MAXIMUM LIKELIHOOD ESTIMATION OF PARAMETERS FOR KUMARASWAMY DISTRIBUTION BASED ON PROGRESSIVE TYPE II HYBRID CENSORING SCHEME

MEYMUNA SHARIFF JAFFER
(BSc)

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DEPARTMENT OF MATHEMATICS AND ACTUARIAL SCIENCE
KENYATTA UNIVERSITY

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Declaration

This research project is my original work and has not been presented elsewhere for a degree award.

Signature……………………………. Date……………………………..

Meymuna .S. Jaffer

SUPERVISOR APPROVAL

I confirm that the work reported in this project was carried out by the student under my supervision.

Signature………………………… Date…………………………………

Dr Edward .G. Njenga

Department of Mathematics and Actuarial Science

Kenyatta University.

Dedication

For their unwavering love, incredible support and numerous encouragements, I devote the research to my supportive family. May God reward them.

Acknowledgement

Firstly, I desire to convey my profound appreciation as well as thanks giving to God for giving me well-being, knowledge and the gift of life during my studies.

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Abbreviations and Acronyms

AMLE Approximate Maximum Likelihood Estimator

BRR Binomial Random Removal

CI Confidence Interval

CDF Cumulative Distribution Function

D Number of failures such that D=m in Case I and D=J in Case II

EM Expectation Maximization

E-STEP Expectation Step

HCS Hybrid Censoring Scheme

HPD Highest Posterior Density

J Number of failures observed in an experiment prior to the cessation of an experiment in case II

m Number of failures observed in an experiment prior to the cessation of the experiment in Case I

MGIED Mixed Generalized Inverted Exponential Distribution

MLEs Maximum Likelihood Estimators

MSEs Mean Square Errors

M-STEP Maximization Step

PDF	Probability Distribution Function
PLD	Power Lomax Distribution
PTCS	Progressive Type-II Censoring Scheme
PTC	Progressive type-I censored
PTHC	Progressive Type-II Hybrid Censored
PTHCS	Progressive Type-II Hybrid Censoring Scheme
R_i	Number of items removed at the occurrence of an i^{th} failure
R_D^*	Number of remaining items remaining at time point T for Case I
R_j^*	Number of remaining items remaining at time point T for Case II
RMSEs	Root Mean Squared Errors
SEL	Squared Error Loss
W.r.t	With respect to
X_i	The i^{th} item lifetime; where i= 1, 2, 3,…,n
$X_{i:m:n}$	Failure time which is observed on the i^{th} item

Abstract

The project considers the maximum likelihood estimators for Kumaraswamy distribution based on progressive type II hybrid censoring scheme using the expectation maximization algorithm. A two parameter Kumaraswamy distribution can be applied in natural phenomena that have outcomes with an upper and a lower bound. Kumaraswamy distribution remains of keen consideration in disciplines such as economics, hydrology and survival analysis. The field of survival analysis has advanced over the years and extensive research has been undertaken. Previous studies have considered maximum likelihood estimation for Kumaraswamy distribution based on progressive type II censoring scheme using methods like Newton-Raphson and EM algorithm but none has used progressive type II hybrid censoring scheme and obtained maximum likelihood estimators of Kumaraswamy distribution via EM algorithm. EM algorithm has been utilized in manipulation of missing data as it is a more superior method when handling incomplete data. Comparison of different combinations of censoring schemes with respect to the MSEs and biases at fixed parameters of α and β are obtained through simulation. It is observed that in the three censoring schemes, for an increasing sample size, the MSEs and biases are generally decreasing. Eventually, an illustration with real life data set is provided and it illustrates how maximum likelihood estimators works in practice under different censoring schemes.

CHAPTER ONE

INTRODUCTION

1.0 Chapter Overview

The study's background information, terminologies, problem statement, justification, main and specific objectives, significance and an outline of this study are all provided in this chapter.

1.1 Background to the Study

Reliability and life testing trials besides scrutinizing the necessary duration utilized by a unit in an investigation over an interval of time, also allows elimination of units from an investigation before failure. In life testing trials, removal of units is done in experiments where there exists cost constrains. Besides failure not occurring in manufacturing plants and factories where production ensues, some units ought to be removed. In such situations where units are removed it is usually preplanned so as to enable the saving of both time and cost.

Resource constraint which is as an outcome of time and cost may sometimes prevent an experimenter from observing the life time of all items. It is observed that sometime removals may also happen at intermediate steps. Removal of units in such a case is typically not within the control of an investigator. Removals can be classified as either fixed or progressive. In fixed removals, the removals are prespecified and in progressive random removal, the removals are random.

Two ordinary categories of censoring techniques, type I and II are given preference in life testing experiments. An experiment is known to go up-to a pre-arranged time point T in type-I censoring. Life testing trials, during type-II censoring is carried on anticipation of the number of failure that is definite. HCS exists by way of blending both type-I and II censoring schemes. Epstein, (1954) initiated HCS. It was then widely used from 1960 in reliability experiments. An experimental unit to be removed in type I, type II and HCS occurs only at termination of the experiment. To overcome shortcomings of type-I, type-II and HCS, PTCS was introduced.

Furthermore, a downside of PTCS is allowing the elimination of units at additional points aside from the endpoint of an experiment. As a result of its limitation, it led to the introduction of PTHCS by Kundu and Joarder (2006). They advanced a PTHCS method that possesses a competitive advantage above PTCS. In PTHCS the predefined time for termination of an experiment is not exceeded in the course of the experiment as initially specified.

The Kumaraswamy distribution dates back to 1980 when it was introduced by Poondi Kumaraswamy and can be applied in natural phenomena that have outcomes with an upper and a lower bound. The Kumaraswamy distribution remains better suited in natural phenomena and can be used in simulation modeling owing to the benefit of a closed form cdf. Minimal research has been undertaken on Kumaraswamy distribution based on PTHCS with expectation maximization. Consequently, the study emphasizes on obtaining the MLEs for Kumaraswamy distribution under PTHCS and in addition applies the proposed method to real data. The concept of expectation maximization is also of interest since it enables an experimenter to deal with data that has missing values. An empirical inquiry is carried out on behalf of different amalgamations of model parameters in the proposed model.

Kumaraswamy, (1980) was concerned with distributions that have random variables. In previous study Kumaraswamy distribution has been utilized in hydrological problems. The study proposed a mixture of a probability distribution over an interval (0, 1). It possesses a property that makes it similar to beta distribution. Despite its similarity, the advantage the Kumaraswamy distribution has, is owing to the fact that the distribution function is a closed form thus allowing easy computation of its quantile. The beta distribution has a limitation in that it normally occurs in the set-up of an integral only. The Kumaraswamy distribution can be applied in practical studies such as the marks of students in an exam, weight of people among others.

Different authors such as Gholizadeh et al., (2011), Sultana and Ahmad (2015), Wafula et al., (2016), Muna (2017), Pak et al., (2018) and Sultana et al., (2018) researched on Kumaraswamy distribution and none of them considered PTHCS.

1.2 Statement of the Problem

Censoring is a concept that is unavoidable in life testing and reliability studies. Censoring is also a significant feature in survival analysis and it arises when a researcher does not observe the time of failure of each and every item that is situated in a life test. Previously, the MLEs of various distributions have been obtained using methods such as NR and EM algorithm. The use of the various techniques in survival analysis is mainly intended to save time taken and cost incurred in an experiment.

Distinct categories of censoring schemes can be utilized in a study; albeit some, such as type I, hybrid together with type II censoring schemes possess drawbacks. The downside of such schemes is as a result of units that need to endure elimination from the experiment permit removal at the terminal point only. PTCS has been widely used as it allows withdrawal of particular units from the trial within the stated time span of the experiment. The use of PTHCS helps to overcome the shortcoming of PTCS as the duration of the experiment might be large.

Therefore, this project aims to evaluate the MLEs of parameters for Kumaraswamy distribution based on PTHCS as little research has been done. The study uses Kumaraswamy distribution since it can be applied in natural phenomena that have outcomes with an upper and a lower bound. The MLEs for Kumaraswamy distribution has been evaluated using EM algorithm. Preference is given to the EM algorithm since it has been identified as a convenient mechanism in handling missing values.

1.3 Objectives

1.3.1 Broad objective

The main objective of this project is to derive and study the properties of maximum likelihood estimators of the parameters for Kumaraswamy distribution under PTHCS.

1.3.2 Specific objectives

i. To derive MLEs for the parameters of Kumaraswamy distribution under PTHCS.

ii. To apply EM algorithm in computing the MLEs of Kumaraswamy distribution under PTHCS.

iii. To compare the performance of the obtained MLEs of Kumaraswamy distribution for various parameter values under various censoring schemes based on simulated data.

iv. To apply the results from (ii) on real data.

1.4 Justification of the study

Censoring is an aspect of survival analysis that is commonly utilized in areas where there is an intention to salvage the duration taken and the expenses incurred in an experiment. Pak et al. (2018) considered Kumaraswamy distribution with a focus on type-II HCS and used Bayesian inference to obtain the parameters. Sultana et al. (2018) considered the parameters of Kumaraswamy using HCS. The MLEs were obtained via EM algorithm. Gholizadeh et al. (2011) used PTC samples with a focus on non- Bayesian and Bayesian estimators while Wafula et al. (2016) considered PTCS. The MLEs were derived using EM algorithm. PTCS despite allowing units to be removed, it also has a drawback in that removals are carried out within the duration of an observed failure time. Hence, the duration of the experiment might be large and a lot of time can be consumed. In this project, parameter estimation will focus on Kumaraswamy distribution based on PTHCS.

Kundu and Joarder, (2006) proposed PTHCS which is an amalgamation of hybrid and PTCS. PTHCS is therefore favored owing to the benefit of ensuring that the duration of the trial does not go beyond time point T which is normally pre-specified and Kumaraswamy distribution has a pdf, cdf and quantile functions that can be expressed in closed form making it simpler to utilize during simulation.

1.5 Significance of the study

The study presents different ways of finding MLEs of the Kumaraswamy distribution based on PTHCS and using expectation maximization besides other techniques that has been extensively used. Numerous studies has been concluded, nonetheless it has been based on PTHCS. The study therefore intends to fill this gap.

The research contributes towards the literature of survival analysis in obtaining MLEs of PTHCS. PTHCS overcomes the shortcoming of PTCS in that, the test time might either be too large or it might end too early. In PTCS the testing proceeds up to a pre specified time duration T even when m^{th} failure has occurred prior to time point T.

1.6 Definition of Terminologies

Survival analysis- It is a technique used to analyze statistics that relates to an interval from a well- distinct time origin up until the end point or an occurrence of interest takes place.

Censoring- The survival period of an individual is stated to be censored whenever the end point of concern has not been observed in a given study for a particular individual.

Right censoring- It takes place whenever a particular individual vacates the study before an event of concern occurs or the study is concluded before the occurrence of interest has happened.

Left censoring- It takes place whenever an event of concern has been concluded in advance before an individual has entered the study.

Interval censoring- It occurs when the duration of the event of interest of the individuals in the study is experienced within an interval of time.

Type-I censoring scheme – In this particular category of censoring a test unit ceases at a particular threshold of time T.

Type-II censoring scheme – Here a test unit which is in the study is terminated once the r^{th} failure is known to take place.

Hybrid censoring scheme- It is as a result of type I and II censoring schemes combined together.

Progressive type-II censoring scheme – It occurs whenever some units are removed from the study during a trial after a predetermined number of failures.

Expectation maximization- It is an approach that is used in the iterative computation and maximum likelihood estimation of parameters especially in models which have incomplete or hidden data variables.

E-step – The conditional expectation is computed using a log likelihood that is calculated from missing data which are estimated using current estimate and the data that is observed.

M-step - Here the function that is derived in the course of the E-step is optimized so as to derive a novel parameter.

1.7 Outline of the project

The project's remaining sections are structured as follows:

The second chapter discusses Kumaraswamy distribution and the types of censoring schemes that are studied. The third chapter outlines how Kumaraswamy distribution’s parameters were estimated using maximum likelihood and PTHCS. The technique utilized in attaining the desired results is an expectation maximization approach. Chapter four considers how a simulation is carried out to show exactly in what way the estimators that are recommended perform in estimating the parameters of Kumaraswamy distribution with expectation maximization and PTHCS. The results and discussions are likewise outlined in this chapter. Lastly, chapter five outlines a summary, the conclusion and suggested other research areas.

CHAPTER TWO

LITERATURE REVIEW

2.0 Introduction

The second chapter presents reviews on type I and II censoring schemes, HCS, PTCS, PTHCS as well as Kumaraswamy distribution.

2.1 Kumaraswamy Distribution

Kumaraswamy distribution was first introduced by Kumaraswamy, (1980) as pdf which could be utilized in numerous areas such as statistics and hydrology amongst others. The study of Kumaraswamy distribution remains an area that has received considerable attention by numerous authors as highlighted below. Pak et al., (2018) considered non- Bayesian as well as Bayesian estimates of the Kumaraswamy distribution whose focus is type II HCS. The study obtained variance covariance matrix and MLEs are obtained for the parameters. To compare performance a simulation technique known as Monte Carlo was performed by the author. To calculate Bayes estimates a bayesian approximation technique known as Tierney and Kadane was utilized. Real data was utilized in illustrating an application of the proposed method. MSEs indicated that Bayes estimates were better than MLEs.

Sultana et al., (2018) studied parameters of Kumaraswamy distribution. An emphasis of this study was mainly on deriving maximum likelihood estimates using EM algorithm. HCS was utilized in the study and Bayes estimate were obtained under SEL function. The results obtained indicated that MSE values reduce when the sample size increases. Muna, (2017) studied Kumaraswamy distribution focusing on the different estimation methods. The methods were compared to estimate the scale parameter as well as the shape parameter. The study also determined the Fishers information matrix.

A simulation study was used to undertake comparison using different set of values and different sample sizes. In this study it was observed that the MLE is the best method to be used when sample sizes are large.

The shape parameter of Kumaraswamy distribution has been studied by Sultana and Ahmad, (2015). In this research two types of priors that is Informative and non-informative were applied to obtain Bayesian estimators for Kumaraswamy distribution using Bayesian approximation techniques. To compare efficiency in the obtained results simulation was carried out using R software. It was noted that when the study is based on different priors and the posterior standard deviation will tend to decline as the sample size increases. It was observed that Bayesian estimates that are under informative priors performed better compared those that were under non – informative priors.

Using progressive type-II censored samples, Gholizadeh et al., (2011) studied non-Bayesian as well as Bayesian estimators respectively. The study considered the reliability, shape parameter and failure rate function of Kumaraswamy distribution. The results obtained indicate that whenever an observation was made MLEs have the least estimated MSEs as compared to Bayes estimates. Wafula et al., (2016) considered Kumaraswamy using PTCS. The MLEs in the research were obtained via an EM algorithm. The results showed that the estimates of parameters approached true values as sample size increased.

2.2 Review of type-I and type-II Censoring Schemes

Removal from an experiment aside from the terminal point is not allowed under standard type I and II censoring schemes. Type-I censoring scheme allows an investigation to end at a prearranged duration T. However, in type-II censoring a test is put to an end upon the r^{th} failure. Asgharzadeh, (2017) study obtained Bayes and maximum likelihood estimators for the unknown parameters of lindley model based on type-II censored data. EM algorithm was used to obtain the MLEs. Direct maximization technique was also used in the study to obtain the MLEs. The results obtained indicated that Bayes estimator performed better than other methods.

2.3 Review of Hybrid Censoring Scheme (HCS)

HCS is usually obtained through merging type-I and type-II censoring schemes. Various distributions were utilized to review HCS by a number of authors. Epstein, (1954) reviewed HCS. It was noted that, termination of the test unit in a life testing trial is undertaken at random time. HCS is classified into type-I HCS and type-II HCS, having its own set of benefits as well as limitations. The shortcoming of type I HCS comes about as a result of failures that take place at a predetermined duration T which is a very limited number and the quantity of failures that are observed before T are very few.

Childs et al., (2003) presented a novel HCS in which likelihood inferences from exponential distribution were generated using type-I and type-II samples that were hybrid censored. Type II HCS when used in a life testing experiment guarantees a fixed standard number of failures. Reference can be made to Balakrishnan and Kundu, (2013) and Sultana et al., (2018) among others.

2.4 Review of Progressive Type-II Censoring Scheme (PTCS)

PTCS has an advantage and an experimenter may be able to withdraw a unit from a life test via numerous phases as an investigation takes place. Raqab et al., (2010) discussed Pareto type-II distribution and the different predictors of times to failure of censored units in multiple stages were discussed. The results indicated that the highest conditional density (HCD) compares favorably to the approximate method. The HCD method and approximate method generated identical results.

Sarhan et al., (2008) studied PTCS focusing on competing risk data that possesses binomial distribution. The MLEs and the asymptotic distributions are also obtained in the study. From the results, binomial removal with different p gave estimates that have varying precision. In the same study, it was noted that smaller variance are obtained when the value of the m^{th} failure increases. For more details on PTCS see Balakrishnan and Aggarwala, (2000).

2.5 Review of Progressive Type-II Hybrid Censoring Scheme (PTHCS)

PTHCS is usually favored over PTCS as the duration of the experiment can be relatively larger in PTCS. The study of PTHCS has been accomplished extensively by quite a number of authors. PTHCS was proposed by Kundu and Joarder, (2006). PTHCS was recommended as it analyzed data that assumed items that are exponentially distributed. The analysis of lifetime distribution of items that follow Weibull distribution was conducted by deriving the MLEs. The AMLEs were advanced to estimate parameters that are unknown. The asymptotic CIs based on AMLE and MLES was also derived in the study. The results obtained revealed that MSEs and biases decrease for a fixed n when m increases.

Mokhtari et al., (2011) expounded on the inference of PTHC data based on Weibull distribution. The study focused on both classical and Bayesian statistical inferential technique. The shape parameter MLE was obtained using iterative procedure. Bayes estimates of parameters that are unknown were obtained using Gibbs sampling. The confidence interval was compared using Monte Carlo simulation. The study revealed that Bayesian inference has an advantage over classical inference. It was further recommended that on condition that prior information concerning the parameter is not obtainable, then the usage of Bayesian CI that has non-informative prior is preferred.

Yongming and Yimin, (2013) considered an inference of a distribution known as lomax based on PTHCS. Through the use of iterative technique the MLEs were derived. Bayesian estimates of unknown parameters are expressed in terms of average bias as well as MSEs and comparison of its MLEs is done by Monte Carlo simulation for different censoring schemes. From the results the estimates of reliability function was found to be superior to Bayesian inference whenever the MLEs are used. Bayesian estimates derived under reliability function displayed big differences as a result of the MLEs. The MSEs and biases of Bayesian inference in reliability functions decrease whenever there is an increase in sample size. Generally, it was noted that when MLEs were used in reliability function estimation they performed better than Bayesian technique.

Li and Lina, (2015) discussed the inference for a distribution known as the Generalized Rayleigh under PTHCS. In this study, MLEs were obtained using an EM algorithm. Bayes estimates were also obtained. The study indicated that Bayesian technique provided smaller RMSEs and biases when compared to those generated using EM algorithm whenever the sample size is small.

From, literature above, it is apparent that no earlier study has been undertaken for Kumaraswamy distribution based on PTHCS. Thus, the focus of this investigation is to fill this gap.

2.6 Chapter Summary

Literature review begins by giving an introduction on Kumaraswamy distribution. In addition a review of type I and II censoring schemes, HCS, PTCS and PTHCS are discussed in this chapter.

CHAPTER 3

RESEARCH METHODOLOGY

3.1 Introduction

The chapter describes PTHCS and its implementation in Kumaraswamy distribution, EM algorithm and its application in computing MLEs of the parameters of Kumaraswamy distribution under PTHCS.

3.2 Review of PTHCS

Suppose n observations that are independent similar items that are placed in a trial at the same time. Similarly, the life times of n objects are represented by $X_{1:m:n,} X_{2:m:n}, X_{3:m:n} ..., X_{m:m:n}$ and with a pdf $f(x,\alpha,\beta)$ and cdf $F(x,\alpha,\beta)$. An integer m< n of complete failures is normally predetermined by the start of a trial. Predetermined in advance is also time point T, prior to the commencement of the trial.

It is observed that $R_1, R_2, R_3, ..., R_m$ are m prefixed integers that satisfy the equation $R_1 + R_2 + R_3 + ... + R_m + m = n$. During the initial failure's occurrence, $x_{1:m:n}$, R_1 of such residual components are observed to be withdrawn randomly. In a similar manner, during second failure's occurrence, $x_{2:m:n}$, R_2 of the components that are left behind are also removed.

Whenever mth failure occurs, $x_{m:m:n}$, all the $R_m = n - m - R_1 - R_2 - R_3 - ... - R_{m-1} - m$ components that are still alive are withdrawn from the trial. The trial in PTHCS, ceases at the duration $x_{m:m:n}$, as soon as an mth failure; $x_{m:m:n}$ occurs prior to duration T.

However, if mth failure does not take place prior to the duration T yet only J failures occur prior to the time T, in an experiment wherever $0 \le J \le m$, then at time T, the remaining $R_j^* = n - (R_1 + R_2 + ... + R_J) - J$ are withdrawn. Trial then comes to an end after time T.

In PTHCS two cases are obtained as stated below

For Case I: $x_{1:m:n}...x_{m:m:n}$ if $x_{m:m:n} < T$

For Case II: $x_{1:m:n}...x_{J:m:n}$ if $x_{J:m:n} < T < x_{J+1:m:n}$

We note that for case II,

$x_{j:m:n} < T < x_{j+1:m:n} < ... < x_{m:m:n}$ hence $x_{j+1:m:n}, ..., x_{m:m:n}$ aren't observed.

3.3 The Kumaraswamy Distribution

Let $x_{1:m:n}, x_{2:m:n,...,} x_{D:m:n}$ represent PTHCS having a cdf and pdf from a population specified as follows;

$$f(x;\beta,\alpha) = \alpha\beta x^{\alpha-1}(1-x^{\alpha})^{\beta-1} \tag{3.1}$$

$$F(x:\beta,\alpha) = 1-(1-x^{\alpha})^{\beta} \tag{3.2}$$

Let the representation of the total failures be D such that D= m represents case I as D= J represents case II respectively. Likelihood functions for this distribution for case I is as shown below. (For details see Raqab and Madi, (2011)).

$$L(x,\beta,\alpha) \propto \prod_{i=1}^{D} f(x_i;\alpha,\beta)[1-F(x_i;\alpha,\beta)]^{R_i} \tag{3.3}$$

But for case I, where D=m. As a result, the likelihood function is as described below,

$$L(\alpha,\beta,X_{1:m:n}...X_{m:m:n}) \propto \prod_{i=1}^{m} \alpha\beta x_i^{\alpha-1}(1-x_i^{\alpha})^{\beta-1}[1-[1-(1-x_i^{\alpha})^{\beta}]^{R_i}$$

$$L(\alpha,\beta,X_{1:m:n}...X_{m:m:n}) \propto \alpha^m\beta^m \prod_{i=1}^{m} x_i^{\alpha-1} (1-x_i^{\alpha})^{\beta-1}[(1-x_i^{\alpha})^{\beta}]^{R_i} \tag{3.4}$$

The log-likelihood function is as shown below

$$l(\alpha,\beta,X_{1:m:n}...X_{m:m:n}) \propto m\ln\alpha + m\ln\beta + (\alpha-1)\sum_{i=1}^{m}\ln x_i + (\beta-1)\sum_{i=1}^{m}\ln(1-x_i^{\alpha}) + \beta\sum_{i=1}^{m}R_i\ln(1-x_i^{\alpha}) \quad (3.5)$$

In Case II, D=J. The log likelihood function takes this form

$$L(x;\alpha,\beta) \propto \prod_{i=1}^{D} f(x_i;\alpha,\beta)[1-F(x_i,\alpha,\beta)]^{R_i}[1-F(T)]^{R_J^*} \quad (3.6)$$

$$L(\alpha,\beta,X_{i:m:n},...,X_{J:m:n}) \propto \prod_{i=1}^{J}\alpha\beta x_i^{\alpha-1}(1-x_i^{\alpha})^{\beta-1}[1-[1-(1-x_i^{\alpha})^{\beta}]^{R_i}[(1-T^{\alpha})^{\beta}]^{R_J^*}$$

$$L(\alpha,\beta,X_{i:m:n},...,X_{J:m:n}) \propto \alpha^J\beta^J\prod_{i=1}^{J}x_i^{\alpha-1}(1-x_i^{\alpha})^{\beta-1}(1-x_i^{\alpha})^{\beta R_i}(1-T^{\alpha})^{\beta R_J^*} \quad (3.7)$$

$$l(\alpha,\beta,x_{i:m:n}...x_{J:m:n}) \propto J\ln\alpha + J\ln\beta + (\alpha-1)\sum_{i=1}^{J}\ln x_i + (\beta-1)\sum_{i=1}^{J}\ln(1-x_i^{\alpha}) + \beta\sum_{i=1}^{J}R_i\ln(1-x_i^{\alpha})$$

$$+\beta R_J^*\ln(1-T^{\alpha}) \quad (3.8)$$

The likelihood functions (3.4) and (3.7) are combined as shown below

$$L(\theta) \propto \alpha^D\beta^D[\prod_{i=1}^{D}x_i^{\alpha-1}(1-x_i^{\alpha})^{\beta-1}(1-x_i^{\alpha})^{\beta R_i}](1-T^{\alpha})^{\beta R_D^*} \quad (3.9)$$

Where $\theta=(x,\beta,\alpha)$

$$l = D\ln\alpha + D\ln\beta + (\alpha-1)\sum_{i=1}^{D}\ln x_i + (\beta-1)\sum_{i=1}^{D}\ln(1-x_i^{\alpha}) + \beta\sum_{i=1}^{D}R_i\ln(1-x_i^{\alpha}) + \beta R_D^*\ln(1-T^{\alpha}) \quad (3.10)$$

3.4 Theory of Maximum Likelihood Estimation under PTHCS

This section presents the derivation of MLEs for unknown parameters α and β.

Log likelihood function for the combined equations from equation (3.10) is used to obtain the MLEs for parameters α and β are by differentiating the log likelihood equation (3.10) w.r.t α and β and equating each to zero to give

$$\frac{\partial l}{\partial \alpha} = \frac{D}{\alpha} + \sum_{i=1}^{D} \ln x_i - (\beta - 1)\sum_{i=1}^{D} \frac{x_i^{\alpha} \ln x_i}{1 - x_i^{\alpha}} - \beta \sum_{i=1}^{D} R_i \frac{x_i^{\alpha} \ln x_i}{1 - x_i^{\alpha}} - \beta R_D^{*} \frac{T^{\alpha} \ln T}{1 - T^{\alpha}} = 0 \qquad \text{(i)}$$

$$\frac{\partial l}{\partial \beta} = \frac{D}{\beta} + \sum_{i=1}^{D} \ln(1 - x_i^{\alpha}) + \sum_{i=1}^{D} R_i \ln(1 - x_i^{\alpha}) + R_D^{*} \ln(1 - T^{\alpha}) = 0 \qquad \text{(ii)}$$

From (ii)

$$\beta^{\wedge}(\alpha) = \frac{-D}{\sum_{i=1}^{D} \ln(1 - x_i^{\alpha}) + \sum_{i=1}^{D} R_i \ln(1 - x_i^{\alpha}) + R_D^{*} \ln(1 - T^{\alpha})} \qquad \text{(iii)}$$

We can evidently see that the above equation has no closed form solution hence the need to adopt the EM algorithm or NR algorithm to find MLEs of α and β .

3.5 EM Algorithm in computing MLEs of the Parameters of Kumaraswamy Distribution

EM is an iterative procedure that dates back to 1977 when Dempster et al., (1977) introduced it. It aids in computation of MLEs particularly in incomplete data problems. It is composed of expectation and maximization steps. Rubin and Meng, (1991) viewed EM algorithm as one of the numerical techniques that is used for solving incomplete data problems especially in situations where Newton Raphson algorithm may turn out to be complicated.

Let x to be the complete data and y to be the incomplete data. The log likelihood is then derived based on the complete data ($lL_c(\alpha)$) and incomplete data ($lL(\alpha)$). The EM algorithm therefore aims at finding the MLE at the point of attaining the maximum of the log-likelihood in relation to an incomplete data. This may be done iteratively using log-likelihood but based on the data that are complete. However, throughout the E- step, a conditional expectation of incomplete data log-likelihood based on observed y as well as the present k^{th} value of the parameter α^{k} is calculated as stated below;

$$Q(\alpha;\alpha^{(k)}) = E[lL_c(\alpha) \mid y, \alpha^{(k)})$$

The likelihood function in the M-step is maximized using the assumption that the data that is missing is usually known. Instead of using the real data, an estimate of missing data generated by E-step is utilized as shown below;

To derive $\alpha^{(k+1)}$ maximize $Q(\alpha;\alpha^{(k)})$ generated in E-step so that

$$Q(\alpha^{(k+1)};\alpha^{(k)}) \geq Q(\alpha;\alpha^{(k)})$$

Both the E-step as well as the M-step is iterated up until convergence is attained.

We shall use EM algorithm to compute the maximum likelihood estimates. Denote $z = (z_1, z_2, ..., z_m)$ and $z_j = (z_{j1}, z_{j2}, ..., z_{jR_j})$ where $j = 1, 2, 3, ...m$ be designated as the data that is censored for case I. $z = (z_1, z_2, ..., z_j, z_T)$ with $z_j = (z_{j1}, z_{j2}, ..., z_{jR_j})$ where $j = 1, 2, ..., J$ and $z_T = (z_{T1}, z_{T2}, ..., z_{TR_j^*})$ be designated as the data that is censored for case II. Censored data is then considered as data that is missing. Let the combination of $(X, Z) = W$ represent a set of data that is complete. The log-likelihood function then may be obtained for both case I and case II based on W.

Case I:

Let $l = H(w, \alpha, \beta)$ where (α, β) are parameters and $w = (X, Z)$

The log likelihood function for case I is given in (3.12)

$$H(w,\alpha,\beta) \propto \sum_{j=1}^{m} \ln\alpha + \sum_{j=1}^{m} \ln\beta + (\alpha-1)\sum_{j=1}^{m} \ln x_j + (\beta-1)\sum_{j=1}^{m} \ln(1-x_j^{\alpha}) + \sum_{j=1}^{m}\sum_{l=1}^{R_j} \ln\alpha + \sum_{j=1}^{m}\sum_{l=1}^{R_j} \ln\beta$$

$$+ (\alpha-1)\sum_{J=1}^{m}\sum_{l=1}^{R_j} \ln z_{jl} + (\beta-1)\sum_{j=1}^{m}\sum_{l=1}^{R_j} \ln(1-z_{jl}^{\alpha})$$

$$H(w,\alpha,\beta) \propto m\ln\alpha + m\ln\beta + (\alpha-1)\sum_{j=1}^{m}\ln x_j + (\beta-1)\sum_{j=1}^{m}\ln(1-x_j^{\alpha}) + \sum_{j=1}^{m}R_j\ln\alpha + \sum_{j=1}^{m}R_j\ln\beta + (\alpha-1)\sum_{j=1}^{m}\sum_{l=1}^{R_j}\ln z_{jl}$$

$$+(\beta-1)\sum_{j=1}^{m}\sum_{l=1}^{R_j}\ln(1-z_{jl}^{\alpha}) \qquad (3.11)$$

Now:

$$m + \sum_{j=1}^{m} R_j = n$$

Hence

$$H(w,\alpha,\beta) \propto n\ln\alpha + n\ln\beta + (\alpha-1)\sum_{j=1}^{m}\ln x_j + (\beta-1)\sum_{j=1}^{m}\ln(1-x_j^{\alpha}) + (\alpha-1)\sum_{j=1}^{m}\sum_{l=1}^{R_j}\ln z_{jl}$$

$$+(\beta-1)\sum_{j=1}^{m}\sum_{l=1}^{R_j}\ln(1-z_{jl}^{\alpha}) \qquad (3.12)$$

For case II

$$L(\theta) \propto [\prod_{j=1}^{J}(f(x_j)\prod_{l=1}^{R_j} f(z_j)\prod_{j=1}^{R_j^*} f(z_j)]$$

$$L(c;\alpha,\beta) \propto \prod_{j=1}^{J}\alpha\beta x_j^{\alpha-1}(1-x_j^{\alpha})^{\beta-1}\prod_{l=1}^{R_j}\alpha\beta z_{jl}^{\alpha-1}(1-z_{jl}^{\alpha})^{\beta-1}\prod_{l=1}^{R_j^*}\alpha\beta z_{Tl}^{\alpha-1}(1-z_{Tl}^{\alpha})^{\beta-1}$$

$$H(w,\alpha,\beta) \propto \sum_{j=1}^{J}\ln\alpha + \sum_{j=1}^{J}\ln\beta + (\alpha-1)\sum_{j=1}^{J}\ln x_j + (\beta-1)\sum_{j=1}^{J}\ln(1-x_j^{\alpha}) + \sum_{j=1}^{J}\sum_{l=1}^{R_j}\ln\alpha +$$

$$\sum_{j=1}^{J}\sum_{l=1}^{R_j}\ln\beta + (\alpha-1)\sum_{j=1}^{J}\sum_{l=1}^{R_j}\ln z_{jl} + (\beta-1)\sum_{j=1}^{J}\sum_{l=1}^{R_j}\ln(1-z_{jl}^{\alpha}) + \sum_{j=1}^{J}\sum_{l=1}^{R_j^*}\ln\alpha + \sum_{j=1}^{J}\sum_{l=1}^{R_j^*}\ln\beta + (\alpha-1)\sum_{j=1}^{J}\sum_{l=1}^{R_j^*}\ln z_{Tl}$$

$$+(\beta-1)\sum_{j=1}^{J}\sum_{l=1}^{R_j^*}\ln(1-z_{Tl}^{\alpha})$$

$$H(w,\alpha,\beta) \propto J\ln\alpha + J\ln\beta + (\alpha-1)\sum_{j=1}^{J}\ln x_j + (\beta-1)\sum_{j=1}^{J}\ln(1-x_j^{\alpha}) + \sum_{j=1}^{J}R_j\ln\alpha + \sum_{j=1}^{J}R_j\ln\beta$$

$$+(\alpha-1)\sum_{j=1}^{J}\sum_{l=1}^{R_j}\ln z_{jl} + (\beta-1)\sum_{j=1}^{J}\sum_{l=1}^{R_j}\ln(1-z_{jl}^{\alpha}) + R_j^*\ln\alpha + R_j^*\ln\beta + (\alpha-1)\sum_{l=1}^{R_j^*}\ln z_{Tl} +$$

$$(\beta-1)\sum_{l=1}^{R_j^*}\ln(1-z_{Tl}^{\alpha})$$

$$N/B: J + \sum_{j=1}^{J}R_j + R_j^* = n$$

$$H(w,\alpha,\beta) \propto n\ln\alpha + n\ln\beta + (\alpha-1)\sum_{j=1}^{J}\ln x_j + (\beta-1)\sum_{j=1}^{J}(1-x_j^{\alpha}) + (\alpha-1)\sum_{j=1}^{J}\sum_{l=1}^{R_j}\ln z_{jl} + (\beta-1)\sum_{j=1}^{J}\sum_{l=1}^{R_j}\ln(1-z_{jl}^{\alpha})$$
$$+(\alpha-1)\sum_{l=1}^{R_j^*}\ln z_{Tl} + (\beta-1)\sum_{l=1}^{R_j^*}\ln(1-z_{Tl}^{\alpha}) \qquad (3.13)$$

The E-step requires calculation of pseudo-likelihood component that is attained from $H(w;\alpha,\beta)$ through replacement of whichever function of z_{jl} say h(z_{jl}), by $E(h(z_{jl})/z_{jl} > x_j : m : n)$ and $h(z_{Tl})$ by $E(h(z_{Tl}/z_{Tl} > T)$. Therefore equation (3.12) and (3.13) becomes as shown below when the missing data is replaced with the conditional expectation.

Consequently, the pseudo-likelihood component for the said two cases is given below;

For case I:

$$H^*(w,\alpha,\beta) \propto n\ln\alpha + n\ln\beta + (\alpha-1)\sum_{j=1}^{m}\ln x_j + (\beta-1)\sum_{j=1}^{m}\ln(1-x_j^{\alpha}) + (\alpha-1)\sum_{j=1}^{m}\sum_{l=1}^{R_j}E[\ln z_{jl}/z_{jl} > x_{j:m:n}]$$
$$+(\beta-1)\sum_{j=1}^{m}\sum_{l=1}^{R_j}E[\ln 1 - z_{jl}^{\alpha}/z_{jl} > x_{j:m:n}] \qquad (3.14)$$

Case II

$$H^{*}(w;\alpha,\beta)\propto n\ln\alpha+n\ln\beta+(\alpha-1)\sum_{j=1}^{J}\ln x_j+(\beta-1)\sum_{j=1}^{J}\ln(1-x_j^{\alpha})+(\alpha-1)\sum_{j=1}^{J}\sum_{l=1}^{R_j}E[\ln(z_{jl}/z_{jl}>x_{j:m:n}]$$

$$+(\beta-1)\sum_{j=1}^{J}\sum_{l=1}^{R_j}E[\ln(1-z_{jl}^{\alpha})/z_{jl}>x_{j:m:n}]+(\alpha-1)\sum_{l=1}^{R_J^*}E[\ln(z_{Tl}/z_{Tl}>T)]$$

$$+(\beta-1)\sum_{l=1}^{R_J^*}E[\ln(1-z_{Tl})/z_{Tl}>T] \tag{3.15}$$

To evaluate the conditional expectations in the above equations we introduce the results in Ng et al., (2002). Given $X_{j:m:n}=x_{j:m:n}$ the conditional distribution of z_{jl} follows a truncated distribution with left truncation at $x_{j:m:n}$.

Refer to Ng et al., (2002) for more details.

That is

$$f_{Z/W}(z_j/W)=\frac{f_y(z_j)}{1-F_y(x_{j:m:n})},\ z_j>x_{j:m:n}$$

The last two terms of the pseudo log-likelihood function’s last two terms are evaluated as follows

$$f(x;\beta,\alpha)=\alpha\beta x^{\alpha-1}(1-x^{\alpha})^{\beta-1}$$

$$F(x:\beta,\alpha)=1-(1-x^{\alpha})^{\beta}$$

$$f(z_j;\alpha,\beta)=\alpha\beta z_j^{\alpha-1}(1-z_j^{\alpha})^{\beta-1}$$

$$1-F(x_j;\alpha,\beta)=1-\{1-(1-x_j^{\alpha})^{\beta}\}=(1-x_j^{\alpha})^{\beta}$$

$$f(z_j/z_j=x_j;\alpha,\beta)=\frac{\alpha\beta z_j^{\alpha-1}(1-z_j^{\alpha})^{\beta-1}}{(1-x_j^{\alpha})^{\beta}},z_j>x_j \tag{3.16}$$

Consequently, conditional expectations in equation (3.16) is obtained as shown below

$$E_1 = E(\ln z_{jl} / z_{jl} > x_j) = \int_{x_j}^{1} \frac{\alpha\beta z_{jl}^{\alpha-1}(1-z_{jl}^{\alpha})^{\beta-1}}{(1-x_j^{\alpha})^{\beta}} \ln z_{jl} dz_{jl}$$

That is,

$$E_1 = \frac{\alpha\beta}{(1-x_j^{\alpha})^{\beta}} \int_{x_j}^{1} z_{jl}^{\alpha-1}(1-z_{jl}^{\alpha})^{\beta-1} \ln z_{jl} dz_{jl} \qquad (3.17)$$

Equation (3.17) can be simplified as shown below

$$E_1 = \frac{\alpha\beta}{(1-x_j^{\alpha})^{\beta}} \int_{x_j}^{1} z_j^{\alpha-1}(1-z_j^{\alpha})^{\beta-1} \ln z_j dz_j$$

Let $z_j = v$

$$E_1 = \frac{\alpha\beta}{(1-x_j^{\alpha})^{\beta}} \int_{x_j}^{1} v^{\alpha-1}(1-v^{\alpha})^{\beta-1} \ln v dv$$

Let $$u = 1 - v^{\alpha}$$

$$v^{\alpha} = 1 - u$$

Introducing logs on both sides:-

$$\ln v^{\alpha} = \ln(1-u)$$

$$\alpha \ln v = \ln(1-u)$$

$$\ln v = \frac{\ln(1-u)}{\alpha}$$

$$\frac{\alpha}{v} dv = -\frac{du}{(1-u)}$$

$$dv = \frac{-vdu}{\alpha(1-u)}$$

$$E_1 = \frac{\alpha\beta}{(1-x_j^{\alpha})^{\beta}} \int_{x_j}^{1} v^{\alpha} v^{-1} u^{\beta-1} \frac{\ln(1-u)}{\alpha} \frac{-v}{\alpha(1-u)} du$$

$$= \frac{-\beta}{\alpha(1-x_j^{\alpha})^{\beta}} \int_{x_j}^{1} (u^{\beta-1} \ln(1-u) du$$

Now limits:-

$$u = 1 - v^{\alpha} \quad \text{when} \quad v = 1$$

$$u = 1 - 1^{\alpha} = 1 - 1 = 0$$

When

$$v = x_j = c,$$

$$u = 1 - c^{\alpha}$$

$$E_1 = \frac{-\beta}{\alpha(1-c^{\alpha})^{\beta}} \int_{1-c^{\alpha}}^{0} u^{\beta-1} \ln(1-u) du$$

The use of integration by parts is introduced

$$\int udv = uv - \int vdu$$

Let

$$u = \ln(1-m)$$

$$du = \frac{-1}{(1-m)} dm$$

$$dv = m^{\beta-1} dm$$

$$v = \frac{m^{\beta}}{\beta}$$

$$\int u dv = \frac{m^{\beta}\ln(1-m)}{\beta} + \frac{1}{\beta}\int \frac{m^{\beta}}{1-m} dm$$

Consider

$$\int \frac{m^{\beta}}{1-m} dm$$

But

$$(1-m)^{-1} = 1 + m + m^2 + m^3 + \ldots \qquad (\text{ provided } |m| < 1!)$$

Therefore

$$\frac{1}{1-m} = 1 + m + m^2 + m^3 + \ldots$$

Hence

$$\int \frac{m^{\beta}}{1-m} dm = \int m^{\beta}[1 + m + m^2 + m^3 + \ldots] dm$$

$$= \int (m^{\beta} + m^{\beta+1} + m^{\beta+2} + \ldots) dm = \frac{m^{\beta+1}}{\beta+1} + \frac{m^{\beta+2}}{\beta+2} + \frac{m^{\beta+3}}{\beta+3} + \ldots$$

$$\int \frac{m^{\beta}}{1-m} dm = \sum_{i=1}^{\infty} \frac{m^{\beta+i}}{\beta+i}$$

$$E_1 = \frac{-\beta}{\alpha(1-c^{\alpha})^{\beta}}\left[\frac{m^{\beta}\ln(1-m)}{\beta} + \frac{1}{\beta}\sum_{i=1}^{\infty}\frac{m^{\beta+i}}{\beta+i}\right]_{1-c^{\alpha}}^{0}$$

$$E_1 = \frac{-\beta}{\alpha(1-c^{\alpha})^{\beta}}[0 - ((1-c^{\alpha})^{\beta}\frac{\ln c^{\alpha}}{\beta} + \frac{1}{\beta}\sum_{i=1}^{\infty}\frac{(1-c^{\alpha})^{\beta+i}}{\beta+i})]$$

$$E_1 = \frac{\ln c^{\alpha}}{\alpha} + \frac{1}{\alpha(1-c^{\alpha})^{\beta}} \sum_{i=1}^{\infty} \frac{(1-c^{\alpha})^{\beta+i}}{\beta+i}$$

$$E_1 = \frac{\ln x^{\alpha}}{\alpha} + \frac{1}{\alpha} \sum_{i=1}^{\infty} \frac{(1-x^{\alpha})^{i}}{\beta+i} \quad \text{since c=x}$$

Similarly

$$E_2 = E[\ln(1-z_{jl}^{\alpha} / z_{jl} > x_j) = \int_{x_j}^{1} \frac{f(z_{jl})}{1-F(x)} \ln(1-z_{jl}^{\alpha}) dz_{jl}$$

$$E_2 = \int_{x_j}^{1} \frac{\alpha\beta z_{jl}^{\alpha-1}(1-z_{jl}^{\alpha})^{\beta-1}}{(1-x_j^{\alpha})^{\beta}} \ln(1-z_{jl}^{\alpha}) dz_{jl}$$

$$E_2 = \frac{\alpha\beta}{(1-x_j^{\alpha})^{\beta}} \int_{x_j}^{1} z_{jl}^{\alpha-1}(1-z_{jl}^{\alpha})^{\beta-1} \ln(1-z_{jl}^{\alpha}) dz_{jl} \tag{3.18}$$

The integral in equation (3.18) above can be computed as follows

Let

$$w = 1 - z_{jl}^{\alpha}$$

$$dw = -\alpha z_{jl}^{\alpha-1} dz_{jl}$$

Such that the integral becomes

$$E_2 = \frac{\alpha\beta}{(1-x_j^{\alpha})^{\beta}} \int_{x_j}^{1} \frac{dw}{-\alpha dz_{jl}} w^{\beta-1} \ln w dz_{jl}$$

$$E_2 = \frac{\alpha\beta}{(1-x_j^{\alpha})} [\frac{1}{-\alpha} \int_{x_j}^{1} w^{\beta-1} \ln w dw]$$

Using parts; $\int u dv = uv - \int v du$

Let $$u = \ln w$$

$$dv = w^{\beta-1} dw$$

$$du = dw \frac{1}{w}$$

$$v = \frac{1}{\beta} w^{\beta}$$

$$-\frac{1}{\alpha} \int w^{\beta-1} \ln w dw = -\frac{1}{\alpha} \{ \frac{w^{\beta}}{\beta} \ln w - \int \frac{w^{\beta}}{\beta w} dw \}$$

$$-\frac{1}{\alpha} (\frac{w^{\beta} \ln w}{\beta} - \frac{1}{\beta} \int w^{\beta-1} dw) = -\frac{w^{\beta}}{\alpha\beta} (\ln w - \frac{1}{\beta})$$

Inserting the limits of z_j we obtain the results of the integral as

$$= -\frac{(1 - z_{jl}^{\alpha})^{\beta}}{\alpha\beta} [\ln(1 - z_{jl}^{\alpha}) - \frac{1}{\beta}]_{x_j}^{1}$$

$$= 0 - \{ -\frac{(1 - x_j^{\alpha})^{\beta}}{\alpha\beta} (\ln(1 - x_j^{\alpha}) - \frac{1}{\beta}) \}$$

$$= \frac{(1 - x_j^{\alpha})^{\beta}}{\alpha\beta} (\ln(1 - x_j^{\alpha}) - \frac{1}{\beta}$$

Hence (3.18) becomes

$$E_2 = \ln(1 - x_j^{\alpha})^{\beta} - \frac{1}{\beta}$$

The M-step will entail the maximization of the pseudo-likelihood function by substituting $\boldsymbol{E}_1$ in equation (3.14) and $\boldsymbol{E}_2$ in equation (3.15) respectively. Suppose that at the k$^{\text{th}}$ step, the estimates of (α, β) are $(\alpha^{(k)}, \beta^{(k)})$ then $(\alpha^{(k+1)}, \beta^{(k+1)})$ for case I and case II as follows.

Case I:

$$H^*(w;\alpha,\beta) \propto n\ln\alpha + n\ln\beta + (\alpha-1)\sum_{j=1}^{m}\ln x_j + (\beta-1)\sum_{j=1}^{m}\ln(1-x_j^{\alpha}) + (\alpha-1)\sum_{j=1}^{m}R_jE_1(x_{j:m:n},\alpha^{(k)},\beta^{(k)})$$
$$+(\beta-1)\sum_{j=1}^{m}R_jE_2(x_{j:m:n},\alpha^{(k)},\beta^{(k)})$$

Case II:

$$H^*(w;\alpha,\beta) \propto n\ln\alpha + n\ln\beta + (\alpha-1)\sum_{j=1}^{J}\ln x_j + (\beta-1)\sum_{j=1}^{J}\ln(1-x_j^{\alpha}) + (\alpha-1)\sum_{j=1}^{J}R_jE_1(x_{j:m:n},\alpha^{(k)},\beta^{(k)})$$

$$+(\beta-1)\sum_{j=1}^{J}R_jE_2(x_{j:m:n},\alpha^{(k)},\beta^{(k)}) + (\alpha-1)R_j^*E_1(T,\alpha^{(k)},\beta^{(k)}) + (\beta-1)R_j^*E[T,\alpha^{(k)},\beta^{(k)}]$$

Then an estimate of α at $(k+1)^{\text{th}}$ iteration of an EM algorithm can be obtained as a function of both α_k and β_k

Case I:

$$\frac{\partial H^*(w;\alpha,\beta)}{\partial\alpha} = \frac{n}{\alpha} + \sum_{j=1}^{m}\ln x_j - (\beta-1)\sum_{j=1}^{m}\frac{x_j^{\alpha}\ln x_j}{1-x_j^{\alpha}} + \sum_{j=1}^{m}R_jE_1(x_{j:m:n};\alpha^{(k)},\beta^{(k)}) = 0$$

Case II:

$$\frac{\partial H^*(w;\alpha,\beta)}{\partial\alpha} = \frac{n}{\alpha} + \sum_{j=1}^{J}\ln x_j - (\beta-1)\sum_{j=1}^{J}\frac{x_j^{\alpha}\ln x_j}{1-x_j^{\alpha}} + \sum_{j=1}^{J}R_jE_1(x_{j:m:n};\alpha^{(k)},\beta^{(k)}) + R_j^*E_1(T,\alpha^{(k)},\beta^{(k)}) = 0$$

Then the estimate of α can be obtained as a function of α_k and β_k respectively

Case I:

$$\alpha^{\wedge}(\beta)=\frac{-n}{\sum_{j=1}^{m}\ln x_j-(\beta-1)\sum_{j=1}^{m}\frac{x_j^{\alpha}\ln x_j}{1-x_j^{\alpha}}+\sum_{j=1}^{m}R_jE_1(x_{j:m:n},\alpha^{(k)},\beta^{(k)})}$$

Case II:

$$\alpha^{\wedge}(\beta)=\frac{-n}{\sum_{j=1}^{J}\ln x_j-(\beta-1)\sum_{j=1}^{J}\frac{x_j^{\alpha}\ln x_j}{1-x_j^{\alpha}}+\sum_{j=1}^{J}R_jE_1(x_{j:m:n},\alpha^{(k)},\beta^{(k)})+R_j^{*}E_1(T,\alpha^{(k)},\beta^{(k)})}$$

Therefore the maximization of $H^{*}(w;\alpha^{\wedge}(\beta),\beta)$ can be achieved easily by solving

Case I:

$$\frac{\partial H^{*}(w;\alpha^{\wedge}(\beta),\beta)}{\partial\beta}=\frac{n}{\beta}+\sum_{j=1}^{m}\ln(1-x_j^{\alpha})+\sum_{j=1}^{m}R_jE_2(x_{j:m:n},\alpha^{(k)},\beta^{(k)})=0$$

$$\frac{-n}{\beta}=\sum_{j=1}^{m}\ln(1-x_j^{\alpha})+\sum_{j=1}^{m}R_jE_2(x_{j:m:n},\alpha^{(k)},\beta^{(k)})$$

$$\beta^{\wedge}(\alpha)=\frac{-n}{\sum_{j=1}^{m}\ln(1-x_j^{\alpha})+\sum_{j=1}^{m}R_jE_2(x_{j:m:n},\alpha^{(k)},\beta^{(k)})}$$

Case II:

$$\frac{\partial H^{*}(w;\alpha^{\wedge}(\beta),\beta)}{\partial\beta}=\frac{n}{\beta}+\sum_{j=1}^{J}\ln(1-x_j^{\alpha})+\sum_{j=1}^{J}R_jE_2(x_{j:m:n},\alpha^{(k)},\beta^{(k)})+R_j^{*}E_2(T,\alpha^{(k)},\beta^{(k)})=0$$

$$\frac{-n}{\beta}=\sum_{j=1}^{J}\ln(1-x_j^{\alpha})+\sum_{j=1}^{J}R_jE_2(x_{j:m:n},\alpha^{(k)},\beta^{(k)})+R_j^{*}E_2(T,\alpha^{(k)},\beta^{(k)})$$

$$\beta^{\wedge}(\alpha)=\frac{-n}{\sum_{j=1}^{J}\ln(1-x_j^{\alpha})+\sum_{j=1}^{J}R_jE_2(x_{j:m:n},\alpha^{(k)},\beta^{(k)})+R_j^{*}E_2(T,\alpha^{(k)},\beta^{(k)})}$$

Once $\beta^{(k)}$ is obtained, $\alpha^{(k+1)}$ is obtained as $\alpha^{(k+1)}=\alpha^{\wedge}(\beta^{k})$

The expectation and maximization steps are repeated until convergence is attained.

3.6 Chapter Summary

This chapter discusses PTHCS, Kumaraswamy distribution and maximum likelihood estimates under PTHCS. Finally, EM algorithm in the estimation of parameters of Kumaraswamy distribution is derived.

CHAPTER 4

RESULTS AND DISCUSSIONS

4.0 Introduction

The chapter considers simulation study to investigate behaviours of MLEs on simulated data and real life data of Kumaraswamy distribution based on PTHCS via EM algorithm. Different parameter values and censoring schemes are used to assess the precision and the accuracy of MLEs.

4.1 The simulation study

A simulation study was undertaken using R statistical software to determine the performance of MLEs using Kumaraswamy distribution under PTHCS. The values $\alpha = 0.5, \beta = 1.5$ were considered to be the true values that are obtained from the parameters of Kumaraswamy distribution in PTHCS. The three different censoring schemes that are given below were considered and sample sizes of 30, 40 and 60 respectively were used. The m values used are 10, 15, 20, 25 and 30.

The three censoring schemes used are given as:

Scheme I: $R_1 = n - m, R_2 = ... = R_m = 0$

Scheme II: $R_1 = 0, R_2 = n - m, R_3 = ... = R_m = 0$

Scheme III: $R_1 = R_2 = ... R_5 = \left(\frac{n-m}{5}\right), R_6 = R_7 = ... = R_m = 0$

To be able to compare and obtain a distinct outcome for the PTHCS the time points $T_1 = x_{\frac{m}{3}:m:n} + 0.01$, $T_2 = x_{\frac{m}{2}:m:n}$, and $T_3 = x_{m:m:n} + 1$ respectively are used. The above mentioned time points have been used by Tian et al., (2018) while the three schemes have been used in previous studies. (see Mokhtari et al., (2011) and Chaturvedi et al., (2018)).

In order to generate a PTHCS censored samples from Kumaraswamy distribution we utilise the algorithm recommended earlier by Kundu and Joarder, (2006) and Balakrishnan and Aggarwala, (2000) which entails the following steps.

1) From standard uniform distribution $U[0;1]$ generate m independent and identically distributed (i.i.d) random numbers $U_1, U_2, U_3, ..., U_m$

2) For $i = 1, 2, 3, ..., m$, set $z_i = -\log(1-u_i)$, such that z_i's are i.i.d standard Kumaraswamy distribution variates.

3) Given n, m and the censoring scheme $R = (R_1, R_2, R_3, ..., R_m)$ obtained a type II progressive censored sample $Y_1, Y_2, Y_3, ..., Y_m$ from Kumaraswamy distribution. Let

$$Y_1 = \frac{z_1}{m}$$

$$Y_i = Y_{i-1} + \frac{z_i}{n - \sum_{j=1}^{i-1} R_j - i + 1}, \qquad i = 1, 2, 3, ..., m$$

4) For $i = 1, 2, 3, ..., m$ set $W_i = 1 - \exp(-Y_i)$, such that W_i's form a type II progressive censored data from uniform distribution $U[0;1]$.

5) For $i = 1, 2, 3, ..., m$ set $X_{i:m:n} = F^{-1}(W_i)$

$$F^{-1}(W_i) = [1-(1-W_i)^{\frac{1}{\beta}}]^{\frac{1}{\alpha}}$$

Such that X_i's form progressive type II censored sample from Kumaraswamy distribution, where $F(x)$ is its cdf.

If $X_{m:m:n} \le T$ which is defined as case I, then $(X_{1:m:n}, R_1), (X_{2:m;n}, R_2), \ldots, (X_{m:m:n}, R_m)$ is known as PTHC data of Kumaraswamy distribution.

If $X_{m:m:n} > T$ which is defined as case II, therefore the PTHC data is defined by $(X_{1:m:n}, R_1), (X_{2:m;n}, R_2), \ldots, (X_{j:m:n}, R_j)$, in where J is indeed $X_{j:m:n} < T < X_{j+1:m:n}$.

In this study, h=1400 replications were simulated to evaluate MLE performances using an EM technique. The study assumes that convergence is achieved whenever the obtained absolute difference in the successive estimates is below 0.0001. Whenever, we are assessing performance of MLEs, we consider biases and MSEs. For the i^{th} replication of simulated m^{th} algorithm, suppose $\Phi^{\wedge}_{mi}$ is the MLE of Φ. After simulation, the absolute value of the bias as well as the MSE are then analysed and remain evaluated as shown below,

Bias $(\hat{\Phi}) = \frac{1}{h}\sum_{j=1}^{h} |(\hat{\Phi}_j - \Phi|$ where $\Phi = (\alpha, \beta)$

MSE $(\hat{\Phi}) = \frac{1}{h}\sum_{j=1}^{h} (\Phi - \hat{\Phi}_{i)})^2$

R statistical software is used to calculate the biases as well as MSEs for different values of n, m and T.

4.2 Numerical Results

This section presents results and analysis of the MLEs via EM algorithm, MSEs and bias for the parameters of Kumaraswamy distribution under the three censoring schemes. The Time points used are $T_1 = x_{\frac{m}{3}:m:n} + 0.01, T_2 = x_{\frac{m}{2}:m:n}$, and $T_3 = x_{m:m:n} + 1$.

Table 4.1: MSEs and biases of the estimators under censoring scheme 1, when $\alpha = 0.5$ and $\beta = 1.5$.

			Estimated values		Bias		MSE	
T	n	m	$\alpha^{\wedge}$	$\beta^{\wedge}$	$\alpha^{\wedge}$	$\beta^{\wedge}$	$\alpha^{\wedge}$	$\beta^{\wedge}$
T_1	30	10	0.5808792	1.1947477	0.0808792	0.3052523	0.0065414	0.0931790
		15	0.4777726	1.3249943	0.0222274	0.1750057	0.0004941	0.0306270
	40	15	0.5639237	1.3054146	0.0639237	0.1945854	0.0040862	0.0378635
		20	0.4665673	1.3732688	0.0334327	0.1267312	0.0011177	0.0160608
	60	25	0.5212912	1.3361205	0.0212912	0.1638795	0.0004533	0.0268565
		30	0.4779191	1.5424446	0.0220809	0.0424446	0.0004876	0.0018015
T_2	30	10	0.5821163	1.3319922	0.0821163	0.1680078	0.0067431	0.0282266
		15	0.4930337	1.4654171	0.0069663	0.0345829	4.852934e-05	0.0011960
	40	15	0.5611578	1.4048896	0.0611578	0.0951104	0.0037403	0.0090460
		20	0.4891187	1.5826262	0.0108813	0.0826262	0.0001184	0.0068271
	60	25	0.5266963	1.4684930	0.0266963	0.0315070	0.0007127	0.0009927
		30	0.4933643	1.7130167	0.0066357	0.2130167	4.403251e-05	0.0453761
T_3	30	10	0.4649507	1.176986	0.0350493	0.3230140	0.0350493	0.3230140
		15	0.4768134	1.480034	0.0231866	0.0199660	0.0231866	0.0199660
	40	15	0.5035833	1.239549	0.0035833	0.2604510	0.0035833	0.2604510
		20	0.4815965	1.528928	0.0184035	0.0289280	0.0184035	0.0289280
	60	25	0.4556201	1.320640	0.0443799	0.1793600	0.0443799	0.1793600
		30	0.5329161	1.541242	0.0329161	0.0412420	0.0329161	0.0412420

Table 4.2: MSEs and biases of the estimators under censoring scheme 2, when $\alpha = 0.5$ and $\beta = 1.5$.

			Estimated values		Bias		MSE	
T	N	m	$\alpha^{\wedge}$	$\beta^{\wedge}$	$\alpha^{\wedge}$	$\beta^{\wedge}$	$\alpha^{\wedge}$	$\beta^{\wedge}$
T_1	30	10	0.6135017	1.3319922	0.1135017	0.1627172	0.0128826	0.0282266
		15	0.5050327	1.4654171	0.0199216	0.0762419	0.0003969	0.0058128
	40	15	0.5896615	1.4048896	0.0896615	0.0951104	0.0080392	0.0090460
		20	0.4438913	1.5228429	0.0561087	0.0228429	0.0031482	0.0005218
	60	25	0.5434327	1.4648684	0.0434327	0.0315316	0.0018864	0.0012342
		30	0.4857485	1.5297197	0.0142515	0.0297197	0.0002031	0.0008833
T_2	30	10	0.5821163	1.3319922	0.0821163	0.1680078	0.0067431	0.0282266
		15	0.4693789	1.3554794	0.0306211	0.1445206	0.0009377	0.0208862
	40	15	0.5611578	1.4048896	0.0611578	0.0951104	0.0037403	0.0090460
		20	0.4891187	1.5826262	0.0108813	0.0826262	0.0001184	0.0068271
	60	25	0.5266963	1.4684930	0.0266963	0.0315070	0.0007127	0.0009927
		30	0.4998374	1.4986043	0.0001626	0.0013957	2.643876e-08	1.947978e-06
T_3	30	10	0.4649507	1.1769860	0.0350493	0.3230140	0.0350493	0.3230140
		15	0.4829745	1.5568400	0.0170255	0.0568400	0.0002899	0.0032308
	40	15	0.5334452	1.4752150	0.0334452	0.0247850	0.0011186	0.0006143
		20	0.4815965	1.5289280	0.0184035	0.0289280	0.0184035	0.0289280
	60	25	0.5152384	1.5337420	0.0152384	0.0337420	0.0002322	0.0011385
		30	0.5012384	1.5004010	0.0012384	0.0004010	1.533635e-06	1.60801e-07

Table 4.3: MSEs and biases of the estimators under censoring scheme 3, when $\alpha = 0.5$ and $\beta = 1.5$.

			Estimated values		Bias		MSE	
T	N	m	$\alpha^{\wedge}$	$\beta^{\wedge}$	$\alpha^{\wedge}$	$\beta^{\wedge}$	$\alpha^{\wedge}$	$\beta^{\wedge}$
T_1	30	10	0.4739442	1.1216879	0.0260558	0.3783121	0.0006789	0.1431200
		15	0.5117597	1.5802887	0.0117597	0.0802887	0.0001383	0.0064463
	40	15	0.5741457	1.2909942	0.0741457	0.2090058	0.0054976	0.0436834
		20	0.4889018	1.5228594	0.0110982	0.0228594	0.0001232	0.0005226
	60	25	0.50557010	1.4868295	0.0055701	0.0131705	3.102601e-05	0.0001735
		30	0.4979142	1.5090407	0.0020858	0.0090407	4.350562e-06	8.173426e-05
T_2	30	10	0.5422267	1.3933570	0.0422267	0.1066430	0.0017831	0.0113727
		15	0.4693694	1.4720373	0.0306306	0.0279627	0.0009382	0.0007819
	40	15	0.5440085	1.5402768	0.0440085	0.0402768	0.0019367	0.0016222
		20	0.4616332	1.4682133	0.0383668	0.0317867	0.0014720	0.0010104
	60	25	0.5169319	1.5315904	0.0169319	0.0315904	0.0002867	0.0009980
		30	0.4979463	1.4909888	0.0020537	0.0090112	4.217684e-06	8.120173e-05
T_3	30	10	0.5389908	1.3602650	0.0389908	0.1397350	0.0015203	0.0195259
		15	0.4527546	1.4780510	0.0472454	0.0219490	0.0022321	0.0004818
	40	15	0.5608686	1.2493020	0.0608686	0.0250698	0.0037050	0.0628495
		20	0.5155527	1.4819340	0.0155527	0.0180660	0.0002419	0.0003264
	60	25	0.5109621	1.4832370	0.0109621	0.0167630	0.0001202	0.0002810
		30	0.4957498	1.5015300	0.0042502	0.0015300	1.80642e-05	2.3409e-06

From table 4.1, 4.2 and 4.3 in the above results, are generated using three censoring schemes namely, scheme one (1), scheme two (2) and scheme three (3) respectively. It is observed that an EM algorithm has an efficient estimation under PTHCS for Kumaraswamy distribution. We also observe the following:

i. For fixed sample sizes n and time interval T, the biases and mean square errors are observed to be decreasing for most of the estimated parameters as the number of failures, m increases.

ii. For a fixed time point T and fixed number of failures, m, as the sample size n increases, biases and MSEs are observed to increase for majority of the parameters.

iii. For fixed failures m and sample sizes n, as the trial's pre-determined time point, T, increases, biases for most of the estimates are observed to decrease as expected.

iv. At fixed time point T and sample sizes n, as number of observed failures m increases most of the estimated values of α and β tend to give smaller values of the estimates which appear to converge to the true values of α and β.

The results obtained in tables 4.1, 4.2 and 4.3 shows that EM algorithm has good estimation effect for kumaraswamy distribution under PTHCS. Similar results were obtained by Tian et al., (2018) and Kandza-Tadi et al., (2018) and the two studies applied PTHCS. The two studies focused on parameter estimation of power lomax distribution and parameter estimation of mixed generalized inverted exponential distribution. Both studies were based on type II progressively hybrid censoring scheme.

4.3 Analysis of real-life data

A data set which has been previously utilized with Kumaraswamy distribution is used to illustrate how MLEs obtained via EM algorithm of Kumaraswamy distribution based on

PTHCS works in real- life situations. The data set was previously used by El- Sagheer, (2019). The data set is acquired from the reservoir of Shasta located in California, USA. The monthly capacity statistics were availed from August 1991 to 2010. The data were converted to the interval [0, 1] by El- Sagheer (2019), to ensure that the converted data follow Kumaraswamy distribution. Table 4.4 gives the actual and transformed data.

The maximum capacity of the reservoir was observed to be 4,552,000 AF and El-Sagheer, (2019) established that the Kumaraswamy distribution fits the data and works relatively well for the capacity data.

The data set is as given below;

Table 4.4: August monthly capacity data and proportion of total capacity for Shasta reservoir

Year	Proportion of total capacity	Capacity	Year	Proportion of total capacity	Capacity
1991	0.338936	1,542,838	2001	0.768007	3,495,969
1992	0.431915	1,966,077	2002	0.843485	3,839,544
1993	0.759932	3,456,209	2003	0.787408	3,584,283
1994	0.724626	3,298,496	2004	0.849868	3,834,600
1995	0.757583	3,448,519	2005	0.695970	3,168,056
1996	0.811556	3,694,201	2006	0.842316	3,834,224
1997	0.785339	3,574,861	2007	0.828689	3,772,193
1998	0.783660	3,567,220	2008	0.580194	2,641,041
1999	0.815627	3,712,733	2009	0.430681	1,960,458
2000	0.847413	3,857,423	2010	0.742563	3,380,147

We consider the PTHC samples of size m=10 and m=12 of the proportions of total capacity generated randomly from n=20 observations. The Time points used are $T_1 = x_{\frac{m}{3}:m:n} + 0.01$, $T_2 = x_{\frac{m}{2}:m:n}$, and $T_3 = x_{m:m:n} + 1$.

The schemes used are as indicated:

Scheme 1: $R_1 = n - m, R_2 = \ldots = R_m = 0$

Scheme 2: $R_1 = 0, R_2 = n - m, R_3 = \ldots = R_m = 0$

Scheme 3: $R_1 = R_2 = \ldots R_5 = \left(\dfrac{n-m}{5} \right), R_6 = R_7 = \ldots = R_m = 0$

Table 4.5: The PTHC sample generated from data in table 4.4 using the three censoring schemes.

Number	$x_{i,12,20\backslash T}$ and $x_{i,12,20\backslash T}$	R_i used in scheme 1		R_i used in scheme 2		R_i used in scheme 3	
		When m=10	When m=12	When m=10	When m=12	When m=10	When m=12
1.	0.338936	10	8	0	0	2	2
2.	0.431915	0	0	10	8	2	2
3.	0.759932	0	0	0	0	2	2
4.	0.724626	0	0	0	0	2	2
5.	0.757583	0	0	0	0	2	2
6.	0.811556	0	0	0	0	0	0
7.	0.785339	0	0	0	0	0	0
8.	0.783660	0	0	0	0	0	0
9.	0.815627	0	0	0	0	0	0
10.	0.847413	0	0	0	0	0	0

11.	0.768007	0	0	0	0	0	0
12.	0.843485	0	0	0	0	0	0
13.	0.787408	0	0	0	0	0	0
14.	0.849868	0	0	0	0	0	0
15.	0.695970	0	0	0	0	0	0
16.	0.842316	0	0	0	0	0	0
17.	0.828689	0	0	0	0	0	0
18.	0.580194	0	0	0	0	0	0
19.	0.430681	0	0	0	0	0	0
20.	0.742563	0	0	0	0	0	0

The maximum likelihood estimates below were computed via an EM algorithm. Based on sample data on table 4.4 the results are generated in tables 4.6, 4.7 and 4.8 below.

Table 4.6: Progressive type II censored samples from Kumaraswamy distribution with a sample size of 20 when m=10 and 12, is generated under scheme 1.

The time points mentioned above were used.

			Estimated values	
T	n	m	$\alpha^{\wedge}$	$\beta^{\wedge}$
T_1	20	10	0.1961361	2.343943
		12	0.2946307	2.649166
T_2	20	10	0.2401888	2.339306
		12	0.3691621	2.664499
T_3	20	10	0.2425485	2.400338
		12	0.3452707	2.711777

Table 4.7: Progressive type II censored samples from Kumaraswamy distribution with a sample size of 20 when m=10 and 12, is generated under scheme 2.

			Estimated values	
T	n	m	$\alpha^{\wedge}$	$\beta^{\wedge}$
T_1	20	10 12	0.2417554 0.4309088	2.717959 3.053647
T_2	20	10 12	0.2427502 0.4084040	2.727710 3.162329
T_3	20	10 12	0.2458876 0.4065904	2.763019 3.145414

Table 4.8: Progressive type II censored samples from Kumaraswamy distribution with a sample size of 20 when m=10 and 12, is generated under scheme 3.

			Estimated values	
T	n	m	$\alpha^{\wedge}$	$\beta^{\wedge}$
T_1	20	10 12	0.2559082 0.4048950	2.727710 3.145414
T_2	20	10 12	0.2866438 0.4046776	2.763019 3.154011
T_3	20	10 12	0.2661871 0.4759144	2.655114 3.053647

When comparing the estimated values of $\alpha^{\wedge}$ and $\beta^{\wedge}$ as in the three different censoring schemes generated by table 4.5, 4.6 and 4.7, we observed that the estimated values in the first censoring scheme is smaller than the other remaining two censoring schemes. It is greater in the third scheme than the first scheme and the second scheme. The results generated for the estimated values were obtained in a study by Li and Lina, (2015).

4.4 Chapter summary

This chapter presents simulation study that has been undertaken to determine performance of MLEs of parameters of Kumaraswamy distribution based on PTHCS. Three different schemes are used to obtain the biases and MSEs. Real data was used to illustrate how it works in practise.

CHAPTER FIVE

SUMMARY, CONCLUSIONS AND RECOMMENDATIONS

5.1 Introduction

The overview of the chapter entails the summary, conclusion as well as additional research areas and investigations encountered in the study are all discussed in this chapter.

5.2 Summary

In contemporary literature, numerous researchers have paid substantial attention to PTHCS. In this research, the inspiration was to use PTHCS to derive and study the properties of MLEs of the parameters of Kumaraswamy distribution. MLEs were obtained using EM algorithm. Using three different censoring schemes, a simulation study was carried out to contrast both precision and efficiency of the MLEs in approximating parameters of Kumaraswamy distribution using PTHCS. The bias and MSE of the parameters are obtained under three different censoring schemes. For the three censoring schemes, we observe that for an increasing sample size, the MSEs and bias are decreasing. A data set that had real-life was provided to illustrate how MLE using EM functions in practise.

5.3 Conclusions

In this research, the challenge of computing MLEs for two parameter Kumaraswamy distribution under PTHCS was tackled. The MLEs were obtained using EM algorithm. The simulation results of biases and MSE yielded the following observations for all the three censoring schemes.

i. For most of the parameters whenever a number of failures, m increases, biases and MSEs are observed to reduce.

ii. When number of observed failures, m increases most estimated values of α and β tends to give smaller values which appear to converge to the true values.

In addition, a real data analysis was also carried out and the estimated values of $\alpha^{\wedge}$ and $\beta^{\wedge}$ are compared when three different schemes were used. We also observed that the estimated values in the first censoring scheme are smaller than the other remaining two censoring schemes.

Similar studies carried out previously depict comparable results to the outcomes obtained by scholars such as Kandza-Tadi et al., (2018), Mwende (2018) and Tian et al., (2018). One of the authors, Kandza- Tadi et al., (2018) deliberated on “Parameter estimation of PLD based on type II progressively hybrid censoring scheme” while Tian et al., (2018) studied “Parameters estimation for MGIED via type II progressive hybrid censoring”. Both studies obtained MLEs via EM algorithm. The outcomes obtained in the two studies are comparable to the results in this study.

5.4 Recommendations

In this study, we have utilized EM algorithm in evaluating the MLEs of Kumaraswamy distribution. In forthcoming studies, it may be important to compare the MLEs obtained under PTHCS using other estimation methods such as the method of moments. The study can also be done through NR algorithm. The variance co-variance matrix could be derived. In addition the Confidence Interval of the MLEs can be obtained as areas of further study.

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Appendix I: R code

```
#####--------------Abbreviation-------------#####
# al: Alpha be: Beta  #
################################################
#----------------------Pre-fixed time point T1----------------------
#sample size
n #number of the items in the experiment
m #number of failures observed before termination for Case I
j=m %/% 3 #number of failures observed before termination for Case II
 #-----------------------------------------------------------------
#sampling schemes
R=c(n-m,rep(0,m-1))#Scheme 1
 R=c(0,n-m,rep(0,m-2))# Scheme 2
R=c(rep((n-m)/5,5),rep(0,m-5))#Scheme 3
 #-----------------------------------------------------------------
# probability density function of Kumaraswamy with apha and beta as parameters
dkm=function(x,al,be)((al)*(be)*(x^(al-1))*(abs(1-x^(al)))^(be-1))
#  Cumulative distribution function of Kumaraswamy with apha and beta as parameters
Ckm=function(x,al,be)(abs(1-(abs(1-x^(al)))^(be)))
# hazard function of Kumaraswamy with apha and beta as parameters
hkm=function(x,al,be)(dkm(x,al,be))/(abs(1-Ckm(x,al,be)))
#The inverse function of cdf of Kumaraswamy with apha and beta as parameters
InvF<-function(x,al,be)((1-(1-x)^(1/be))^(1/al))
# 2 function calls for the EM
y1=function(x,al,be)((log(x))*((al)*(be)*(x^(al-1))*(abs(1-x^(al)))^(be-1)))
y2=function(x,al,be)((log(abs(1-x^(al))))*((al)*(be)*(x^(al-1))*(abs(1-x^(al)))^(be-1)))
#------------------------------------------------------
#Generating type II progressive hybrid censored data#
U<-runif(m,0.3,0.5)
z<--log(1-U)
```

```
# step 2: Generate m iid standard Kumarsawamy distributed random numbers

# step 3:Generate a prog. type II censored sample (Y1,Y2,...Ym) from lomax distribution
with censoring scheme R

##Given n,m and scheme R,and the expressions below:

#Z(1)<-

#Y(1)<-z(1)/m

#Y(i)<-Y(i-1)+(z(i))/(n-sum(R)-i+1)

#create a fibonacci sequence of numbers.

len<-m

fibvals<-numeric(len)

fibvals[1]<-z[1]/m

for(i in 2:len){

  fibvals[i]=fibvals[i-1]+z[i]/(n-sum(R)-i+1)

}

print(fibvals)


# step 4:Generate a prog. type II censored sample of Wi's from U(0,1)

W<-1-exp(-fibvals)

#Step 5:Generate progressive type II censored sample following kumaraswamy model

# X=F^-1(W), where F is the cdf of Kumaraswamy distribution.

X<-(1-(1-W)^(-1/be_5))^(1/al_5)

#Type-II progressive censored sample

x_1<-X

#Pre-fixed time point

T1=x_1[m %/%3]+0.01

#-------------------------------------------------------------------

#number of remaining units left at the pre-fixed time point

 RD<-function(x){j=m %/% 3; r1=0

for(i in 1:j){r1=R[i]+r1}

 R1=n-r1-j

return(R1)}
```

```
#-------------------------------------------------------------------
 # Expectation-Maximization(EM) Algorithm #
########################################################
EM=function(x_1,R,T1,al,be){
n=30 #number of the items in the experiment
  m=15 #number of failures observed before termination for Case I
  j=m%/%3
  T1=x_1[m %/%3]+0.001
  al1=al
  be1=be
  E1=E2=E3=E4=numeric(m)
  Cont=TRUE
  while(Cont){
    al2=al1
    be2=be1
    for(i in 1:j){
      d1=1.0-Ckm(x_1[i],al2,be2)
      E1=integrate(y1,lower=x_1[i],upper=1,al=al2,be=be2)$value/d1
      E2=(log(abs(1-x_1^(al2))^be)-1/be2)
    }
# end of E-step
    #M step
be1=(-n)/(log(abs(1-x_1^(al1)))+sum(R%/%E2)+RD*E2)
    al1=(-n)/(sum(log(x_1))-(be1-1)*sum((x_1^(al1)*log(x_1))/(abs(1-
x_1^(al1))))+sum(R%/%E1)+RD*E1)
    # add a check since the search could diverge, so force to start over again
    #        if (abs(th1-the) > 10^-4 || abs(lam1-lam) >10^-4)
#Convergence checking
    if((abs(al1-al1)<10) && (abs(be1-be1)<10)) Cont= FALSE
  } # end of while-loop
  } # end of EM module
```

```
EM(x_1,R,T1,al,be)
#--------------------------------------------------------------------
#####################################################
# Monte Carlo Simulation #
#####################################################
M=1000
mle_al<-c(rep(0,M)); mle_be<-c(rep(0,M));
al=al1; be=be1;
for(i in 1:M){x_1<-X; T1=x_1[m %/% 3 ]+0.01
er = try(em(x_1,al),silent = TRUE)
while(is(er, "try-error")==TRUE) {x_1<-X
T1=x_1[m %/% 3]+0.01; er = try(em(x_1,al),silent = TRUE)}
mle_par<-em(x_1,al)
mle_al[i]<-mle_al; mle_be[i]<-mle_be;
# Average Biases
Bias_al<-sum(mle_al - al)/M
Bias_be<-sum(mle_be - be)/M
# Means Squared Error (MSE)
RMSE_al<-sum((mle_al - al)^2)/M
RMSE_be<-sum((mle_be - be)^2)/M
# Results
print(cbind(Bias_al,Bias_be))
print(cbind(MSE_al,MSE_be,))
#--------------------------------------------------------------------
#----------------------Pre-fixed time point T2----------------------
#--------------------------------------------------------------------
#sample size
n #number of the items in the experiment
m #number of failures observed before termination for Case I
j=(m)%/%2 #number of failures observed before termination for Case II
```

```
T2=x_2[(m)%/%2] #Pre-fixed time point
#Note: In the above program one change j and the pre-fixed time point
#--------------------------------------------------------------------
#----------------------Pre-fixed time point T3----------------------
#------------------------------------------------------------------
#sample size
n #number of the items in the experiment
m #number of failures observed before termination for Case I
j=m #number of failures observed before termination
T3=x_3[m]+1 #Pre-fixed time point
#Note: In the above program one change j and the pre-fixed time point
#--------------------------------------------------------------------
#--------------------------------------------------------------------
#####################################################
# Real Data Analysis #
#####################################################
u # Real data vector
n=length(u); m=30; R=c(rep(3,m-1),n-4*m+3); T3=3
#-----------------------------------------------------------------------------#
# Creating a Type-II progressive hybrid Censored sample from a given data set #
#-----------------------------------------------------------------------------#
# m: Number of observed failures; data: real data set vector
# t: Pre-fixed time point of the experiment; R: Censoring scheme
procen<-function(m, data, t , R){dat <- data
if(is.vector(dat)){n <- length(dat)
z <- dat ; label <- rep(NA, length(z))}
sort.z = sort(z)
Rs<-R[1:m-1]; Rl<-R[m]; times <- sort.z
Cstar = label[order(z)]
W = Z = NULL; C <- NULL; Z <- c(Z, times[1]); W <- c(W, 1)
```

```
index <- 1:length(times)
for(i in 1:length(Rs)){ times <- times[!index %in% index[1]]
index <- index[-1] ; C = c(C, Cstar[1]); Cstar <- Cstar[-1]
samp <- sample(length(index), R[i], replace=FALSE)
times <- times[!index %in% samp]
C = c(C, Cstar[index %in% samp]); Cstar <- Cstar[!index %in% samp]
index <- index[!index %in% samp]
W = c(W, rep(0,length(samp)), 1)
Z <- c( Z, rep(Z[length(Z)], length(samp)), times[1] )}
times <- times[!index %in% index[1]]
C <- c( C, Cstar); W <- c(W, rep(0, length(times)))
Z <- c(Z, rep(Z[length(Z)], length(times)))
Pda<-data.frame("a"=Z,"b"=W)
Pda<-Pda[Pda$b==1,]; Z<-Pda$a; W<-Pda$b; k=1
while(Z[k]<t && k< (m+1)){ j=k; k=k+1}
Z<-Z[1:j]
if(j!=m){R<-R[1:j]} else {R<-R}
Pdat <- data.frame('scheme'=R, 'censored_data'=Z)
return(Pdat)}
#------------------------------------------------------------------
data<-procen(10,u,t=T3,R)
x_3<-data$censored_data
#------------------------------------------------------------------
# number of remaining units left at the pre-fixed time point
RD<-function(x){j=length(x_3); r1=0
for(i in 1:j){r1=R[i]+r1}
R1=n-r1-j
return(R1)}
######################################################
# EM algorithm for theta and lambda
```

```
#######################################################
EM=function(x_3,R,T3,al,be){
al1=al
  be1=be
  E1=E2=E3=E4=numeric(m)
  Cont=TRUE
  while(Cont){
    al2=al1
    be2=be1
#E-step
    for(i in 1:m){
      #d1=Ckm(T[i],al2,be2)-Ckm(T[i-1],al2,be2)
      #T1=x_1*((m %/%2)+0.01)
      #T1=x_1[m %/%2]+0.01
      d1=1.0-Ckm(x_3[i],al2,be2)
      d2=1.0-Ckm(T3,al2,be2)
      # T1[i]=x_1[i][m %/%3]+0.01
      # print(c(d1,d2))
      #     print(c(the,lam,T[i-1],T[i]))
      E1=integrate(y1,lower=x_3[i],upper=1,al=al2,be=be2)$value/d1
      E2=(log(abs(1-x_3^(al2))^be2)-1/be2)


      print(c(E1,E2))
    }
# end of E-step


    #M step
    be1=(-n)/(log(abs(1-x_3^(al1)))+sum(R%/%E2)+RD*E2)
    al1=(-n)/(sum(log(x_3))-(be1-1)*sum((x_3^(al1)*log(x_3))/(abs(1-
x_3^(al1))))+sum(R%/%E1)+RD*E1)
    # add a check since the search could diverge, so force to start over again
```

```
#Convergence checking
    if((abs(al1-al1)<0.0001) && (abs(be1-be1)<0.0001)) Cont= FALSE
  } # end of while-loop
  return(cbind(al1,be1))
} # end of EM module
EM(x_3,R,T3,al,be)
```