

**GRAPHS WHOSE AUTOMORPHISM GROUPS CONTAIN OR
REPRESENT THE ALTERNATING GROUPS, A_n AND
SYMMETRIC GROUPS, S_n**

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DECLARATION:

I declare that this work is my original work and has not been to the best of my knowledge presented for the award of a degree in any other university.

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DEDICATION

To my dear wife Grace Hosea and my sons Hosea Perfect Hosea and Yan Vitorre Hosea,

I dedicate this work that you may be inspired.

ACKNOWLEDGEMENT

Glory, honor and thanksgiving are to God in enabling me accomplish this work.

To my supervisor Dr. Fredrick Odonde Olum, I render my sincere gratitude for his presence to inspire and guide me throughout the project. Indeed, you have been available for me even on your tight schedules.

To my wife and sons, I am very much indebted for your unending support and encouragement even during the difficult times.

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SYMBOLS & NOTATIONS

D_n The Dihedral Group of degree n , cardinality $2n$

A_n Group of all even permutations degree n or simply alternating group order $\frac{n!}{2}$
and degree n

Δ Sub-orbit of G

$X \times Y$ For any two sets X, Y ; this is the Cartesian product

$\mathcal{G} \times \mathcal{H}$ Given two groups \mathcal{G}, \mathcal{H} ; this represents their direct product

$\binom{n}{r}$ n combination r i.e. $\frac{n!}{(n-r)! r!}$

$n \nmid r$ Number n Isn't multiple of r ($\nexists a \in \mathbb{N}$ so that $ar = n$; for $r, n \in \mathbb{N}$)

$N_G(H)$ Normalizer in G for H

$|Fix(G)|$ given a group G , this is the cardinality of points that its elements fix.

π Permutation representation character of G upon its action on a set X

n / r r a factor for n ($\exists a \in \mathbb{N}$ s.t $ar = n$; $r, n \in \mathbb{N}$)

$R(G)$ it's represents rank.

G_x at a point x , this is the Stabilizer in G

S_n Symmetric Group

ABSTRACT

There have been many investigations on the combinatorial structures and invariants over the group actions on the subsets of its elements. Studies on Group Theory have yielded varied and important results in the advancement of Algebra. Several studies have also been made on Graph Theory. Some Mathematicians have studied the concept of automorphisms on graphs thereby yielding important results. Automorphism groups from graphs containing the cyclic and dihedral groups, C_n and D_n respectively have been constructed using Schur's Algorithm. In this project, we have extended the work to graphs whose Automorphism groups contain the Alternating Group A_n as well as those representing Symmetric group S_n . The graphs whose Automorphism groups contain or represent the Alternating Group A_n and Symmetric group S_n respectively have been constructed. Schur's algorithm has been employed to construct these graphs. The actions of the Alternating Group A_n and the Symmetric group S_n have been shown to be transitive using the Cauchy- Frobenius Lemma and the Orbit-Stabilizer Theorem. The Automorphism graphs for the groups A_n and S_n have been constructed for $n = 3, 4, \text{ And } 5$. The number of graphs whose groups of Automorphism contain A_n is 2 being the null and the complete graphs. The number of graphs whose groups of Automorphism represent S_n is 2 being the null and the complete graphs. We have presented the results of our findings from our workings as theorems and constructed the applicable graphs.

CHAPTER ONE

INTRODUCTION

This is the introductory chapter focusing on necessary definitions, theorems and algorithms useful in the course of our project. It clearly highlights the justification for our study, together with objectives and a clear problem statement of the study.

We have divided our introduction into four key sections named 1.1 up to 1.4 starting with background information as section 1.1 laying the basic background for our study. The background information also covers definitions of some important terms and review of important theorems and algorithms that enlists some of the methods we used to come up with our results. The second section 1.2 is on the problem statement and justification stating clearly our problem statement and highlighting on our justification. In section 1.3, the objectives of the study are expressed and finally section 1.4 underpins the significance of the study to show the contribution that the study brings with it.

1.1 Background Information

We have divided this section into three sub sections namely; groups and product actions, graphs and digraphs and Schur's Algorithm reflection respectively. In the sub sections, we give the various definitions and important theorems that will be useful in the course of our study. Finally, in the third sub section, Schur's algorithm has been reviewed.

Different elements of a group correspond to transformations of spaces while preserving the structures of these same spaces. Group actions are formal ways of interpreting and representing these transformations.

The theory of group actions has over the years contained highly significant theorems which in turn have been extensively used in several fields, for example the Orbit Stabilizer theorem.

Group actions of different groups have resulted in various properties.

A group G partitions a given a set X it acts on into subsets referred to as orbits. Overtime, the orbits' numbers have been counted using the Cauchy- Frobenius Lemma stated in theorem 1.1.1.2.

With the growth of algebra, Mathematicians have studied the actions of different transitive and non-transitive permutation groups on a range of sets.

1.1.1 Group and Group Actions

Definition 1.1.1.1

For a set $X \neq \emptyset$, a mapping that is both 1-1 and onto on the set X itself is called a permutation of X . The permutation is said to be odd or even if it can be factored into a product of an odd or even numbers of transpositions (2-cycles) respectively.

Definition 1.1.1.2

The set of all permutations of the set $X = \{1,2,3, \dots, n\}$ under the binary operation of composition of maps is a symmetric group. This group is of order $n!$

Definition 1.1.1.3

In S_n , we may form a group of order $\frac{n!}{2}$ known as the alternating group (A_n) from the set of all permutations that are even.

Definition 1.1.1.4

Suppose X is a set and G , a group. G acts on the left of X if $\forall g \in G, x \in X$ with $gx \in X$ and $g_1, g_2 \in G$, then;

- i. $(g_1g_2)x = g_1(g_2x)$
- ii. $Ix = x$, I being the identity element of G .

Action on the right is similarly defined.

Definition 1.1.1.5

Let a group G act on a set $X \neq \emptyset$, the orbit of $x \in X$ in G is defined to be the set

$$Orb_G(x) = \{ gx \mid g \in G \}$$

Definition 1.1.1.6

A group G acts on a set X transitively if there's only one orbit for the action of G on X .

Equivalently, it acts transitively if $\forall x, y \in X \exists g \in G$ s. t. $gx = y$.

Definition 1.1.1.7

Suppose a group G acts on a set X and $x \in X$, the stabilizer of x in G denoted by $Stab_G$ or G_x , consists of all $g \in G$ fixing x

$$\Rightarrow Stab_G(x) = \{g \in G \mid gx = x\}$$

Theorem 1.1.1.1: (Orbit-Stabilizer Theorem)

Let G , be a group acting on a set X and $x \in X$, then the Orbit-Stabilizer theorem states that:

$$|Orb_G(x)| = |G:Stab_G(x)| \quad (\text{Rose, 1978})$$

Definition 1.1.1.8

Suppose G is a group acting transitively on a set X and G_x is the stabilizer of the point $x \in X$. We refer to orbits $\Delta_0 = \{x\}$, $\Delta_1, \Delta_2, \dots, \Delta_{k-1}$ of G_x on X as sub-orbits of G .

Definition 1.1.1.9

Let G be a group acting transitively on a set X , then the number of fixed points in G denoted $\text{Fix}(g)$ is given by: $\text{Fix}(g) = \{x \in X \mid gx = x, g \in G\}$

Definition 1.1.1.10

The tally of sub-orbits of G is its rank. The “lengths” of the sub-orbits of G are the sub degrees of G . Ranks k and sub degrees Δ_i don't depend on the choices of $x \in X$.

Theorem 1.1.1.2: (Cauchy- Frobenius Lemma)

For any group G , which acts on a finite set X , the number of G - orbits is established by

$$\frac{1}{|G|} \sum_{g \in G} |Fix(g)| \text{ in } X. \quad (\text{Rose, 1978})$$

Definition 1.1.1.11

Assume a group G acts on a set X transitively and Δ is orbit of G_x on X .

Let $\Delta^* = \{gx \mid g \in G, x \in g\Delta\}$, then Δ^* is also known as the G_x -orbit (or G -sub orbit) paired with Δ (Wielandt, 1964).

$\Delta^{**} = \Delta$ and $|\Delta| = |\Delta^*|$. Δ is self-paired If $\Delta = \Delta^*$.

G_x^Δ (The transitive constituent of G on Δ) forms a permutation group obtained when you restrict the elements of G_x to Δ .

Definition 1.1.1.12

The Character π for the permutation representation of a group G on acting on a set X is defined by

$$\pi(g) = |Fix(g)| \quad \forall g \in G.$$

Theorem 1.1.1.3

The number of self-paired sub-orbits of G is given by:

$$n_{\pi} = \frac{1}{|G|} \sum_{g \in G} \pi(g^2), g \in G. \quad (\text{Cameron, 1974})$$

Definition 1.1.1.14

An $n \times n$ matrix obtained from permuting the columns of an $n \times n$ identity matrix I_n is referred to as a permutation matrix.

1.1.2 Graphs and digraphs**Definition 1.1.2.1**

Suppose V is a set of points known as vertices while E is set of vertices in twos not in any definite order referred to as edges. A diagram with the sets V and E is called a graph denoted $\mathcal{G}(V, E)$ or sometimes \mathcal{G} given no uncertainty on V and E .

Definition 1.1.2.2

Let V be a set of vertices and E be a set of directed edges that is ordered pairs of vertices.

A graph consisting of the sets V and E is said to be directed.

Definition 1.1.2.3

Given a graph \mathcal{G} , girth measures the short most cycle in \mathcal{G}

Definition 1.1.2.4

Let \mathcal{G} be a graph. A walk in \mathcal{G} consists of a finite array of the edges mainly represented as $v_0v_1, v_1v_2, v_2v_3, \dots, v_{m-1}v_m$. The length of the walk, m is simply the number of edges. If all vertices in a particular walk are distinct then it's a path. If a path is closed, then it's known as a circuit or cycle.

Definition 1.1.2.5

Let \mathcal{G} be a graph. If each combination of the 2 vertices of \mathcal{G} is linked by some path, it is connected otherwise it is disconnected.

A connected component in \mathcal{G} is the maximal connected sub graph in \mathcal{G} .

Let v be any vertex and e , any edge of \mathcal{G} , then each v and e is in a specific connected component of \mathcal{G} .

Definition 1.1.2.6

Suppose \mathcal{G} is a graph. A permutation α of the vertex set of \mathcal{G} , $\mathcal{V}(\mathcal{G})$ is an automorphism of \mathcal{G} if $\forall u, v \in \mathcal{V}(\mathcal{G})$;

$$\{u, v\} \in E(\mathcal{G}) \text{ if and only if } \{\alpha(u), \alpha(v)\} \in E(\mathcal{G})$$

Theorem 1.1.2.1

Let G be transitive group on X and let the sub orbit Δ_i ($i = 0, 1, \dots, r - 1$) correspond to suborbital O_i . Then the corresponding suborbital graph \mathcal{G}_i is undirected if Δ_i is self-paired and directed if Δ_i is not self-paired. (Sims, 1967)

Definition 1.1.2.7

An Automorphism group of a graph \mathcal{G} denoted $\text{Aut}(\mathcal{G})$ is the set of all automorphisms of the graph \mathcal{G} , under the compositions of functions. Ideally, it forms a subgroup of S_n on $\mathcal{V}(\mathcal{G})$.

Definition 1.1.2.8

A graph where $E = \emptyset$ i.e. with no edge is said to be a null graph.

Definition 1.1.2.9

Suppose \mathcal{G} is a graph. If every two unique vertices $v_1, v_2 \in V$ in \mathcal{G} are connected by a specific edge $e \in E$, then \mathcal{G} is complete.

Definition 1.1.2.10

Given a graph \mathcal{G} with n vertices labeled 1 up to n , an adjacency matrix $A(\mathcal{G})$ is the matrix defined by $A(\mathcal{G}) = (a_{ij}), i, j = 1, 2, 3, \dots, n$ such that;

$$a_{ij} = \begin{cases} 1; & \text{if } \exists \text{ an edge between the vertices } i \text{ and } j \\ 0; & \text{otherwise} \end{cases}$$

Definition 1.1.2.11

An $n \times n$ matrix whose entries are +1 or -1 and with mutually orthogonal rows is referred to as a Hadamard matrix

1.1.3 Schur's Algorithm Reflection (Chao, 1965)

Schur's algorithm is constituted as below;

Consider a transitive permutation group G acting on n elements say $\{1, 2, 3 \dots n\}$ and G_1 the stabilizer of 1, then the orbits of G_1 are given as;

$$\Delta_1 = \{1\}, \Delta_2, \Delta_3, \dots, \Delta_k.$$

Associate each Δ_m with an $n \times n$ matrix as;

$$B(\Delta_m) = (b_{ij}), i, j = 1, 2, 3, \dots, n \text{ such that;}$$

$$b_{ij} = \begin{cases} 1; & \text{if } \exists a g \in G \text{ and } x \in \Delta_m \text{ where } g \cdot 1 = j \text{ and } gx = i \\ 0; & \text{otherwise} \end{cases}$$

$B(\Delta_m)$ is a symmetric matrix if and only if Δ_m is self-paired. The Identity matrix, I_n is clearly observed as $B(\Delta_1)$, hence we only have to find $B(\Delta_i)$ for $i = 2, 3, \dots, k$.

Consider each $B(\Delta_i)$ for $i = 2, 3, \dots, k$ separately. If $B(\Delta_i)$ is a symmetric matrix, a graph X_i can be constructed whose adjacency matrix is given by $A(X_i) = B(\Delta_i)$.

We ignore $B(\Delta_i)$ for a moment if asymmetric.

Now, proceed to sum $B(\Delta_i) + B(\Delta_j), i \neq j, i, j = 2, 3, \dots, k$. Construct the graph for the sum if symmetric and ignore briefly if asymmetric.

We then repeat the process of addition to find all the possible sums for 3, 4... k-1 different $B(\Delta_i)$ matrices.

We use every symmetric matrix from the results as adjacency matrices to construct respective graphs.

Finally, the null graphs are constructed with n vertices.

This process gives all the Automorphism groups of Graphs containing the transitive group G

1.2 Problem Statement

Some studies have been made on groups of automorphisms of graphs. Olum (2018) while investigating Automorphism Groups from Graphs containing the Dihedral and Cyclic Groups used Schur's algorithm. We seek to continue in this concept and study graphs whose groups of automorphisms contain the Alternating group A_n and those representing the symmetric group S_n .

The characteristics of these group actions like transitivity among others will be investigated.

1.3 Objectives

1.3.1 General Objective:

The general objective of the study is to establish Automorphism groups of Graphs containing or representing the Alternating and Symmetric groups, A_n and S_n respectively.

1.3.2 Specific Objectives:

The Specific objectives are:

- i. To construct Graphs whose groups of automorphisms contain A_n .
- ii. To derive a formula for finding the tally of Graphs whose groups of automorphisms contain A_n .
- iii. To construct Graphs whose groups of automorphisms represent S_n .
- iv. To derive a formula for finding the tally of Graphs whose groups of automorphisms represent S_n .

1.4 Significance of the study:

This study yields to important results for graph theory that will indeed go a long way in establishing the relationship between Schur's Algorithm and Graph theory. The theory of graphs is severally applicable in many fields. The algorithms on graph groups are as well useful in computer and software programming.

The theory of graphs is applicable in astronomy, geography, navigation among the other life Sciences e.g. calculation of the shortest paths. Other Sciences like Chemistry and Physics have employed graph theory in the study of molecular structures. In Physics, for example, in condensed matter, statistical analysis of theoretical graph characteristics in relation to atoms' topology is used to quantitatively examine the 3-dimensional structures of atoms. In Chemistry, atoms are represented by vertices and bonds by edges for the natural model of molecules.

The theory of graphs has also been employed in Euler tours to figure out whether an object may move from one location and then make its way back e.g. a cargo ship taking cargo to different countries.

CHAPTER TWO

LITERATURE REVIEW

Various Mathematicians have studied the properties of group actions on both ordered and un-ordered subsets of different groups. Many Mathematicians have also considered the concepts of transitivity, primitivity, ranks and sub degrees during their studies. The theory of graphs has also received some kind of consideration lately. Of interest has been the structure of sub-orbital graphs of different groups of permutations and their actions on the sets of their respective vertices. Graph automorphisms have also received quite a bit of attention.

Frucht (1939) found that all groups are also groups' automorphisms of a graph. Sabidussi (1957) while studying graph products established that the products of connected graphs is also connected and, the product there of for a disconnected graph with any graph is disconnected. Chao (1965) used an algorithm developed from Schur's theorem to determine the graphs whose automorphism groups contain transitive groups.

Bouwer (1969) showed that if G is any given permutation group that is finite, then exists infinitely several undirected and directed graphs which are not isomorphic and whose groups of automorphisms has G as it's sub direct component. A classification of all groups of permutations G having a sub orbit Δ of length 4 where we have that $G_x \cong A_4$ or $G_x \cong S_4$ is faithful was done by Quirin (1971).

Leon (1979) described an algorithm for computing the automorphism group of a Hadamard matrix. He showed how to modify the algorithm for determining the equivalence of any two Hadamard matrices. The algorithm yielded the order of the automorphism group, the orbits of the automorphism groups on the rows and columns of the matrix and a set of permutations generating the automorphism group. Kim *et al.* (1980) looked into graph algebras with a finding that two graph algebras are isomorphic if and only if their graphs are isomorphic. Servatius (1989) on his study of graph groups improved on that result working on the centralizer problem for graph groups.

Babai *et al.* (1991) built a framework useful for studying the minimum number of edge orbits and showed that a bounded total number of edge-orbits are admitted as a representation of large classes of groups. In this case, if the group of automorphisms of X is isomorphic to G then a graph X is said to represent the group G .

Cameron (2004) did a survey on finite graphs' automorphisms, especially the symmetry of the typical graphs. He dealt mainly with identifying automorphism groups as either abstract or permutation groups. He came up with a number of key findings among them being that a graph and its complement have the same automorphism group. He further discussed a finding by Frucht (1939) that we highlighted earlier above.

Greco & Iannazzo (2010) presented an algorithm based on Schur decomposition of a nonsingular matrix A with an order of lower complexity than the Smith Algorithm which is also based on Schur' s. The algorithm computes the principal root of a matrix with no non positive real Eigen values.

Babai & Cameron (2015) proved that for a permutation group G acting on a set X of size n , other than alternating group, the probability that $\text{Aut}(X, Y^G) = G$ for a random subset Y of X , tends to 1 as $n \rightarrow \infty$.

Olum (2018) used the concept of the same algorithm as Chao (1965) mentioned earlier developed from Schur's theorem to determine a formula for finding the tally of graphs whose automorphism groups contain given finite cyclic and dihedral groups. In this project, we extend this study to alternating and symmetric groups. The generalized Schur Algorithm is useful in many more computations. Laudadio *et al* (2018) discusses the application of Schur algorithm in computing the classical decompositions of matrices such as QR and LU factorizations.

CHAPTER THREE
AUTOMORPHISMS GROUPS OF GRAPHS CONTAINING
ALTERNATING GROUPS

Throughout this chapter, we shall let G be the Alternating Group A_n of degree n and X be the set of vertices of a regular n -gon ordered cyclically from 1 to n . The chapter is divided into three Sections. Section 3.1 investigates transitivity of the action of G on X . In Section 3.2, we illustrate Graphs whose Automorphisms Groups contain the Alternating group, A_n of degree n , $n \geq 3$. The main result in this section are given in the form of theorems.

3.1 Transitivity of the alternating, A_n group

Let G be an alternating group A_n acting on a set $X = \{1, 2, 3, \dots, n\}$.

Given $g \in G$ and $x \in X$, each element $x \in X$ is fixed by exactly $\frac{|G|}{n} = \frac{(n-1)!}{2}$.

$$\Rightarrow |\text{Stab}_G(x)| = \frac{(n-1)!}{2} \quad \forall x \in X,$$

$$\Rightarrow \sum_{x \in X} |\text{Stab}_G(x)| = \frac{n!}{2}.$$

$|G| = \frac{n!}{2}$, since G is an alternating group of degree n .

$\Rightarrow |\text{orb}_G(x)| = 1$ by Theorem 1.1.1.1

Hence G acts transitively X .

3.2 Graphs whose automorphisms groups contain the Alternating group, A_n of degree n ; $n \geq 3$.

Employing Schur's algorithm, we construct all regular simple graphs whose automorphisms groups contain alternating groups. We derive a formula for finding the number of graphs for any A_n .

3.2.1 Graphs automorphisms groups that contain A_3

The elements of $G = A_3$ are $\{1, (123), (132)\}$,

The stabilizer, $\text{Stab}_G 1$, given by $G_1 = \{1\}$.

The orbits of G_1 are: $\Delta_1 = \{1\}$, $\Delta_2 = \{2\}$, $\Delta_3 = \{3\}$,

Using Schur's algorithm, we enumerate $B(\Delta_i)$, $i = 2, 3$ as below;

$$B(\Delta_2) = \begin{pmatrix} 0 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix}, \quad B(\Delta_3) = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{pmatrix}$$

By summing $B(\Delta_2)$ and $B(\Delta_3)$, the result is a symmetric matrix,

$$B(\Delta_2) + B(\Delta_3) = \begin{pmatrix} 0 & 1 & 1 \\ 1 & 0 & 1 \\ 1 & 1 & 0 \end{pmatrix}$$

Setting $A(X_1) = B(\Delta_1)$ and $A(X_2) = B(\Delta_2) + B(\Delta_3)$ as adjacency matrices for the graphs X_1 and X_2 respectively.

$$A(X_1) = B(\Delta_1) = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

$$A(X_2) = B(\Delta_2) + B(\Delta_3) = \begin{pmatrix} 0 & 1 & 1 \\ 1 & 0 & 1 \\ 1 & 1 & 0 \end{pmatrix}$$

The two graphs are constructed as below:

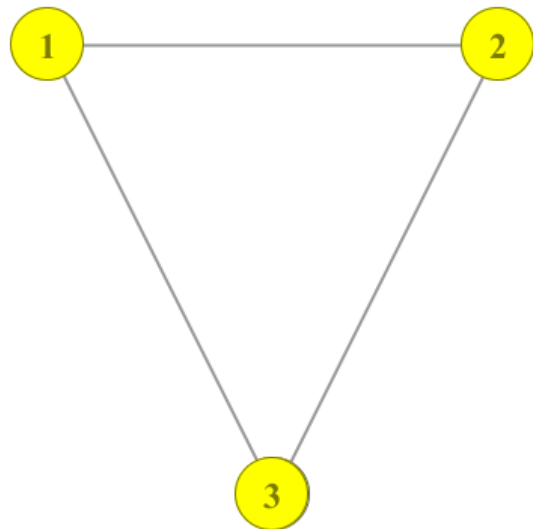


Figure 3.1 (a) X_1

Figure 3.1 (b) X_2

3.2.2 Graphs automorphisms groups that contain A_4

The elements of $G = A_4$ are $\{1, (123), (124), (132), (134), (142), (143), (234), (243), (12)(34), (13)(24), (14)(23)\}$

The stabilizer, $\text{Stab}_G 1$, given by $G_1 = \{1, (234), (243)\}$

The orbits of G_1 are; $\Delta_1 = \{1\}$, $\Delta_2 = \{2, 3, 4\}$

Using Schur's algorithm, we establish $B(\Delta_2)$ as below;

$$B(\Delta_2) = \begin{pmatrix} 0 & 1 & 1 & 1 \\ 1 & 0 & 1 & 1 \\ 1 & 1 & 0 & 1 \\ 1 & 1 & 1 & 0 \end{pmatrix}$$

Setting $A(X_1) = B(\Delta_1)$ and $A(X_2) = B(\Delta_2)$ as adjacency matrices for the graphs X_1 and X_2 respectively.

$$A(X_1) = B(\Delta_1) = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

$$A(X_2) = B(\Delta_2) = \begin{pmatrix} 0 & 1 & 1 & 1 \\ 1 & 0 & 1 & 1 \\ 1 & 1 & 0 & 1 \\ 1 & 1 & 1 & 0 \end{pmatrix}$$

The two graphs are constructed as below:

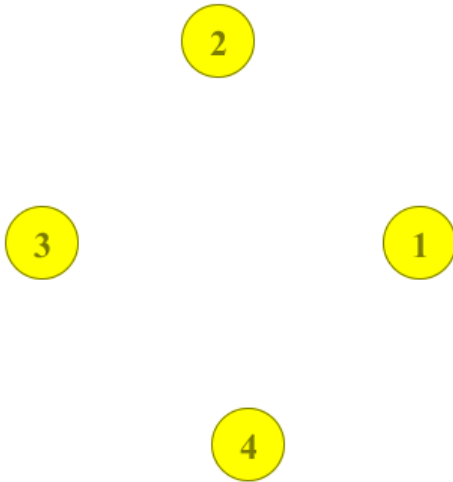


Figure 3.2 (a) X_1

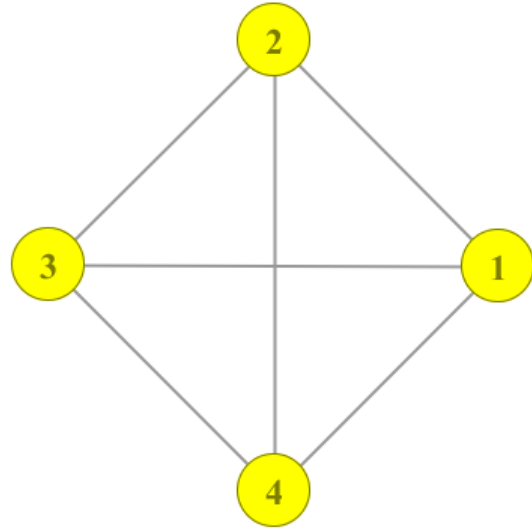


Figure 3.2 (b) X_2

3.2.3 Graphs automorphisms groups that contain A_5

The elements of $\mathcal{G} = A_5$ are $\{1, (123), (124), (125), (132), (134), (135), (142), (143), (145), (152), (153), (154), (234), (235), (243), (245), (253), (254), (345), (354), (12345), (12354), (12435), (12453), (12534), (12543), (13245), (13254), (13425), (13452), (13524), (13542), (14235), (14253), (14325), (14352), (14523), (14532), (15234), (15243), (15324), (15342), (15423), (15432), (12)(34), (12)(35), (12)(45), (13)(24), (13)(25), (13)(45), (14)(23), (14)(25), (14)(35), (15)(23), (15)(24), (15)(34), (23)(45), (24)(35), (25)(34)\}$

The stabilizer, G_1

$= \{1, (234), (235), (243), (245), (253), (254), (345), (354), (23)(45), (24)(35), (25)(34)\}$

The orbits of G_1 are; $\Delta_1 = \{ 1 \}$, $\Delta_2 = \{ 2, 3, 4, 5 \}$

Using Schur's algorithm, we form $B(\Delta_2)$ as below;

$$B(\Delta_2) = \begin{pmatrix} 0 & 1 & 1 & 1 & 1 \\ 1 & 0 & 1 & 1 & 1 \\ 1 & 1 & 0 & 1 & 1 \\ 1 & 1 & 1 & 0 & 1 \\ 1 & 1 & 1 & 1 & 0 \end{pmatrix}$$

Setting $A(X_1) = B(\Delta_1)$ and $A(X_2) = B(\Delta_2)$ as adjacency matrices for the graphs X_1 and X_2 respectively.

$$A(X_1) = B(\Delta_1) = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{pmatrix}$$

$$A(X_2) = B(\Delta_2) = \begin{pmatrix} 0 & 1 & 1 & 1 & 1 \\ 1 & 0 & 1 & 1 & 1 \\ 1 & 1 & 0 & 1 & 1 \\ 1 & 1 & 1 & 0 & 1 \\ 1 & 1 & 1 & 1 & 0 \end{pmatrix}$$

The two graphs are constructed as below:

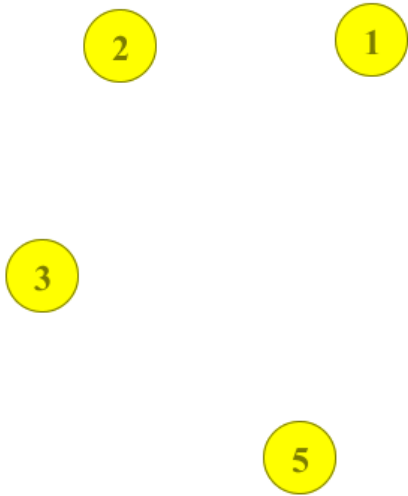


Figure 3.3 (a) X_1

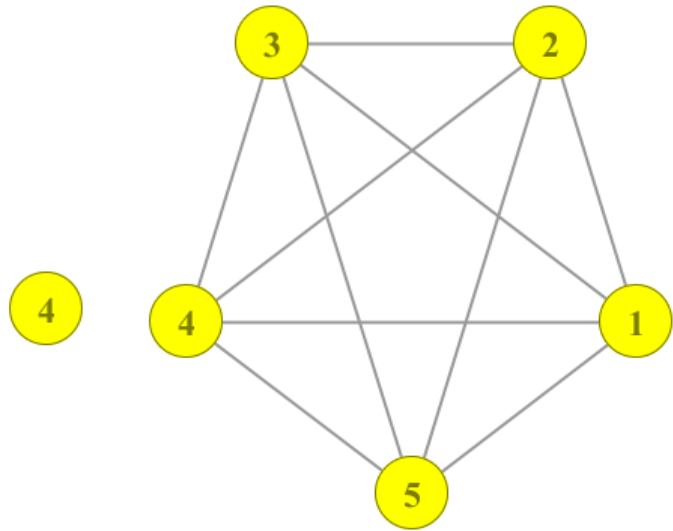


Figure 3.3 (b) X_2

Theorem 3.1

Suppose G is an alternating group of degree n , then the number of regular graphs whose groups of automorphisms contain G is 2, I.e. the null and the complete graphs.

Proof

In general, the Stabilizer of 1 is given by:

$$G_1 = \{1, (234), (235), \dots, (n-2 \ n \ n-1), \dots, (n-3 \ n)(n-2 \ n-1)\} \text{ such that ;}$$

$$|G_1| = \frac{(n-1)!}{2}$$

For $n = 3$, The orbits of G_1 are: $\Delta_1 = \{1\}$, $\Delta_2 = \{2\}$, $\Delta_3 = \{3\}$

Hence, Using Schur's algorithm, we have that;

$$B(\Delta_2) + B(\Delta_3) = \begin{pmatrix} 0 & 1 & 1 \\ 1 & 0 & 1 \\ 1 & 1 & 0 \end{pmatrix}$$

Setting $A(X_1) = B(\Delta_1)$ and $A(X_2) = B(\Delta_2) + B(\Delta_3)$ as adjacency matrices for the graphs X_1 and X_2 respectively.

$$A(X_1) = B(\Delta_1) = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

$$A(X_2) = B(\Delta_2) + B(\Delta_3) = \begin{pmatrix} 0 & 1 & 1 \\ 1 & 0 & 1 \\ 1 & 1 & 0 \end{pmatrix}$$

The graphs are constructed with X_1 and X_2 being a null graph and a complete graph respectively of 3 vertices.

The orbits of G_1 are $\Delta_1 = \{ 1 \}$, $\Delta_2 = \{ 2, 3, 4, \dots, n \}$ for $n > 3$

Hence, Using Schur's algorithm, we establish $B(\Delta_2)$, as below;

$$B(\Delta_2) = \begin{pmatrix} 0 & 1 & 1 & 1 & \dots & \dots & \dots & 1 \\ 1 & 0 & 1 & 1 & \dots & \dots & \dots & 1 \\ 1 & 1 & 0 & 1 & \dots & \dots & \dots & 1 \\ 1 & 1 & 1 & 0 & \dots & \dots & \dots & 1 \\ \dots & \dots & \dots & \dots & \dots & \dots & \dots & \dots \\ \dots & \dots & \dots & \dots & \dots & \dots & \dots & \dots \\ \dots & \dots & \dots & \dots & \dots & \dots & \dots & \dots \\ \dots & \dots & \dots & \dots & \dots & \dots & \dots & \dots \\ 1 & 1 & 1 & 1 & \dots & \dots & 0 & 1 \\ 1 & 1 & 1 & 1 & \dots & \dots & \dots & 0 \end{pmatrix}$$

Setting $A(X_1) = B(\Delta_1)$ and $A(X_2) = B(\Delta_2)$ as adjacency matrices for the graphs X_1 and X_2 respectively.

$$A(X_1) = B(\Delta_1) = \begin{pmatrix} 1 & 0 & 0 & 0 & \dots & \dots & \dots & 0 \\ 0 & 1 & 0 & 0 & \dots & \dots & \dots & 0 \\ 0 & 0 & 1 & 0 & \dots & \dots & \dots & 0 \\ 0 & 0 & 0 & 1 & \dots & \dots & \dots & 0 \\ \dots & \dots & \dots & \dots & \dots & \dots & \dots & \dots \\ \dots & \dots & \dots & \dots & \dots & \dots & \dots & \dots \\ \dots & \dots & \dots & \dots & \dots & \dots & \dots & \dots \\ \dots & \dots & \dots & \dots & \dots & \dots & \dots & \dots \\ 0 & 0 & 0 & 0 & \dots & \dots & 1 & 0 \\ 0 & 0 & 0 & 0 & \dots & \dots & \dots & 1 \end{pmatrix}$$

$$A(X_2) = B(\Delta_2) = \begin{pmatrix} 0 & 1 & 1 & 1 & \dots & \dots & \dots & 1 \\ 1 & 0 & 1 & 1 & \dots & \dots & \dots & 1 \\ 1 & 1 & 0 & 1 & \dots & \dots & \dots & 1 \\ 1 & 1 & 1 & 0 & \dots & \dots & \dots & 1 \\ \dots & \dots & \dots & \dots & \dots & \dots & \dots & \dots \\ \dots & \dots & \dots & \dots & \dots & \dots & \dots & \dots \\ \dots & \dots & \dots & \dots & \dots & \dots & \dots & \dots \\ \dots & \dots & \dots & \dots & \dots & \dots & \dots & \dots \\ 1 & 1 & 1 & 1 & \dots & \dots & 0 & 1 \\ 1 & 1 & 1 & 1 & \dots & \dots & \dots & 0 \end{pmatrix}$$

The graphs are constructed with X_1 and X_2 being a null graph and a complete graph respectively of n vertices.

■

Alternative Proof for $n > 3$

In general, the Stabilizer of 1 is given by:

$$G_1 = \{ 1, (234), (235), \dots, (n-2 \ n \ n-1), \dots, (n-3 \ n)(n-2 \ n-1) \} \text{ such that ;}$$

$$|G_1| = \frac{(n-1)!}{2}$$

For $n > 3$, the orbits of G_1 are $\Delta_1 = \{ 1 \}$, $\Delta_2 = \{ 2, 3, 4, \dots, n \}$

\Rightarrow There are 2 orbits for the stabilizer of 1.

Clearly, the 2 orbits of G_1 are self-paired by definition 1.1.1.11.

This implies that the matrices associated with the sub-orbits of G are symmetric.

But the number of graphs whose groups of automorphisms contain G is equal to the total number of symmetric matrices, then;

The number of graphs whose groups of automorphisms contain $G = 2$.

Since the identity matrix yields the null graph while the other symmetric matrix yields a complete graph, the proof is complete.

■

Example 3.1

Taking $G = A_{15}$, find the number of regular graphs whose groups of Automorphism contains G .

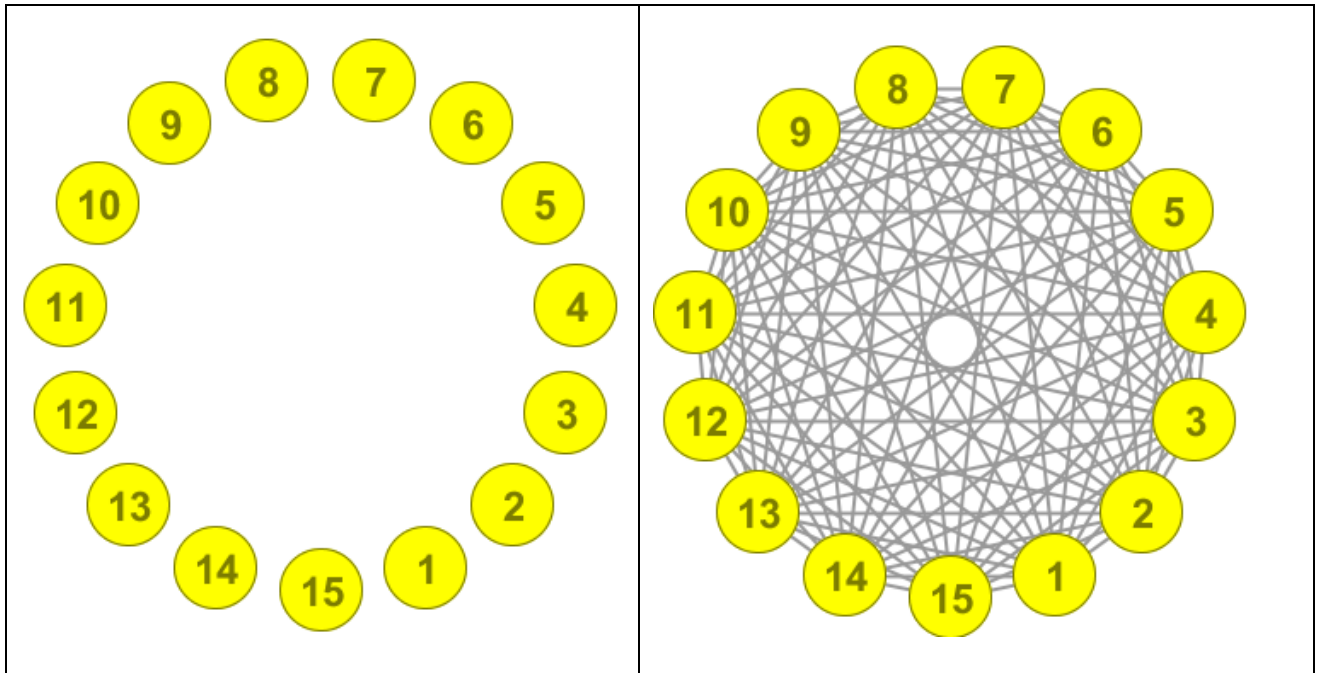
Solution

The orbits of A_{15} are $\Delta_1 = \{ 1 \}$, $\Delta_2 = \{ 2, 3, 4, 5, 6, 7, 8, 9, 10, 11, 12, 13, 14, 15 \}$

The 2 orbits are self- paired and hence they form symmetric matrices.

The 2 symmetric matrices may be used to form 2 adjacency matrices for our graphs

⇒ There are 2 graphs whose groups of automorphisms contains A_{15} .



CHAPTER FOUR

AUTOMORPHISMS GROUPS OF GRAPHS REPRESENTING SYMMETRIC GROUPS

In this chapter, we shall let G be the Symmetric Group S_n of degree n and X be the set of vertices of a regular n -gon ordered cyclically from 1 to n . The chapter is divided into three sections. Section 4.1 investigates transitivity of the action of G on X . In Section 4.2, we illustrate Graphs whose Automorphisms Groups represent the Symmetric Group, S_n of degree n , $n \geq 3$. The main result in this section are given in the form of theorems.

4.1 Transitivity of the symmetric group of degree n , S_n

Let G be a symmetric group S_n acting on a set $X = \{1, 2, 3, \dots, n\}$.

Given $g \in G$ and $x \in X$, any element $x \in X$ is fixed by exactly $\frac{|G|}{n} = (n-1)!$.

$$\Rightarrow |\text{Stab}_G x| = (n-1)! \quad \forall x \in X, \text{ and so}$$

$$\sum_{x \in X} |\text{Stab}_G x| = n!$$

$|G| = n!$, The degree of G being n we have that

$$|\text{orb}_G x| = 1.$$

Hence

G acts transitively X .

4.2 Automorphisms groups of Graphs representing the Symmetric group, degree n ; $n \geq 3$.

By Schur's algorithm, we construct all regular simple graphs whose Automorphism groups represent the symmetric groups. We establish a formula for finding the tally of graphs for any S_n .

4.2.1 Automorphisms groups of Graphs representing S_3

The elements of $G = S_3$ are $\{1, (12), (13), (23), (123), (132)\}$

The stabilizer, $\text{Stab}_G 1$, given by $G_1 = \{1, (23)\}$.

The orbits of G_1 are: $\Delta_1 = \{1\}$, $\Delta_2 = \{2,3\}$

Using Schur's algorithm, we find $B(\Delta_2)$ as below;

$$B(\Delta_2) = \begin{pmatrix} 0 & 1 & 1 \\ 1 & 0 & 1 \\ 1 & 1 & 0 \end{pmatrix}$$

Setting $A(X_1) = B(\Delta_1)$ and $A(X_2) = B(\Delta_2)$ as adjacency matrices for the graphs X_1 and X_2 respectively.

$$A(X_1) = B(\Delta_1) = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

$$A(X_2) = B(\Delta_2) = \begin{pmatrix} 0 & 1 & 1 \\ 1 & 0 & 1 \\ 1 & 1 & 0 \end{pmatrix}$$

The two graphs are constructed as below:

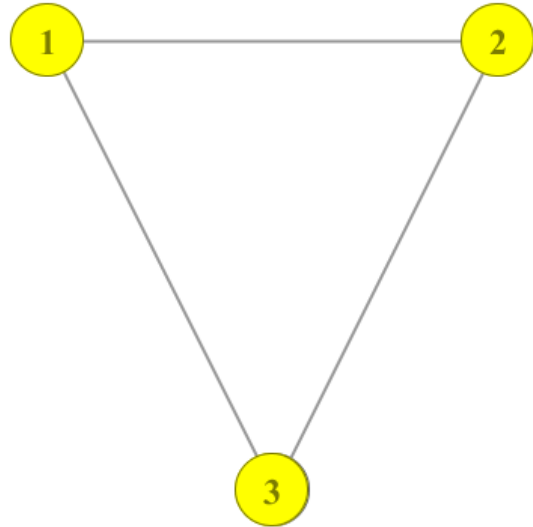


Figure 4.1 (a) X_1

Figure 4.1 (b) X_2

4.2.2 Automorphisms groups of Graphs representing S_4

The elements of $G = S_4$ are $\{1, (12), (13), (14), (23), (24), (34), (123), (124), (132), (134), (142), (143), (234), (243), (12)(34), (13)(24), (14)(23), (1234), (1243), (1324), (1342), (1423), (1432)\}$

The stabilizer, $\text{Stab}_G 1$, given by $G_1 = \{1, (23), (24), (34), (234), (243)\}$.

The orbits of G_1 are; $\Delta_1 = \{1\}$, $\Delta_2 = \{2, 3, 4\}$

Using Schur's algorithm, we establish $B(\Delta_2)$ as below;

$$B(\Delta_2) = \begin{pmatrix} 0 & 1 & 1 & 1 \\ 1 & 0 & 1 & 1 \\ 1 & 1 & 0 & 1 \\ 1 & 1 & 1 & 0 \end{pmatrix}$$

Setting $A(X_1) = B(\Delta_1)$ and $A(X_2) = B(\Delta_2)$ as adjacency matrices for the graphs X_1 and X_2 respectively.

$$A(X_1) = B(\Delta_1) = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

$$A(X_2) = B(\Delta_2) = \begin{pmatrix} 0 & 1 & 1 & 1 \\ 1 & 0 & 1 & 1 \\ 1 & 1 & 0 & 1 \\ 1 & 1 & 1 & 0 \end{pmatrix}$$

The two graphs are constructed as below:

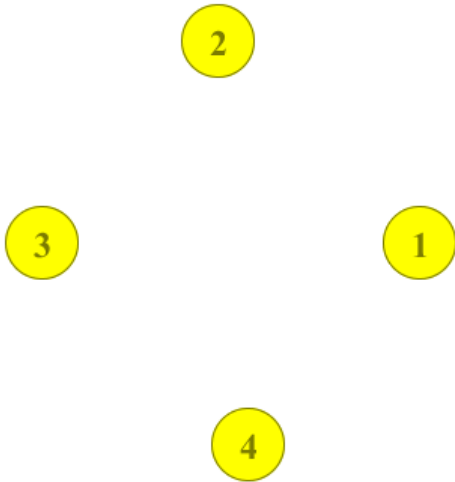


Figure 4.2 (a) X_1

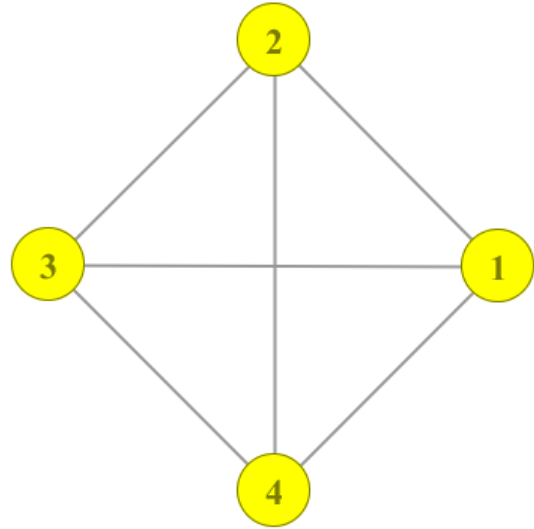


Figure 4.2 (b) X_2

4.2.3 Automorphisms groups of Graphs representing S_5

The elements of $G = S_5$ are $\{1, (12), (13), (14), (15), (23), (24), (25), (34), (35), (45), (123), (124), (125), (132), (134), (135), (142), (143), (145), (152), (153), (154), (234), (235), (243), (245), (253), (254), (345), (354), (1234), (1235), (1243), (1245), (1253), (1254), (1324), (1325), (1342), (1345), (1352), (1354), (1423), (1425), (1432), (1435), (1452), (1453), (1523), (1524), (1532), (1534), (1542), (1543), (2345), (2354), (2435), (2453), (2534), (2543), (12345), (12354), (12435), (12453), (12534), (12543), (13245), (13254), (13425), (13452), (13524), (13542), (14235), (14253), (14325), (14352), (14523), (14532), (15234), (15243), (15324), (15342), (15423), (15432), (12)(34), (12)(35), (12)(45), (13)(24), (13)(25), (13)(45), (14)(23), (14)(25), (14)(35), (15)(23), (15)(24), (15)(34), (23)(45), (24)(35), (25)(34), (12)(345), (12)(354), (13)(245), (13)(254), (14)(235), (14)(253), (15)(234), (15)(243), (23)(145), (23)(154), (24)(135), (24)(153), (25)(134), (25)(143), (34)(125), (34)(152), (35)(124), (35)(142), (45)(123), (45)(132)\}$

The stabilizer, $\text{Stab}_G 1$, given by

$$G_1 = \{1, (23), (24), (25), (34), (35), (45), (234), (235), (243), (245), (253), (254), (345), (354), (2345), (2354), (2435), (2453), (2534), (2543), (23)(45), (24)(35), (25)(34)\}$$

The orbits of G_1 are: $\Delta_1 = \{1\}$, $\Delta_2 = \{2, 3, 4, 5\}$

Using Schur's algorithm, we form $B(\Delta_2)$ as below;

$$B(\Delta_2) = \begin{pmatrix} 0 & 1 & 1 & 1 & 1 \\ 1 & 0 & 1 & 1 & 1 \\ 1 & 1 & 0 & 1 & 1 \\ 1 & 1 & 1 & 0 & 1 \\ 1 & 1 & 1 & 1 & 0 \end{pmatrix}$$

Setting $A(X_1) = B(\Delta_1)$ and $A(X_2) = B(\Delta_2)$ as adjacency matrices for the graphs X_1 and X_2 respectively.

$$A(X_1) = B(\Delta_1) = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{pmatrix}$$

$$A(X_2) = B(\Delta_2) = \begin{pmatrix} 0 & 1 & 1 & 1 & 1 \\ 1 & 0 & 1 & 1 & 1 \\ 1 & 1 & 0 & 1 & 1 \\ 1 & 1 & 1 & 0 & 1 \\ 1 & 1 & 1 & 1 & 0 \end{pmatrix}$$

The two graphs are constructed as below:

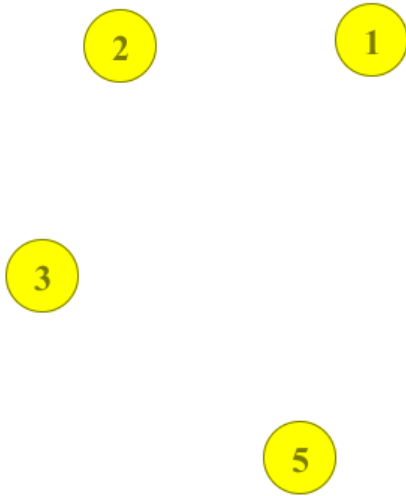


Figure 4.3 (a) X_1

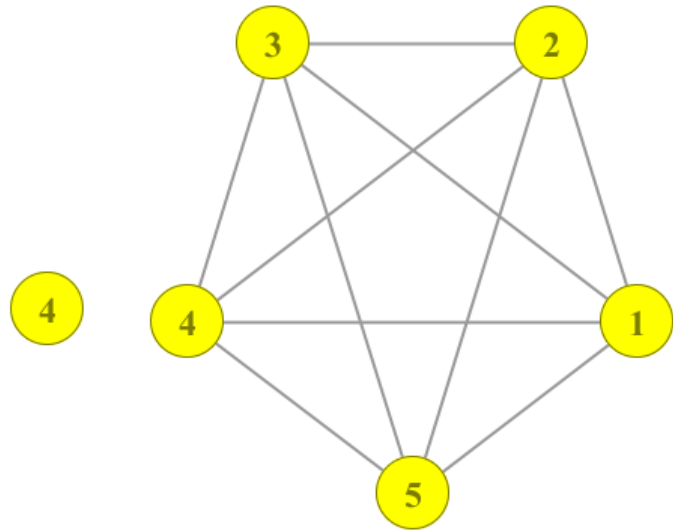


Figure 4.3 (b) X_2

Theorem 4.1

Suppose G is S_n , the symmetric group of degree n , the number of regular graphs whose groups of automorphisms represent G is 2 that is the null and the complete graphs.

Proof

In general, the Stabilizer of 1 is given by:

$G_1 = \{1, (23), (24)\dots (n-1 n), (234), (235)\dots (n-2 n n-1)\dots (n-3 n)(n-2 n-1)\}$ Such that;

$$|G_1| = (n - 1)!$$

The orbits of G_1 are $\Delta_1 = \{1\}$, $\Delta_2 = \{2, 3, 4, \dots, n\}$

Hence, Using Schur's algorithm, we establish $B(\Delta_2)$, as below;

$$B(\Delta_2) = \begin{pmatrix} 0 & 1 & 1 & 1 & \dots & \dots & \dots & 1 \\ 1 & 0 & 1 & 1 & \dots & \dots & \dots & 1 \\ 1 & 1 & 0 & 1 & \dots & \dots & \dots & 1 \\ 1 & 1 & 1 & 0 & \dots & \dots & \dots & 1 \\ \dots & \dots & \dots & \dots & \dots & \dots & \dots & \dots \\ \dots & \dots & \dots & \dots & \dots & \dots & \dots & \dots \\ \dots & \dots & \dots & \dots & \dots & \dots & \dots & \dots \\ \dots & \dots & \dots & \dots & \dots & \dots & \dots & \dots \\ 1 & 1 & 1 & 1 & \dots & \dots & 0 & 1 \\ 1 & 1 & 1 & 1 & \dots & \dots & \dots & 0 \end{pmatrix}$$

Setting $A(X_1) = B(\Delta_1)$ and $A(X_2) = B(\Delta_2)$ as adjacency matrices for the graphs X_1 and X_2 respectively.

$$A(X_1) = B(\Delta_1) = \begin{pmatrix} 1 & 0 & 0 & 0 & \dots & \dots & \dots & 0 \\ 0 & 1 & 0 & 0 & \dots & \dots & \dots & 0 \\ 0 & 0 & 1 & 0 & \dots & \dots & \dots & 0 \\ 0 & 0 & 0 & 1 & \dots & \dots & \dots & 0 \\ \dots & \dots & \dots & \dots & \dots & \dots & \dots & \dots \\ \dots & \dots & \dots & \dots & \dots & \dots & \dots & \dots \\ \dots & \dots & \dots & \dots & \dots & \dots & \dots & \dots \\ \dots & \dots & \dots & \dots & \dots & \dots & \dots & \dots \\ 0 & 0 & 0 & 0 & \dots & \dots & 1 & 0 \\ 0 & 0 & 0 & 0 & \dots & \dots & \dots & 1 \end{pmatrix}$$

$$A(X_2) = B(\Delta_2) = \begin{pmatrix} 0 & 1 & 1 & 1 & \dots & \dots & \dots & 1 \\ 1 & 0 & 1 & 1 & \dots & \dots & \dots & 1 \\ 1 & 1 & 0 & 1 & \dots & \dots & \dots & 1 \\ 1 & 1 & 1 & 0 & \dots & \dots & \dots & 1 \\ \dots & \dots & \dots & \dots & \dots & \dots & \dots & \dots \\ \dots & \dots & \dots & \dots & \dots & \dots & \dots & \dots \\ \dots & \dots & \dots & \dots & \dots & \dots & \dots & \dots \\ \dots & \dots & \dots & \dots & \dots & \dots & \dots & \dots \\ 1 & 1 & 1 & 1 & \dots & \dots & 0 & 1 \\ 1 & 1 & 1 & 1 & \dots & \dots & \dots & 0 \end{pmatrix}$$

Two graphs may be constructed with X_1 and X_2 being a null graph and a complete graph respectively of n vertices.

■

Alternative Proof

In general, the Stabilizer of 1 is given by:

$$G_1 = \{ 1, (234), (235), \dots, (n-2 \ n \ n-1), \dots, (n-3 \ n)(n-2 \ n-1) \} \text{ such that ;}$$

$$|G_1| = \frac{(n-1)!}{2}$$

The orbits of G_1 are $\Delta_1 = \{ 1 \}$, $\Delta_2 = \{ 2, 3, 4, \dots, n \}$

\Rightarrow There are 2 orbits for the stabilizer of 1.

Clearly, the 2 orbits of G_1 are self-paired by definition 1.1.1.11.

This implies that the matrices associated with the sub-orbits of G are symmetric.

But the number of graphs whose groups of automorphisms represent G is equal to the total number of symmetric matrices, then; the number of graphs whose groups of automorphisms represent $G = 2$.

Since the identity matrix yields the null graph while the other symmetric matrix yields a complete graph, the proof is complete.

■

Example 4.1

Taking $G = S_{15}$, find the number of regular graphs whose groups of automorphism represents G .

Solution

The orbits of S_{15} are $\Delta_1 = \{ 1 \}$, $\Delta_2 = \{ 2, 3, 4, 5, 6, 7, 8, 9, 10, 11, 12, 13, 14, 15 \}$

The 2 orbits are self- paired and hence they form symmetric matrices.

The 2 symmetric matrices may be used to form 2 adjacency matrices for our graphs

\Rightarrow There are 2 graphs whose groups of automorphisms represents S_{15} .

CHAPTER FIVE**CONCLUSIONS AND RECOMMENDATIONS****5.1 Conclusions**

From this project, we have employed Schur's algorithm to construct graphs whose groups of automorphisms contains or represents the Alternating and Symmetric groups respectively. We have as well calculated the number of graphs whose Automorphism groups contain or represent the Alternating and Symmetric groups respectively.

It has been shown that the actions of the groups A_n and S_n on X are transitive. We have constructed the varied graphs for the corresponding group actions. The number of graphs whose groups of automorphisms contain A_n is 2 i.e. the null and the complete graphs. Similarly, the number of graphs whose groups of automorphisms represent S_n is 2 that is the null and the complete graphs.

5.2 Recommendations

We have investigated the graphs whose Automorphism groups contain or represent the Alternating and Symmetric groups using Schur's algorithm. Schur's algorithm is an interesting area of study that may give more for research purposes. One may employ this study to Graphs whose Groups of Automorphisms contain or represent other finite permutation groups.

We further recommend study to be done on product graphs and their constructions as well. Aside from the above areas identified for further research, one may want to develop algorithms necessary for computer program that may be used to design and employ these kinds of graphs.

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