial V_m}{\partial \dot{x}^j} H^i_{kh} + \frac{\partial V_m}{\partial \dot{x}^k} H^i_{hj} \tag{4.1.29}$$

respectively.

Definition 4.1.3: Bi-recurrence

In a non-flat Finsler space F_n, if there exists a non-zero tensor A_{lm} such that the curvature tensor field satisfies the relation

$$W^i_{jkh(l)(m)} = A_{lm} W^i_{jkh} \tag{4.1.30}$$

where

$$A_{lm} = K_{l(m)} - K_l K_m \tag{4.1.31}$$

then the Finsler space is called a bi-recurrent Finsler space or a recurrent space of order two (Sinha and Singh, 1971).

The tensor field A_{lm} defined by (4.1.30) is called bi-recurrence tensor field. We denote such a Finsler space by $2R - F_n$.

Transvection of (4.1.30) successively by $\dot{x}^h$ and $\dot{x}^k$ we obtain

$$W^i_{jkh(l)(m)} \dot{x}^h = W^i_{jk(l)(m)} \tag{4.1.32}$$

and

$$W^i_{jk(l)(m)} \dot{x}^k = W^i_{j(l)(m)} \tag{4.1.33}$$

respectively.

In a bi-recurrent Finsler space $2R - F_n$ the projective tensor W^i_{jk} and the deviation tensor W^i_{jk} satisfy the recurrence relations

and

$$W^i_{jk(l)(m)} = A_{lm} W^i_{jk} \tag{4.1.34}$$

$$W^i_{j(l)(m)} = A_{lm} W^i_j \tag{4.1.35}$$

respectively.

The projective curvature tensor field W^i_{jkh} has the Ricci curvature tensor which is obtained by contracting the contravariant index with the third covariant index and given by

$$W^i_{jki} = W_{jk} \tag{4.1.36}$$

and the scalar curvature tensor given by

$$W = g^{jk} W_{jk} \tag{4.1.37}$$

The projective curvature tensor and its Ricci tensor satisfy the recurrence relation

$$W^i_{jkh(l)} = W_{kh(j)} - W_{jh(k)} \tag{4.1.38}$$

The Finsler space is said to be a Ricci-recurrent space if it satisfies the recurrence relations

$$W_{jk(l)} = K_l W_{jk} \tag{4.1.39}$$

and on multiplying by g^{jk} we have

$$W_{(l)} = K_l W \tag{4.1.40}$$

4.1.3 Lie-derivatives (L_v) of tensors and (L_V) of $W^i_{jkh}(x, \dot{x})$

We consider the infinitesimal transformation point given by

$$\bar{x}^i = x^i + V^i(x)\delta t \tag{4.1.41}$$

where δt is an infinitesimal point constant and $V^i(x)$ is a contravariant vector field independent of directional arguments and dependent on positional coordinates x^i only (Yano 1957). Infinitesimal method is a tool that leads to Lie-derivatives. Lie derivatives arise naturally and they evaluate the rate of change of a scalar function, a vector field or a tensor field along the flow of another vector or tensor field. They provide a congruence of curves that we use to compute the derivatives of a tensor. The Lie-derivative is defined on any differentiable manifold since the change in a tensor field is co-ordinate invariant. The vector $V^i(x)$ may assume any of the forms below:

$$V^i_{(l)} = 0, \quad V^i_{(l)} = c\delta^i_j\,, \quad V^i_{(l)} = \beta\delta^i_j \tag{4.1.42}$$

Let us denote the Lie-differentiation operator with respect to the transformation (4.1.41) by L_v.

Definition 4.1.4: Lie-derivative

The Lie-derivative of a vector field X^i is given by

$$L_v X^i = V^i X^i_{(j)} - X^i V^h_{(j)} + \dot{\partial}_j X^i V^h_{(l)} \dot{x}^j \tag{4.1.43}$$

(Takano, 1961).

The Lie-derivative of a general mixed tensor field $T^i_j(x,\dot{x})$ and the connection coefficient $G^i_{jk}(x,\dot{x})$with respect to equation (4.1.41) are expressed in the form:

$$L_v T^i_j = V^h T^i_{j(h)} - T^i_h V^h_{(j)} + (\dot{\partial}_h T^i_j) V^h_{(s)} \dot{x}^s \tag{4.1.44}$$

and

$$L_v G^i_{jk} = V^i_{(j)(k)} + H^i_{jkh} V^h + G^i_{ljk} V^l_{(s)} \dot{x}^s \tag{4.1.45}$$

respectively, where indices in brackets represent covariant differentiation in the sense of Berwald.

For the Berwald curvature tensor field H^i_{jkh}, the Lie-derivative is given by

$$L_v H^i_{jkh} = V^l H^i_{jkh(l)} - H^i_{jkh} V^i_{(l)} + H^i_{lkh} V^i_{(j)} + H^i_{jlh} V^l_{(k)} + H^i_{jkl} V^l_{(h)} + \dot{\partial}_l H^i_{jkh} V^l_{(m)} \dot{x}^m \tag{4.1.46}$$

The commutation formulae involving Lie-derivatives and other derivatives for tensors T_j^i and T_{jk}^i are given by:

$$L_v(\dot{\partial}_l T_j^i) - \dot{\partial}_l(L_v T_j^i) = 0 \tag{4.1.47}$$

and

$$L_v(T_{jk}^i{}_{(m)}) - (L_v T_{jk}^i)_{(m)} = T_{jk}^l L_v G_{lm}^i - T_{lk}^i L_v G_{jm}^l - T_{jl}^i L_v G_{km}^l - \dot{\partial} T_{jk}^i L_v G_{sm}^l \dot{x}^s \tag{4.1.48}$$

respectively.

The Lie-derivative of the projective curvature tensor W_{jkh}^i is given by

$$L_v W_{jkh}^i = V^l W_{jkh(l)}^i - W_{jkh}^i V_{(l)}^i + W_{lkh}^i V_{(j)}^i + W_{jlh}^i V_{(k)}^l + W_{jkl}^i V_{(h)}^l + \dot{\partial}_l W_{jkh}^i V_{(m)}^l \dot{x}^m \tag{4.1.49}$$

4.2 W-curvature inheritance in bi-recurrent Finsler space

In this section the curvature inheritance property of the projective curvature tensor W_{jkh}^i in a bi-recurrent Finsler space has been studied and the results obtained stated in form of Theorems.

Definition 4.2.1: W-curvature inheritance

In a bi-recurrent Finsler space $2R - F_n$, if the Lie derivative of the projective curvature tensor W_{jkh}^i is proportional to the curvature itself, thus satisfies the relation

$$L_v W_{jkh}^i = \alpha(x) W_{jkh}^i \tag{4.2.1}$$

with respect to the vector field $V^i(x)$ where $\alpha(x)$ is a non-zero scalar function, the infinitesimal transformation (4.1.41) is called an W-curvature inheritance in $2R - F_n$.

On transvection, in view of identities (4.1.4b), (4.1.4c) and equation (4.2.1), we have

$$\text{a)} L_v W_{kh}^i = \alpha(x) W_{kh}^i \qquad \text{b)} L_v W_h^i = \alpha(x) W_h^i \tag{4.2.2}$$

Singh (2003) has established that in bi-recurrent Finsler space $\bar{F}_n$, the Lie derivatives involving the Berwald curvature tensor field also satisfy the relations

$$\text{a) } L_v H^i_{jkh} = \alpha(x) H^i_{kh} \quad \text{b) } L_v H^i_{kh} = \alpha(x) H^i_{kh} \tag{4.2.3}$$

Hence considering the properties satisfied by (1.2.29), (1.2.30), (3.7.9) and (4.2.3) and the fact that the operations of contraction and Lie derivative are commutative, we obtain

$$\text{a) } L_v H^r_{rkh} = \alpha(x) H^r_{rjk} \quad \text{b) } L_v H_{jk} = \alpha(x) H_{jk} \text{ and c) } L_v H = \alpha(x) H \tag{4.2.4}$$

In equation (4.1.1) the projective curvature tensor field W^i_{jkh} is defined in terms of the Berwald curvature tensor field H^i_{jkh} and its derivatives and hence on applying the Lie-derivative operator to the equation (4.1.1), (Rund, 1959)

we have

$$L_v W^i_{jkh} = L_v H^i_{jkh} + L_v \frac{\delta^i_j}{n+1}(H_{kh} - H_{hk}) + L_v \frac{\dot{x}^i}{n+1}(\dot{\partial}_j H_{kh} - \dot{\partial}_j H_{hk})$$

$$+ L_v \frac{\delta^i_k}{n^2-1}(nH_{jh} + H_{hj} + \dot{x}^r \partial_j H_{hr})$$

$$- L_v \frac{\delta^i_h}{n^2-1}\left(nH_{jk} + H_{kj} + \dot{x}^r \partial_j H_{kr}\right)$$

Using the equation (3.2.10) the Kronecker delta $\delta^i_l = 0$ in the terms which contain it in the equation above, the differential operator $\dot{x}^i$ is Lie- invariant thus $L_v \dot{x}^i = 0$ and furthermore Lie-derivative and partial derivative are commutative.

$$L_v W^i_{jkh} = L_v H^i_{jkh} + L_v \frac{1}{n+1}(H_{kh} - H_{hk}) + L_v \frac{1}{n^2-1}(nH_{jh} + H_{hj})$$

$$- L_v \frac{1}{n^2-1}\left(nH_{jk} + H_{kj}\right)$$

$$L_v W^i_{jkh} = L_v H^i_{jkh} + \frac{1}{n^2 - 1} L_v (H_{kh} - nH_{hk} - H_{kh} + H_{hk} + nH_{jh} + H_{hj} - nH_{jk} - H_{kj})$$

In view of equations (4.2.3) and (4.2.4) the above becomes

$$L_v W^i_{jkh} = \alpha(x)\, H^i_{jkh} + \frac{1}{n^2 - 1} (\alpha(x) H_{kh} - \alpha(x) nH_{hk} - \alpha(x) H_{kh} + \alpha(x) H_{hk} + \alpha(x) nH_{jh} + \alpha(x) H_{hj} - \alpha(x) nH_{jk} - \alpha(x) H_{kj})$$

Since $\alpha(x)$ is a scalar function we obtain

$$L_v W^i_{jkh} = \alpha(x) W^i_{jkh}$$

W-Curvature inheritance may also be called W- Lie recurrence. In the definition (4.2.1) we have assumed that the projective curvature tensor is non-zero, $W^i_{jkh} \neq 0$.

Hence we state

Theorem 4.2.1: A bi-recurrent Finsler space $2R - F_n$ which admits H-curvature inheritance is also W-curvature inheritance.

Singh (2011) has proved that in a Finsler space F_n, every motion admitted is a W-curvature inheritance if the space is an isotropic and thus does not depend on orientation.

In view of the commutation formula for covariant derivatives, equation (3.7.2) and the above theorem we conclude that

Theorem 4.2.2: Every motion admitted in a bi-recurrent Finsler space $2R - F_n$ is also a W-curvature inheritance if the space is isotropic.

Differentiating (4.2.1) covariantly with respect to x^l, and using the recurrence equation (4.1.21) we obtain

$$(L_v W^i_{jkh})_{(l)} = (\alpha W^i_{jkh})_l = \alpha_l W^i_{jkh} + \alpha W^i_{jkh(l)}$$

$$(L_v W^i_{jkh})_{(l)} = \alpha_l W^i_{jkh} + \alpha K_l W^i_{jkh}$$

hence

$$(L_v W^i_{jkh})_{(l)} = (\alpha_{(l)} + \alpha K_l) W^i_{jkh} \tag{4.2.5}$$

where $\alpha(x) = \alpha$.

Taking the Lie-derivatives on both sides of (4.1.21) and using the equation (4.2.1), we obtain

$$L_v(W^i_{jkh(l)}) = L_v(K_l\, W^i_{jkh})$$

$$L_v(W^i_{jkh(l)}) = L_v K_l\, W^i_{jkh} + K_l L_v\, W^i_{jkh}$$

$$L_v(W^i_{jkh(l)}) = L_v K_l\, W^i_{jkh} + K_l \alpha\, W^i_{jkh}$$

hence

$$L_v(W^i_{jkh(l)}) = (L_v K_l + \alpha K_l)\, W^i_{jkh} \tag{4.2.6}$$

We now subtract the equation (4.2.6) from the equation (4.2.5), we get

$$(L_v W^i_{jkh})_{(l)} - L_v(W^i_{jkh(l)}) = (\alpha_{(l)} + \alpha K_l - L_v K_l - \alpha K_l) W^i_{jkh}$$

$$(L_v W^i_{jkh})_{(l)} - L_v(W^i_{jkh(l)}) = (\alpha_{(l)} - L_v K_l)\, W^i_{jkh} \tag{4.2.7}$$

From equation (4.2.7), we have

Theorem 4.2.3: In a recurrent Finsler space $R - F_n$, W-Curvature inheritance and covariant differentiation for the connection G^i_{jk} commute if and only if $L_v K_l = \alpha_{(l)}$.

Applying commutation formula (4.1.47) for W^i_{jkh} and using (4.2.1), we obtain

$$L_v(\dot{\partial}_l W^i_{jkh}) = \alpha(x)(\dot{\partial}_l W^i_{jkh}) \tag{4.2.8}$$

since $\alpha(x)$ is a scalar function.

Lemma 4.2.1

In a recurrent space $R - F_n$ which admits the W- curvature inheritance the partial derivative of the projective curvature tensor W^i_{jkh} satisfies the inheritance property (4.2.8).

We now consider a bi-recurrence case as defined by the equation (4.1.31).

Using commutation formula (3.7.24) for the curvature tensor W^i_{jkh} and applying equation (4.1.31), we get

$$(A_{lm} - A_{ml})W^i_{jkh} = -\dot{\partial}_r W^i_{jkh} H^r_{lm}\dot{x}^r + W^r_{jkh} H^i_{rlm} - W^i_{rkh} H^r_{jlm} - W^i_{jrh} H^r_{klm} - W^i_{jkr} H^r_{hlm} \tag{4.2.9}$$

since

$$A_{lm} W^i_{jkh} = W^i_{jkh(l)(m)} = -A_{ml} W^i_{jkh} \tag{4.2.10}$$

we get

$$2W^i_{jkh[l][m]} = \dot{\partial}_r W^i_{jkh} H^r_{lm}\dot{x}^r + W^i_{jkh} H^l_{hjk} - W^i_{jkh} H^l_{jm} \tag{4.2.11}$$

Now taking Lie-derivative of both sides of equation (4.2.9) and using equations (4.2.1) and (4.2.3) we have

$$[(L_v A_{lm} - L_v A_{ml}) + \alpha(A_{lm} - A_{ml})]W^i_{jkh} = 2\alpha[\dot{\partial}_r W^i_{jkh} H^r_{lm} + W^r_{jkh} H^i_{rlm} - W^i_{rkh} H^r_{jlm} - W^i_{jrh} H^r_{klm} - W^i_{jkr} H^r_{hlm}]$$

From equation (4.2.9), the above equation reduces to

$$(L_v A_{lm} - L_v A_{ml}) + \alpha(A_{lm} - A_{ml}) = 2\alpha(A_{lm} - A_{ml}) \tag{4.2.12}$$

which implies $(L_v A_{lm} - L_v A_{ml}) = \alpha(A_{lm} - A_{ml})$ and hence we conclude

$$L_v A_{[lm]} = \alpha A_{[lm]} \tag{4.2.13}$$

where $[lm]$ represents skew-symmetric part.

Thus we state

Theorem 4.2.4: In a bi-recurrent Finsler space $2R - F_n$ which admits a W-curvature inheritance, the recurrence tensor field A_{lm} satisfies the inheritance identity equation (4.2.13).

If we commute the indices l and m in the equation (4.1.35), it yields

$$(A_{lm} - A_{ml})W^i_{kh} = -W^i_{rkh}H^r_{lm} + W^r_{kh}H^i_{rlm} - W^i_{rh}H^r_{klm} - W^i_{kr}H^r_{hlm} \tag{4.2.14}$$

in view of equation (3.7.20).

Using the recurrence property of Berwald curvatures (Sinha and Singh 1971), equations (4.1.21) and (4.1.22) in the covariant derivative of (4.2.14), we obtain

$$(A_{lm} - A_{ml})_{(n)} + K_n(A_{lm} - A_{ml}) = 2K_n(A_{lm} - A_{ml}) \tag{4.2.15}$$

which implies $(A_{lm} - A_{ml})_{(n)} = K_n(A_{lm} - A_{ml})$ and therefore

$$A_{[lm](n)} = K_n[lm] \tag{4.2.16}$$

Applying equation (4.2.13) in the Lie-derivative of both sides of (4.2.14), we find

$$L_v A_{[lm](m)} = L_v K_n A_{[lm]} + \alpha K_n A_{[lm]} \tag{4.2.17}$$

which gives

$$L_v A_{[lm](n)} = (L_v K_n + \alpha K_n) A_{[lm]} \tag{4.2.18}$$

Accordingly we state

Theorem 4.2.5: In a bi-recurrent Finsler space $2R - F_n$ which admits a W-curvature inheritance, the recurrence tensor field A_{lm} satisfies the identity (4.2.18).

Assuming that the recurrence vector K_l satisfies the inheritance property $L_v K_l = -\alpha K_l$, then in that case the equation (4.2.18) reduces to

$$L_v A_{[lm](n)} = 0 \tag{4.2.19}$$

which also implies that

$$L_v A_{[lm](n)} + L_v A_{[mn](l)} + L_v A_{[nl](m)} = 0 \tag{4.2.20}$$

Conversely, if equation (4.2.19) is true, then the equation (4.2.18) reduces to

$$(L_v K_n + \alpha K_n) A_{[lm]} = 0 \tag{4.2.21}$$

Since A_{lm} is non-zero recurrence tensor field, the equation (4.2.21) yields

$$L_v K_n + \alpha K_n = 0 \tag{4.2.22}$$

which implies

$$L_v K_n = -\alpha K_n \tag{4.2.23}$$

Hence we have

Theorem 4.2.6: In a bi-recurrent Finsler space $2R - F_n$ which admits a W- Curvature inheritance the necessary and sufficient condition for identity

$$L_v A_{[lm](n)} + L_v A_{[mn](l)} + L_v A_{[nl](m)} = 0$$

to be true is that the recurrence vector K_l satisfies the inheritance property (4.2.23).

Now we consider covariant differentiation of equation (4.2.16) with respect to x^s to have

$$A_{[lm](n)(s)} = K_{n(s)} A_{[lm]} + K_n A_{[lm](s)}, \tag{4.2.24}$$

which yields

$$A_{[lm](n)(s)} - A_{[lm](s)(n)} = \left(K_{n(s)} - K_{s(n)}\right)A_{[lm]} = (K_{(s)(n)})A_{[lm]} \qquad (4.2.25)$$

In view of commutation formula (3.7.24), it yields

$$\dot{\partial}_r A_{[lm]} W^r_{ns} + A_{[rm]} W^r_{lns} + A_{[lr]} W^r_{mns} = (-K_{n(s)} + K_{(s)(n)})A_{[lm]} \qquad (4.2.26)$$

Applying Lie-derivative to equation (4.2.24) and noting equations (4.2.3) and (4.2.23), we get

$$L_v(\dot{\partial}_r A_{[lm]} W^r_{ns} + \alpha\left(\dot{\partial} A_{[lm]}\right) W^r_{ns} + 2\alpha\left[A_{[lm]} W^r_{lns} + A_{[lr]} W^r_{mns}\right] = L_v\left(K_{s(n)} - K_{(n)(s)}\right)A_{[lm]} + \alpha(K_{s(n)} - K_{n(s)})A_{[lm]} \qquad (4.2.27)$$

From (4.2.27), we get

$$L_v A_{lm} = \alpha A_{lm} \qquad (4.2.28)$$

This leads to

Theorem 4.2.7: The recurrence tensor field A_{lm} is Lie- recurrent with respect to W-curvature inheritance.

Applying equations (4.1.47), (4.2.13) and (4.2.26) to the above equation, we obtain

$$2\alpha\left(K_{s(n)} - K_{n(s)}\right)A_{[lm]} = L_v\left(K_{s(n)} - K_{n(s)}\right)A_{[lm]} + \alpha(K_{s(n)})A_{[lm]}, \qquad (4.2.29)$$

which reduces to

$$\alpha\left(K_{s(n)} - K_{n(s)}\right) = L_v(K_{s(n)} - K_{n(s)}) \qquad (4.2.30)$$

since α is a scalar function and A_{lm} is a non-zero recurrence tensor.

The equation (4.2.29) can be expressed as

$$L_v K_{[s(n)]} = \alpha K_{[s(n)]} \qquad (4.2.31)$$

Accordingly, we have

Theorem 4.2.8: In a bi-recurrent Finsler space $2R - F_n$ which admits a W-curvature inheritance, the recurrence vector field K_l satisfies the inheritance identity (4.2.31).

Hiramatu (1954) has established that the infinitesimal transformation (4.1.41) is a homothetic transformation if $L_v g_{ij} = 2C g_{ij}$, where C is a constant. If the Finsler space F_n admits a homothetic transformation then $L_v G^i_{jk} = 0$ is necessarily true. In such a case for affine motion the curvature H^i_{jkh} is Lie-invariant which automatically implies that W^i_{jkh} is also Lie-invariant.

Hence we conclude

Theorem 4.2.9: Every homothetic transformation admitted in bi-recurrent Finsler space $2R - F_n$ is also W-curvature inheritance if the space is isotropic.

4.3 W- Curvature collineation in bi-recurrent Finsler space $2R - F_n$

In this section we have defined and studied the collineation property of projective curvature tensor W^i_{hjk} in a bi-recurrent Finsler space $2R - F_n$.

Definition 4.3.1: W- Curvature collineation

In a bi-recurrent Finsler space $2R - F_n$ if the projective curvature tensor field W^i_{hjk} satisfies the relation

$$L_v W^i_{jkh} = 0 \qquad (4.3.1)$$

with respect to vector field $V^i(x)$, then we call such infinitesimal transformation (4.1.41) a W- curvature collineation.

W- Curvature inheritance cannot be a curvature collineation. Thus the sets of W-curvature collineations and curvature inheritances are disjoint.

By transvecting the equation (4.3.1) successively with respect to $\dot{x}^j$ and $\dot{x}^h$ we also obtain

$$L_v W^i_{kh} = 0\,, \quad L_v W^i_k = 0 \tag{4.3.2}$$

We have observed that the infinitesimal transformation (4.1.41) is called an affine motion if it satisfies the relation $L_v g_{ij} = 0$. In this case it is necessary and sufficient that $L_v G^i_{jk} = 0$. In such case $L_v H^i_{jkh} = 0$. Singh (2013) has also observed that $L_v W^i_{jkh} = 0$ in a recurrent space. In a bi-recurrent space it is also true. Thus an affine motion is a W-curvature collineation.

Hence we state

Theorem 4.3.1: Every motion admitted in a bi-recurrent Finsler space $2R - F_n$ is a W-curvature collineation.

A necessary and sufficient condition that the infinitesimal transformation (4.1.41) to be a homothetic transformation Hiramatu (1954) is that the relation $L_v g_{ij} = 2C g_{ij}$ where C is a constant holds.

Singh (2013) has proved that every homothetic transformation admitted in a Finsler space F_n is a W- curvature collineation. It is also true in a bi-recurrent Finsler space $2R - F_n$.

Hence we state

Theorem 4.3.2: Every homothetic transformation admitted in a bi-recurrent Finsler space $2R - F_n$ is also a W- curvature collineation.

Applying (4.1.47) for W^i_{jkh} and noting the result (4.3.1), we have

$$L_v(\dot{\partial}_l W^i_{jkh}) = 0 \tag{4.3.3}$$

In view of commutation formula (3.7.21) for W^i_{jkh}, we obtain

$$2W^i_{jkh[(lm)]} = -\dot{\partial}_p W^i_{jkh} H^p_{lm} + W^p_{jkh} H^i_{plm} - W^i_{pkh} H^p_{jlm} - W^i_{jph} H^p_{klm} - W^i_{jkp} H^p_{hlm} \tag{4.3.4}$$

In a bi-recurrent Finsler space $2R - F_n$, the above equation assumes the form

$$(A_{lm} - A_{ml}) W^i_{jkh} = -\dot{\partial}_p W^i_{jkh} H^p_{lm} + W^p_{jkh} H^i_{plm} - W^i_{pkh} H^p_{jlm} - W^i_{jph} H^p_{klm} - W^i_{jkp} H^p_{hlm} \tag{4.3.5}$$

Taking Lie-derivative of both sides of (4.3.5), we get

$$(L_v A_{lm} - L_v A_{ml}) W^i_{jkh} = 0 \tag{4.3.6}$$

in view of (4.3.1), (4.3.3) and the fact that $L_v H^i_{jkh} = 0$ in an affine motion.

Since $W^i_{jkh} \neq 0$, the equation (4.3.6) implies that

$$L_v A_{[lm]} = L_v A_{lm} - L_v A_{ml} = 0 \tag{4.3.7}$$

Hence we have

Theorem 4.3.3: In a bi-recurrent Finsler space $2R - F_n$ which admits a W-curvature collineation, the recurrent tensor field A_{lm} satisfies the identity (4.3.7).

The recurrence tensor field is also expressed in recurrence vector field K_l as $A_{lm} = K_{l(m)} + K_l K_m$. From definition of recurrence tensor field we have $A_{[lm]} = K_{[l(m)]}$ which yields

$$L_v K_{[l(m)]} = 0 \tag{4.3.8}$$

in view of (4.3.7). Thus we have

Corollary 4.3.1: In a bi-recurrent Finsler space $2R - F_n$ which admits a W-curvature collineation the recurrence vector field K_l satisfies the identity $L_v K_{[l(m)]} = 0$.

The covariant differentiation of the equation (4.3.5) and use of the equation (4.1.21) together with recurrence property of H^i_{jkh}, we obtain

$$(A_{lm} - A_{ml})_{(n)} + K_n(A_{lm} - A_{ml}) = 2K_n((A_{lm} - A_{ml}) \tag{4.3.9}$$

which implies

$$A_{[lm](n)} = K_n A_{[lm]} \tag{4.3.10}$$

Taking Lie- derivative of both sides of (4.3.10) we find

$$L_v A_{[lm](n)} = L_v K_n A_{[lm]} \tag{4.3.11}$$

in view of (4.3.7).

Accordingly we state

Theorem 4.3.4: In a bi-recurrent Finsler space $2R - F_n$ which admits a W-curvature collineation, the bi-recurrence tensor A_{lm} satisfies the Lie-recurrence relation (4.3.11).

Let us assume that the recurrent vector K_n satisfies the collineation property then the result (4.3.11) reduce to

$$L_v A_{[lm](n)} = 0 \tag{4.3.12}$$

which also implies that

$$L_v A_{[lm](n)} + L_v A_{[mn](l)} + A_{[nl](m)} = 0 \tag{4.3.13}$$

Conversely if the identity (4.3.13) is true it implies that (4.3.12) is also true. Then in view of (4.1.22), we obtain

$$L_v[K_{l(m)} + K_l K_m]_{(n)} = 0 \tag{4.3.14}$$

or equivalently

$$L_v[K_{l(m)(n)} + K_{l(n)} K_m + K_l K_{m(n)}] = 0 \tag{4.3.15}$$

In view of corollary (4.3.1) and equation (4.3.8), the equation (4.3.15) reduces to

$$K_{l(n)}(L_v K_m) + K_{m(n)}(L_v K_l) = 0 \qquad (4.3.16)$$

since Lie derivative and covariant differentiation are commutative.

The equation (4.3.16) implies that

$$L_v K_l = 0 \qquad (4.3.17)$$

that is the recurrence vector K_l satisfies the collineation property.

Hence we state

Theorem 4.3.5: In a bi-recurrent Finsler space $2R - F_n$ which admits a W-curvature collineation, the necessary and sufficient condition is for the recurrence tensor A_{lm} to satisfy the identity (4.3.13), is that the recurrence vector K_l also satisfies the collineation property.

Definition 4.3.2: Contra field

In a bi-recurrent Finsler space $2R - F_n$, the vector field $V^i(x)$ of equation (4.1.41) determines a contra field if it satisfies the relation

$$V^i_{(j)} = 0 \qquad (4.3.18)$$

Let us consider a special W- Curvature collineation

$$\bar{x}^i = x^i + V^i(x)dt, \quad V^i_{(j)} = 0 \qquad (4.3.19)$$

In this special case if (4.3.19) is a motion, then equation (4.3.1) for the Lie derivative of W^i_{jkh} given by equation (4.1.49) results into

$$L_V W^i_{jkh} = W^i_{jkl} V^l = 0 \qquad (4.3.20)$$

Hence we state

Theorem 4.3.6: In a bi-recurrent Finsler space $2R - F_n$ which admits a special W-curvature collineation given by the equation (4.3.19) becomes a motion and the condition (4.3.20) holds.

Since L_v denotes the derivative, we be denote the reverse process by L^v and the integrability condition of (4.3.20) becomes

$$L^v W^i_{jkh} = W^i_{jkh(l)} V^l = 0 \tag{4.3.21}$$

Accordingly we state

Theorem 4.3.7: In a bi-recurrent Finsler space $2R - F_n$ which admits a W-Curvature collineation, if the vector field $V^i(x)$ spans a contra field, the conditions $W^i_{jkl} V^i = 0$ and $W^i_{jkh(l)} V^l = 0$ necessarily hold.

From equations (4.1.30) and (4.3.21) we find that in a bi-recurrent Finsler space $2R - F_n$,

$$W^i_{jkh(l)(m)} V^l = A_{lm} W^i_{jkh} V^l$$

$$A_{lm} V^l W^i_{hjk} = 0 \tag{4.3.22}$$

Since $2R - F_n$ is not isotropic, thus $W^i_{hjk} \not\equiv 0$ equation (4.3.22) yields

$$A_{lm} V^l = 0 \tag{4.3.23}$$

which is a necessary and sufficient condition.

Hence we state

Theorem 4.3.8: In a bi-recurrent Finsler space, $2R - F_n$ which admits a W-Curvature collineation for the vector field $V^i(x)$ to span a contra-field it is necessary and sufficient that $W^i_{jkh} V^l = 0$ and $A_{lm} V^l = 0$.

Definition 4.3.3: Concurrent field

In a bi-recurrent Finsler space $2R - F_n$, if the vector field $V^i(x)$ satisfies the relation

$$V^i_{(j)} = \lambda \delta^i_j \tag{4.3.24}$$

where λ is a non-zero constant, the vector field $V^i(x)$ is said to determine a concurrent field.

Let us consider the W-curvature collineation of the form

$$\bar{x}^i = x^i + V^i(x)\delta t,\ V^i_{(j)} = \lambda \delta^i_j \tag{4.3.25}$$

where δ^i_j is the Kronecker delta defined by the equation (3.2.10).

Applying the second part of (4.3.25) in equation (4.3.20), we obtain $W^i_{khl}V^l = 0$. The covariant differentiation of this relation in view of equations (4.3.25) and (4.3.24) gives $\lambda W^i_{jkh} = 0$.

However, λ is a non-zero constant hence the relation yields $W^i_{jkh} = 0$. This contradicts the assumption we made earlier that the Finsler space $2R - F_n$ is non flat.

Accordingly we state

Theorem 4.3.9: A general bi-recurrent Finsler space $2R - F_n$ does not permit a W-curvature collineation of the form (4.3.25).

We now investigate the relationship between curvature inheritance and curvature collineation for a contra field. Using the first part of the equation (4.3.20) and the equation (4.1.20) we have

$$L_v W^i_{jkh} = W^i_{jkh(l)}V^l = K_l W^i_{jkh}V^l \tag{4.3.26}$$

In view of the definition given by the equation (4.2.1) for W- curvature inheritance if the space is recurrent then we

$$\alpha(x)W^i_{jkh(l)} = \alpha(x)K_l W^i_{jkh}$$

$$L_v W^i_{jkh} = K_l W^i_{jkh} V^l$$

By the definition for W-curvature collineation given by equation (4.3.1) $L_v W^i_{jkh} = 0$ which implies

$$K_l W^i_{jkh} V^l = K_l V^l W^i_{jkh} = 0$$

Since $W^i_{jkh} \neq 0$, we obtain

$$K_l V^l = 0 \qquad (4.3.27)$$

which implies that the vector field (V^l) and the recurrence vector (K_l) are orthogonal.

Hence we state

Theorem 4.3.10: In a recurrent Finsler space $R - F_n$ which admits a special curvature collineation of the form (4.3.19), the necessary and sufficient condition for a curvature inheritance to become a curvature collineation is that the recurrence vector behaves like a gradient vector.

CHAPTER FIVE

RESULTS AND DISCUSSION ON DECOMPOSITION OF $W^i_{jkh}(x,\dot{x})$ IN RECURRENT AND BI-RECURRENT FINSLER SPACE

5.1 Introduction

In this chapter different types of decomposition tensors have been used to break down the projective curvature tensor field $W^i_{jkh}(x,\dot{x})$ in recurrent Finsler space $(R-F_n)$ and bi-recurrent Finsler space $(2R-F_n)$. The chapter is presented in six sections. In section 5.2 well known results and previously established results to be used later in the chapter have been highlighted. Section 5.3 contains decomposition of $W^i_{jkh}(x,\dot{x})$ using a covariant decomposition vector and a mixed tensor while section 5.4 contains use of a contravariant decomposition vector and a covariant decomposition tensor. In section 5.5 the decomposition of $W^i_{jkh}(x,\dot{x})$ as a product of two vector fields and a tensor field have been studied and finally in Section 5.6 $W^i_{jkh}(x,\dot{x})$ has been decomposed in terms of a mixed tensor of rank 2 and a covariant tensor of rank 2.

5.2 Preliminary results

Let us consider an n-dimensional Finsler space, F_n in which the projective curvature tensor field $W^i_{jkh}(x,\dot{x})$, projective torsion tensor $W^i_{jk}(x,\dot{x})$ and the deviation tensor field W^i_j are as previously defined by equations (4.1.1) and (4.1.3) respectively (Rund, 1959). In a recurrent Finsler space $(R-F_n)$ the projective curvature tensor satisfies the equation (4.1.21) and in a bi-recurrent Finsler space $(2R-F_n)$ it satisfies the equation (4.1.30) (Sinha and Singh, 1971).

Definition 5.2.1: Tensor decomposition is a way of breaking down higher order tensors into contravariant and covariant tensors with lower orders or ranks than the original tensor. The projective curvature tensor $W^i_{jkh}(x,\dot{x})$ is a mixed tensor of order (1, 3) and by Theorem 3.1.2 it may be written as a tensor product of a vector or tensor of rank 1 (X) and a tensor of rank 3 (Ψ) with the possibilities as

a) $W^i_{jkh} = X_j \Psi^i_{kh}$

b) $W^i_{jkh} = X^i \Psi_{jkh}$

c) $W^i_{jkh} = X_k \Psi^i_{jh}$

d) $W^i_{jkh} = X_h \Psi^i_{jk}$.

It may also be written as a tensor product of two tensors, one mixed tensor of the form (1, 1) denoted by (X) and the other is a covariant tensor of the form (0, 2) denoted by (Ψ) as

a) $W^i_{jkh} = X^i_j \Psi_{kh}$

b) $W^i_{jkh} = X^i_k \Psi_{jh}$

c) $W^i_{jkh} = X^i_h \Psi_{jk}$.

The projective curvature tensor field can also be decomposed into two vector fields and a tensor field in the forms

a) $W^i_{jkh} = P^i X_j \Psi_{kh}$

b) $W^i_{jkh} = P^i X_k \Psi_{jh}$

c) $W^i_{jkh} = P^i X_h \Psi_{jk}$

5.3 Decomposition using a covariant vector $X_j(x,\dot{x})$ and mixed tensor $\Psi^i_{kh}(x,\dot{x})$

We consider the decomposition of the projective curvature tensor W^i_{jkh} in the form

$$W^i_{jkh} = X_j \Psi^i_{kh} \tag{5.3.1}$$

where X_j is a non-zero covariant tensor and Ψ^i_{kh} is a skew-symmetric decomposition tensor.

We shall denote the space with such decomposition of the W- curvature tensor by $R - F_n$ for recurrent space and $2R - F_n$ for bi-recurrent as given in the previous chapter by definitions (4.1.2 and 4.1.3) respectively.

Differentiating equation (5.3.1) covariantly with respect to x^l in the sense of Berwald we get

$$W^i_{jkh(l)} = X_{j(l)} \Psi^i_{kh} + X_j \Psi^i_{kh(l)} \tag{5.3.2}$$

Using equation (4.1.21) and (5.3.1) on equation (5.3.2) above, we obtain

$$W^i_{jkh(l)} = K_l W^i_{jkh} = X_{j(l)} \Psi^i_{kh} + X_j \Psi^i_{kh(l)} = K_l X_j \Psi^i_{kh}$$

$$W^i_{jkh(l)} = \beta_l X_j \Psi^i_{kh} + X_j \Psi^i_{kh(l)} \tag{5.3.3}$$

where

$$X_{j(l)} = \beta_l X_j \tag{5.3.4}$$

In view of equations (4.1.21) and (5.3.4) we get

$$X_j \Psi^i_{kh(l)} = K_l X_j \Psi^i_{kh} - X_{j(l)} \Psi^i_{kh}$$

$$X_j \Psi^i_{kh(l)} = K_l X_j \Psi^i_{kh} - \beta_l X_j \Psi^i_{kh}$$

$$X_j \Psi^i_{kh(l)} = X_j (K_l \Psi^i_{kh} - \beta_l \Psi^i_{kh})$$

hence

$$\Psi^i_{kh(l)} = (K_l - \beta_l)\Psi^i_{kh} \tag{5.3.5}$$

Assuming $K_l \neq \beta_l$, equation (5.3.5) may be written as

$$\Psi^i_{kh(l)} = \alpha_l \Psi^i_{kh} \tag{5.3.6}$$

where $\alpha_l = (K_l - \beta_l)$.

Conversely if the equation (5.3.5) above is true then we use equation (5.3.6) to transform equation (5.3.3) and we have

$$\beta_l X_j \Psi^i_{kh} + X_j \Psi^i_{kh(l)} = K_l X_j \Psi^i_{kh}$$

$$\beta_l \Psi^i_{kh} + \Psi^i_{kh(l)} = K_l \Psi^i_{kh}$$

$$\beta_l \Psi^i_{kh} + \alpha_l \Psi^i_{kh} = K_l \Psi^i_{kh}$$

or

$$K_l \Psi^i_{kh} = (\beta_l + \alpha_l)\Psi^i_{kh} \tag{5.3.7}$$

Hence we state

Theorem 5.3.1: In a $R - F_n$ the necessary and sufficient condition for the decomposition tensor Ψ^i_{kh} to be recurrent is that the recurrence vector K_l is not equal to the recurrence vector β_l.

Let us assume that the recurrent vectors K_l and β_l are equal such that

$$K_l = \beta_l \tag{5.3.8}$$

then in view of equation (5.3.8) since the decomposition tensor $\Psi^i_{kh} \neq 0$, equation (5.3.5) simplifies to

$$\Psi^i_{kh(l)} = 0 \tag{5.3.9}$$

Using equation (5.3.9) in equation (5.3.3), we obtain

$$W^i_{jkh(l)} = X_{j(l)}\Psi^i_{kh}$$

which on using (5.3.4) becomes

$$W^i_{jkh(l)} = \beta_l X_j \Psi^i_{kh} \tag{5.3.10}$$

Performing cyclic change with respect to indices k, h, l on equation (5.3.10) we have

$$W^i_{jhl(k)} = \beta_k X_j \Psi^i_{hl}$$

$$W^i_{jlk(h)} = \beta_h X_j \Psi^i_{lk}$$

and adding the results we have

$$W^i_{jkh(l)} + W^i_{jhl(k)} + W^i_{jlk(h)} = X_j(\beta_l \Psi^i_{kh} + \beta_k \Psi^i_{hl} + \beta_h \Psi^i_{lk}) \tag{5.3.11}$$

In view of the identity satisfied by the projective curvature in equation (4.1.27) and the recurrence relation $W^i_{jkh(l)} = K_l W^i_{jkh}$, equation (5.3.11) reduces to

$$X_j\left(\beta_l \Psi^i_{kh} + \beta_k \Psi^i_{hl} + \beta_h \Psi^i_{lk}\right) = 0 \tag{5.3.12}$$

Since X_j is a non-zero covariant tensor of rank one (vector field) equation (5.3.12) implies that

$$\left(\beta_l \Psi^i_{kh} + \beta_k \Psi^i_{hl} + \beta_h \Psi^i_{lk}\right) = 0 \tag{5.3.13a}$$

or

$$\left(K_l \Psi^i_{kh} + K_k \Psi^i_{hl} + K_h \Psi^i_{lk}\right) = 0 \tag{5.3.13b}$$

Therefore we state

Theorem 5.3.2: In $2R - F_n$ under decomposition (5.3.1), if the recurrence vector K_l is equal to β_l, the decomposition tensor Ψ^i_{kh} satisfies the identity (5.3.13).

Differentiating equation (5.3.10) covariantly with respect to x^m in the sense of Berwald

we have $W^i_{jkh(l)(m)} = (\beta_l X_j \Psi^i_{kh})_{(m)}$

$$W^i_{jkh(l)(m)} = \beta_{l(m)} X_j \Psi^i_{kh} + \beta_l (X_j \Psi^i_{kh})_{(m)}$$

$$W^i_{jkh(l)(m)} = \beta_{l(m)} X_j \Psi^i_{kh} + \beta_l (X_{j(m)} \Psi^i_{kh} + X_j \Psi^i_{kh(m)})$$

applying equation (5.3.9), we obtain

$$W^i_{jkh(l)(m)} = \beta_{l(m)} X_j \Psi^i_{kh} + \beta_l X_{j(m)} \Psi^i_{kh} \tag{5.3.14}$$

In view of the relation $X_{j(l)} = \beta_l X_j$ in equation (5.3.4), we have

$$W^i_{jkh(l)(m)} = \beta_{l(m)} X_j \Psi^i_{kh} + \beta_l \beta_m X_j \Psi^i_{kh}$$

and the equation above becomes

$$W^i_{jkh(l)(m)} = (\beta_{(l)(m)} + \beta_l \beta_m) X_j \Psi^i_{kh} \tag{5.3.15}$$

On applying equations (5.3.10) and (5.3.1), we get

$$W^i_{jkh(l)(m)} = A_{lm} W^i_{jkh} = A_{lm}(X_j \Psi^i_{kh})$$

$$A_{lm}(X_j \Psi^i_{kh}) = (\beta_{(l)(m)} + \beta_l \beta_m) X_j \Psi^i_{kh} \tag{5.3.16}$$

From equations (5.3.15) and (5.3.16), we obtain

$$A_{lm} = \beta_{(l)(m)} + \beta_l \beta_m \tag{5.3.17}$$

Therefore we state

Theorem 5.3.3: In $2R - F_n$, under decomposition (5.3.1), if the vector K_l is equal to β_l for which recurrence vector field β_l satisfies the condition $\beta_{(l)(m)} + \beta_l \beta_m \neq 0$, then the equality (5.3.17) holds true.

Commuting indices l and m in equation (5.3.15) we have

$$W^i_{jkh(m)(l)} = (\beta_{(m)(l)} + \beta_m \beta_l) X_j \Psi^i_{kh}$$

and subtracting the equations obtained, we have

$$W^i_{jkh(l)(m)} - W^i_{jkh(m)(l)} = \left(\beta_{(l)(m)} + \beta_l \beta_m\right) X_j \Psi^i_{kh} - (\beta_{(m)(l)} + \beta_m \beta_l)(X_j \Psi^i_{kh}\big)$$

Using equation (5.3.17), we have

$$W^i_{jkh(l)(m)} - W^i_{jkh(m)(l)} = (A_{lm} - A_{ml}) X_j \Psi^i_{kh} \qquad (5.3.18a)$$

or

$$W^i_{jkh(l)(m)} - W^i_{jkh(m)(l)} = \left(\beta_{l(m)} - \beta_{m(l)}\right) X_j \Psi^i_{kh} \qquad (5.3.18b)$$

Hence we state

Corollary 5.3.1

In $2R - F_n$ under the decomposition (5.3.1), if the vectors K_l and β_l are equal the projective curvature tensor satisfies the identity (5.3.18).

Differentiating equation (5.3.6) covariantly with respect to x^m in the sense of Berwald, and using $\alpha_l = (K_l - \beta_l)$, we obtain

$$\Psi^i_{kh(l)(m)} = \alpha_{l(m)} \Psi^i_{kh} + \alpha_l \Psi^i_{kh(m)} = \left(K_{l(m)} - \beta_{l(m)}\right) \Psi^i_{kh} + (K_l - \beta_l) \Psi^i_{kh(m)}$$

(5.3.19)

On applying equation (5.3.6), the equation (5.3.19) takes the form

$$\Psi^i_{kh(l)(m)} = \left(K_{l(m)} - \beta_{l(m)}\right) \Psi^i_{kh} + (K_l - \beta_{l)}) \Psi^i_{kh(m)} \qquad (5.3.20)$$

$$\left(K_l - \beta_{l)}\right)\Psi^i_{kh(m)} = K_l\Psi^i_{kh(m)} - \beta_l\Psi^i_{kh(m)} = K_lK_m\Psi^i_{kh} + K_l\beta_m\Psi^i_{kh} - \beta_lK_m\Psi^i_{kh} - \beta_l\beta_m\Psi^i_{kh}$$

$$\Psi^i_{kh(l)(m)} = \left(K_{l(m)} - \beta_{l(m)}\right)\Psi^i_{kh} + (K_m - \beta_m)\left(K_l - \beta_{l)}\right)\Psi^i_{kh} \qquad (5.3.21)$$

The equation above transforms to

$$\Psi^i_{kh(l)(m)} = (K_{l(m)} - \beta_{l(m)} + K_mK_l - K_m\beta_l + K_l\beta_m - \beta_l\beta_m)\Psi^i_{kh} \qquad (5.3.22)$$

Hence we conclude

Theorem 5.3.4:

In a bi-recurrent Finsler space $2R - F_n$, under the decomposition (5.3.1) the second order derivative of decomposition tensor Ψ^i_{kh} satisfies the relation (5.3.22).

In view of (5.3.8) equation (5.3.22) is reduced to

$$\Psi^i_{kh(l)(m)} = 0 \qquad (5.3.23)$$

Hence we state

Corollary 5.3.2

In a bi-recurrent Finsler space $2R - F_n$, the second order covariant derivative of the decomposition tensor vanishes if the vector K_l is equal to β_l.

5.4 Decomposition using a contravariant vector $X^i(x,\dot{x})$ and covariant tensor $\Psi_{jkh}(x,\dot{x})$

In this section we consider the decomposition of the projective curvature tensor $W^i_{jkh}(x,\dot{x})$ in the form

$$W^i_{jkh} = X^i \Psi_{jkh} \tag{5.4.1}$$

where $X^i(x,\dot{x})$ is any contravariant vector field which will be called the decomposition vector and $\Psi_{jkh}(x,\dot{x})$ is a covariant tensor field of rank three and will be called the decomposition tensor.

The decomposition vector $X^i(x,\dot{x})$ should satisfy the relation

$$K_i X^i = 1 \tag{5.4.2}$$

where K_i is a recurrence vector as defined in section 4.1.2.

The covariant derivative of $X^i(x,\dot{x})$ with respect to $\dot{x}^j$ and using the notation $\frac{\partial}{\partial \dot{x}^j} = \dot{\partial}_j$ is given by

$$X^i{}_{(j)} = \dot{\partial}_j X^i - (\dot{\partial}_m X^i)\Gamma^m_{kj}\dot{x}^k + X^k \Gamma^i_{kj}$$

where $\Gamma^i_{kj}(x,\dot{x})$ is a with properties described in section 3.5.

Using equations (4.1.4a) and (4.1.15) the following relations hold for the decomposition tensor Ψ_{jkh}

$$\Psi_{jkh}\dot{x}^j = \Psi_{kh} \tag{5.4.3}$$

$$\dot{\partial}_l \Psi_{kh} = \Psi_{khl} \tag{5.4.4}$$

Transvecting (5.4.1) by $\dot{x}^j$ and using (5.4.3) and (5.4.4) we obtain

$$W^i_{jkh}\dot{x}^j = X^i\Psi_{jkh}\dot{x}^j$$

$$W^i_{kh} = X^i\Psi_{kh} \qquad (5.4.5)$$

where $\Psi_{kh} \neq 0$ is a homogeneous tensor of degree one in $\dot{x}^j$.

Differentiating (5.4.1) covariantly with respect to $\dot{x}^s$

$$W^i_{jkh(s)} = X^i_{(s)}\Psi_{jkh} + X^i\Psi_{jkh(s)} \qquad (5.4.6)$$

where $\Psi_{jkh} = \dot{\partial}_j\Psi_{kh}$ in relation to equation (5.4.4).

Using (4.1.21), (5.4.1) and (5.4.6), we obtain

$$W^i_{jkh(s)} = \alpha_s\, W^i_{jkh} = \alpha_s X^i\Psi_{jkh}$$

$$\alpha_s X^i\Psi_{jkh} = X^i_{(s)}\Psi_{jkh} + X^i\Psi_{jkh(s)} \qquad (5.4.7)$$

Assuming that the contravariant decomposition vector X^i is covariant constant its derivative $(X^i_{(s)})$ will be zero and hence equation (5.4.6) takes the form

$$\alpha_s X^i\Psi_{jkh} = X^i\Psi_{jkh(s)} \qquad (5.4.8)$$

On dividing both sides of (5.4.8) we write this equation as

$$\Psi_{jkh(s)} = \alpha_s\Psi_{jkh} \qquad (5.4.9)$$

Transvecting equation (5.4.9) by $\dot{x}^j$ and using (5.4.3), we get

$\Psi_{jkh(s)}\dot{x}^j = \alpha_s\Psi_{jkh}\dot{x}^j$

$$\Psi_{kh(s)} = \alpha_s\Psi_{kh} \qquad (5.4.10)$$

Using equations (5.4.7), (5.4.9) and (5.4.10), we state

Theorem 5.4.1: In a recurrent Finsler $(R-F_n)$, the necessary and sufficient condition for the decomposition vector $\Psi_{jkh}(x,\dot{x})$ to be recurrent is that the decomposition vector X^i is assumed to be a covariant constant.

Theorem 5.4.2: In a recurrent Finsler space $(R-F_n)$, if the decomposition tensor field Ψ_{jkh} of rank (0,3) is recurrent , then the tensor field Ψ_{kh} of rank (0,2) is also recurrent if the recurrence vector α_s is independent of the directional argument.

Differentiating (5.4.2) covariantly with respect to x^s we get,

$$K_i X^i_{(s)} + X^i K_{i(s)} = 0 \tag{5.4.11}$$

Corollary 5.4.1 In a recurrent Finsler space $(R-F_n)$ if the decomposition vector field X^i is covariant constant then the recurrence vector field K_i is also a covariant constant.

In view of (5.4.1) and (5.4.2) and definition (3.9) the identities (4.1.7) and (4.1.9) have the forms

$$\Psi_{jkh} = -\Psi_{jhk} \tag{5.4.12}$$

and

$$\Psi_{jkh} + \Psi_{khj} + \Psi_{hjk} = 0 \tag{5.4.13}$$

Hence we state,

Theorem 5.4.3: In a recurrent Finsler space $(R-F_n)$ the decomposition tensor Ψ_{jkh} field is symmetric and satisfies the identity (5.4.12) and (5.4.13).

In view of definition (5.2.1) we decompose the decomposition tensor Ψ_{jkh} in equation (2.1) using a covariant decomposition vector field K_j as

$$\Psi_{jkh} = K_j \Psi_{kh} \tag{5.4.14}$$

The identity (5.4.12) therefore takes the form

$$\Psi_{kh} = -\Psi_{hk} \tag{5.4.15}$$

We then use equations (5.4.14) and (5.4.15) to obtain

$$\Psi_{khj} = K_k \Psi_{hj} = -K_k \Psi_{jh} ,$$
$$\Psi_{hjk} = K_h \Psi_{jk} = -K_h \Psi_{kj} \tag{5.4.16}$$

which we use to write equation (5.4.13) as

$$\Psi_{khj} - K_k \Psi_{jh} + K_h \Psi_{jk} = 0 \tag{5.4.17}$$

Transvecting (5.4.17) by $\dot{x}^j$ we have

$$\Psi_{jkh}\dot{x}^j - K_k \Psi_{jh}\dot{x}^j + K_h \Psi_{jk}\dot{x}^j = 0$$

$$\Psi_{jkh}\dot{x}^j = K_k \Psi_{jh}\dot{x}^j - K_h \Psi_{kj}\dot{x}^j$$

$$\Psi_{jkh}\dot{x}^j = \Psi_j (K_k \Psi_h + K_h \Psi_k)\dot{x}^j$$

Using the decomposition equation (5.4.1) for W^i_{jkh} we write

$$W^i_{jkh} = 2\Psi_{\,j\,[h} K_{k]}\, X^i \tag{5.4.18}$$

in which the square brackets are used to represent the skew symmetric part of the equation.

Hence we state,

Theorem 5.4.4: In a recurrent Finsler space $(R - F_n)$ the projective curvature tensor W^i_{jkh} can be expressed in form of the equation (5.4.18).

We now consider the decomposability of the projective curvature W^i_{jkh} in the form of equation (5.4.1) in a birecurrent Finsler space. This space is defined by definition (4.1.3) and denoted by $2R - F_n$.

Differentiating (5.4.1) successively with respect to x^s and x^m we obtain

$$W^i_{jkh(sm)} = X^i_{(sm)}\Psi_{jkh} + X^i\Psi_{jkh(sm)} \tag{5.4.19}$$

We use equations (4.1.30) and (5.4.1) in equation (2.18) we have

$$W^i_{jkh(sm)} = \beta_{sm} W^i_{jkh} \tag{5.4.20}$$

$$W^i_{jkh(sm)} = X^i\Psi_{jkh(sm)} \tag{5.4.21}$$

and hence

$$\beta_{sm} X^i \Psi_{jkh} = X^i_{(sm)}\Psi_{jkh} + X^i\Psi_{jkh(sm)} \tag{5.4.22}$$

If we assume that the decomposition vector X^i is a covariant constant then the covariant derivative $X^i_{(sm)} = 0$. On dividing both sides of equation (5.4.22) we get

$$\Psi_{jkh(sm)} = \beta_{sm}\Psi_{jkh} \tag{5.4.23}$$

Equation (5.4.23) gives an assumption that the decomposition vector Ψ_{jkh} is birecurrent.

Transvecting equation (5.4.23) by $\dot{x}^j$ and using (5.4.3)

$$\Psi_{jkh(sm)}\dot{x}^j = \beta_{sm}\Psi_{jkh}\dot{x}^j$$

$$\Psi_{kh(sm)} = \beta_{sm}\Psi_{kh} \tag{5.4.24}$$

Using equations definitions (4.1.2) and (4.1.3) and equations (4.1.21) and (4.1.30), the projective curvature tensor satisfies the relations

$$W^i_{jkh(sm)} = \alpha_m W^i_{jkh(s)} \quad (5.4.25a)$$

$$W^i_{jkh(sm)} = \alpha_s W^i_{jkh(m)} \quad (5.4.25b)$$

On using equation (5.4.23) in equation (5.4.19) we obtain

$$\alpha_m W^i_{jkh(s)} = X^i_{(sm)}\Psi_{jkh} + X^i\Psi_{jkh(sm)} \quad (5.4.26a)$$

$$\alpha_s W^i_{jkh(m)} = X^i_{(sm)}\Psi_{jkh} + X^i\Psi_{jkh(sm)} \quad (5.4.26b)$$

The presence of the recurrence vector α_m or α_s in equations (5.4.25) and (5.4.26) makes us assume that the space under consideration is also recurrent and we write

$$\alpha_m W^i_{jkh(s)} = v_{ms} W^i_{jkh}$$

or

$$\alpha_s W^i_{jkh(m)} = v_{sm} W^i_{jkh}$$

and equation (5.4.26) results into

$$v_{ms} W^i_{jkh} = X^i_{(sm)}\Psi_{jkh} + X^i\Psi_{jkh(sm)} \quad (5.4.27a)$$

$$v_{ms} W^i_{jkh} = X^i_{(ms)}\Psi_{jkh} + X^i\Psi_{jkh(ms)} \quad (5.4.27b)$$

where $v_{sm} = \alpha_s\alpha_m$.

Considering that the decomposition given by equation (5.4.1) is possible then equation (5.4.27) yields

$$v_{ms} X^i\Psi_{jkh} = X^i_{(sm)}\Psi_{jkh} + X^i\Psi_{jkh(sm)} \quad (5.4.28a)$$

$$v_{sm} X^i\Psi_{jkh} = X^i_{(ms)}\Psi_{jkh} + X^i\Psi_{jkh(ms)} \quad (5.4.28b)$$

Let us assume that decomposition vector X^i is a covariant constant then $X^i_{(sm)} = X^i_{(ms)} = 0$ and from (5.4.27) we get

$$\Psi_{jkh(sm)} = v_{ms}\Psi_{jkh} \tag{5.4.29a}$$

$$\Psi_{jkh(ms)} = v_{sm}\Psi_{jkh} \tag{5.4.29b}$$

Therefore we state

Theorem 5.4.5: In a bi-recurrent Finsler space $2R - F_n$ the decomposition tensor fields are not bi-recurrent in general but are birecurrent only when the contravariant decomposition vector is a covariant constant.

Theorem 5.4.6: In a bi-recurrent Finsler space $2R - F_n$, the necessary and sufficient condition for the decomposition tensor $\Psi_{jkh}(x, \dot{x})$ to be recurrent is that the decomposition vector X^i is a covariant constant and has a vanishing covariant derivative.

5.5 The decomposition of the projective curvature tensor using two vector fields $P^i X_j(x, \dot{x})$ and a tensor $\Psi_{kh}(x, \dot{x})$

We consider the decomposition of the projective curvature tensor W^i_{jkh} in the form

$$W^i_{jkh} = P^i X_j \Psi_{kh} \tag{5.5.1}$$

where P^i is a contravariant decomposition vector as in section 5.3, X_j is covariant decomposition vector as in section (5.4) and Ψ_{kh} is a covariant decomposition tensor of rank 2.

The decomposition vector field $P^i(x, \dot{x})$ satisfies the relation given by equation (5.4.2) and therefore

$$K_i P^i = 1 \tag{5.5.2}$$

In an n- dimensional Finsler space the projective curvature tensor W^i_{jkh} satisfies the Bianchi identities given by equations (4.1.9) and (4.1.27).

From equations (4.1.9) and (5.4.1), we obtain

$$W^i_{jkh} + W^i_{khj} + W^i_{hjk} = 0$$

$$P^i X_j \Psi_{kh} + P^i X_k \Psi_{hj} + P^i X_h \Psi_{jk} = 0$$

or

$$P^i (X_j \Psi_{kh} + X_k \Psi_{hj} + X_h \Psi_{jk}) = 0 \tag{5.5.3}$$

Multiplying equation (5.5.3) by the recurrence vector λ_i and using equation (5.5.2), we get (5.5.4)

$$X_j \Psi_{kh} + X_k \Psi_{hj} + X_h \Psi_{jk} = 0$$

Using equation (4.1.27), equation (4.1.21) and equation (5.5.1), we have

$$W^i_{jkh(l)} + W^i_{jhl(k)} + W^i_{jlk(h)} = 0$$

$$K_i W^i_{jkh} + K_k W^i_{jhl} + K_h W^i_{jlk} = 0$$

$$K_i P^i X_j \Psi_{kh} + K_k P^i X_j \Psi_{hl} + K_h P^i X_j \Psi_{lk} = 0$$

hence

$$P^i X_j (K_i \Psi_{kh} + K_k \Psi_{hl} + K_h \Psi_{lk}) = 0 \tag{5.5.5}$$

Multiplication of equation (5.5.5) by K_i and using equation (5.5.2) gives

$$X_j (K_l \Psi_{kh} + K_k \Psi_{hl} + K_h \Psi_{lk}) = 0 \tag{5.5.6}$$

Since $X_j \neq 0$ equation (5.5.6) yields

$$K_l \Psi_{kh} + K_k \Psi_{hl} + K_l \Psi_{lk} = 0 \tag{5.5.7}$$

Hence we state

Theorem 5.5.1: In a recurrent Finsler space under the decomposition (5.5.1) the Bianchi identities for W^i_{jkh} take the forms of equation (5.5.4) and equation (5.5.7).

The projective curvature tensor W^i_{jkh} has the Ricci tensor given by equation (4.1.37) and the scalar curvature given by equation (4.1.38) respectively. These tensors satisfy the identities given by equations (4.1.39) to (4.1.42). We use these to establish a relation between the Ricci tensor and the decomposition tensor.

With the help of equations (4.1.39), (4.1.21) and (4.1.43) we have

$$W^i_{jkh(l)} = W_{kh(j)} - W_{jh(k)}$$

$$K_l W^i_{jkh} = K_j W_{kh} - K_k W_{jh} \tag{5.5.8}$$

Multiplying equation (5.5.1) by the vector K_i and using the relation (5.5.2) we get

$$K_i W^i_{jkh} = K_i P^i X_j \Psi_{kh}$$

$$K_i W^i_{jkh} = X_j \Psi_{kh} \tag{5.5.9}$$

From equation (5.5.8) and equation (5.5.9), we obtain the relation

$$K_l W^i_{jkh} = K_j W_{kh} - K_k W_{jh} = X_j \Psi_{kh} \tag{5.5.10}$$

Therefore, we can state the following

Theorem 5.5.2: Under the decomposition (5.5.1) the tensor fields W^i_{jkh}, W_{jk} and Ψ_{kh} satisfy the relation (5.5.10).

Differentiating equation (5.5.8) covariantly with respect to x^m and using equation (5.5.1), we have

$$K_{l(m)} W^i_{jkh} = K_{j(m)} W_{kh} - K_{k(m)} W_{jh}$$

$$K_{l(m)} P^l X_j \Psi_{kh} = K_{j(m)} W_{kh} - K_{k(m)} W_{jh} \tag{5.5.11}$$

Multiplying (5.5.11) k_l and using (5.5.1) and (5.5.9), we have

$$K_{l(m)} K_l P^l X_j \Psi_{kh} = K_{j(m)} K_l W_{kh} - K_{k(m)} K_l W_{jh}$$

$$K_{l(m)} X_j \Psi_{kh} = K_{j(m)} K_l W_{kh} - K_{k(m)} K_l W_{jh}$$

$$K_{l(m)} (K_j W_{kh} - K_k W_{jh}) = K_{j(m)} K_l W_{kh} - K_{k(m)} K_l W_{jh}$$

hence

$$K_{l(m)} (K_j W_{kh} - K_k W_{jh}) = K_l (K_{j(m)} W_{kh} - K_{k(m)} W_{jh}) \tag{5.5.12}$$

Multiplying equation (5.5.12) by a vector K_i, we have

$$K_{l(m)} (K_j W_{kh} - K_k W_{jh}) K_i = K_l K_i (K_{j(m)} W_{kh} - K_{k(m)} W_{jh}) \tag{5.5.13}$$

Let us consider the expression on the right hand side of equation (5.5.13). This expression is symmetric in *l* and *i* provided $(K_j W_{kh} - K W_{jh}) \neq 0$.

The vector $K_l \neq 0$ and so we choose a proportional vector V_m such

$$K_{l(m)} = V_m K_l \tag{5.5.14}$$

Hence we state

Theorem 5.5.3: In a recurrent Finsler space $R - F_n$, under the decomposition (5.5.1), the quantity K_l behaves like the recurrent vector and the recurrent form is given by equation (5.5.14).

Differentiating equation (5.5.2) covariantly with respect to x^m we have

$$K_i P^i_{(m)} + P^i K_{i(m)} = 0 \tag{5.5.15}$$

Using (5.5.14) in (5.5.15), we get

$$K_i P^i_{(m)} + V_m K_i P^i = 0 \tag{5.5.16}$$

Since $K_i \neq 0$ equation (5.5.16) may be written as

$$P^i_{(m)} = -V_m P^i \tag{5.5.17}$$

Therefore we state

Theorem 5.5.4: In a recurrent Finsler space $R - F_n$, under the decomposition (5.5.1), the quantity P^i behaves like a contravariant vector and satisfies the equation (5.5.17).

5.6 Decomposition using a mixed tensor $X^i_j(x, \dot{x})$ and a covariant tensor $\Psi_{kh}(x, \dot{x})$

Let us consider the decomposition of the projective curvature tensor $W^i_{jkh}(x, \dot{x})$ in the form

$$W^i_{jkh} = X^i_j \Psi_{kh} \tag{5.6.1}$$

where X^i_j is a non-zero mixed tensor of rank 2 and Ψ_{kh} is a covariant decomposition tensor of rank 2.

The space equipped with such decomposition has been defined in section (4.1.2) and projective curvature tensor satisfies the relation given by equation (4.1.21).

Differentiating (5.6.1) covariantly with respect to x^l in the sense of Berwald, we obtain

$$W^i_{jkh(l)} = X^i_j \Psi_{kh(l)} + X^i_{j(l)} \Psi_{kh} \tag{5.6.2}$$

Using equation (4.1.21) and equation (5.6.1) in (5.6.2), we obtain

$$W^i_{jkh(l)} = X^i_j \Psi_{kh(l)} + \beta_l X^i_j \Psi_{kh} \tag{5.6.3}$$

where

$$X^i_{j(l)} = \beta_l X^i_j \tag{5.6.4}$$

Using equation (5.6.1) and equation (5.6.3)

$$W^i_{jkh(l)} = K_l X^i_j \Psi_{kh} = X^i_j \Psi_{kh(l)} + \beta_l X^i_j \Psi_{kh}$$

$$X_j^i \Psi_{kh(l)} = K_l X_j^i \Psi_{kh} - \beta_l X_j^i \Psi_{kh}$$

or

$$\Psi_{kh(l)} = (K_l - \beta_l)\Psi_{kh} \tag{5.6.5}$$

Let us assume that $\beta_l \neq K_l$ and $\alpha_l = (K_l - \beta_l)$ then we may write the equation (5.6.5) as

$$\Psi_{kh(l)} = \alpha_l \Psi_{kh} \tag{5.6.6}$$

If the equation (5.6.6) above is true, then the equation (5.6.3) yields

$$W^i_{jkh(l)} = K_l X_j^i \Psi_{kh} = \beta_l X_j^i \Psi_{kh} + X_j^i \Psi_{kh(l)}$$

$$K_l \Psi_{kh} = \beta_l \Psi_{kh} + (K_l - \beta_l)\Psi_{kh}$$

$$K_l \Psi_{kh} = \beta_l \Psi_{kh} + \alpha_l \Psi_{kh}$$

hence

$$K_l \Psi_{kh} = (\beta_l + \alpha_l)\Psi_{kh} \tag{5.6.7}$$

Thus we have

Theorem 5.6.1: In $R - F_n$ under the decomposition (5.6.1) the necessary and sufficient condition for the decomposition tensor Ψ_{kh} to be recurrent is that the recurrent vector K_l is not equal to the recurrent vector β_l.

Now let us assume that the recurrence vectors K_l and β_l are equal such that

$$K_l = \beta_l \tag{5.6.8}$$

By considering equation (5.6.8), the equation (5.6.6) is reduced to

$$\Psi_{kh(l)} = \alpha_l \Psi_{kh} = 0 \tag{5.6.9}$$

Using equation (5.6.9) in equation (5.6.3), we obtain

$$W^i_{jkh(l)} = \beta_l X^i_j \Psi_{kh}$$

or

$$W^i_{jkh(l)} = X^i_{j(l)} \Psi_{kh} \tag{5.6.10}$$

Performing cyclic change of equation (5.6.10) with respect to the indices k, h and l we get

$$W^i_{jhl(k)} = X^i_{j(k)} \Psi_{hl}$$

$$W^i_{jlk(h)} = X^i_{j(h)} \Psi_{lk}$$

Adding the expressions we obtain

$$W^i_{jkh(l)} + W^i_{jhl(k)} + W^i_{jlk(h)} = X^i_{j(l)} \Psi_{kh} + X^i_{j(k)} \Psi_{hl} + X^i_{j(h)} \Psi_{lk}$$

which we can rewrite as

$$W^i_{jkh(l)} + W^i_{jhl(k)} + W^i_{jlk(h)} = X^i_j (\beta_l \Psi_{kh} + \beta_k \Psi_{hl} + \beta_h \Psi_{lk}) \tag{5.6.11}$$

In view of the identity 4.1.27, thus

$$W^i_{jkh(l)} + W^i_{jhl(k)} + W^i_{jlk(h)} = 0$$

equation (5.6.11) simplifies to

$$X^i_j (\beta_l \Psi_{kh} + \beta_k \Psi_{hl} + \beta_h \Psi_{lk}) = 0 \tag{5.6.12}$$

Since from equation (5.6.1) the tensor X^i_j is described a non zero, it follows that

$$\beta_l \Psi_{kh} + \beta_k \Psi_{hl} + \beta_h \Psi_{lk} = 0 \tag{5.6.13}$$

alternatively by using equation (5.6.8), we write

$$K_l \Psi_{kh} + K_k \Psi_{hl} + K_h \Psi_{lk} = 0$$

Therefore we state

Theorem 5.6.2: In a recurrent Finsler space $R-F_n$ under the decomposition (5.6.1), if the recurrence vector K_l is equal to the recurrence vector β_l the decomposition tensor satisfies the identity (5.6.13).

Now let us consider a case of bi-recurrence where we use recurrence tensor of order two.

Differentiating equation (5.6.10) covariantly with respect to x^m in the sense of Berwald

$$W^i_{jkh(l)} = \beta_l X^i_j \Psi_{kh}$$

$$W^i_{jkh(l)(m)} = \beta_{l(m)} X^i_j \Psi_{kh} + \beta_l X^i_{j(m)} \Psi_{kh} + \beta_l X^i_j \Psi_{kh(m)}$$

Using equation (5.6.9) the term containing $\Psi_{kh(m)}$vanishes and we have

$$W^i_{jkh(l)(m)} = \beta_{l(m)} X^i_j \Psi_{kh} + \beta_l X^i_{j(m)} \Psi_{kh} \tag{5.6.14}$$

By considering equation (5.6.4)

$$W^i_{jkh(l)(m)} = \beta_{l(m)} X^i_j \Psi_{kh} + \beta_l \beta_m X^i_j \Psi_{kh}$$

and we write equation (5.6.14) as

$$W^i_{jkh(l)(m)} = (\beta_{l(m)} + \beta_l \beta_m) X^i_j \Psi_{kh} \tag{5.6.15}$$

In view of the recurrence relation for the projective curvature tensor and the projective deviation tensor given by equation (4.1.30) and (4.1.34) respectively and using equation (5.6.1), we have

$$W^i_{jkh(l)(m)} = A_{lm} W^i_{jkh}$$

$$W^i_{jk(l)(m)} = A_{lm} W^i_{jk}$$

hence

$$A_{lm} X^i_j \Psi_{kh} = (\beta_{l(m)} + \beta_l \beta_m) X^i_j \Psi_{kh} \tag{5.6.16}$$

From equation (5.6.15) and (5.6.16)

$$A_{lm} = (\beta_{l(m)} + \beta_l \beta_m) \tag{5.6.17}$$

Therefore we conclude

Theorem 5.6.3: In a bi-recurrent Finsler space $2R - F_n$ under the decomposition (5.6.1) if the recurrence vector K_l is equal to the recurrence vector β_l then the recurrence vector field satisfies the relation $\left(\beta_{l(m)} + \beta_l \beta_m\right) \neq 0$.

Interchanging the indices l and m in equation (5.6.15) we get

$$W^i_{jkh(m)(l)} = (\beta_{m(l)} + \beta_m \beta_l) X^i_j \Psi_{kh}$$

which on subtracting from equation (5.6.15) gives

$$W^i_{jkh(l)(m)} - W^i_{jkh(m)(l)} = ((\beta_{l(m)} + \beta_l \beta_m) - (\beta_{m(l)} + \beta_m \beta_l)) X^i_j \Psi_{kh}$$

$$W^i_{jkh(l)(m)} - W^i_{jkh(m)(l)} = (\beta_{l(m)} - \beta_{m(l)}) X^i_j \Psi_{kh}$$

Using equation (5.6.17) we have

$$W^i_{jkh(l)(m)} - W^i_{jkh(m)(l)} = (A_{lm} - A_{ml}) X^i_j \Psi_{kh} \tag{5.6.18}$$

Accordingly we state

Corollary 5.6.1: In a bi-recurrent Finsler space $2R - F_n$ under the decomposition (5.6.1) if the vector K_l is equal to β_l the projective curvature tensor W_{jkh} satisfies the identity (5.6.18).

Differentiating equation (5.6.6) covariantly with respect to x^m in the sense of Berwald

$$\Psi_{kh(l)(m)} = \alpha_{l(m)} \Psi_{kh} + \alpha_l \Psi_{kh(m)}$$

We consider that $\alpha_l = (K_l - \beta_l)$ and so we have

$$\Psi_{kh(l)(m)} = (K_l - \beta_l)_{(m)}\Psi_{kh} + (K_l - \beta_l)\Psi_{kh(m)}$$

$$\Psi_{kh(l)(m)} = (K_{l(m)} - \beta_{l(m)})\Psi_{kh} + (K_l - \beta_l)\Psi_{kh(m)} \qquad (5.6.19)$$

Using equation (5.6.6) we write the equation (5.6.19) in the form

$$\Psi_{kh(l)(m)} = (K_{l(m)} - \beta_{l(m)})\Psi_{kh} + \alpha_l\Psi_{kh} \qquad (5.6.20)$$

alternatively

$$\Psi_{kh(l)(m)} = (K_{l(m)} - \beta_{l(m)})\Psi_{kh} + (K_m - \beta_m)(K_l - \beta_l)\Psi_{kh} \qquad (5.6.21)$$

$$\Psi_{kh(l)(m)} = (K_{l(m)} - \beta_{l(m)} + K_m K_l - K_m\beta_l - K_l\beta_m + \beta_l\beta_m)\Psi_{kh} \qquad (5.6.22)$$

Hence we state

Theorem 5.6.4: In a bi-recurrent Finsler space $2R - F_n$, under the decomposition (5.6.1) the second order covariant derivative of the decomposition tensor Ψ_{kh} satisfies the relation (5.6.22).

In view of equation (5.6.9) $\Psi_{kh} = 0$ and equation (5.6.22) reduces to

$$\Psi_{kh(l)(m)} = 0 \qquad (5.6.23)$$

Corollary 5.6.2: In a recurrent Finsler space $R - F_n$, under the decomposition (5.6.1) the second order covariant derivative of decomposition tensor Ψ_{kh} vanishes if the recurrence vector K_l is equal to the recurrence vector β_l.

Now let us write equation (5.2.4) for the non zero mixed tensor X^i_j as

$$\beta_l X^i_j = K_l X^i_j \qquad (5.6.24)$$

Differentiating equation (5.6.24) covariantly with respect to x^m in the sense of Berwald, we have

$$\beta_{l(m)} X_j^i = \mathrm{K}_{l(m)} X_j^i + K_l X_{j(m)}^i$$

$$\beta_m \beta_l X_j^i = (\beta_m \mathrm{K}_l) X_j^i + K_l \beta_m X_j^i \tag{5.6.25}$$

Using equation (5.6.24) in equation (5.6.25), we obtain

$$\beta_m \beta_l X_j^i = (\beta_m \mathrm{K}_l) X_j^i + K_l K_m X_j^i$$

$$\beta_m \beta_l X_j^i = (\mathrm{K}_l \beta_m + K_l K_m) X_j^i \tag{5.6.26}$$

Theorem 5.6.5: In a recurrent Finsler space $R - F_n$, under the decomposition (5.6.1) the second order covariant derivative of the tensor X_j^i satisfies equation (5.6.26).

CHAPTER SIX

CONCLUSIONS AND RECOMMENDATIONS

The conclusion and recommendations in this chapter are based on the results obtained in Chapter Four and Chapter Five. It is worth noting that some of the results of this study have already been published in the International Japanese Journal, T*ensor New Series*.

6.1 Conclusions

The aim of this thesis was to use a Finslerian framework and geometrical methods to investigate the inheritance symmetry property, collineation symmetry property and decomposition property for the projective curvature tensor $W^i_{jkh}(x,\dot{x})$. The results obtained have been presented in form of new Theorems and equations which have geometrical and physical applications in various fields.

In Chapter Four the curvature inheritance property has been defined by $L_v W^i_{jkh} = \alpha(x) W^i_{jkh}$, $\alpha(x) \neq 0$ and the curvature collineation property by $L_v W^i_{jkh} = 0$ and from the results in chapter 4, we make the following conclusions:

i) An infinitesimal transformation in a Finsler space given by the equation the $\bar{x}^i = x^i + V^i(x)\delta t$ is a Lie-recurrence if and only if the W-projective curvature tensor is Lie-recurrent in the same space.

ii) A Finsler space which admits a W- curvature inheritance must necessarily be isotropic so that its geometry remains the same regardless of the direction.

iii) Every motion admitted in a bi-recurrent Finsler space is also a W-curvature inheritance if the space is isotropic otherwise it is a W-curvature collineation.

iv) The projective curvature tensor $W^i_{jkh}(x,\dot{x})$ exhibits different recurrence and inheritance relations.

v) W-curvature inheritance cannot be W-curvature collineation.

vi) No contra vector field $(V^i_{(j)} = 0)$ or concurrent vector field $(V^i_{(j)} = \lambda\delta^i_j)$ can generate a W-curvature inheritance.

vii) Both contra vector field and concurrent vector field generate W-curvature collineation if $\lambda = 0$ otherwise $W^i_{jkh}V^l = 0$ and $A_{lm}V^l = 0$ hold true.

viii) In the contra field the recurrence vector K_l behaves like a gradient vector and is orthogonal to the vector field V^l thus, $K_lV^l = 0$.

In Chapter Five, the projective curvature $W^i_{jkh}(x,\dot{x})$ has been decomposed in four different ways that is $X_j\Psi^i_{kh}$, $X^i\Psi_{jkh}$, $P^iX_j\Psi_{kh}$ and $X^i_j\Psi_{kh}$. The decomposition tensors denoted by Ψ are of ranks (1, 2), (0, 3), (0, 2) and (1, 1) and hence smaller than the projective curvature tensor whose rank is (1, 4). The results in this chapter have led to the following conclusions:

i) The decomposition tensors have some properties similar to those of the original tensor including recurrence relations involving the recurrence vector K_l and the recurrence tensor A_{lm}.

ii) A decomposition tensor $\Psi(x, \dot{x})$ which is still reducible has properties similar to those of the original $W^i_{jkh}(x, \dot{x})$.

iii) Tensor decomposition can be viewed as a process which will compress models to produce smaller devices for example those that can fit in the pocket but still have properties similar to those of a larger device.

iv) In three of the decomposition algorithms there is a decomposition vector of rank either (1, 0) or (0, 1) and the study has shown that the decomposition vector must be a covariant constant so that it has a vanishing covariant derivative.

v) The decomposition tensor X^i_j of rank (1, 1) is an irreducible tensor and cannot be decomposed further so it remains intact in the decomposition algorithm.

vi) Tensor decomposition is only possible under specified conditions and this greatly determines the properties of the decomposition tensor and possible applications of the decomposition algorithm.

6.2 Recommendations

The results obtained in this study open the door to further work in both theoretical investigation and practical applications of tensors. We therefore recommend extension of this work by considering the following areas.

i) Development of geometric algorithms for higher order recurrence tensor fields $(n > 2)$ and possibility of obtaining recurrence relations, inheritance properties and collineation properties of $W^i_{jkh}(x,\dot{x})$ for recurrence tensor $A_{l_1\dots l_n}$ of order *n*.

ii) Investigation of the recurrence, inheritance and collineation properties of the associate tensor $W_{ijkh}(x,\dot{x})$ which is of rank (0, 4) in relation to those of $W^i_{jkh}(x,\dot{x})$ which is of rank (1, 3).

iii) Decomposition of projective curvature tensor $W^i_{jkh}(x,\dot{x})$ as a sum of simplest possible tensors ($W = \sum_!^N \dots$). The results obtained could be compared with those from Canoniccal Polyadic (CP) decomposition and Tucker decomposition (TD) which have recently gained popularity in studying how different tensor items interact with each other.

6.3 Recommendations for Further Research

i) In this study we worked with recurrence tensor fields K_l and A_{lm} only and therefore further work could be done on development of geometric algorithms for higher order recurrence tensor fields and possibly obtain recurrence relations, inheritance properties and collineation properties for recurrence tensor $A_{l_1\dots l_n}$ of tensor of order *n* for other tensors such as *U*, *H*, *K* and *R* which have different geometrical meanings and applications.

ii) Use of tensor decomposition technique in Machine Learning to process, organize and analyze relational data. For instance currently in the fight against Corona Virus Disease 2019 (COVID-19) pandemic medical researchers and practitioners are faced with large volumes of data to analyze for relationships between rate of infection and factors such as age, race, gender and existing condition. The factors can be tensorized by creating a multi-dimensional array containing each factor and Machine Learning technology together with existing or newly developed tensor software used to identify relations and patterns in the spread of COVID-19 and future pandemics.

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APPENDICES

APPENDIX I

PUBLISHED PAPER

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W-CURVATURE INHERITANCE IN BI-RECURRENT FINSLER SPACE.

By Mary A. OPONDO and Surendra Pratap SINGH.

Abstract. S. P. Singh [6][1] has investigated curvature inheritance in Finsler spaces. C. K. Mishra and D. D. S. Yadav [2] have studied curvature inheritance in an $NP-F_n$. S. P. Singh [7] has developed curvature inheritance in bi-recurrent Finsler space $\bar{F}_n$. Recently S. P. Singh [8] has studied W-curvature inheritance in Finsler spaces F_n. The objective of this paper is to define and study the concept of W-curvature inheritance in bi-recurrent Finsler space $\bar{F}_n$.

Introduction. In an n-dimensional Finsler space F_n, the connection parameters $G^i_{jk}(x,\dot{x})$ are positively homogeneous of degree zero in the directional arguments by Euler's theorem

$$\dot{x}^h G^i_{jkh} = \dot{x}^h \dot{\partial}_h G^i_{jk} = 0.$$

The covariant derivative of a tensor X^i, in the sense of Berwald [3] is given by

$$X^i_{(h)} = \partial_h X^i - \dot{\partial}_k X^i \dot{\partial}_h G^k + X^k G^i_{hk},$$

where $G^k(x,\dot{x})$ is a positively homogeneous function of degree two in $\dot{x}^i$ and

$$\partial_h = \frac{\partial}{\partial x^h}, \qquad \dot{\partial}_k = \frac{\partial}{\partial \dot{x}^k}.$$

The commutation formulae involving the Berwald covariant derivatives are expressed as [3]

$$\dot{\partial}_h (T^i_{j(k)}) - (\dot{\partial}_h T^i_j)_{(k)} = T^l_j G^i_{lhk} - T^i_l G^l_{jhk},$$

$$(1) \qquad 2T^i_{j[(h)(k)]} = -\dot{\partial}_l T^i_j H^l_{hk} + T^l_j H^i_{lhk} - T^i_l H^l_{jhk},$$

where

$$H^i_{hjk} = 2(\partial_{[k} G^i_{j]h} + G^l_{h[j} G^i_{k]l} + G^i_{lh[k} G^l_{j]})$$

is Berwald's curvature tensor field.

The above Berwald's curvature tensor H_{hjk^i} satisfies the following properties:

$$(2) \qquad \text{a)}\ H^r_{rhj} = 2H_{[hj]}, \qquad \text{b)}\ H^r_{hjr} = H_{hj}, \qquad \text{c)}\ H^i_{jk}\dot{x}^j = H^i_k,$$

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1) Numbers in brackets refer to the references at the end of the paper.

where

$$2H_{[hj]} = H_{hj} - H_{jh}.$$

In Finsler space F_n, the corresponding projective curvature field $W^i_{hjk}(x, \dot{x})$ is defined as

$$W^i_{hjk} = H^i_{hjk} + \frac{1}{n+1}(\delta^i_h H^r_{rkj} + \dot{x}^i \dot{\partial}_h H^r_{rkj} + 2\delta^i_{[j}(H^r_{(h)k} + \dot{\partial}_{k]}\dot{\partial}_h H), \tag{3}$$

where H^i_{hjk} is Berwald curvature tensor.

The above curvature tensor W^i_{jkh} is skew-symmetric in the last two indices j, k and homogeneous of degree zero in $\dot{x}^i$ and so satisfies the following identities:

(4) a) $W^i_{hjk} = -W^i_{hkj}$, b) $W^i_{hjk}\dot{x}_h = W^i_{jk}$, c) $W^i_{hjk}\dot{x}^h\dot{x}^j = W^i_k$.

In a non-flat Finsler space F_n, if there exists a non-zero vector K_l whose components are positively homogeneous functions of degree zero in $\dot{x}^i$, such that the curvature tensor field W^i_{jkh} satisfies the relation

$$W^i_{jkh(l)} = K_l W^i_{jkh} \tag{5}$$

then it is called a recurrent Finsler space [4].

In view of (4)(c), we also get

$$W^i_{kh(l)} = K_l W^i_{kh}. \tag{6}$$

Similarly, in a non-flat Finsler space F_n, if there exists a non-zero tensor A_{lm} such that the curvature tensor field satisfies the relation

$$W^i_{jkh(l)(m)} = A_{lm} W^i_{jkh}, \tag{7}$$

where

$$A_{lm} = K_{l(m)} - K_l K_m \tag{8}$$

then the Finsler space is called a bi-recurrent Finsler space [5].

The tensor field A_{lm} defined by (8) is called recurrence tensor field whereas the non-zero vector K_l is called recurrence vector field. We denote such a Finsler space by $\bar{F}_n$.

Transvection of (7) by $\dot{x}^j$ also yields

$$W^i_{kh(l)(m)} = A_{lm} W^i_{hk}. \tag{9}$$

We consider the infinitesimal transformation point given by Yano [9]

$$\bar{x}^i = x^i + V^i(x)dt, \tag{10}$$

where $V^i(x)$ is any vector field and dt is an infinitesimal point constant.

The Lie-derivative of any tensor field $T^i_j(x, \dot{x})$ and the connection coefficient $G^i_{jk}(x, \dot{x})$ are expressed as under:

$$L_v T^i_j = V^h T^i_{j(h)} - T^h_j V^i_{(h)} + T^i_h V^h_{(j)} + (\dot{\partial}_h T^i_j) V^h_{(s)} \dot{x}^s$$

and

$$L_v G^i_{jk} = V^i_{(j)(k)} + H^i_{jkh} V^h + G^i_{ljk} V^l_{(s)} \dot{x}^s$$

respectively.

The commutation formulae involving Lie-derivative and other derivatives for any tensor T^i_{jk} are given by:

$$(11) \qquad L_v(\dot{\partial}_l T^i_j) - \dot{\partial}_l (L_v T^i_j) = 0$$

and

$$L_v(T^i_{jk(m)}) - (L_v T^i_{jk})_{(m)} = T^l_{jk} L_v G^i_{lm} - T^i_{lk} L_v G^l_{jm} - T^i_{jl} L_v G^l_{km} - \dot{\partial} T^i_{jk} L_v G^l_{sm} \dot{x}^s.$$

§ 1. W-Curvature Inheritance in $\bar{F}_n$. In this section the authors wish to study the curvature inheritance property of a projective curvature W^i_{jkh} in bi-recurrent Finsler space $\bar{F}_n$. In order to do so, we first define W-curvature inheritance in bi-recurrent Finsler space $\bar{F}_n$ in a similar way as [7].

Definition 1.1. In a bi-recurrent Finsler space $\bar{F}_n$, if the projective curvature W^i_{jkh} satisfies the relation

$$(1.1) \qquad L_v W^i_{jkh} = \alpha(x) W^i_{jkh}$$

with respect to the vector field $V^i(x)$ where $\alpha(x)$ is a non-zero scalar function, the infinitesimal transformation (10) is called an W-curvature inheritance in $\bar{F}_n$.

In view of identities (4)(b), (c) and equation (1.1) we also have

$$\text{a)} \quad L_v W^i_{kh} = \alpha(x) W^i_{kh}, \qquad \text{b)} \quad L_v W^i_h = \alpha(x) W^i_h.$$

S. P. Singh [6] has established that in bi-recurrent Finsler space $\bar{F}_n$, the Berwald curvature tensor field also satisfies the relations

$$(1.2) \qquad \text{a)} \quad L_v H^i_{jkh} = \alpha(x) H^i_{jkh}, \qquad \text{b)} \quad L_v H^i_{kh} = \alpha(x) H^i_{kh}.$$

Hence in view of (2) and (1.2) we find

$$(1.3) \quad \text{a)} \quad L_v H^r_{rjk} = \alpha(x) H^r_{rjk}, \quad \text{b)} \quad L_v H_{jk} = \alpha(x) H_{jk}, \quad \text{c)} \quad L_v H = \alpha(x) H.$$

Applying Lie-derivative operator to the equation (3), we obtain

$$L_v W^i_{jkh} = \alpha(x) W^i_{jkh}$$

in view of (1.2) and (1.3) since $\dot{x}^i$ is Lie-invariant and $\alpha(x)$ is a scalar function as well as the fact that Lie derivative and partial derivative are commutative.

Theorem 1.1. *A bi-recurrent Finsler space which admits H-curvature inheritance is also W-curvature inheritance.*

S. P. Singh [7] has proved that in a Finsler space F_n, every motion admitted is a W-curvature inheritance if the space is an isotropic. In view of the above theorem we conclude that

Theorem 1.2. *Every motion admitted in a bi-recurrent Finsler space $\bar{F}_n$ is also a W-curvature inheritance if the space is isotropic.*

Applying commutation formula (11) for W^i_{jkh} and using (1.1), we obtain

$$L_v(\dot{\partial}_l W^i_{jkh}) = \alpha(x)(\dot{\partial}_l W^i_{jkh})$$

since $\alpha(x)$ is a scalar function.

Using commutation formula (1) to the curvature tensor W^i_{jkh} and applying (7), we find

$$(1.4)\qquad \begin{aligned}(A_{lm} - A_{ml})W^i_{jkh} = &-\dot{\partial}_r W^i_{jkh} H^r_{lm} + W^r_{jkh} H^i_{rlm} \\ &- W^i_{rkh} H^r_{jlm} - W^i_{jrh} H^r_{klm} - W^i_{jkr} H^r_{hlm}.\end{aligned}$$

Now taking Lie-derivative of both sides of (1.4) and using (1.1) and (1.2) we have

$$[(L_v A_{lm} - L_v A_{ml}) + \alpha(A_{lm} - A_{ml})]W^i_{jkh} =$$
$$2\alpha[-\dot{\partial}_r W^i_{jkh} H^r_{lm} + W^r_{jkh} H^i_{rlm} - W^i_{rkh} H^r_{jlm} - W^i_{jrh} H^r_{klm} - W^i_{jkr} H^r_{hlm}].$$

From (1.4), the above equation reduces to

$$(L_v A_{lm} - L_v A_{ml}) + \alpha(A_{lm} - A_{ml}) = 2\alpha(A_{lm} - A_{ml}),$$

which implies

$$(1.5)\qquad L_v A_{[lm]} = \alpha A_{[lm]},$$

where $[lm]$ represents skew-symmetric part.

Thus, we state

Theorem 1.3. *In a bi-recurrent Finsler space $\bar{F}_n$, which admits a W-curvature inheritance, the recurrence tensor field A_{lm} satisfies the inheritance identity* (1.5).

If we commute the indices l and m in the equation (9), it yields

$$(1.6)\qquad (A_{lm} - A_{ml})W^i_{kh} = -W^i_{rkh} H^r_{lm} + W^r_{kh} H^i_{rlm} - W^i_{rh} H^r_{klm} - W^i_{kr} H^r_{hlm}$$

in view of equation (1).

Using the recurrence property of Berwald curvatures [4], equations (5) and (6) in the covariant derivative of (1.6), we obtain

$$(A_{lm} - A_{ml})_{(n)} + K_n(A_{lm} - A_{ml}) = 2K_n(A_{lm} - A_{ml}),$$

which implies

$$A_{[lm](n)} = K_n A[lm]. \tag{1.7}$$

Applying (1.5) in the Lie-derivative of both sides of (1.7), we find

$$L_v A_{[lm](n)} = L_v K_n A_{[lm]} + \alpha K_n A_{[lm]},$$

which gives

$$L_v A_{[lm](n)} = (L_v K_n + \alpha K_n) A_{[lm]}. \tag{1.8}$$

Accordingly, we have

Theorem 1.4. *In a bi-recurrent Finsler space $\bar{F}_n$ which admits a W-curvature inheritance, the recurrence tensor field A_{lm} satisfies the identity* (1.8).

Here we assume that the recurrence vector K_l satisfies the inheritance property $L_v K_l = -\alpha K_l$, then in that case the equation (1.8) reduces to

$$L_v A_{[lm](n)} = 0, \tag{1.9}$$

which also implies that

$$L_v A_{[lm](n)} + L_v A_{[mn](l)} + L_v A_{[nl](m)} = 0.$$

Conversely, if (1.9) is true, then the equation (1.8) reduces to

$$(L_v K_n + \alpha K_n) A_{[lm]} = 0. \tag{1.10}$$

But A_{lm} is non-zero recurrence tensor field, therefore (1.10) yields

$$L_v K_n + \alpha K_n = 0,$$

which implies

$$L_v K_n = -\alpha K_n. \tag{1.11}$$

Hence we have

Theorem 1.5. *In a bi-recurrent Finsler space $\bar{F}_n$, the necessary and sufficient condition for identity*

$$L_v A_{[lm](n)} + L_v A_{[mn](l)} + L_v A_{[nl](m)} = 0$$

to be true is that the recurrence vector K_l satisfies the inheritance property (1.11).

Now we consider covariant differentiation of (1.7) with respect to x^s to have

$$A_{[lm](n)(s)} = K_{n(s)} A_{[lm]} + K_n A_{[lm](s)},$$

which yields

$$A_{[lm](n)(s)} - A_{[lm](s)(n)} = (K_{n(s)} - K_{(s)(n)})A_{[lm]}.$$

In view of commutation formula (1), it yields

$$\dot{\partial}_r A_{[lm]} H^r_{ns} + A_{[rm]} H^r_{lns} + A_{[lr]} H^r_{mns} = (-K_{n(s)} + K_{(s)(n)})A_{[lm]}. \tag{1.12}$$

Applying Lie-derivative to (1.12) and noting (1.2) and (1.5), we get

$$L_v(\dot{\partial}_r A_{[lm]} H^r_{ns} + \alpha(\dot{\partial} A_{[lm]}) H^r_{ns} + 2\alpha[A_{[lm]} H^r_{lns} + A_{[lr]} H^r_{mns}]$$
$$= L_v(K_{s(n)} - K_{(n)(s)})A_{[lm]} + \alpha(K_{s(n)} - K_{n(s)})A_{[lm]}.$$

Observing (11), (1.5) and (1.12) in the above equation, we obtain

$$2\alpha(K_{s(n)} - K_{n(s)})A_{[lm]} = L_v(K_{s(n)} - K_{n(s)})A_{[lm]} + \alpha(K_{s(n)} - K_{n(s)})A_{[lm]},$$

which reduces to

$$\alpha(K_{s(n)} - K_{n(s)}) = L_v(K_{s(n)} - K_{n(s)}) \tag{1.13}$$

since α is a scalar function and A_{lm} is a non-zero recurrence tensor.

The equation (1.13) can be expressed as

$$L_v K_{[s(n)]} = \alpha K_{[s(n)]}. \tag{1.14}$$

Accordingly, we have

Theorem 1.6. *In a bi-recurrent Finsler space $\bar{F}_n$, which admits a W-curvature inheritance, the recurrence vector field K_l satisfies the inheritance identity* (1.14).

H. Hiramatu [1] has established that the infinitesimal transformation (10) be a homothetic transformation if $L_v g_{ij} = 2C g_{ij}$, where C is a constant, holds good. If the Finsler space F_n admits a homothetic transformation then $L_v G^i_{jk} = 0$ is necessarily true. In such case for affine motion the curvature H^i_{jkh} is Lie-invariant which automatically implies that W^i_{jkh} is also Lie-invariant.

Hence we conclude

Theorem 1.7. *Every homothetic transformation admitted in a bi-recurrent Finsler space $\bar{F}_n$ is also W-curvature inheritance if the space is isotropic.*

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