Finding The Distribution Of A Random Variable From Its Moment Function

By

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DECLARATION

This is my original work and has not been presented for a degree award in any other University in part or whole.

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This work has been presented with my approval as the University Supervisor.

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DEDICATION

I dedicate this work to my parents Mr. and Mrs. Otwombe for their tireless effort in sponsoring my studies at the university upto this level.

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ABSTRACT

Consider the problem of $r + 1$ randomly distributed points in a unit n-ball and the convex hull created by these points. Let Δ_n be $r!$ times the r-content of an r-simplex whose p vertices are in the interior and $r + 1 - p$ vertices on the boundary of a unit n-ball. Explicit expressions for the exact distribution functions of Δ_n are given when $r + 1$ points are independently, and identically distributed according to the Uniform distribution. The exact distributions are obtained using the technique of Inverse Mellin transforms with the help of the moment functions. The technique is illustrated for the general case $p = r + 1$ and a particular case $p = 3$, $r = 2$. Various representations of the distributions in *psi* and the generalized *zeta* functions are given. These representations are also given in the most general case as an H-function distribution.

SYMBOLS AND NOTATIONS

The following is a list of symbols and notations, with meanings indicated on the right that will frequently occur in this Research.

$\binom{n}{m} = \frac{n!}{m!(n-m)!}$	Binomial Coefficient
$(\alpha)_m = \prod_{j=0}^{m-1} (\alpha + j), \quad (\alpha)_0 = 1$	Pochhammer Symbol
$\Gamma(\alpha)$	Gamma Function
pdf	Probability density function
cdf	The cumulative distribution function
$Log\lvert(.)\rvert$	The natural logarithm of $\lvert(.)\rvert$
Δ	*r*-content of the *r*-simplex generated by $r + 1$ points
$\Delta_n = r!\Delta$	$r!$ times the $r -$ content of the simplex
R^n, E^n	Euclidian *n*-space
Re(.)	The real part of (.)
arg(.)	Argument of (.)

GRAPHS AND TABLES

Fig. 4.1 Theoretical *cdf* plot for ***r=2, n=2, p=3***

Fig. 4.2 Density plot for ***r=2, n=2, p=3***

Table 1 Table of Theoretical moments and the exact Moments from Equation (4.1.1)

CHAPTER ONE

INTRODUCTION AND REVIEW

1.1 Purpose And Scope

In the theory of geometric probability the random elements are not quantities but geometrical objects such as points, lines, planes and rotations. The determination of a measure to such elements is not quite an obvious procedure and a number of "paradoxes" can be produced by failure to distinguish the reference set. The choice of a measure for a problem in geometrical probability can create problems in general, and it is the failure to recognise this that leads to paradoxes. The 18^{th} Century researchers (CROFTON and his contemporaries) perfected a technique for computing probabilities of the occurrence of simple geometrical phenomena, and the expectations of simple geometrical enumerations associated with uniform patterns of independent random lines in the plane. The vital adjectives used were *uniform* and *independent.* The plane (or a higher-dimensional Euclidean space) is thought of as carrying the Euclidean group and the stochastic structures envisaged are all invariant under this group, while geometrical questions asked are relative to the geometry implied by the group. Further the stochastic elements mentioned above such as lines are not only randomly distributed but also in some sense (never made quite clear) "independent". Thus the procedure can be summed up in three steps:

(i) An assumption is made about the randomness of the geometrical quantity (or the number of geometrical objects)

(ii) The distribution of the geometrical object conditional on their number is found.

(iii) Any two characteristics of the geometrical objects are assumed uniformly distributed and independent, for example the perpendicular distance of a line from the origin and its orientation with respect to some arbitrary axis.

So once all the practicable problems about lines and other geometrical objects have been solved, one proceeds to formulate similar problems about random geodesics on manifolds and so on. Professor D.G. KENDALL referred to this approach as " the chicken coming before the Egg" since problems about random lines in the plane are solved by looking first at parallel problems involving random great circles on sphere and then letting the radius of the sphere tend to infinity. In recent years the introduction of Integral Geometry has brought about changes where the plane has been replaced by a homogeneous space and letting the active group play the role formerly assigned to the Euclidian group so that the Harr Measure replaces the Lebesque Measure and so on.

The probability distribution of the r-Content (Volume) of a random r-simplex in a compact n-space has been studied by many authors using different techniques (An 0-simplex is a point, 1-simplex is an interval, 2-simplex is a triangle and 3-simplex is a tetrahedron and so on).Some of the techniques involve

(i) solution of first order differential equations resulting from the applications of Crofton's Theorem in two dimensions

(ii) application of the theory of Mellin transforms and Calculus of residues and

(iii) multiple series solutions of Wilk's type B and C Integral equations.

Most of these results are for particular cases when $1 \leq r \leq 2$ and $n \geq 2$,where the compact n-space is either a unit n-ball or an n-ball of unit Volume.The merits and demerits of these and other techniques are well documented in the literature, some of which is cited in subsequent sections of this chapter.

The main purpose about this research is to derive the cummulative distribution function (*cdf*) of a constant multiple of Δ the r-content of a random r-simplex whose vertices are independent and identically distributed random variables. Some of these may be in the interior and some on the boundary of a unit n-ball.We consider the case when the points are Uniformly distributed.

To achieve our objective, we use the Mellin Inversion and Residue theorems. There are two approaches to this problem ;

(i) apply the Mellin Inversion Integral to the h^{th} moment of Δ, $(E(\Delta^h))$ (or a constant multiple of Δ) and express the *cdf* as a series with the help of calculus of residues or

(ii) define the random variable under consideration as an H-function random variable whose *cdf* is an H-function, and then evaluate its series representation.

The existence of the distributions in (i) is guaranteed by the properties of Mellin transforms and in (ii) by the existence conditions of H-functions, while convergence of the resulting inversion integrals is also guaranteed by the contour of integration considered, and in this case a suitable contour is either a Mellin-Barnes contour or the Bromwich contour over the Bromwich path.To this respect we have cited the relevant references at the appropriate sections and hence details are omitted at this stage.Series representations of H-functions in the literature are inadequate for our problem as they are not valid when the integrand of the inversion integral has higher order poles. Failure to recognise this fact has often led to incorrect results as pointed out below.Our approach has been to evaluate the Mellin inversion integral after expressing the moment function in its simplest possible form by cancelling out identical gamma factors in both the numerator and the denominator of the integrand. This technique has enabled us to give for the first time :

(i) exact and explicit polynomial forms of the *cdf* of Δ (or its constant multiple) for general parameter values involving higher order poles and

(ii) show for the first time that the *cdf* of Δ can be evaluated directly from the Mellin transform of $f(\Delta)$ the *pdf* of Δ without any prior knowledge of $f(\Delta)$.

These results were then expressed in the most general form as *H*-functions of lower orders.

The presentation of this work is organised as follows:

The material in chapter one is of an expository type. It begins with an introduction and embodies achievements in geometrical and mathematical statistics, exemplifying those concepts that are intended as prerequisites to the theory of subsequent chapters.Chapter two deals with the theory of Mellin transform and the inversion. In chapter three we expound the theory of an H-function variate and its probability distributions. An exposition of some special cases of well known mathematical functions and probability distributions are given. Chapter four deals with the derivation of the cummulative distribution function using the techniques of chapters two and three. An illustrative example and a particular case is given. Exact and theoretical moments of the particular case are compared.Most results derived in chapters two to four should be considered as new unless otherwise stated. We conclude this research in chapter five by citing a few applications of the theory of the previous chapters.

1.2 Historical Background of The Problem

The theory of geometric probability dates back to the 18^{th} century and noteworthy at the time is the classical Buffon's Needle Problem. Buffon investigated a game already in practice in the 18^{th} century known as "Clean Tile".

In a room tiled or paved with equal tiles, of any shape, a coin is thrown upwards. There are three players in the game. Player number one bets that the coin will rest cleanly, that is on one tile only. Player number number two bets that the coin will rest on two tiles, that is, it will cover one of the cracks that separate the tiles, and player number three bets that the coin will rest on 3, 4, or 6 cracks (depending on whether the shape of the tile is equilateral, triangle, hexagonal or diamond shaped). The problem was to calculate the chances for each of the three players and Buffon calculated the ratio of the diameter of the coin to the equal sides of the particular shaped tile that provides a fair game for each player.

The analogous problem considered by Buffon was that of a needle or headless pin of length l, thrown upwards in a room merely divided by parallel lines at a distance *d* units apart (or a needle placed randomly on a plane on which are ruled parallel lines at a distance *d* units apart). We are interested in the probability that the needle intersect these lines. A summary of Buffon's contributions to geometric probability can be seen in ROGERS (1977 pp.29-35). Various extensions and analogues of the needle problem have been developed. A complete summary of these extensions can be found in CHU (1990 pp.5-6). A detailed exposition of the subject and the many references given therein can be found in the monographs of KENDALL and MORAN (1963), SOLOMON (1973) and BADDELEY (1977). For the development of integral geometry and its connection with geometric probability, see for instance SANTALO (1976). This single volume by Professor LUIS .A. SANTALO is a classical, and to date is the main source of reference on the subject of Integral geometry Geometric probability.

1.3 Some Achievements In Geometric Probability

The publication of the monograph KENDALL and MORAN (1963) was the first unified approach to problems of geometric probabilities under five categories; namely problems concerned with random distribution of points, lines and planes, random rotations and problems of coverage. Until then a considerable body of research in geometric probability had been published in different scientific journals. A Bibliography at the end of the book covers most of the five categories above. An important source of additional information is to be found in the series of "NOTES ON RECENT RESEARCH IN GEOMETRIC PROBABILITY". The first Note of these series MORAN (1966) gives additional 94 sources of references to those found in KENDALL and MORAN (1963) including developments in the subject up to 1966. The second Note, MORAN (1969) is an exposition of all research achievements up to 1969 and the third Note, LITTLE (1974) gives extensions to the subjects of stereology, pattern recognition and search. Both surveys give extensive

Bibliographies. The fourth Note, BADDELEY (1977) reviews investigations in geometric probability up to that period. The phrase " stochastic geometry" was supposedly introduced by D. G. KENDALL in June 1969 when planning the OBERWOLFACH conference in integral geometry and Geometrical probability. A book STOCHASTIC GEOMETRY, KENDALL and HARDING (1974) had been published in the memory of Cambridge mathematician ROLLO DAVIDSON (1944-1970). STOYAN et al.(1987) is an extension of Stochastic geometry into the subject of Stereology and random tessellations. MOLLER (1989), MOLLER(1994) and others are good references in Voronoi tessellations.

Geometric probability and the closely related subjects of stereology, pattern recognition and spatial statistics has generated much interest recently because of its applicability to real life problems ranging from archaeology to engineering and many other scientific fields.

Random Simplices In a Unit *n*-Ball

A detailed investigation of random simplices in a unit n-ball has been carried out by MILES (1971). He gives a new proof of Blaschke and Petkantschin's formular in integral geometry (SANTALO 1976 pp. 201

equation 12.22), and applies it to determine all the moments of Δ, the random volume of various isotropic random r-dimensional simplices in $E^n(r = 1,\dots,n)$. The complexity of an analytic determination of the moments of Δ has been discussed in KLEE (1969). KINGMAN (1969) has obtained the first moment of Δ for $r = n-1$. GROEMER (1973) establishes that the mean value of Δ is a minimum if and only if the simplex is contained in an ellipsoid. MILES (1971), also investigates a set of random points whose distributions possess a certain rotation invariance property. RUBEN and MILES (1980) gives a canonical decomposition of the probability measure of a set of isotropic random points, independent but not necessarily identically distributed in R^n, and shows that the moments $E(\Delta^k)$ of Δ (the r - content of the r - simplex) can be determined only if the distribution of the random points fall into one of the following three categories: Gaussian type or beta type distributions. RUBEN (1979) arrives at the same results by using a geometrical approach to determine the moments of $\Delta_{p,n}$.

Here $\Delta_{p,n}$ is the p - content of the isotropic random p - paralleltope determined by p points $1 \le p \le n$ in R^n. $\Delta_{p,n}$ is in fact equal to $(p!\Delta)$, where Δ is the p - content of the isotropic p - simplex $(1 \le p \le n)$ with one vertex at the origin, determined by (p +1) points, p of which determine the paralleltope.

The bulk of this research deals with determination of the distributions of Δ. We have elected to revisit the objectives of this subsection in chapter four after establishing some pre-requisite concepts in chapters two and three.

1.4 Uniformly Distributed Random Points

Consider a set $X_0, X_1, \ldots, X_r$ of $(r+1)$ independent and identically distributed random points $(1 \leq r \leq n)$ of which p are uniform in the interior and $r+1-p$ are uniform on the boundary of a unit n-ball.These $r+1$ points determine a unique r-simplex almost surely via its convex hull.Let Δ_n denote $r!$ times the r-content of the r-simplex generated by these points.Several authors have considered the exact and asymptotic distributions of $U = \frac{r^{r} \Delta^{2}_{n}}{(r+1)^{r+1}}$ using various techniques.A review of the work in this area until 1987 is available from MATHAI and PEDERZOLI (1990).The probability density function *(pdf)*,*f(u)* of U for particular cases *(r,p)=(1,0),(1,1),(1,2),(2,0),(2,1),(2,2)* and *(2,3)* is available in terms of elementary functions.See for example MILES(1971), MATHAI (1982), MATHAI and TRACY(1983).Once the *pdf* $f(u)$ is available, the cummulative distribution function *(cdf)*

$F(u)=\int_0^u f(t)dt$ can easily be obtained.Often this is not the case.

The purpose of this article is to show that if the h^{th} moment, $\mathrm{E}\left(X^h\right)$ of U is known, then $F(u)$ can be obtained in computable terms with no prior knowledge of $f(u)$. This is achieved by evaluating the inversion integral of $F(u)$ with the help of calculus of residues. The general result is an infinite series of the form

$$F(u)=Cu^k\sum_{j=0}^{\infty}d_j u^{g_j}(-\log u)^{f_j},\quad 0<u<1 \tag{1.4.1}$$

where C, k, d_j, g_j, and $0<f_j\le m-1$ are constants free of U.

CHAPTER TWO

2.0 Integral Transforms and Transform Pairs

Integral transforms are frequently encountered in mathematics, probability and statistics. In probability and statistics the Laplace, Fourier and Mellin transforms play an important role in distribution theory. Since they are related to the moment generating function, characteristic function and the s^{th} moment respectively of a continuous random variable.The definitions of integral transforms in the current literature are not standard as will be shown below. It is important therefore, to explicitly state the form of the definition to be used.Listed below are the definitions of the three transforms mentioned above. Each transform and its corresponding inverse transform form a transform pair. More information about Laplace, Fourier and Mellin transforms can be found in ERDELYI (1954,Vol 1 and 2), LUKE (1969,Vol 1 and 2), SPRINGER (1979) and others.

In this research, emphasis will be on the Mellin transform.

2.1 The Laplace Transform

Consider a function $f(u)$ which is sectionally continuous and defined for all positive values of the variable u with $f(u)=0$ for $u \leq 0$.A sectionally continuous function may not have an infinite number of discontinuities nor any positive vertical asymptotes. If $f(u)$ grows no faster than an exponential function, then the Laplace transform of $f(u)$ will exist. There must exist two positive numbers M and U such that for all $u > U$ and for some real number α,

$$\left|\frac{f(u)}{e^{\alpha u}}\right| \leq M \qquad (2.1.1)$$

The definition of the Laplace transform of the function $f(u)$, $L_s\{f(u)\}$ is

$$L_s\{f(u)\} = \int_0^{\infty} e^{-su} f(u) du \qquad (2.1.2)$$

In general s is a complex variable. The Laplace transform of $f(u)$ will exist for the real part of s greater than α $(\mathrm{Re}(s) > \alpha)$. Note that (2.1.2) is the moment generating function of a positive random variable with $-t$ replacing s. The inversion integral or inverse Laplace transform is given by

$$f(u) = \frac{1}{2\pi i} \int_{c-i\infty}^{c+i\infty} e^{su} L_s\{f(u)\} ds \qquad (2.1.3)$$

where $L_s\{f(u)\}$ is an analytic function for Re(s)>c. A function is analytic at $s = s_0$

if its derivative exists at s_0 and at every point in some neighbourhood of s_0.

The Taylor series expansion of an analytic function of a complex variable will exist, converge and equal the function evaluated at the argument. For all practical purposes a function *f(u)* and its Laplace transform (if it exists) uniquely determine each other.

2.2 The Fourier Transform.

There are two forms of the definition of Fourier transforms in the literature.

$$F_s\{f(u)\} = \int_{-\infty}^{\infty} e^{isu} L_s\{f(u)\} du \tag{2.2.1}$$

and

$$f(u) = \frac{1}{2\pi} \int_{-\infty}^{\infty} e^{isu} F_s\{f(u)\} du \tag{2.2.2}$$

or

$$F_s\{f(u)\} = \frac{1}{2\pi i} \int_{-\infty}^{\infty} e^{isu} f(u) du \tag{2.2.3}$$

and

$$f(u) = \frac{1}{2\pi i} \int_{-\infty}^{\infty} e^{isu} F_s\{f(u)\} du \tag{2.2.4}$$

where

$$f(u) = \frac{1}{2}\{(\lim)_{u \to u_0, u < u_0} + (\lim)_{u \to u_0, u > u_0}\} \tag{2.2.5}$$

If $f(u)$ exists and is continuous at u_0, the inverse Fourier transform of $F_s\{f(u)$ will give x_0. If $f(u)$ is not continuous at u_0, the inverse Fourier transform of $F_s\{f(u)$ will produce the average of the limits of $f(u)$ from the left of u_0. Notice that $s = t$ at (2.2.1) and $s = -t$ in (2.2.3) gives the characteristics function of the random variable u, if $f(u)$ is a *pdf*. The theory of Fourier transforms is easily extended to complex variables. See for example SPRINGER (1979, pp29).

2.3 The Mellin Transform and Inversion Integral

A real valued function $f(u)$ that is defined and single valued almost everywhere for $u \geq 0$,is said to be Mellin transformable if for u a real variable

$\int_0^\infty u^{h-1}|f(u)|du$ converges for some real value h (SNEDDON(1951)p.51, and TITCHMARSH (1937)p.2).Then the Mellin transform

$$M_s\{f(u)\} = \int_0^\infty u^{s-1} f|(u)|du \tag{2.3.0}$$

and the inverse Mellin transform or inversion integral is

$$f(u)=\frac{1}{2\pi i}\int_{c-i\infty}^{c+i\infty} u^{-s} M_s\{f(u)\}ds \tag{2.3.1}$$

If the right hand side of equation (2.3.1) is analytic in the relevant strip or half-plane, then $f(u)$ is expressible as a laurent series.This is a sufficient (but not necessary) condition to ensure the uniqueness of $f(u)$.

Hence $f(u)$ can be determined by the method of residues, provided at least a single pole exists in the integrand of the inversion integral. Let U be a random variable with *pdf* $f(u)$ and *cdf* $F(u) = \int_0^u f(t)dt$.

And let also $G(u) = 1 - F(u)$ where $F'(u) = \frac{d}{du}\{F(u)\} = f(u)$

Assuming that the derivative exists for $0 < u < \infty$. Then from SNEDDON (1951,pp.42) and ERDELYI (1951 pp.307) we have

$$M_s^r\{f(x)\} = \frac{(-1)^r \Gamma(s)}{\Gamma(s-r)} M_{s-r}\{f(x) \qquad (2.3.2)$$

where

$M_s^r\{f(x)\}$ denotes the Mellin transform of the r^{th} derivative $\frac{d^r f(x)}{dx^r}$.

Now put $r = 1$ in (2.3.2) to get

$$M_s^1\{f(x)\} = -(s-1)M_{s-1}\{f(x)\} \qquad (2.3.3)$$

Analogously we have

$$M_s\{G'(u)\} = -(s-1)M_{s-1}\{G(u)$$

$$-M_s\{F'(u)\} = -(s-1)M_{s-1}\{G(u)$$

$$= -(s-1)M_{s-1}\{1-F(u)$$

and

$$s^{-1}M_{s+1}\{F'(u)\} = M_s\{1-F(u)\} \qquad (2.3.4)$$

From equation (2.3.1) we get

$$1\text{-}F(u) = \frac{1}{2\pi i}\int_{c-i\infty}^{c+i\infty} u^{-s}s^{-1}M_{s+1}\{f(u)\}ds$$

or equivalently

$$F(u) = 1\text{-}\frac{1}{2\pi i}\int_{c-i\infty}^{c+i\infty} u^{-s}s^{-1}M_{s+1}\{f(u)\}ds$$

$$=1\text{-}G(u),\text{say.} \qquad (2.3.4)$$

Suppose $U = X_1, X_2, ..., X_n$ where X_i, i=1,….,n are independent , positive random variables with *pdf*, $f(x_i)$, $X_i>0$ for $i=1,\dots,n$.

Let $h(u)$ be the *pdf* of U. Then by definition (2.3.0),

$$\mathrm{M}_s\{h(u)\} = E[u^{s-1}] = E[(X_1, X_2, ..., X_n)^{s-1}]$$

$$= E[X_1^{s-1}, X_2^{s-1}, ...X_n^{s-1}]$$

$$= \prod_{i=1}^{n} E[X_i^{s-1}] = \prod_{i=1}^{n} M_s\{f(x_i) \qquad (2.3.5)$$

Hence by definition (2.3.1)

$$h(u) = \frac{1}{2\pi i}\int_{c-i\infty}^{c+i\infty} U^{-s}\prod_{i=1}^{n} M_s\{f(x_i)\}ds \qquad (2.3.6)$$

and if

$H(u) = \int_0^u h(t)dt$ is the *cdf* of $U = \prod_{i=1}^{n} X_i$ then by definition (2.3.4),

$$H(u)=1-\frac{1}{2\pi i}\int_{c-i\infty}^{c+i\infty}U^{-s}S^{-1}M_{s+1}\{h(u)\}ds \tag{2.3.7a}$$

and

$$H(u)=1-\frac{1}{2\pi i}\int_{c-i\infty}^{c+i\infty}U^{-s}S^{-1}M_{s}\{uh(u)\}ds \tag{2.3.7b}$$

since

$$M_{s+1}\{h(u)\}=M_{s}\{uh(u)\}. \tag{2.3.8}$$

See for example SPRINGER (1979, pp.35).

One can look upon the integral in (2.3.7b) to be of the form

$$\frac{1}{2\pi i}\int_{c-i\infty}^{c+i\infty}F(s)G(s)U^{-s}ds=\int_{0}^{\infty}f\left(\frac{u}{x}\right)g(x)\frac{dx}{x} \tag{2.3.9}$$

where $F(s)$ and $G(s)$ are the Mellin transforms of the functions $g(u)=uh(u)$ and $f(u)$ respectively. This is the *Faltung theorem* for the product of Mellin transforms.Details of the proof of equation (2.3.9) including special cases of the theorem are given in SNEDDON (1951, pp.42).Note that the right hand side of equation (2.3.9) can be identified as a Mellin convolution integral.

It is clear that the Fourier and Laplace transforms are of the exponential type. That the Mellin integral transform is also of the exponential type becomes evident if one notes that any non-negative variable x is expressible in the form $u = e^{\ln u}$ Thus

$$M_s(f(x)) = \int_0^\infty u^{s-1} f(u) du$$

$$= \int_0^\infty e^{(s-1)\ln u} f(u) du \tag{2.3.10}$$

which is clearly a function of the exponential type. This is of considerable importance when Jordan's Lemma is invoked in the process of inverting a Mellin transform to obtain the *pdf f(x)*.

CHAPTER THREE

3.1 The *H-function* Distribution

There are two definitions of FOX'S *H-function* appearing in the literature.

(PRUDNIKOV et al 1990, BODENSCHATZ, 1992, pp.25-27, and others).

One such definition is

$$H(z) = H_{p,q}^{m,n}\left[z \mid \begin{matrix} (a_1,\alpha_1), & \ldots, & (a_p,\alpha_p) \\ (b_1,\beta_1), & \ldots, & (b_q,\beta_q) \end{matrix} \right]$$

$$= \frac{1}{2\pi i}\int_L \Phi(s) z^{-s} ds \qquad (3.1.0)$$

where

$$\Phi(s) = \frac{\prod_{j=1}^{m}\Gamma(b_j+\beta_j s)\prod_{i=1}^{n}\Gamma(1-a_j-\alpha_j s)}{\prod_{j=m+1}^{q}\Gamma(1-b_j-\beta_j s)\prod_{j=n+1}^{p}\Gamma(a_j+\alpha_j s)}. \qquad (3.1.1)$$

where L is a suitable contour of the Mellin-Barnes type, which is indented if necessary, inorder to separate the points

$$s = \left[\frac{-b_j - v}{\beta_j}\right], j=1,\ldots,m;\ v=0,1,\ldots.$$

which are poles of

$$\Gamma\lfloor b_j + \beta_j s \rfloor, j=1,\ldots,m$$

from the points

$$s = \left[\frac{1-a_j-v}{\alpha_j}\right], j=1,\ldots,n;\ v=0,1,\ldots..$$

which are poles of

$$\Gamma[1-a_j-\alpha_j s],\ j=1,\ldots,n.$$

In the above notation an empty product is interpreted as unity, the non-negative integers m, n, p and q satisfy the inequalities $1 \le m \le q$ and $0 \le n \le p$. The coefficients $\alpha_1,\ldots,\alpha_p$ and $\beta_1,\ldots,\beta_q$ are real positive numbers and $a_1,\ldots,a_p;b_1,\ldots,b_q$ are complex numbers .

The H-function is an analytic function of z, the point $z = 0$ being tacitly excluded and

$$[z]^{-s} = \exp[s[LOG|z| + i\arg z]] \qquad (3.1.2)$$

in which $LOG|z|$ represents the natural logarithm of $|z|$ and arg z is not necessarily the principal value.

The *H*-function makes sense if the following existence conditions are satisfied.

Case (i) :

For all $z \ne 0$ if $\mu > 0$

Case(ii) :

For $0 < |z| < R^{-1}$ if $\mu = 0$ (3.1.3)

Where

$$\mu = \sum_{j=1}^{q} [\beta_j] - \sum_{j=1}^{p} [\alpha_j] \tag{3.1.4}$$

$$R = \prod_{j=1}^{p} [\alpha_j]^{\alpha_j} \prod_{j=1}^{q} [\beta_j]^{-\beta_j} \tag{3.1.5}$$

Due to the occurrence of the factor z^{-s} in the integrand of (3.1.0) it is , in general, multiple- valued but one-valued on the Riemann surface of LOG z. Further discussions on the *H*-function can be found in books of MATHAI & SAXENA (1978) and SRIVASTAVA & MANOCHA (1984).

A second definition of the *H-function* is

$$H(z) = H_{p,q}^{m,n}\left[z \,\middle|\, \begin{matrix} (a_1,\alpha_1), & . \ . \ ., & (a_p,\alpha_p) \\ (b_1,\beta_1), & . \ . \ ., & (b_q,\beta_q) \end{matrix} \right]$$

$$= \frac{1}{2\pi i} \int_L \Phi^*(s) z^s ds$$

where

$$\Phi^*(s) = \frac{\prod_{j=1}^{m} \Gamma(b_j - \beta_j s) \prod_{j=1}^{n} \Gamma(1 - a_j - \alpha_j s)}{\prod_{j=m+1}^{q} \Gamma(1 - b_j + \beta_j s) \prod_{j=n+1}^{p} \Gamma(a_j + \alpha_j s)} \tag{3.1.6}$$

Here $i = \sqrt{-1}$, *m*, *n* ,*p* and *q* are integers such that $0 \le n \le p, \alpha_j (j = 1,....,p), \beta_j (j = 1,...,q)$ are positive numbers and $a_j (j = 1,...,p), b_j (j = 1,...,q)$ are complex numbers.The path of integration *L* is a contour in the complex *s*-plane from $c - i\infty$ to $c + i\infty$ such that all the

Left Half Plane (LHP) poles of $\prod_{j=1}^{n}\Gamma(b_j+\beta_j s)$ lie to the left of L and all Right Half-Plane(RHP) poles of $\prod_{j=1}^{n}\Gamma(1-a_j-\alpha_j s)$ lie to the Right. Similarly the contour L is such that the poles of $\prod_{j=1}^{n}\Gamma(1-a_j+\alpha_j s)$ lie to the left.In equations (3.1.1 and 3.1.6) $\Phi(s)$ and $\Phi^{*}(s)$ can be seen as the Mellin transform of *H(z)*.

Definition

A random variable U is known as an *H-function* variate (SPRINGER 1979, pp.200; BODENSCHATZ 1992, pp.87) if it has the following probability density function

$$f(u)=kH_{p,q}^{m,n}\left[u\,\middle|\,\begin{matrix}(a_1,\alpha_1), & .\;.\;., & (a_p,\alpha_p)\\ (b_1,\beta_1), & .\;.\;., & (b_q,\beta_q)\end{matrix}\right]\qquad u>0$$

$$=0 \text{ elsewhere} \qquad (3.1.7)$$

Here k is a normalizing constant such that $\int_0^\infty f(u)du=1$.The h^{th} moment of the random variable is given by

$$\mu_h=\int_0^\infty u^h f(u)du \qquad (3.1.8)$$

provided the integral exists. Then by equation (2.3.0) we have

$$\mu_h = M_{h+1}\{f(u)\}$$

$$= k\frac{\prod_{j=1}^{m}\Gamma(b_j+\beta_j h)\prod_{i=1}^{n}\Gamma(1-a_i-\alpha_i h)}{\prod_{j=m+1}^{q}\Gamma(1-b_j-\beta_j h)\prod_{i=n+1}^{p}\Gamma(a_i+\alpha_i h)} \tag{3.1.9}$$

A similar expression for μ_h can be derived from equation (2.3.0).Note that

$\mu_{s-1} = M_s\{f(u)\}$ when $h = s-1$ in equation (3.1.10), and

$k = \dfrac{1}{M_s\left[H_{p,q}^{m,n} \mid u\right]}$ at $s = 1$

$$= \frac{\prod_{i=n+1}^{p}\Gamma(a_i+\alpha_i)\prod_{j=m+1}^{q}\Gamma(1-b_j-\beta_j)}{\prod_{j=1}^{m}\Gamma(b_j+\beta_j)\prod_{i=1}^{n}\Gamma(1-a_i-\alpha_i)} \tag{3.1.11}$$

Equation (3.1.11) is valid provided the integrand in equation (2.3.0) has no pole at $s=1$.If the Mellin transform (2.3.0) is zero or undefined at $s = 1$ $(h = 0)$ then μ_0 would not be unit as required for a valid probability density function, and no constant k would exist to create a valid *pdf*. A commonly met condition of nearly all *H-function* distributions which guarantees no pole at $s = 1$ is that $\frac{-b_j}{\beta_j}<1$ for $j=1,\dots,m$ and $\frac{1-a_i}{\alpha_i}>1$ for $i=1,\dots,n$.Now replace s with $s = 1$ in equation (3.1.1) and *F(u)* in equation (2.3.4) can be expressed in the form of an *H-function* inversion integral as

$$F(u) = 1 - kH^{m+1,n}_{p+1,q+1}\left[u \mid \begin{matrix} (a_1+\alpha_1,\alpha_1), & . & ., & (a_p+\alpha_p,\alpha_p), & (1,1) \\ (0,1), & (b_1+\beta_1,\beta_1), & . & ., & (b_q+\beta_q,\beta_q) \end{matrix}\right] \tag{3.1.12}$$

3.2 Some Special Cases

3.2 (i) The Exponential Function

$e^x = H^{1,0}_{0,1}[-x \mid (0,1)], -\infty < x \le 0$

This is easily verified from definition (3.1.1), one has for $b_1 = 0, \beta_1 = m = q = 1, n = p = 0.$

So

$$H(-x) = H^{1,0}_{0,1}[-x \mid (0,1)]$$

$$= \frac{1}{2\pi i}\int_L (-x)^s \Gamma(-s)ds$$

$$= \frac{1}{2\pi i}\left(2\pi i \sum_{j=0}^{\infty} R_j\right), \tag{3.2.1}$$

Where R_j denotes the residue at the pole $s = j$. That is

$$R_j = (-s+j)(-x)^s \Gamma(-s)\big|_{s=j}$$

$$= \frac{(-s+j)(-x)^s \Gamma(-s+j+1)}{-s(-s+1)\ldots(-s+j-1)}\Big|_{s=j}$$

for $j = 0,1,2,\ldots$ and the denominator is understood to be 1 when $j = 0$.

Hence $R_j = \frac{x^j}{j!}, j = 0,1,2,\ldots$

So that $H_{0,1}^{1,0}[-x \mid (0,1)] = \sum_{j=1}^{n} \frac{x^j}{j!} = e^x$

3.2 (ii) The Gamma Distribution

$$f(x) = \frac{x^{\theta-1} e^{\frac{-x}{\phi}}}{\phi^{\theta}\Gamma(\theta)}, x > 0; \theta, \phi > 0$$

$$= \frac{1}{\phi^{\theta}} H_{0,1}^{1,0}\left[\frac{1}{\phi}x \mid (\theta-1,1)\right] \qquad (3.2.2)$$

To establish this result, note that the Mellin transform of

$f(x) = e^{-\frac{x}{\phi}}, 0 \le x < \infty$ is $M_s(f(x)) = \phi^s \Gamma(s)$

so that the inversion integral yielding $f(x)$ is, by definition,

$$f(x) = \frac{1}{2\pi i} \int_L \left(\frac{x}{\phi}\right)^{-s} \Gamma(s) ds$$

$$= H_{0,1}^{1,0}\left[\frac{1}{\phi}x \mid (0,1)\right]$$

So that (3.1.6) with $b_1 = 0, \beta_1 = -1, m = q = 1, n = p = 0,$ then

$$\frac{1}{\phi\Gamma(\theta)}\left(\frac{x}{\phi}\right)^{\theta} e^{\frac{-x}{\phi}} = \frac{1}{\phi\Gamma(\theta)} H_{0,1}^{1,0}\left[\frac{1}{\phi}x \mid (\theta-1,1)\right] \qquad (3.2.3)$$

CHAPTER FOUR

4.1 The Cummulative Distribution Function

The h^{th} moment of U is given in MATHAI and PEDERZOLI (1990) as

$$E\left(u^{h}\right)=C\Phi(h) \qquad (4.1.1)$$

where

$$\Phi(h)=\frac{\prod_{j=0}^{r}\Gamma\left(h+\frac{n}{2}-\frac{(r-p)}{r+1}+\frac{j}{r+1}\right)\prod_{j=0}^{r-1}\Gamma\left(h+\frac{(n-j)}{2}\right)}{\prod_{j=0}^{r-1}\Gamma\left(h+\frac{n(r+1)}{2r}-\frac{(r-p)}{r}+\frac{j}{r}\right)\Gamma^{p}\left(h+\frac{n}{2}+1\right)\Gamma^{r+1-p}\left(h+\frac{n}{2}\right)} \qquad (4.1.2)$$

$U=\dfrac{r^{r}\Delta_{n}^{2}}{(r+1)^{r+1}}$ and C is such that

$E\left(u^{h}\right)=1$ when $h=0$. On account of (2.3.4) and (4.1.1) we can write

$F(u)=1-G(u)$ where

$$G(u)=C'u^{\frac{n-2m-1}{2}}\frac{1}{2\pi i}\int_{L}u^{-\alpha}\left(\alpha-\left(\frac{n-2m-1}{2}\right)\right)^{-1}\Phi_{1}(\alpha)d\alpha \text{ ,for } r=2m \qquad (4.1.3(a))$$

$$=C'u^{\frac{n-2m}{2}}\frac{1}{2\pi i}\int_{L'}u^{-\alpha}\left(\alpha-\left(\frac{n-2m}{2}\right)\right)^{-1}\Phi_{2}(\alpha)d\alpha \text{ , for } r=2m-1 \qquad (4.1.3(b))$$

Here C' is such that $E\left(U^{h+1}\right)=1$ when $h=-1$ and

$\alpha=h+\frac{n}{2}-m+\frac{1}{2}$ for $r=2m$

$= h + \frac{n}{2} - m + 1$ for $r = 2m - 1$ and $m = 1,2,\ldots; i = \sqrt{-1}$,L and L' are appropriate contours. In writing (4.1.3(a)) and (4.1.3(b)) we consider that

$$E(U^{h+1}) = M_{h+2}\{f(u)\}$$
$$= M_{s+1}\{f(u)$$

by the definition of the Mellin transform given in (2.3.0).

The integrals (4.1.3(a)) and (4.1.3(b)) are of the Mellin-Barnes type, where $\Phi_1(\alpha)$ and $\Phi_2(\alpha)$ are such that no gamma factors can be cancelled between numerator and denominator. This is achieved with the help of the following lemmas.

Lemma 1

Let $\Phi(\alpha) = \Gamma(\alpha + b_1)\Gamma(\alpha + b_2)\ldots\Gamma(\alpha + b_m)$

Where

$b_2 - b_1 = n_1, \ldots, b_m - b_{m-1} = n_{m-1}$ and $n_j, j = 1,2,\ldots,m-1$

are positive integers. Then

(a) $$(\alpha + b_k + v)^k \Phi(\alpha) = \frac{(n_k + n_{k+1} + \ldots + n_{m-1} - v - 1)!(n_k + \ldots + n_{m-2} - v - 1)!\ldots(n_k - v - 1)!\Gamma^k(1)}{(-1)^{kv + \sum_{i=1}^{k-1} n_{k-i}(k-i)}(n_1 + n_2 + \ldots + n_{k-1} + v)!\ldots(n_{k-1} + v)!v^k!}$$

(4.1.4)

at $\alpha = -b_k - v, v = 0,1\ldots,k \leq m$. In particular if $n_1 = n_2 = \ldots = n_{m-1} = 1$ and $b_1 = 0$

then (4.1.4) becomes

$$(\alpha+k-1-v)^k\Phi(\alpha)=\frac{(m-k-v-1)!(m-k-1-v-1)!\ldots(1-v)!(-v!)\Gamma^k(1)}{(-1)^{kv+\sum_{i=1}^{k-1}(k-i)}(k-1+v)!(k-2+v)!\ldots(1-v)!\ldots(1+v)!v^k!}$$

(4.1.5)

(b)

$$\frac{d^t}{d\alpha^t}\log(\alpha+b_k+v)^k\Phi(\alpha)=k\Psi(\alpha+b_k+v+1)+\sum_{j=k+1}^{m}\Psi(\alpha+b_j)-k\sum_{j=1}^{v}(\alpha+b_k+v-j)^{-1}-$$

$$\sum_{i=1}^{k-1}(k-i)\sum_{j=1}^{n_{k-1}}(\alpha+b_{k-i+1}-j)^{-1}$$

for $t=1$, and

$$=(-1)^t(t-1)!\left\{k\varepsilon(t,\alpha+b_k+v+1)+\sum_{j=k+1}^{m}\varepsilon(t,\alpha+h_j)-k\sum_{j=1}^{v}(\alpha+b_k+v-j)^{-t}-\sum_{i=1}^{k-1}(k-i)\sum_{j=1}^{n_{k-1}}(\alpha+b_{k-i+1}-\right.$$

for $t\geq 2$ (4.1.6)

proof

$$(\alpha+b_k+v)^k\prod_{j=1}^{m}\Gamma(\alpha+b_j)=\frac{\Gamma^k(\alpha+b_k+v+1)\prod_{j=k+1}^{m}\Gamma(\alpha+b_j)}{\prod_{j=1}^{v}(\alpha+b_k+v-j)^k\prod_{i=1}^{k-1}\prod_{j=1}^{n_{k-i}}(\alpha+b_{k-i+1}-j)^{k-i}} \qquad (4.1.7)$$

since $z\Gamma(z)=\Gamma(z+1)$ and $\Gamma(z)=(z-1)!$ for $z\geq 1$.

(a) Now put $\alpha=-b_k-v$, $b_k=b_1+n_1+n_2+\ldots+n_{k-1}$ in (4.1.7) and the lemma is proved.

(b) This result follows directly from the definitions of *psi* and generalized *zeta* functions (MATHAI, 1971, pp.73) and the fact that

$\frac{d^t}{d\alpha^t}(-\ln\alpha)=(-1)^t(t-1)!\alpha^t$ with the convention that 0!=1.

Lemma 2

$$\text{Let } \Phi(\alpha)=\frac{\Gamma(\alpha+b_1)\Gamma(\alpha+b_2)\ldots\Gamma(\alpha+b_m)}{\Gamma^m(\alpha+b_m+k)} \tag{4.1.8}$$

Where $b_j,\ j=1,\ldots,m$ are as in lemma 1 and $k=0,1,\ldots$. Then for a fixed j

(a) $$(\alpha+b_m+k-j)^m\Phi(\alpha)=\left[\prod_{t>1}^{j}(t-1)^m\{(-1)(-2)\ldots(j-k)\}^m\prod_{l=1}^{m-1}\prod_{i=1}^{n_{m-l}}(j-k-i)^{m-l}\right]^{-1}$$

$$\text{as } \alpha\to -b_m-k+j,\ j=1,\ldots,k. \tag{4.1.9}$$

(b) In particular if $k=l, n_1=n_2=\ldots=n_{m-1}=1$ and $b_1=\frac{1}{2}$ then

$$\left(\alpha+m-\frac{1}{2}+k-j\right)^m\Phi(\alpha)=\left[\prod_{l=1\neq k''}^{m}(k'-l)^{k'}\right]^{-1} \tag{4.1.10}$$

as $\alpha\to -m+\frac{1}{2}-k+j.$

An empty product is interpreted as unity.

proof

After cancelling all the possible gammas in (4.1.8) we get

$$\Phi(\alpha)=\left[\prod_{j=1}^{k}(\alpha+b_m+k-j)^m\prod_{l=1}^{m-1}\prod_{i=1}^{n_{m-l}}(\alpha+b_1+n_1+\ldots+n_{m-l})^{m-l}\right]^{-1} \tag{4.1.11}$$

(a) put $\alpha=-b_m-k+j$ in $(\alpha+b_m+k-j)^m\Phi(\alpha)$ and the result follows.

(b) If $k=1, b_1=\frac{1}{2}, n_1=n_2=\ldots=n_{m-1}=1$, it is easy to see that

$$\Phi(\alpha)=\left[\left(\alpha+m-\frac{1}{2}\right)^{m}\prod_{l=1}^{m-1}\left(\alpha+m-l-\frac{1}{2}\right)^{m-l}\right]^{-1}$$

$$=\left[\left(\alpha+m-\frac{1}{2}\right)^{m}\left(\alpha+m-1-\frac{1}{2}\right)^{m-1}\ldots\left(\alpha+1-\frac{1}{2}\right)\right]^{-1}$$

$$=\left[\prod_{l=1}^{m}\left(\alpha+l-\frac{1}{2}\right)^{l}\right]^{-1} \qquad (4.1.12)$$

Hence

$$\left(\alpha+l-\frac{1}{2}\right)^{l}\left[\prod_{l=1}^{m}\left(\alpha+l-\frac{1}{2}\right)^{l}\right]^{-1}=\left[\prod_{j=1\neq k'}^{m}(k'-j)^{k'}\right]^{-1} \qquad (4.1.13)$$

as $\alpha \to -l+\frac{1}{2}$. An empty product being interpreted as unit.

Theorem 1

Let $G(u)=\int_{u}^{\infty} f(t)dt$ be as defined in (4.1.3(a)) and (4.1.3(b)), then

$$G(u)=1-C'u^{k}\sum_{j=0}^{\infty}d_{j}u^{g_j}(-\log u)^{f_j} \qquad (4.1.14)$$

Where C', k, d_j, g_j and $0\leq f_j\leq m_j-1$ are constants free of u. f_j is an integer and m_j is the j^{th} pole considered, and d_j's are functions of gamma if $m_j=1$ for all j, functions of *gamma* and *psi* if $m_j=2$ for some j, and of *gamma*, *psi* and *generalised zeta* functions if there is an $m_j\geq 3$ for some $j=0,1,\ldots$.

Proof

Let $B(\alpha) = u^{-\alpha}\left(\alpha - \frac{n}{2} + m + \frac{1}{2}\right)^{-1}\Phi(\alpha)$, r even

$$= u^{-\alpha}\left(\alpha - \frac{n}{2} + m\right)^{-1}\Phi(\alpha), \quad r \text{ odd} \tag{4.1.15}$$

From the calculus of residues we have

$$G(u) = C'u^{\frac{n}{2}-m-\frac{1}{2}}(\text{Sum of residues of } B(\alpha)), \quad \text{r even}$$

$$= C'u^{\frac{n}{2}-m}(\text{Sum of residues of } B(\alpha)), \quad \text{r odd.}$$

It is well known (MATHAI 1970, 1971) that if $\Phi(\alpha)$ is a gamma product with a pole of order m at $\alpha = a$, then the residue of $\Phi(\alpha)u^{-\alpha}$ at $\alpha = a$ is given by

$$R_{ma} = \frac{u^{-a}}{(m-1)!}\sum_{r=0}^{m-1}\binom{m-1}{r}(-\log u)^{m-1-r}\left\{\sum_{r_1=0}^{r-1}\binom{r-1}{r_1}A_0^{r-1-r_1}\ldots\right\}B_0 \tag{4.1.16}$$

where

$$B_0 = (\alpha - a)^m\Phi(\alpha) \text{ as } \alpha \to a$$

$$A_0^{(t)} = \frac{d^t}{d\alpha^t}\left(\frac{d}{d\alpha}\log(\alpha - a)^m\Phi(\alpha)\right) \text{ as } \alpha \to a \text{ and } t \geq 0. \tag{4.1.17}$$

$A_0^{(t)}$ is available in terms of *psi* functions for $t = 0$ and *generalised zeta* functions for $t \geq 1$. When $m = 1$, $A_0^{(t)} \equiv 0$ and B_0 is available in terms of gamma functions on account of the definitions of residue and *psi* functions. Also since C' is such that $E(u^{h+1}) = 1$ when $h = -1$, then residue at $h=-1$ or (equivalently at $\alpha=\frac{n}{2}-m-\frac{1}{2}$ when r is even or $\alpha=\frac{n}{2}-m$ when r is odd) is $(C')^{-1}$ and this accounts for the presence of the constant 1 in (4.1.14) and the absence of the same constant in (1.4.1). It is worth noting that the convenient form of the residue in (4.1.15) is made possible by the differential operator

$$G_v=\left\{\frac{\delta}{\delta\alpha}+(-\ln u)\right\}^v \tag{4.1.18}$$

operating on a gamma product $\Phi(\alpha)$, (MATHAI and SAXENA 1973).

4.2 An *H-function* Representation of *F(u).*

It is shown in PEDERZOLI (1985) that $U=\prod_{j=1}^{2r}X_j$ where X_j are *Beta type-1* random variables with parameters (α_j,β_j), $j=1,\ldots,2r$. By (3.1.12) above we can write

$$G(u)=\prod_{j=1}^{2r}\frac{\Gamma(\alpha_j+\beta_j)}{\Gamma(\alpha_j)}H^{2r+1,0}_{2r+1,2r+1}\left[u\,\middle|\,\begin{matrix}(\alpha+\beta_{11}), & . & . & , & (\alpha_{2r}+\beta_{2r}), & (1,1)\\ (\alpha_1,1), & . & . & , & (\alpha_{2r},1), & (0,1)\end{matrix}\right] \tag{4.2.1}$$

where $\alpha_j,\beta_j,j=1,\ldots,2r$ are as defined in PEDERZOLI (1985, equations 2.6 to 2.11). Consequently (4.2.1) can be evaluated with the technique of calculus of residues applied on a contour integral (3.1.1).Proof of the validity of the Residue Theorem in evaluating the *H-function* inversion integral can be seen in SPRINGER (1979, pp.431-440).The resulting infinite series is of the form (4.1.14).

However,

$g(u)=\frac{d}{du}(G(u))$ (if the derivative exists)

when expressed in the form (3.1.7), is a convenient form in the characterization theory of distributions. See for example MATHAI and SAXENA (1973, chap.6).

4.3 An Illustrative Example.

Consider the case $p=r+1$. Replace h with $h+1$ in (4.1.1) and let

$\alpha=h+\frac{n}{2}-\frac{r}{2}+\frac{1}{2}$ to get

Case(i) $r=2m$

$$\Phi_1(\alpha)=\frac{\prod_{j=1}^{m}\Gamma(\alpha+j)\prod_{j=1}^{2m}\Gamma\left(\alpha+m+\frac{1}{2}+\frac{j}{2m+1}\right)}{\prod_{j=1}^{m}\left(\alpha+j+\frac{1}{2}\right)^{j}\Gamma^{m}\left(\alpha+m+\frac{3}{2}\right)\prod_{j=1}^{2m}\Gamma\left(\alpha+m+\frac{1}{2}+\frac{n}{4m}+\frac{j}{2m}\right)} \qquad (4.3.1)$$

and *Case(ii)* $r=2m-1$

$$\Phi_2(\alpha)=\frac{\prod_{j=1}^{m}\Gamma\left(\alpha+j+\frac{1}{2}\right)\prod_{j=1\neq m}^{2m-1}\Gamma\left(\alpha+m+\frac{j}{2m}\right)}{\prod_{j=1}^{m}(\alpha+j)^{j}\Gamma^{m-1}(\alpha+m+1)\prod_{j=1}^{2m-1}\Gamma\left(\alpha+m+\frac{n}{4m-2}+\frac{j}{2m-1}\right)} \qquad (4.3.2)$$

In *Case(i)* the poles are at

$$\alpha + j \text{ of order } j,\ j = 1, \ldots, m-1$$

$$\alpha + m + v \text{ of order } m,\ v = 0, 1, \ldots.$$

$$\alpha + m + \frac{1}{2} + \frac{j}{2m+1} + v \text{ of order } 1, j = 1, \ldots, 2m,\ v = 0, 1, \ldots \tag{4.3.3}$$

Using the relation

$\Gamma(z)\Gamma(1-z) = \pi \csc \pi z$ LUKE (1969, pp.11)

it can easily be shown that the denominator gamma factors

$\Gamma^{m}\left(\alpha + m + \frac{3}{2}\right)$ and $\prod_{j=1}^{2m} \Gamma\left(\alpha + m + \frac{1}{2} + \frac{n}{4m} + \frac{j}{2m}\right)$ are bounded from below, SPRINGER (1979, pp.434-435) and hence poles coming from such factors are no cause for concern.

4.4 Computation Of The Residues

Denote the residues coming from the poles (4.3.3) by R_j, R_{mv}, R'_j and R_{jv}, respectively. Then from (4.1.16) we have

$$R_j = \frac{u^j}{(j-1)!} \sum_{r=0}^{j-1} \binom{j-1}{r} (-\log u)^{j-1-r} \left\{ \sum_{r_1=0}^{r-1} \binom{r-1}{r_1} A_{0j}^{(r-1-r_1)} \ldots \right\} B_{0j}, j = 1, \ldots, m-1$$

$$R_{mv} = \frac{u^{m+v}}{(m-1)!} \sum_{r=0}^{m-1} \binom{m-1}{r} (-\log u)^{m-1-r} \left\{ \sum_{r_1=0}^{r-1} \binom{r-1}{r_1} A_{0mv}^{(r-1-r_1)} \ldots \right\} B_{0mv}, v = 0, 1, \ldots.$$

$$R'_j = \frac{u^{j+\frac{1}{2}}}{(j-1)!} \sum_{r=0}^{j-1} \binom{j-1}{r} (-\log u)^{j-1-r} \left\{ \sum_{r_1=0}^{r-1} \binom{r-1}{r_1} A'^{(r-1-r_1)}_{0j} \ldots \right\} B'_{0j}, j = 1, \ldots, m \tag{4.4.1}$$

and

$$R_{jv} = u^{m+\frac{1}{2}+\frac{j}{2m+1}+v} B_{ojv}, j = 1, \ldots, 2m; v = 0, 1, \ldots$$

where for example

$$R_2 = u^2\{-\ln u + A_{02}\}B_{02}$$

on account of (4.1.18). Here

$$B_{02} = \frac{\Gamma^2(1)\prod_{i=3}^{m}\Gamma(i-2)\prod_{i=1}^{2m}\Gamma\left(m-\frac{3}{2}+\frac{i}{2m+1}\right)}{(-1)\prod_{i=1}^{m}\left(i-\frac{3}{2}\right)^{i}\Gamma^{m}\left(m-\frac{1}{2}\right)\prod_{i=1}^{2m}\Gamma\left(m-\frac{3}{2}+\frac{n}{4m}+\frac{i}{2m}\right)} \qquad (4.4.2)$$

and

$$B_{0j} = \frac{\Gamma^{j}(1)\prod_{i=j+1}^{m}\Gamma(i-j)\prod_{i=1}^{2m}\Gamma\left(m-j+\frac{1}{2}+\frac{i}{2m+1}\right)}{\prod_{i=1}^{m-1}(-1)^{m-i}\prod_{i=1}^{m}\left(i-j+\frac{1}{2}\right)^{i}\Gamma^{m}\left(m-j+\frac{3}{2}\right)\prod_{i=1}^{2m}\Gamma\left(m-j+\frac{1}{2}+\frac{n}{4m}+\frac{i}{2m}\right)} \qquad (4.4.3)$$

for $j = 1,\ldots,m-1$, by putting $b_k = k = j, n_k = 1, k = 1,\ldots,m-1$ and $v = 0$ in (4.1.10) above.

$$A_{0j} = j\Psi(1)+\sum_{i=j+1}^{m}\Psi(i-j)+\sum_{i=1}^{2m}\Psi\left(m-j+\frac{1}{2}+\frac{i}{2m+1}\right)-i\sum_{i=1}^{m}\left(i-j+\frac{1}{2}\right)^{-1}-m\Psi\left(m-j+\frac{3}{2}\right)-$$

$$(j-i)\sum_{i=1}^{j-i}(-i)^{-1}-\sum_{i=1}^{2m}\Gamma\left(m-j+\frac{1}{2}+\frac{n}{4m}+\frac{i}{2m}\right), j=2 \qquad (4.4.4)$$

for $j > 2$.

Here $\Psi(\cdot)$ and $\xi(\cdot)$ are the *psi* and *zeta* functions respectively.See for example MATHAI (1971, pp.73). The residues coming from (4.3.2) can be similarly determined on account of (4.1.5) and (4.1.6) above.

4.5 A Particular Case:

As mentioned before-hand (section 1.4) the cummulative distribution function $F(u)=\int_0^u f(u)du$ is easily obtained once the probability density function $f(u)$ is known. In our problem, the series forms of $f(u)$ can be integrated term by term to get $F(u)$ provided the probability density function is available. Often this is not the case.

In section 4.1 above, we have shown that if the h^{th} moment of a random variable exists and is known, then $F(u)$ is available by the calculus of residues (equation 4.1.18). One illustrative example when the pole is simple is given below.

r=2, p=3 and n=2

From equations (4.1.16) and (4.1.17) set $B(\alpha)=s^{-1}M_{s+1}\{f(u)\}$ to get

$$B(\alpha)=\frac{\Gamma(\alpha+1)\Gamma\left(\alpha+\frac{11}{6}\right)\Gamma\left(\alpha+\frac{13}{6}\right)u^{-\alpha}}{\left(\alpha+m-\frac{n}{2}+\frac{1}{2}\right)\left(\alpha+\frac{3}{2}\right)\Gamma\left(\alpha+\frac{5}{2}\right)\Gamma\left(\alpha+\frac{n}{4}+2\right)\Gamma\left(\alpha+\frac{n}{4}+\frac{5}{2}\right)} \qquad (4.5.1)$$

Note that $\left(\alpha+m-\frac{n}{2}+\frac{1}{2}\right)B(\alpha)$ can be obtained by replacing α with $\alpha+1$ in equation (4.1.2) or equivalently h with $h+1$ in the moment function of u. Proceeding as in section (4.1) let

$B_{01}=(\alpha+1+v)B(\alpha)$ as $\alpha\to -1-v$,

$$B_{02} = \left(\alpha + \frac{11}{6} + v\right)B(\alpha) \text{ as } \alpha \to -\frac{11}{6} - v,$$

$$B_{03} = \left(\alpha + \frac{13}{6} + v\right)B(\alpha) \text{ as } \alpha \to -\frac{13}{6} - v,$$

$$B_{04} = \left(\alpha + \frac{3}{2} - \frac{n}{2}\right)B(\alpha) \text{ as } \alpha \to \frac{n}{2} - \frac{3}{2}$$

and

$$B_{05} = \left(\alpha + \frac{3}{2}\right)B(\alpha) \text{ as } \alpha \to -\frac{3}{2}.$$

The poles of the gamma factors in equation (4.5.1) are simple. So the expressions B_{0j}, $j = 1,\ldots,5$ are the residues at the respective poles.

$$B_{01} = \frac{(-1)^v \Gamma\left(\frac{5}{6} - v\right)\Gamma\left(\frac{7}{6} - v\right)u^{(1+v)}}{v!\left(\frac{1}{2} - \frac{n}{2} - v\right)\left(\frac{1}{2} - v\right)\Gamma\left(\frac{3}{2} - v\right)\Gamma\left(\frac{n}{4} + 1 - v\right)\Gamma\left(\frac{n}{4} + \frac{3}{2} - v\right)} \tag{4.5.2}$$

$$B_{02} = \frac{(-1)^v \Gamma\left(-\frac{5}{6} - v\right)\Gamma\left(\frac{2}{6} - v\right)u^{\left(\frac{11}{6} + v\right)}}{v!\left(-\frac{n}{2} - \frac{1}{3} - v\right)\left(-\frac{2}{6} - v\right)\Gamma\left(\frac{4}{6} - v\right)\Gamma\left(\frac{n}{4} + \frac{1}{6} - v\right)\Gamma\left(\frac{n}{4} + \frac{4}{6} - v\right)} \tag{4.5.3}$$

$$B_{03} = \frac{(-1)^v \Gamma\left(-\frac{7}{6} - v\right)\Gamma\left(-\frac{2}{6} - v\right)u^{\left(\frac{13}{6} + v\right)}}{v!\left(-\frac{n}{2} - \frac{4}{6} - v\right)\left(-\frac{4}{6} - v\right)\Gamma\left(\frac{2}{6} - v\right)\Gamma\left(\frac{n}{4} - \frac{1}{6} - v\right)\Gamma\left(\frac{n}{4} + \frac{2}{6} - v\right)} \tag{4.5.4}$$

$$B_{04} = \frac{\Gamma\left(\frac{n}{2} - \frac{1}{2}\right)\Gamma\left(\frac{n}{2} + \frac{2}{6}\right)\Gamma\left(\frac{n}{2} + \frac{4}{6}\right)u^{\left(\frac{3}{2} - \frac{n}{2}\right)}}{\left(\frac{n}{2}\right)\Gamma\left(\frac{n}{2} + 1\right)\Gamma\left(\frac{3n}{4} + \frac{1}{2}\right)\Gamma\left(\frac{3n}{4} + 1\right)} \tag{4.5.5}$$

and

$$B_{05} = \frac{\Gamma\left(-\frac{1}{2}\right)\Gamma\left(\frac{2}{6}\right)\Gamma\left(\frac{4}{6}\right)u^{\frac{3}{2}}}{\left(-\frac{n}{2}\right)\Gamma(1)\Gamma\left(\frac{n}{4}+\frac{1}{2}\right)\Gamma\left(\frac{n}{4}+1\right)} \tag{4.5.6}$$

Now from (4.1.18) we get

$$F(u) = Cu^{\left(\frac{n}{2}-\frac{3}{2}\right)}\left\{\sum_{j=0}^{\infty}\left(B_{01}+B_{02}+B_{03}\right)+B_{04}+B_{05}\right\} \tag{4.5.7}$$

After evaluating the residues, (4.5.7) becomes

$$F(u) = 4.96196\ u^{\frac{1}{2}} - 13.5\ u + 22.0201\ u^{\frac{4}{3}} - 14.8859\ u^{\frac{3}{2}} + 2.43755\ u^{\frac{5}{3}} - 0.0476626\ u^{\frac{7}{3}} + 0.0156253\ u^{\frac{8}{3}} - 0.00299059\ u^{\frac{10}{3}} + 0.00155755\ u^{\frac{11}{3}} - 0.000524132\ u^{\frac{13}{3}} + 0.000329113\ u^{\frac{14}{3}} - 0.000145478\ u^{\frac{16}{3}} + 0.000101237\ u^{\frac{17}{3}} - 0.0000526398\ u^{\frac{19}{3}} + 0.0000391041\ u^{\frac{20}{3}} - 0.000022654\ u^{\frac{22}{3}} + 0.0000176079\ u^{\frac{23}{3}} - 0.0000110201\ u^{\frac{25}{3}} + 8.855\ 10^{-6}\ u^{\frac{26}{3}} - 5.87307\ 10^{-6}\ u^{\frac{28}{3}} + 4.84093\ 10^{-6}\ u^{\frac{29}{3}} \tag{4.5.8}$$

Differentiating (4.5.8), the *pdf* $f(u)$ becomes:

$$f(u) = -13.5 + 2.48098\, u^{-1} + 29.3602\, u^{\frac{1}{3}} - 22.3288\, u^{\frac{1}{2}} + 4.06259\, u^{\frac{2}{3}} - 0.111213\, u^{\frac{4}{3}} + 0.0416676\, u^{\frac{5}{3}} - 0.00996864\, u^{\frac{7}{3}} + 0.00571101\, u^{\frac{8}{3}} - 0.00227124\, u^{\frac{10}{3}} + 0.00153586\, u^{\frac{11}{3}} - 0.000775883\, u^{\frac{13}{3}} + 0.000573679\, u^{\frac{14}{3}} - 0.000333386\, u^{\frac{16}{3}} + 0.000260694\, u^{\frac{17}{3}} - 0.000166129\, u^{\frac{19}{3}} + 0.000134994\, u^{\frac{20}{3}} - 0.0000918343\, u^{\frac{22}{3}} + 0.0000767433\, u^{\frac{23}{3}} - 0.0000548153\, u^{\frac{25}{3}} + 0.0000467956\, u^{\frac{26}{3}} - 0.0000347114\, u^{\frac{28}{3}} + 0.000030135\, u^{\frac{29}{3}} \qquad (4.5.9)$$

Our procedure was to add the residues until the n^{th} residue so added failed to contribute to the percentage points *F(u)* for particular values of *U*. The version of Mathematica program used had a machine precision of sixteen. With this precision, we found that by taking at least nine terms $\int_0^1 f(u)du = 1$.Also $F(u) \to 1$as $u \to 1$ and $F(u) \to 0$ as $u \to 0$ demonstrating the monotonicity of the function.

Table 1 below compares the first ten theoretical moments computed from $E(u^h) = \int_0^1 u^h f(u)du$ and the exact moments computed from equation (4.1.1) above.

Table 1

Theoretical Moments	Exact Moments
1	1
0.0555577	0.0555556
0.0109759	0.0109739
0.00349463	0.00349286
0.00143999	0.00143835
0.000698155	0.000696639
0.000378733	0.000377318
0.000223115	0.000221788
0.000140024	0.000138774
0.0000924012	0.0000912209
0.000063522	0.0000624038

We note that the above results are similar upto five decimal places.

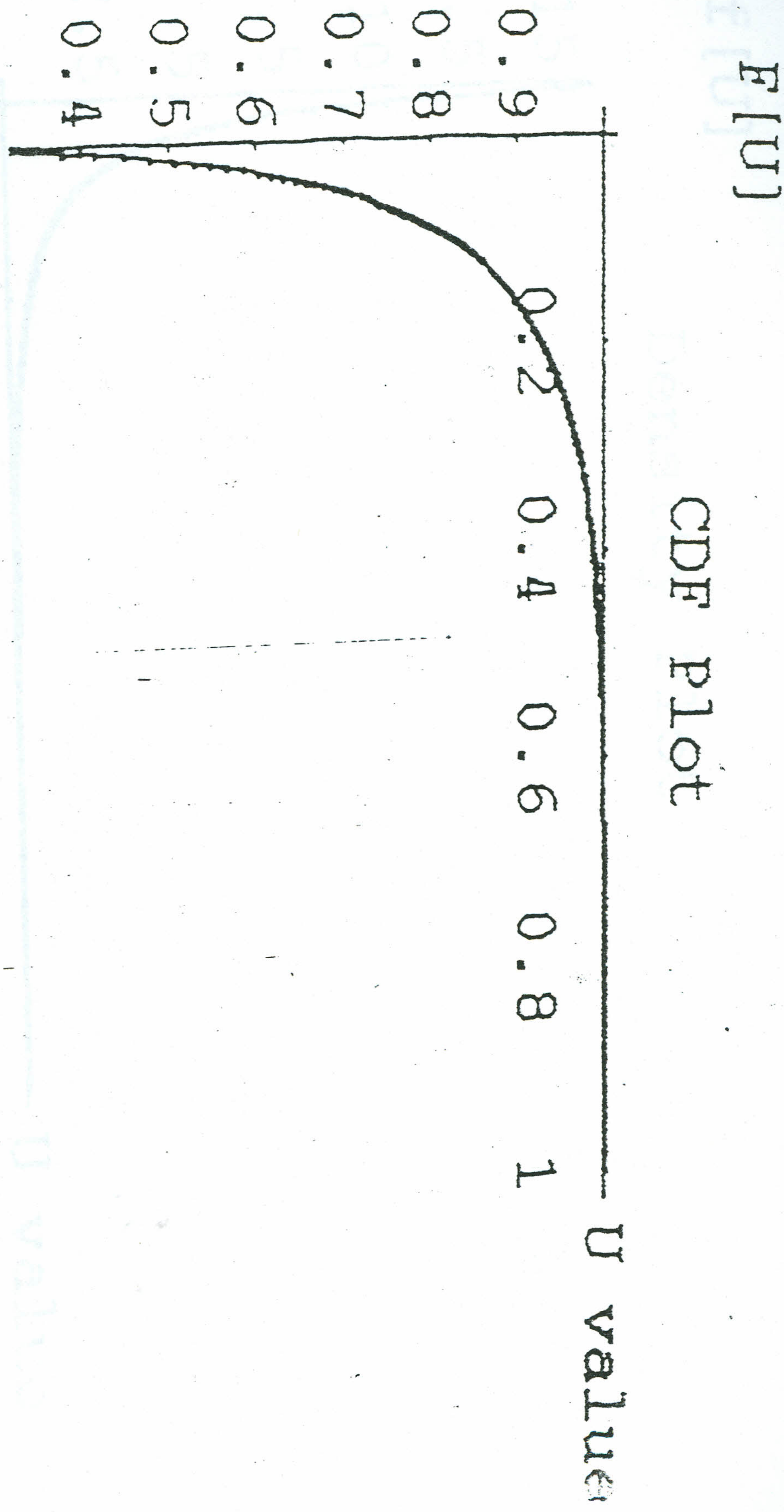

Fig 4.1

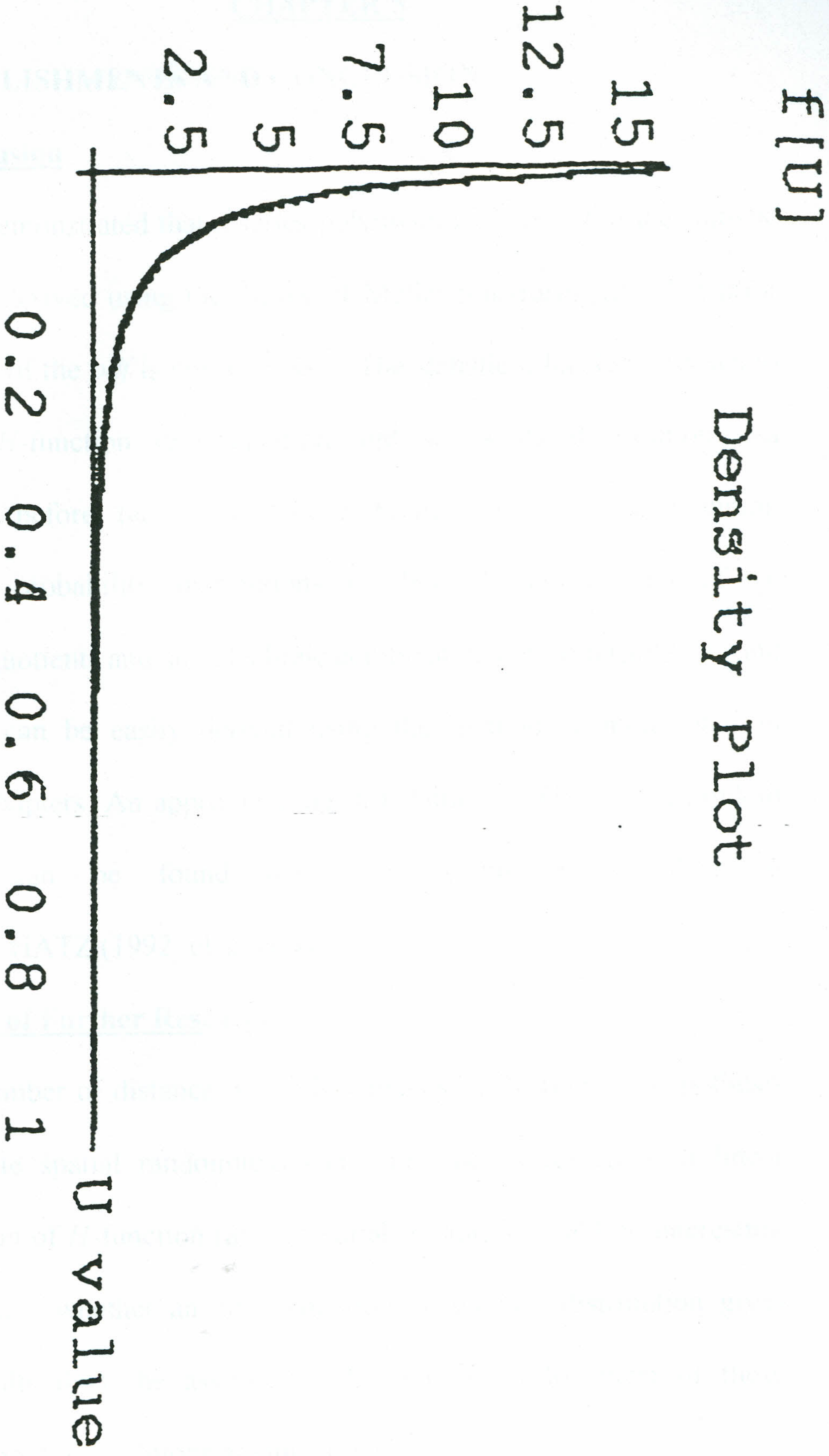

Fig 4.2

CHAPTER 5

ACCOMPLISHMENTS AND CONCLUSION

5.1 Conclusion

We have demonstrated that a series polynomial of the *cdf* of the statistic u is easily derived using the theory of Mellin transforms and that prior knowledge of the *pdf* is not necessary. The statistic u has been shown to have an H-function representation and so is its distribution. As mentioned before the H-function embodies most of the common continuous probability distributions and both the *pdf* and *cdf* of the products, quotients and any algebraic combinations of both (products and quotients) can be easily derived using the methods demonstrated in previous chapters. An approximating distribution of H-function random variables can be found using the technique described in BODENSCHATZ (1992, chapter 4).

5.2 Areas of Further Research

A good number of distance methods statistics for testing the hypothesis of complete spatial randomness can infact be expressed as a linear combination of H-function random variables and it would be interesting to investigate whether an approximating H-function distribution gives better results than the asymptotic distribution as for most of these statistics,exact distributions are unknown.

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