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Ranks, Subdegrees and Suborbital Graphs of Direct Product of the Symmetric Group Acting on the Cartesian Product of Three Sets

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Abstract: Transitivity and Primitivity of the action of the direct product of the symmetric group on Cartesian product of three sets are investigated in this paper. We prove that this action is both transitive and imprimitive for all $n \geq 2$. In addition, we establish that the rank associated with the action is a constant 2^3. Further; we calculate the subdegrees associated with the action and arrange them according to their increasing magnitude.

Keywords: Direct Product, Symmetric Group, Action, Rank, Subdegrees, Cartesian Product, Suborbit

1. Introduction

Group Action of $S_n \times S_n \times S_n$ on $X \times Y \times Z$ is defined as $(g_1, g_2, g_3)(x, y, z) = (g_1 x, g_2 y, g_3 y)$ $\forall$ g 1 , g 2 , g 3$\in$ S$_n$ $x \in X, y \in Y \, and \, z \in Z$. This paper explores the action $S_n \times S_n \times S_n$ on $X \times Y \times Z$.

2. Notation and Preliminary Results

Let X be a set, a group G acts on the left of X if for each and $x \in X$ there corresponds a unique element $gx \in X$ such that $(g_1 g_2)x = g_1(g_2 x)$ $\forall g_1, g_2 \in G$, and $x \in X$ and $1x = x$, $\forall x \in X$, where 1 is the identity in G.

The action of G on X from the right can be defined in a similar way.

Let G act on a set X. Then X is partitioned into disjoint equivalence classes called orbits or transitivity classes of the action. For each $x \in X$ the orbit containing x is called the orbit of x and is denoted by $Orb_G(x)$ Thus

$$Orb_G(x) = \{gx \mid g \in G\}.$$

The action of a group G on the set X is said to be transitive if for each pair of points $x, y \in X$, there exists g ∈ G such that gx=y ; in other words, if the action has only one orbit.

Suppose that G acts transitively on X. Then a subset Y of X, where |Y| is a factor of |X|, is called a block or set of imprimitivity for the action if for each g ∈ G, either gY=Y or gY∩Y= ∅; in other words gY and Y do not overlap partially. In particular, ∅, X and all 1- element subsets of X are obviously blocks. These are called the trivial blocks. If these are the only blocks, then we say that G acts primitively on X. Otherwise, G acts imprimitively.

Theorem 2.1 Orbit-Stabilizer Theorem [10]

Let G be a group acting on a finite set X and x ∈ X. Then $|Orb_G(x)| = |G : Stab_G(x)|$

Theorem 2.2 Cauchy-Frobenius lemma *[4]*

Let G be finite group acting on a set X. The number of orbits of G is $\frac{1}{|G|}\sum_{g \in G}|Fix(g)|$.

Let (G_1, X_1) and (G_2, X_2) be permutation groups. The direct product $G_1 \times G_2$ acts on Cartesian product $X_1 \times X_2$ by rule $(g_1, g_2)(x_1, x_2) = (g_1x_1, g_2x_2)$.

3. Main Results

3.1. Transitivity of $S_n \times S_n \times S_n$ Acting on $X \times Y \times Z$

Theorem 3.1 If n ≥ 2, then $G = S_n \times S_n \times S_n$ acts transitively on $X \times Y \times Z$, where $X = \{1, 2, 3, ..., n\}$, $Y = \{n+1, n+2, ..., 2n\}$ and $Z = \{2n+1, 2n+2, ..., 3n\}$.

Proof. Let G act on $X \times Y \times Z$. It is enough to show that the cardinality of

$Orb_G(x, y, z)$ is equal to $|X \times Y \times Z|$. We determine $|H| = |Stab_G(x, y, z)|$.

Now $(g_1, g_2, g_3) \in S_n \times S_n \times S_n$ fixes $(x, y, z) \in X \times Y \times Z$

If and only if $(g_1, g_2, g_3)(x, y, z) = (x, y, z)$ so that $g_1 x = x$, $g_2 y = y$, $g_3 z = z$. Thus x, y, z comes from a 1- cycle of $g_i\ (i = 1, 2, 3)$.

Hence H is isomorphic to $S_{n-1} \times S_{n-1} \times S_{n-1}$ and so

$$|H| = ((n-1)!)^3$$

By the use Theorem 2.1 $|Orb_G(x, y, z)| = |G : Stab_G(x, y, z)|$

$$= \frac{|G|}{|H|}$$

$$= \frac{(n!)^3}{((n-1)!)^3}$$

$$= \left(\frac{(n!)}{(n-1)\ !}\right)^3$$

$$= \left(\frac{n(n-1)!}{(n-1)!}\right)^3$$

$$= (n)^3$$

$$= |X \times Y \times Z|$$

3.2. Primitivity of $S_n \times S_n \times S_n$ on $X \times Y \times Z$

Theorem 3.2 The action of $S_n \times S_n \times S_n$ *on* $X \times Y \times Z$ is imprimitive for $n \geq 2$.

Proof. This action is transitive by theorem 3.1 Consider $K = X \times Y \times Z$ where

$X = \{1, 2, ..., n\}$, $Y = \{n+1, n+2, ..., 2n\}$ and $Z = \{2n+1, 2n+2, ..., 3n\}$ therefore G acts on K and $|K| = n \times n \times n$. Let L be any non trivial subset of K such that $|L|$ divides $|K|$ by $\frac{n \times n \times n}{n}$. $L = \begin{pmatrix} (1, n+1, 2n+2), (1, n+1, 2n+3), ..., \\ (1, n+1, 3n), (1, n+2, 2n+1) \end{pmatrix}$ therefore $|L| = n$ for $n \geq 2$. For each element of L there exist $(g_1, g_2, g_3) \in G$ with $n-$ cycles permutation $\begin{pmatrix} (1, 2, 3, ..., n), (n+1, n+2, n+3, ..., 2n), \\ (2n+1, 2n+2, 2n+3, ..., 3n) \end{pmatrix}$ for $n \geq 2$ such that $(g_1, g_2, g_3) \in G$ moves an element of L to an element not in L so that $gY \cap Y = \varphi$. This argument shows that L is a block for the action and the conclusion follows.

3.3. Ranks and Subdegrees of $S_2 \times S_2 \times S_2$ on $X \times Y \times Z$

In this section $X = \{1, 2\}$, $Y = \{3, 4\}$ and $Z = \{5, 6\}$

Theorem 3.3 The rank of $G = S_2 \times S_2 \times S_2$ acting on $X \times Y \times Z$ is 2^3.

Proof. Let $G = S_2 \times S_2 \times S_2$ *act* on $X \times Y \times Z$.

$Stab_G(1, 3, 5) = (1, 1, 1)$, the identity. By use of Theorem 2.5 to get the number of orbits of $G_{(1,3,5)}$ on $X \times Y \times Z$

Let $K = X \times Y \times Z$, then

K={(1,3,5),(1,3,6),(2,3,5),(2,3,6),(1,4,5),(1,4,6), (2,4,5),(2,4,6)} .

A permutation in $G_{(1,3,5)}$ is of the form (1,1,1) since it is the identity. The number of elements in $X \times Y \times Z$ fixed by each $(g_1, g_2, g_3) \in G$ is 8 since identity fixes all the elements in $X \times Y \times Z$.

Hence by Cauchy Frobenius Lemma, the number of orbits of $G_{(1,3,5)}$ acting on $X \times Y \times Z$ is

$$\frac{1}{|G_{(1,3,5)}|} \sum_{(g_1, g_2, g_3) \in G_{(1,3,5)}} |Fix(g_1, g_2, g_3)| = \frac{1}{1}(1 \times 8) = 8 = 2^3$$

Let $A = \{1, 3, 3\}$

The 2^3 orbits of $G_{(1,3,5)}$ on $X \times Y \times Z$ are:

a) The Suborbit whose every element contains exactly 3 elements from A $\Delta_0 = Orb_{G(1,3,5)}(1, 3, 5) = \{(1, 3, 5)\}$ -the trivial orbit.

b) Suborbits each of whose every element contains exactly 2 elements from A

$$\Delta_1 = Orb_{G(1,3,5)}(1, 3, 6) = \{(1, 3, 6)\}$$

$$\Delta_2 = Orb_{G(1,3,5)}(1, 4, 5) = \{(1, 4, 5)\}$$

$$\Delta_3 = Orb_{G(1,3,5)}(2, 3, 5) = \{(2, 3, 5)\}$$

c) Suborbits each of whose every element contains exactly 1 element from A

$$\Delta_4 = Orb_{G(1,3,5)}(1,4,6) = \{(1,4,6)\}$$

$$\Delta_5 = Orb_{G(1,3,5)}(2,3,6) = \{(2,3,6)\}$$

$$\Delta_6 = Orb_{G(1,3,5)}(2,4,5) = \{(2,4,5)\}$$

d) Suborbit whose elements contain no element from A

$$\Delta_7 = Orb_{G(1,3,5)}(2,4,6) = \{(2,4,6)\}$$

From the above, the rank of $G = S_2 \times S_2 \times S_2$ acting on $X \times Y \times Z$ is 2^3 and subdegrees are 1,1,...,1.

3.4. Ranks and Subdegrees of $S_3 \times S_3 \times S_3$ on $X \times Y \times Z$

In this section $X = \{1,2,3\}$, $Y = \{4,5,6\}$ and $Z = \{7,8,9\}$

Theorem 3.4 The rank of $G = S_3 \times S_3 \times S_3$ acting on $X \times Y \times Z$ is 2^3.

Proof. Let $G = S_3 \times S_3 \times S_3$ act on $X \times Y \times Z$ and $K = X \times Y \times Z$ then

K = {(1,4,7),(1,4,8),(1,4,9),(1,5,7),
(1,5,8),(1,5,9),(1,6,7),(1,6,8),(1,6,9),
(2,4,7),(2,4,8),(2,4,9),(2,5,7),(2,5,8),
(2,5,9),(2,6,7),(2,6,8),(2,6,9),
(3,4,7),(3,4,8),(3,4,9),(3,5,7),(1,5,8),
(3,5,9),(3,6,7),(3,6,8),(3,6,9)}.

By Theorem 2.1,

$Stab_G$ (1,4,7)= {(1,1,1),(1,1,(89)),(1,(56),1),
(1,(56),(89)),((23),1,1),((23),(1),(89)),
((23),(56),1),((23),(56),(89))}

By applying Cauchy Frobenius Lemma the number of orbits of $G_{(1,4,7)}$ acting on $X \times Y \times Z$ are;

The number of elements in $X \times Y \times Z$ fixed by each $(g_1, g_2, g_3) \in G_{(1,4,7)}$ is given by Table1

Table 1. *Permutations in $G_{(1,4,7)}$ and the number of fixed points.*

Type of ordered triple permutations in $G_{(1,4,7)}$	**Number of ordered triple of permutations**	$\lvert Fix(g_1, g_2, g_3) \rvert$
(1,1,1)	1	27
(1,1,(ab))	1	9
(1,(ab),1)	1	9
(1,(ab),(ab))	1	3
((ab),1,1)	1	9
((ab),(ab),1)	1	3
	1	3
((ab),(ab),(ab))	1	1

Now applying Theorem 2.2, the number of orbits of $G_{(1,4,7)}$ acting on $X \times Y \times Z$ is

$$\frac{1}{|G_{(1,4,7)}|} \sum_{(g_1, g_2, g_3) \in G_{(1,4,7)}} |Fix(g_1, g_2, g_3)| =$$

$$\frac{1}{8}(1\times27)+(1\times9)+(1\times9)+(1\times3)+$$

$$(1\times9)+(1\times3)+(1\times3)+(1\times1) = \frac{64}{8} = 8 = 2^3$$

Let A = { 1,4,7 }

The 2^3 orbits of $G_{(1,4,7)}$ on $X \times Y \times Z$ are:

a) Suborbit whose every element contains exactly 3 elements from A

Δ_0 =$Orb_{G(1,4,7)}$ (1,4,7)={(1,4,7)} -the trivial orbit.

b) Suborbits each of whose every element contains exactly 2 elements from A

Δ_1 =$Orb_{G(1,4,7)}$ (1,4,8)={(1,4,8),(1,4,9)}

Δ_2 =$Orb_{G(1,4,7)}$ (1,5,7) = {(1,5,7),(1,6,7)}

Δ_3 =$Orb_{G(1,4,7)}$ (2,4,7) = {(2,4,7),(3,4,7)}

c) Suborbits each of whose every element contains exactly 1 element from A

Δ_4 =$Orb_{G(1,4,7)}$ (1,5,8)={(1,5,8),(1,5,9),(1,6,8),(1,6,9)}

Δ_5 =$Orb_{G(1,4,7)}$ (2,4,8)={(2,4,8),(2,4,9),
(3,4,8),(3,4,9)}

Δ_6 =$Orb_{G(1,4,7)}$ (2,5,7) = {(2,5,7),(2,6,7),
(3,5,7),(3,6,7)}

d) Suborbit whose elements contain no element from A

Δ_7 =$Orb_{G(1,4,7)}$ (2,5,8)={(2,5,8),(2,5,9),(2,6,8),
(3,5,8),(2,6,9),(3,5,9),(3,6,8),(3,6,9)}

From the above, the rank of $G = S_3 \times S_3 \times S_3$ acting on $X \times Y \times Z$ is 2^3 and subdegrees are 1,2,2,2,4,4,4,8

3.5. Ranks and Subdegrees of $S_n \times S_n \times S_n$ on $X \times Y \times Z$

Theorem 3.5 If $n \ge 2$, the rank of $G = S_n \times S_n \times S_n$ acting on $X \times Y \times Z$ *is* 2^3*, where* $X = \{1,2,...,n\}$, $Y = \{n+1, n+2, ..., 2n\}$ and $Z = \{2n+1, 2n+2, ..., 3n\}$.

Proof. The number orbits of $G = S_n \times S_n \times S_n$ acting on $X \times Y \times Z$ are given as follows;

Let $A = \{1, n+1, 2n+1\}$

Table 2. *The rank of $G = S_n \times S_n \times S_n$ acting on $X \times Y \times Z$.*

Suborbit		Number of suborbits
Orbit containing no element from A	${}^3C_0$	1
Orbits containing exactly 1 element from A	${}^3C_1$	2
Orbits containing exactly 2 elements from A	${}^3C_2$	2
Orbit containing exactly 3 elements from A	${}^3C_3$	1

Hence the rank of the $G = S_n \times S_n \times S_n$ acting on $X \times Y \times Z$ is

$$1 + 3 + 3 + 1 = 8 = 2^3$$

The 2^3 orbits of $G_{(1,n+1,2n+1)}$ on $X \times Y \times Z$ are:

a) Suborbit whose every element contains exactly 3 elements from A

$\Delta_0 = Orb_{G(1,n+1,2n+1)}(1, n+1, 2n+1) = \{(1, n+1, 2n+1)\}$ -the trivial orbit.

b) Suborbits each of whose every element contains exactly 2 elements from A

$$\Delta_1 = Orb_{G(1,n+1,2n+1)}(1, n+1, 2n+2) = \{(1, n+1, 2n+2), (1, n+1, 2n+3), ..., (1, n+1, 3n)\} \quad n \geq 2 \cdot$$

$$\Delta_2 = Orb_{G(1,n+1,2n+1)}(1, n+2, 2n+1) = \{(1, n+2, 2n+1), (1, n+3, 2n+1), ..., (1, 2n, 2n+1)\} \quad n \geq 2 \cdot$$

$$\Delta_3 = Orb_{G(1,n+1,2n+1)}(2, n+1, 2n+1) = \{(2, n+1, 2n+1), (3, n+1, 2n+1), ..., (n, n+1, 2n+1)\} \quad n \geq 2 \cdot$$

c) Suborbits each of whose every element contains exactly 1 element from A

$$\Delta_4 = Orb_{G(1,n+1,2n+1)}(1, n+2, 2n+2) = \{(1, n+2, 2n+2), (1, n+3, 2n+3), ..., (1, 2n, 3n)\}$$

$$n \geq 2 .$$

$$\Delta_5 = Orb_{G(1,n+1,2n+1)}(2, n+1, 2n+2) = \{(2, n+1, 2n+2), (3, n+1, 2n+3), ..., (n, n+1, 3n)\} \quad n \geq 2 \cdot$$

$$\Delta_6 = Orb_{G(1,n+1,2n+1)}(2, n+2, 2n+1) = \{(2, n+2, 2n+1), (3, n+2, 2n+1), ..., (n, 2n, 2n+1)\} \quad n \geq 2 \cdot$$

d) Suborbit whose elements contain no element from A

$$\Delta_7 = Orb_{G(1,n+1,2n+1)}(2, n+2, 2n+2) = \{(2, n+2, 2n+2), (3, n+2, 2n+2), ..., (n, 2n, 3n)\} \quad n \geq 2 \cdot$$

The subdegrees of G are as shown in Table 3 below:

Table 3. *Subdegrees of $G = S_2 \times S_2 \times S_2$ acting on $X \times Y \times Z$ for $n \geq 2$.*

Suborbit length	1	$n-1$	$(n-1)^2$	$(n-1)^3$
number of suborbits	1	3	3	1

4. Conclusions

In this study, some properties of the action of $S_n \times S_n \times S_n$ on $X \times Y \times Z$ were studied. It can be concluded that;

- $S_n \times S_n \times S_n$ acts transitively and imprimitely on $X \times Y \times Z$ for all $n \geq 2$.
- The rank of $S_n \times S_n \times S_n$ on $X \times Y \times Z$ is 2^3 for all $n \geq 2$.

References

[1] Akbas, M. (2001). Suborbital graphs for modular group. *Bulletin of the London Mathematical Society.* 33:647–652.

[2] Cameron, P. J. (1973). Primitive groups with most suborbits doubly transitive. *Geometriae Dedicata* 1:434–446.

[3] Cameron, P. J., Gewurz, D. A., and Merola, F. (2008). Product action. *Discrete Math.* 386–394.

[4] Harary, F. (1969). *Graph Theory*. Addison-Wesley Publishing Company, New York.

[5] Higman, D.G (1970). Characterization of families of rank 3 permutation groups by the subdegrees. *Archiv der Mathematik*, 21.

[6] Higman, D. G. (1964). Finite permutation groups of rank 3. *Math Zeitschriff*, 86:145–156.

[7] Krishnamurthy, V. (1985). *Combinatorics, Theory and Applications*. Affiliated East-West Press Private Limited, New Delhi.

[8] Nyaga, L. N. (2012). *Ranks, Subdegrees and Suborbital Graphs of the Symmetric Group S_n Acting on Unordered r− Element Subsets*. PhD thesis, JKUAT, Nairobi, Kenya.

[9] Rimberia, J. K. (2011). *Ranks and Subdegrees of the Symmetric Group S_n Acting on Ordered r−* Element *Subsets*. PhD thesis, Kenyatta University, Nairobi, Kenya.

[10] Rose, J. S. (1978). *A Course in Group Theory*. Cambridge University Press, Cambridge.

[11] Sims, C. C. (1967). Graphs and finite permutation groups. *Mathematische Zeitschrift*, 95:76–86.

[12] Wielandt, H. (1964). Finite Permutation Groups. Academic Press New York.