RANKS AND SUBDEGREES OF THE SYMMETRIC GROUP S_n ACTING ON ORDERED r-ELEMENT SUBSETS

Rimberia Jane Kagwiria (M.Sc.)

Reg. No.: I84/12768/2009

A THESIS SUBMITTED IN PARTIAL FULFILLMENT OF THE REQUIREMENTS FOR THE AWARD OF THE DEGREE OF DOCTOR OF PHILOSOPHY (MATHEMATICS) IN THE SCHOOL OF PURE AND APPLIED SCIENCES OF KENYATTA UNIVERSITY

NOVEMBER 2011

DECLARATION

This thesis is my original work and has not been presented for a degree award in any other university or for any other award.

Rimberia Jane Kagwiria

Signature.. Date..........................

This thesis has been submitted for examination with our approval as university supervisors.

Dr. Ireri Kamuti

Signature.. Date..........................

Department of Mathematics

Kenyatta University

Dr. Benard Kivunge

Signature.. Date..........................

Department of Mathematics

Kenyatta University

DEDICATION

To my lovely mum ***Zipporah Gacheri Rimberia***

ACKNOWLEDGEMENT

My sincere thanks, glory and honour goes to the Almighty God for His unfailing provision, protection and sustainance throughout the study period.

I am greatly indebted to my supervisors; Dr. Kamuti and Dr. Kivunge who tirelessly and patiently inspired me in undertaking this work. Their timely guidance, encouragement and support have not only made the completion of this research possible but have left an impression which will continue to influence my work.

My sincere gratitude goes to Dr. Malonza, the chairman of Mathematics department for his advice and encouragement throughout this course.

I am most grateful to my mum Zipporah, my sister Lucy, my brothers Elijah and Joseph and my aunties Susan, Harriet and the late Beatrice for their support, prayers and encouragement that they gave me throughout the study period.

TABLE OF CONTENTS

CHAPTER ONE

INTRODUCTION

CHAPTER TWO

LITERATURE REVIEW

CHAPTER THREE

ACTION OF S_n ON ORDERED 2-ELEMENT SUBSETS, 3-ELEMENT SUBSETS AND 4-ELEMENT SUBSETS

CHAPTER FOUR

RANKS AND SUBDEGREES OF S_n ACTING ON $X^{[r]}$

CHAPTER FIVE

SUBORBITAL GRAPHS OF S_n ACTING ON $X^{[r]}$

CHAPTER SIX

CONCLUSION AND RECOMMENDATIONS FOR FURTHER RESEARCH

LIST OF TABLES

LIST OF FIGURES

SYMBOLS

S_n - Symmetric group of degree n

A_n - Alternating group of degree n

D_n - Dihedral group of order $2n$

G_x - Stabilizer of a point x in G

$N_G(H)$ - Normalizer of H in G

$H \leq G$ - H is a subgroup of G

$|X|$ - Cardinality of the set X

$X^{[r]}$ - The set of all ordered r-element subsets from X

$R(G)$ - The rank of a permutation group G acting on the set X

$\mathbb{Z}$ - The set of all integers

$\mathbb{N}$ - The set of all natural numbers

$\mathbb{Q}$ - The set of all rational numbers

$\hat{\mathbb{Q}}$ - The rational projective line

$\mathbb{C}$ - The set of all complex numbers

ϕ - The empty set

$X \times X$ - Cartesian square of the set X

$\mathbb{Z}_2^h,\ h \geq 1$ - Direct product $\mathbb{Z}_2 \times \mathbb{Z}_2 \times \ldots \times \mathbb{Z}_2$ for some $h \geq 1$ number of factors

$\mathrm{Im}(z)$	-	Imaginary part of the complex number *z*
(m, n)	-	greatest common divisor of *m* and *n*
$m \mid n$	-	*m* divides *n*
$\lvert Fix(g) \rvert$	-	Number of points fixed by *g*
$SL(n, q)$	-	Special linear group where, *n* is a positive integer and *q* is a power of a prime
$PGL(n, q)$	-	Projective general linear group
$PSL(n, q)$	-	Projective special linear group
$G \cong P\Gamma L_2(8)$	-	*G* is isomorphic to the projective semilinear group
$\binom{n}{r}$	-	*n* combination *r*
$^{n}P_{r}$	-	*n* permutation *r*
$m(A, B, G)$	-	The mark of subgroup *A* in the representation of *G* on the cosets of subgroup *B*
π	-	Character of permutation representation of *G* acting on *X*
Δ	-	A suborbit of *G*
Δ^*	-	The suborbit of *G* paired with Δ
Γ	-	The modular group
$\Gamma_0(N)$	-	Congruence subgroup of Γ

$Nor(N)$	-	Normalizer of $\Gamma_0(N)$ in $PSL(2,\mathbb{R})$
$\mathscr{G}$	-	Suborbital graph
$\mathscr{F}$	-	The Farey graph
$n(\mathscr{G}_j)$	-	Number of connected components in the suborbital graph $\mathscr{G}_j$

ABSTRACT

The action of the symmetric group S_n on ordered subsets from the set $X = \{1, 2, \ldots, n\}$ is an aspect that seems to have received little attention for a long time. Most studies have focused on the action of S_n on unordered subsets leaving many properties about its action on ordered subsets unknown. This research is set to determine the rank and subdegrees of S_n acting on $X^{[r]}$, the set of all ordered *r*-element subsets from *X*. Particular cases when $r = 2, 3$ and 4 have been considered first and then a generalization has been made for any value of *r* and *n*. In the action of S_n on $X^{[2]}$, $X^{[3]}$ and $X^{[4]}$, the rank is shown to be 7, 34 and 209 respectively. By generalizing these results, we have come up with the formulas for the rank and subdegrees of S_n acting on $X^{[r]}$. This study shows that if $n \geq 2r$, then for a fixed value of *r*, the rank of S_n on $X^{[r]}$ is a constant while the subdegrees vary with *n*. The action of S_n on $X^{[r]}$ has been shown to be both transitive and imprimitive. We have also formulated the conditions for a suborbit of S_n corresponding to this action to be either self-paired or paired with another. A formula for computing the number of self-paired suborbits has also been derived using a theorem from character theory. Finally, the suborbital graphs corresponding to this action have been constructed and their theoretic properties studied. The results show that all these graphs are disconnected. We have also come up with the formulas for computing the number of connected components in these graphs. The girth sizes of the suborbital graphs corresponding to suborbits of S_n containing exactly *r* elements and no element from $A = \{1, 2, ..., r\}$ have also been determined. For the suborbital graphs corresponding to self-paired suborbits of S_n with exactly *r* elements from *A*, the girth is shown to be zero while that of paired suborbits with precisely *r* elements from *A* is shown to be three. This study also reveals that the girth of the suborbital graph corresponding to the suborbit of S_n with no element from A is three provided $n \geq 3r$. The results obtained have been summarized in form of theorems while others are displayed in tables.

CHAPTER ONE

INTRODUCTION

1.1 Definitions and Preliminary Results

Definition 1.1.1

If *X* is a finite set $\{1, 2, \ldots, n\}$, then the symmetric group of degree *n* is the group of all permutations of *X* under the binary operation of composition of maps. It is denoted by S_n and has order $n!$.

Definition 1.1.2

If $\sigma \in S_n$ has α_1 cycles of length 1, α_2 cycles of length 2, …, α_n cycles of length *n*, then the cycle type of σ is the *n*-tuple $(\alpha_1, \alpha_2, \ldots, \alpha_n)$.

Theorem 1.1.3

Two permutations in S_n are conjugate if and only if they have the same cycle type; and if $g \in S_n$ has cycle type $(\alpha_1, \alpha_2, \ldots, \alpha_n)$, then the number of permutations in S_n conjugate to *g* is

$$\frac{n!}{\prod_{i=1}^{n} \alpha_i ! i^{\alpha_i}}.$$ (Krishnamurthy, 1985)

Definition 1.1.4

Let *X* be a set and *G* be a group. We say that *G* acts on the left of the set *X* if for each $x \in X$ and $g \in G$ there corresponds a unique element $gx \in X$ such that, for all $x \in X$ and $g_1, g_2 \in G$

i) $(g_1 g_2)x = g_1(g_2 x)$

ii) $1x = x$, where 1 is the identity in *G*.

Definition 1.1.5

Let *G* act on the set *X*, and let $x \in X$. The orbit of *x* is the set

$Orb_G(x) = \{gx \mid g \in G\}$.

Definition 1.1.6

Let *G* act on the set *X*, and let $x \in X$. The stabilizer of *x* in *G* is the set

$Stab_G(x) = \{g \in G \mid gx = x\}$.

This set is also denoted by G_x. It can easily be shown that $Stab_G(x) \leq G$.

Theorem 1.1.7 (Orbit-Stabilizer Theorem)

Let *G* be a group acting on a finite set *X*, and let $x \in X$. Then

$$|Orb_G(x)| = |G : Stab_G(x)|. \text{ (Rose, 1978)}$$

Definition 1.1.8

The action of a group *G* on a set *X* is said to be transitive if for each pair of points $x, y \in X$, there exists $g \in G$ such that $gx = y$; in other words the action has only one orbit.

Definition 1.1.9

Suppose *G* is a group acting transitively on a set *X* and let G_x be the stabilizer in *G* of a point $x \in X$. The orbits $\Delta_0 = \{x\}, \Delta_1, \Delta_2, \ldots, \Delta_{r-1}$ of G_x on *X* are known as suborbits of *G*. The rank of *G* in this case is *r*. The sizes $n_i = |\Delta_i|$ $(i = 0, 1, 2, \ldots, r-1)$ often called the 'lengths' of suborbits are known as the subdegrees of *G*. It can be shown that both *r* and the cardinalities of the suborbits Δ_i $(i = 0, 1, 2, \ldots, r-1)$ are independent of the choices of $x \in X$ (Ivanov *et al.*, 1983).

Definition 1.1.10

Let G act transitively on a set X and let Δ be an orbit of G_x on X. Define $\Delta^* = \{gx \mid g \in G,\ x \in g\Delta\}$, then Δ^* is also an orbit of G_x and is called the G_x-orbit (or G-suborbit) paired with Δ (Wielandt, 1964). Clearly $\Delta^{**} = \Delta$ and $|\Delta| = |\Delta^*|$. If $\Delta = \Delta^*$, then Δ is said to be self-paired. The transitive constituent of G_x on Δ, denoted G_x^Δ is the permutation group on Δ obtained by restricting elements of G_x to Δ.

Theorem 1.1.11

Let G act transitively on X. Then G_x has an orbit different from $\{x\}$ and self-paired if and only if G has even order. (Wielandt, 1964)

Definition 1.1.12

Let G act on a set X, then the character π of permutation representation of G on X is defined by

$$\pi(g) = |Fix(g)|, \text{ for all } \mathrm{g} \in \mathrm{G}.$$

Theorem 1.1.13

Let G act transitively on a set X, and let $g \in G$. Suppose π is the character of the permutation representation of G on X, then the number of self-paired suborbits of G is given by

$$n_\pi = \frac{1}{|G|}\sum_{g \in G} \pi\left(g^2\right). \text{ (Cameron, 1974)}$$

Definition 1.1.14

Suppose that G acts transitively on a finite set X. Then a subset Y of X is said to be a block for the action if, for each $g \in G$, either $gY = Y$ or $gY \cap Y = \phi$. In particular, ϕ, X

and all 1-element subsets of *X* are obviously blocks: these are called the trivial blocks. The action is said to be primitive if the only blocks are the trivial blocks; otherwise the action is imprimitive.

Theorem 1.1.15

Let G be transitive on X and let G_x be the stabilizer of the point $x \in X$. Let $\Delta_0 = \{x\}, \Delta_1, \Delta_2, \ldots, \Delta_{k-1}$ be orbits of G_x on X of lengths $n_0 = 1, n_1, n_2, \ldots, n_{k-1}$, where $n_0 \leq n_1 \leq n_2 \leq \ldots \leq n_{k-1}$. If there exists an index $j > 0$ such that $n_j > n_1 n_{j-1}$, then G is imprimitive. (Wielandt, 1964)

Definition 1.1.16

A graph is a diagram consisting of a set *V* whose elements are called vertices, nodes or points and a set *E* of unordered pairs of vertices called edges or lines. We denote such a graph by $G(V, E)$ or by *G* if there is no ambiguity of *V* and *E*.

Definition 1.1.17

A directed graph is a diagram consisting of a set *V* whose elements are called vertices, nodes or points and a set *E* of ordered pairs of vertices called directed edges or arcs.

Definition 1.1.18

A walk in a graph *G* consists of a finite sequence of edges of the form $v_0v_1, v_1v_2, v_2v_3, \ldots, v_{m-1}v_m$. The number *m* of edges is called the length of the walk. A path is a walk in which all the vertices are distinct. A cycle or circuit is a closed path.

Definition 1.1.19

The girth of a graph *G* is the length of the shortest cycle in *G*.

Definition 1.1.20

A graph *G* is said to be connected if every pair of vertices of *G* is joined by some path; otherwise, *G* is disconnected. A connected component of *G* is a maximal connected subgraph of *G*. Each vertex and edge of *G* belongs to precisely one connected component of *G*.

Theorem 1.1.21

Let *G* be transitive on *X* and let suborbit $\Delta_i \ \left(i = 0, 1, \ldots, r-1\right)$ correspond to suborbital O_i. Then the corresponding suborbital graph $\mathscr{G}_i$ is undirected if Δ_i is self-paired and directed if Δ_i is not self-paired. (Sims, 1967)

Theorem 1.1.22

Let *G* be transitive on *X* and let G_x be the stabilizer of the point $x \in X$. Suppose $\Delta_0 = \left\{x\right\}, \Delta_1, \Delta_2, \ldots, \Delta_{r-1}$ are the orbits of G_x on *X* and let $\mathrm{O}_i \subseteq X \times X,\ i = 0, 1, \ldots, r-1$ be the suborbital corresponding to $\Delta_i,\ i = 0,\ 1, \ldots, r-1$. Then *G* is primitive if and only if each suborbital graph $\mathscr{G}_i,\ i = 1, 2, \ldots, r-1$ is connected. (Sims, 1967)

1.2 Background information

The rank and subdegrees are very significant characteristics of a group action. In certain cases, knowledge of these properties is sufficient for resolution of the following problems:

i) identification of rank 3 graphs (Hubaut, 1975);

ii) proof of the existence and non-existence of distance-transitive graphs (Ivanov *et al.*, 1984);

iii) computation of the decomposition of the permutation character of the group into irreducibles by means of the character table (Faradzev *et al*., 1994).

There are different approaches to computation of subdegrees of a transitive permutation group. One of these approaches relies on the famous Cauchy-Frobenius Lemma. This lemma is usually but erroneously attributed to Burnside (1911). It is stated as follows:

Theorem 1.2.1 (Cauchy-Frobenius Lemma)

Let *G* be a finite group acting on a finite set *X*. Then the number of *G*-orbits in *X* is

$$\frac{1}{|G|}\sum_{g\in G}\left|Fix(g)\right|,$$

where $\left|Fix(g)\right| = \left|\{x \in X \mid gx = x\}\right|$. (Rose, 1978)

This is one of the classical enumeration techniques and is applicable to groups of finite order only.

The second approach relies on solution of systems of linear equations arising from the computation of marks of a finite group. For any two subgroups *A* and *B* of a group *G*, the mark of *A* in the representation of *G* on the cosets of *B* is the number $m(A, B, G)$ of the cosets of *B* that are fixed by every permutation of *A* (Ivanov *et al.*, 1983). If *G* is transitive on *X* and $\Delta_0 = \{x\}, \Delta_1, \Delta_2, \ldots, \Delta_{r-1}$ are suborbits of G, then the action of *G* on *X* is equivalent to its action on the cosets of $H = G_x,\ x \in X$, while that of *H* on Δ_i $(i = 0, 1, 2, \ldots, r-1)$ is equivalent to its action on the set of cosets of some subgroup $F \leq H$.

Now, let $\{H_1, H_2, \ldots, H_t\}$ be a set of representatives of all distinct conjugacy classes of subgroups of *H* in *G*, ordered such that $|H_1| \leq |H_2| \leq \ldots \leq |H_t| = |H|$. Form a matrix $M = (m_{ij})$, where $m_{ij} = m(H_j, H_i, H)$. We call *M* the table of marks of *H*. If we denote by Q_i the number of suborbits Δ_j on which the action of *H* is equivalent to its action on the cosets of H_i $(i = 1, 2, \ldots, t)$, then by computing all Q_i we get the subdegrees of *G* on *X*.

Theorem 1.2.2

The numbers Q_i satisfy the system of linear equations

$$\sum_{i=j}^{t} Q_i \; m(H_j, H_i, H) = m(H_j, H, G) \ ,$$

for each $j = 1, 2, \ldots, t$. (Ivanov *et a*l., 1983)

This method can be applied not only in concrete groups, but also in infinite series of groups provided detailed structural information about the group under investigation is known.

The final approach relies on the use of combinatorial arguments. In this method, all the possible arrangements of the combinatorial objects are listed down and then the fundamental principles of counting are applied. The technique is applicable to permutation groups of both finite and infinite order. Consequently, in this research we have used this approach to compute the rank and subdegrees of the symmetric group S_n acting on ordered *r*-element subsets from $X = \{1, 2, \ldots, n\}$.

Now, let G be a transitive permutation group acting on X. Then G acts on $X \times X$ by $g(x, y) = (g(x), g(y)),\ g \in G,\ x, y \in X$. The orbits of this action are called suborbitals of G. The orbit containing (x, y) is denoted by $\mathrm{O}(x, y)$. Now, if $\mathrm{O} \subseteq X \times X$ is a G-orbit, then for any $x \in X$, $\Delta = \{ y \in X \mid (x, y) \in \mathrm{O} \}$ is a G_x-orbit on X. Conversely, if $\Delta \subseteq X$ is a G_x-orbit, then $\mathrm{O} = \{ (gx, gy) \mid g \in G,\ y \in \Delta \}$ is G-orbit on $X \times X$. We say that Δ corresponds to O.

From $\mathrm{O}(x, y)$ we can form a suborbital graph $\mathscr{G}(x, y)$: its vertices are elements of X, and there is a directed edge from v to w if and only if $(v, w) \in \mathrm{O}(x, y)$. Clearly $\mathrm{O}(y, x)$ is also a suborbital, and it is either equal to or disjoint from $\mathrm{O}(x, y)$. In the former case, $\mathscr{G}(x, y) = \mathscr{G}(y, x)$ and the graph consists of pairs of oppositely directed edges. It is convenient to replace each such pair by a single undirected edge, so that we have an undirected graph which we call self-paired. In the latter case, $\mathscr{G}(x, y)$ is just $\mathscr{G}(y, x)$ with arrows reversed, and we call $\mathscr{G}(x, y)$ and $\mathscr{G}(y, x)$ paired suborbital graphs. If $x = y$, then $\mathrm{O}(x, x) = \{ (x, x) \mid x \in X \}$ is the diagonal of $X \times X$. The corresponding suborbital graph $\mathscr{G}(x, x)$, called the trivial suborbital graph, is self-paired and consists of a loop based at each vertex $x \in X$.

1.3 Statement of the problem

Although the action of S_n on ordered subsets from the set $X = \{1, 2, \dots, n\}$ has been known for a long time, little has been done in this area. For the last 40 years, a lot of attention has been given to the action of S_n on unordered subsets leaving many properties about its action on ordered subsets unknown. This study is set to determine the rank and subdegrees of the symmetric group S_n acting on $X^{[r]}$, the set of all ordered *r*-element subsets from the set $X = \{1, 2, \dots, n\}$. Suborbital graphs corresponding to this action will also be constructed and their theoretic properties studied. These properties include; self-pairing, pairing, girth sizes, connectivity and number of connected components.

1.4 Hypotheses

i) The rank and subdegrees of the symmetric group S_n acting on $X^{[r]}$ where $X = \{1, 2, \dots, n\}$ are functions of both *n* and *r*.

ii) Some suborbital graphs corresponding to the action of S_n on $X^{[r]}$ are paired and disconnected.

1.5 Objectives

1.5.1 General objective

To study the properties of the action of the symmetric group S_n on $X^{[r]}$, where $X = \{1, 2, \dots, n\}$.

1.5.2 Specific objectives

i) To determine the rank and subdegrees of S_n acting on $X^{[r]}$ where $X = \{1, 2, \dots, n\}$.

ii) To construct the suborbital graphs corresponding to this action.

iii) To study theoretic properties of the suborbital graphs constructed.

1.6 Significance of the Study

This study has generalized some of the existing results in the area of combinatorial enumeration. By so doing, new results have been realized. In addition, the results of this study provide valuable information to the graph theorists. Graphs have several practical applications in real life situations as well as in other fields of study. For instance, they can be used to determine the shortest or longest distance between places on the earth's surface. In Chemistry and Physics, graph theory can be used to study the structure of molecules. In particular, the three-dimensional structure of complicated simulated atoms in condensed matter physics can be studied quantitatively by gathering statistics on graph-theoretical properties related to the topology of the atoms. On the other hand, in Chemistry, a graph makes a natural model for molecules, where vertices represent atoms and edges bonds.

1.7 Outline of the Thesis

This thesis is organized into six chapters. Chapter two gives a literature review on ranks, subdegrees and suborbital graphs. In Chapter three the properties of the action of S_n on ordered 2-element subsets, 3-element subsets and 4-element subsets from

$X = \{1, 2, \ldots, n\}$ are investigated. The results in this chapter form a basis for generalization to be made on ranks and subdegrees in the following chapter. Chapter four is the keystone of this thesis, it provides computations of ranks and subdegrees for the action of S_n on ordered *r*-element subsets. The concluding section of this chapter gives some properties of suborbits of S_n on $X^{[r]}$. In Chapter Five the general suborbital graphs for the action of S_n on $X^{[r]}$ are studied. Chapter six gives the conclusion of this study and a brief discussion of future research.

CHAPTER TWO

LITERATURE REVIEW

2.1 Introduction

In this chapter we consider studies that have already been done on ranks, subdegrees and suborbital graphs. The chapter is divided into three sections. In Section 2.2 we review ranks and subdegrees while in Section 2.3 we review suborbital graphs.

2.2 Ranks and subdegrees

The rank and subdegrees of a permutation group *G* have been considered by several mathematicians before. Higman (1970) investigated finite permutation groups of rank 3. It was proved that the symmetric group S_n on $X = \{1, 2, \ldots, n\}$, $n \geq 4$, acts as a rank 3 group on the set of $\binom{n}{2}$ unordered 2-element subsets of *X*, with subdegrees $1, 2(n-2)$ and $\binom{n-2}{2}$. Higman also proved that if $G \leq S_n$, $n \geq 4$, has rank 3 on the unordered 2-element subsets, then *G* is 4-fold transitive unless $n = 9$ and $G \cong P\Gamma L_2(8)$.

Quirin (1971) classified all primitive permutation groups *G* which have a suborbit Δ of length 4 for which $G_x^\Delta \cong A_4$ or $G_x^\Delta \cong S_4$ and is faithful. Permutation groups *G* which have a suborbit Δ of prime length $p \geq 5$ such that $\left|G_x^\Delta\right| \leq 2p$ were also considered and this reduced the classification problem to that of classifying simple groups with maximal dihedral subgroups of order 2*p*. Finally, it was proved that if *G* is a transitive group on a finite set X and Δ is a suborbit of G of length greater than 1, then for any $x \in X$ the groups $C = G_x^{\Delta}$ and $D = G_x^{\Delta^*}$ have a common non-trivial homomorphic image.

Cameron (1972) studied the aspect of bounding the rank of certain permutation groups. It was shown that if G is a primitive permutation group with subrank 3, then one of the following holds:

i) G has rank at most 3.

ii) G is a Frobenius group with Frobenius complement of order 3.

iii) $G \cong [V_{16}]D_5$, with degree 16 and rank 4.

(Here $[V_{16}]D_5$ denotes a split extension of an elementary abelian group of order 16 by a dihedral group of order 10). The proof depended on graph-theoretic methods and elementary facts about permutation characters.

Cameron (1973) studied primitive groups with most suborbits doubly transitive. It was found that if G is primitive on X and G_x is doubly transitive on all non-trivial suborbits except possibly one, with $|G_x| > 2$, then G has rank at most 4. It was further proved that if the rank is 4, then the two doubly transitive suborbits of G are paired with each other, and the degrees, subdegrees, and intersection numbers are polynomials in a single parameter.

Neumann (1977) extended the work of Higman on finite permutation groups. It was shown that when S_5 acts on unordered 2-element subsets from the set $X = \{1, 2, 3, 4, 5\}$, the rank is 3 and its suborbits are all self-paired. Neumann also constructed the famous Petersen graph corresponding to one of the non-trivial suborbits. This graph was first constructed by Julius Petersen in 1898. The Petersen graph serves as a useful example and counter example for many problems in graph theory.

Numata (1978) studied primitive rank 5 permutation groups. It was proved that if G is a primitive rank 5 permutation group on a finite set X, and the stabilizer G_x of a point $x \in X$ is doubly transitive on $\Delta_1(x)$ and $\Delta_2(x)$, where $\Delta_1(x)$ and $\Delta_2(x)$ are two G_x-orbits with $\Delta_1 \circ \Delta_1^* \neq \Delta_2 \circ \Delta_2^*$, then G is isomorphic to the small Janko simple group and $|X| = 266$.

Praeger and Saxl (1979) studied fixed points in paired suborbits. They showed that if P is a Sylow p-subgroup of G_x for some prime p dividing $|G_x|$ and Φ is the set of fixed points of P in X, and $N = N_G(P)$, then $\Phi \cap \Delta$ and $\Phi \cap \Delta^*$ are paired N_x-orbits provided $\Phi \cap \Delta \neq \phi$. This led to an immediate conclusion that a Sylow subgroup of G_x fixes the same number of points in Δ and Δ^*.

Faradzev and Ivanov (1990) computed the subdegrees of primitive permutation representations of $PSL(2, q)$. They showed that if $G = PSL(2, q)$ acts on the cosets of a maximal subgroup H, then the rank is at least $|G| / |H|^2$ and if $q > 100$, the rank is more than 5.

Marusic and Nedela (1998) characterized transitive permutation groups having a non-self-paired suborbit of length 2 in terms of their point stabilizers. As a consequence, elementary abelian groups were proved to be the only possible abelian point stabilizers arising from such actions, and D_4 was shown to be the only non-abelian group of order 8 with the same property. Constructions of such group actions with point stabilizers isomorphic to D_4 or $\mathbb{Z}_2^h$, $h \geq 1$ were also given there.

Evans (2001) showed how to construct a primitive permutation group which has a finite suborbit paired with a suborbit of size *k*, for every cardinal *k*. The existence of such groups answered a question raised by Neumann (1992) and completed a rough taxonomy of infinite, primitive permutation groups as outlined below:

Case A: There is no (non-trivial) finite suborbit;

Case B: There is a (non-trivial) finite suborbit Δ and Δ^* is finite;

Case C: There is a finite suborbit Δ with Δ^* infinite.

In order to accomplish this, a digraph with finite out-valency and in-valency *k*, whose automorphism group is primitive on vertices and transitive on directed edges was constructed and studied.

Kamuti (2006) computed the subdegrees of primitive permutation representations of $PGL(2,q)$ using a method proposed by Ivanov *et al*. (1983) which uses marks of a permutation group. It was proved that when $PGL(2,q)$ acts on the cosets of its maximal subgroup *H* isomorphic to S_4, the rank is

$$r = \frac{q^3 + 189q - 82}{576},$$

where q is a power of a prime. The subdegrees corresponding to this action were found to be 1, 4, 6, 8, 12 and 24. On the other hand in the action of $PGL(2,q)$ on the cosets of its maximal subgroup H isomorphic to the dihedral group of degree $2(q-1)$, it was found that the rank is

$$r = \begin{cases} \frac{1}{2}(q+3), \text{ if q is odd} \\ \frac{1}{2}(q+2), \text{ if q is even.} \end{cases}$$

If q is odd the subdegrees were shown to be 1, $\frac{1}{2}(q-1)$, $(q-1)$ and $2(\mathrm{q}-1)$ while if q is even the subdegrees were 1, $(q-1)$, and $2(q-1)$.

2.3 Suborbital graphs

The idea of suborbital graphs corresponding to non-trivial suborbits of a group G acting on a set X was first introduced by Sims (1967) when he studied graphs and finite permutation groups. He defined the suborbital graph Γ corresponding to suborbital $O \subseteq X \times X$ as a graph whose vertex set is X and edge set E consists of directed edges $x \to y$, where $(x, y) \in \mathrm{O}$. Some other researchers have studied these graphs;

Jones *et al*. (1991) considered the action of the modular group

$$\Gamma = PSL(2,\mathbb{Z}) = SL(2,\mathbb{Z})/\{\pm I\}$$

on the upper half-plane $U = \{z \in \mathbb{C} \mid \mathrm{Im}(z) > 0\}$ and on the rational projective line $\hat{\mathbb{Q}} = \mathbb{Q} \cup \{\infty\}$. They defined the action of Γ on U as follows;

$$z \mapsto \frac{az+b}{cz+d}, \qquad (a, b, c, d \in \mathbb{Z},\ ad-bc=1).$$

For $\hat{\mathbb{Q}}$, each element was represented as a reduced fraction $\frac{x}{y}$, with $x, y \in \mathbb{Z}$ and $(x, y) = 1$. The action of Γ on $\hat{\mathbb{Q}}$ was defined as follows;

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix}: \frac{x}{y} \rightarrow \frac{ax+by}{cx+dy}, \text{ where } \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \Gamma.$$

This action was shown to be both transitive and imprimitive.

They also constructed the suborbital graphs corresponding to the action of Γ on $\hat{\mathbb{Q}}$. Since Γ acts transitively on $\hat{\mathbb{Q}}$, each suborbital was shown to contain the pair (∞, v) for some $v \in \hat{\mathbb{Q}}$. If $v = {}^{u}\!/_{n}$, with $n \geq 0$, and $(u, n) = 1$, then the corresponding suborbital graph was denoted by $G_{u,n}$. The simplest of these graphs was the Farey graph $\mathcal{F}$. In this graph, the vertex ∞ is joined to integers, while two rational numbers $\frac{r}{s}$ and $\frac{x}{y}$ (in reduced form) are adjacent if and only if $ry - sx = \pm 1$, or equivalently if they are consecutive terms in the Farey sequence $(\mathcal{F}_m)$ consisting of rationals $\frac{x}{y}$ with $|y| \leq m$, arranged in ascending order. They found that $\mathcal{F}$ is connected, contains anti-directed triangles, such as $\infty \rightarrow 1 \leftarrow 2 \rightarrow \infty$ and further conjectured that any other suborbital graph is a forest if and only if it contains no triangles.

Kamuti (1992) devised a method for constructing some of the suborbital graphs of $PSL(2,q)$ and $PGL(2,q)$ acting on the cosets of their maximal dihedral subgroups of orders $(q-1)$ and $2(q-1)$ respectively. This method gave an alternative way of constructing the Coxeter graph which was first constructed by Coxeter (1983). The Coxeter graph is a non-Hamiltonian cubic graph on 28 vertices and 42 edges with girth 7.

Akbas (2001) extended the work of Jones *et al.* (1991) on the action of the modular group on $\hat{\mathbb{Q}}$. It was shown that $G_{u,n}$ contains a directed triangle if and only if $u^2 \mp u + 1 \equiv 0 (\bmod n)$. If $n > 1$, it was found that $G_{u,n}$ contains no anti-directed triangles and for every even natural number *n*, $G_{u,n}$ is a forest. Finally, Akbas proved the conjecture by Jones *et al.* (1991) that if $n > 1$, then the suborbital graph $G_{u,n}$ is a forest if and only if it contains no triangles.

Heng *et al.* (2004) studied primitive permutation groups with small suborbits and their suborbital graphs. Based on the classification result of Quirin (1971) and Wang *et al.* (1992), they came up with a precise list of primitive permutation groups with a suborbit of length 4. In particular, they showed that there exists no example of such groups with the point stabilizer of order $2^4 3^6$, clarifying an open question (since 1970s). They also analyzed the suborbital graphs of primitive permutation groups with a suborbit of length 3 or 4. They were able to obtain a complete classification of vertex-primitive arc-transitive graphs of valency 3 and 4. They also proved that there exists no vertex-primitive half-arc-transitive graph of valency less than 10. Finally, they constructed vertex-primitive half-arc-transitive graphs of valency 2*k* for infinitely many integers *k*, with 14 as the smallest valency.

Keskin (2006) characterized all circuits in the suborbital graph for the normalizer of the congruence subgroup $\Gamma_0(m)$ of the modular group in $PSL(2,\mathbb{R})$ when *m* is a square-free integer. The normalizer of $\Gamma_0(m)$ was taken to consist of matrices of the form

$$\begin{pmatrix} ae & b/h \\ cm/h & de \end{pmatrix},$$

where $e \,\big|\, (m/h^2)$ and h is the largest divisor of 24 for which $h^2 \,\big|\, m$ with the understanding that the determinant of the matrix is $e>0$, and that $(e,\ m/h^2e)=1$. It was shown that if *m* has prime power decomposition $2^{\alpha_1}.3^{\alpha_2}.p_3^{\alpha_3}...p_r^{\alpha_r}$, then the normalizer of $\Gamma_0(m)$ acts transitively on $\hat{\mathbb{Q}}$ if and only if $\alpha_1 \le 7,\ \alpha_2 \le 3,\ \alpha_i \le 1,\ i=3,4,...,r$. Consequently, each non-trivial suborbital graph was seen to contain a pair $\left(\infty, {}^{u}\!/\!{}_{n}\right)$ for some ${}^{u}\!/\!{}_{n} \in \hat{\mathbb{Q}}$ and this graph was denoted by $G\left(\infty, {}^{u}\!/\!{}_{n}\right)$.

Keskin also proved that if $(m,n)>1$, then any circuit in $G\left(\infty, {}^{u}\!/\!{}_{n}\right)$ is in the form

$$v \to T(v) \to T^2(v) \to T^3(v) \to ... \to T^{k-1}(v) \to v$$

for a unique elliptic mapping *T* in the normalizer of $\Gamma_0(m)$ of order *k* and for some $v \in \hat{\mathbb{Q}}$. Finally, it was conjectured that if $n>1$ and the normalizer of $\Gamma_0(m)$ acts transitively on $\hat{\mathbb{Q}}$, then any circuit of length k in the suborbital graph $G\left(\infty, {}^{u}\!/\!_{n}\right)$ is of the form

$$v \rightarrow T(v) \rightarrow T^{2}(v) \rightarrow T^{3}(v) \rightarrow \ldots \rightarrow T^{k-1}(v) \rightarrow v$$

for a unique elliptic element T in the normalizer of $\Gamma_0(m)$ of order k and for some $v \in \hat{\mathbb{Q}}$.

Guler *et al*. (2008) worked on the suborbital graphs of the congruence subgroup $\Gamma_0(N)$ of the modular group Γ. The elements of this subgroup were taken to be matrices of the form;

$$\Gamma_0(N) = \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \Gamma \,\middle|\, c \equiv 0 \left(\operatorname{mod} N\right) \right\}.$$

Without loss of generality, for making calculations easier, N was taken to be a prime p. They showed that the action of $\Gamma_0(p)$ on $\hat{\mathbb{Q}}$ is both intransitive and imprimitive. They also showed that the orbits of $\Gamma_0(p)$ on $\hat{\mathbb{Q}}$ are $\begin{pmatrix} 1 \\ 1 \end{pmatrix}$ and $\begin{pmatrix} 1 \\ p \end{pmatrix}$. Finally, they determined the suborbital graphs for $\Gamma_0(p)$ on $\begin{pmatrix} 1 \\ p \end{pmatrix}$; each suborbital was found to contain a pair (∞, v) for some $v \in \begin{pmatrix} 1 \\ p \end{pmatrix}$, $v = \dfrac{u}{p}$ and was denoted by $O_{u,p}$. The corresponding suborbital graph was denoted by $F_{u,p}$. They proved that $F_{u,p}$ contains a triangle if and only if $u^2 \mp u + 1 \equiv 0 \left(\operatorname{mod} p\right)$.

Keskin and Bahar (2009) proved the conjecture by Keskin (2006) concerning the general form of any circuit of length *k* in the suborbital graph $G\left(\infty, {}^{u}\!/_{n}\right)$. They also showed that if *m* is a square-free positive integer and $(u, n)=1$, then $G\left(\infty, {}^{u}\!/_{n}\right)$ contains a triangle if and only if $m \mid n$, $u^2 \mp u + 1 \equiv 0 \pmod{n}$, a rectangle if and only if $2 \mid m$, $m \mid 2n$ and $2u^2 \mp 2u + 1 \equiv 0 \pmod{n}$, and a hexagon if and only if $3 \mid m$, $m \mid 3n$ and $3u^2 \mp 3u + 1 \equiv 0 \pmod{n}$. They finally concluded that if the normalizer of $\Gamma_0(m)$ is transitive on $\hat{\mathbb{Q}}$, then the length of any circuit in $G\left(\infty, {}^{u}\!/_{n}\right)$ is either 3 or an even natural number.

Besenk *et al*. (2010) examined some suborbital graphs for the normalizer $Nor(N)$ of $\Gamma_0(N)$ in $PSL(2,\mathbb{R})$ when *N* is an integer of the form $2^{\alpha} p^2$, where $\alpha \geq 8$ and *p* is a prime greater than 3. Since $Nor(N)$ is not transitive on $\hat{\mathbb{Q}}$, they considered the action of $Nor\left(2^{\alpha} p^2\right)$ on $\hat{\mathbb{Q}}\left(2^{\alpha} p^2\right)$, which is the set

$$\hat{\mathbb{Q}}\left(2^{\alpha} p^2\right) = \left\{ \bigcup \begin{pmatrix} a \\ b \end{pmatrix}, \text{where } (a, b)=1,\ b = 2^i p^j \text{ or } 2^{\alpha-i} p^j;\ i = 0, 1, 2, 3 \text{ and } j = 0, 2 \right\}.$$

This is the maximal subset of $\hat{\mathbb{Q}}$ on which $Nor\left(2^{\alpha} p^2\right)$ acts transitively. They showed that each non-trivial suborbital graph corresponding to this action contains a pair $\left(\infty, {}^{u}\!/_{2^{\alpha} p^2}\right)$ for some ${}^{u}\!/_{2^{\alpha} p^2} \in \hat{\mathbb{Q}}$ and denoted such a graph by $F\left(\infty, {}^{u}\!/_{2^{\alpha} p^2}\right)$. They proved that $F\left(\infty, {}^{u}\!/_{2^{\alpha} p^2}\right)$ is self-paired if and only if $u^2 \equiv -1 \left(\bmod\ 2^{\alpha-3} p^2\right)$ and that it is a forest. Finally, they conjectured that if N has prime decomposition $2^{\alpha}.3^{\beta}.p_3^{\gamma_3}...p_r^{\gamma_r}$, then among others than the case of the transitive action, also for $\beta \geq 4$, the suborbital graphs of the normalizer would be a forest.

The action of the symmetric group S_n on $X^{[r]}$ is an aspect that seems to have received little attention for a long time. Also the suborbital graphs corresponding to this action have not been studied. This study is aimed at investigating the action of the symmetric group S_n on $X^{[r]}$. Suborbital graphs corresponding to this action will also be constructed and their theoretic properties studied.

CHAPTER THREE

ACTION OF S_n ON ORDERED 2-ELEMENT SUBSETS, 3-ELEMENT SUBSETS AND 4-ELEMENT SUBSETS

3.1 Introduction

The main objective of this chapter is to come up with some results that will form a basis for generalization to be made on rank and subdegrees in the next chapter. We consider particular cases when the symmetric group S_n is acting on ordered 2-element subsets, 3-element subsets and 4-element subsets from the set $X = \{1, 2, \dots, n\}$. Section 3.2 gives some properties of S_n acting on $X^{[2]}$, the set of all ordered pairs from *X*. In Section 3.3 we discuss some properties of S_n acting on $X^{[3]}$ while in Section 3.4 we consider the action of S_n on $X^{[4]}$. Throughout this chapter and the remaining chapters, *G* will represent the symmetric group S_n while *X* will represent the set $\{1, 2, \dots, n\}$.

3.2 Action of G on $X^{[2]}$

The set $X^{[2]}$ has ${}^nP_2 = \frac{n!}{(n-2)!} = n(n-1)$ elements and the action of *G* on $X^{[2]}$ is defined as follows;

$$g[x, y] = [g(x), g(y)], \ \forall\, g \in G,\ [x, y] \in X^{[2]}.$$

3.2.1 Properties of the action of G on $X^{[2]}$

Theorem 3.2.1.1

G acts transitively on $X^{[2]}$.

Proof

Let $[x, y] \in X^{[2]}$, where $x, y \in \{1, 2, \ldots, n\}$. We need to show that the length of the orbit of $[x, y]$ is equal to $\left|X^{[2]}\right|$. We first determine $|H| = |Stab_G[x, y]|$. Now $g \in S_n$ fixes $[x, y]$ if and only if $g[x, y] = [x, y]$ so that $g(x) = x$ and $\mathrm{g}(\mathrm{y}) = y$. Thus *x* and *y* must come from 1-cycles. Hence *H* is isomorphic to the symmetric group S_{n-2} and so $|H| = (n-2)!$.

By Theorem 1.1.7,

$$\begin{aligned}|Orb_G[x, y]| &= |G : Stab_G[x, y]| \\ &= |G| \big/ |Stab_G[x, y]| \\ &= \frac{n!}{(n-2)!} \\ &= {}^{n}P_2 \\ &= \left|X^{[2]}\right|.\end{aligned}$$

Hence *G* acts transitively on $X^{[2]}$. □

Theorem 3.2.1.2

If $n \geq 4$, then the rank of G on $X^{[2]}$ is equal to 7.

Proof

Let G act on $X^{[2]}$. Then $G_{[1,2]}$ has orbits with exactly 2, 1 and no element from $A = \{1, 2\}$. Now, there are $2! = 2$ suborbits with exactly two elements from *A*. Also there are 2 ways of selecting an element from *A* and each can occupy the two positions in 2 ways. Hence there are 4 suborbits with exactly one element from *A*. Finally, there is 1 suborbit with no element from *A*. Hence the rank of *G* on $X^{[2]}$ is equal to $R(G) = 2+4+1=7$. □

The 7 orbits of $G_{[1,2]}$ on $X^{[2]}$ are:

a) Suborbits of *G* containing both 1 and 2;

$$Orb_{G_{[1,2]}}[1,2] = \{[1,2]\} = \Delta_0 \qquad Orb_{G_{[1,2]}}[2,1] = \{[2,1]\} = \Delta_1$$

b) Suborbits of *G* containing exactly one element from $A = \{1,2\}$;

$$Orb_{G_{[1,2]}}[1,3] = \{[1,3],[1,4],[1,5],...,[1,n]\} = \Delta_2$$

$$Orb_{G_{[1,2]}}[3,1] = \{[3,1],[4,1],[5,1],...,[n,1]\} = \Delta_3$$

$$Orb_{G_{[1,2]}}[2,3] = \{[2,3],[2,4],[2,5],...,[2,n]\} = \Delta_4$$

$$Orb_{G_{[1,2]}}[3,2] = \{[3,2],[4,2],[5,2],...,[n,2]\} = \Delta_5$$

c) Suborbit of *G* containing neither 1 nor 2;

$$Orb_{G_{[1,2]}}[3,4] = \{[3,4],[3,5],...,[3,n],[4,3],[4,5],...,[n,n-1]\} = \Delta_6$$

The subdegrees of G are as shown in Table 3.1 below:

Table 3.1: Subdegrees of G acting on $X^{[2]}$ for $n \geq 4$

Suborbit length	1	$(n-2)$	$(n-2)(n-3)$
Number of suborbits	2	4	1

3.2.2 Suborbital graphs corresponding to the action of $G = S_n \ (n \geq 4)$ on $X^{[2]}$

The suborbital graph corresponding to Δ_0 is a null graph which is of little interest in our case. We then remain with suborbits $\Delta_1, \Delta_2, \Delta_3, \Delta_4, \Delta_5$ and Δ_6. Since $|G|$ is even, then by Theorem 1.1.11, $G_{[1,2]}$ has at least an orbit different from Δ_0 which is self-paired. In fact the suborbits $\Delta_1, \Delta_2, \Delta_5$ and Δ_6 are self-paired by Definition 1.1.10. Thus by Theorem 1.1.21 the corresponding suborbital graphs have undirected edges. On the other hand, Δ_3 and Δ_4 are paired and so their corresponding suborbital graphs have directed edges.

Now, let *V* and *W* be any two distinct ordered pairs from *X*. Then we construct these graphs as follows;

a) The suborbital O_1 corresponding to suborbit Δ_1 is $\mathrm{O}_1 = \left\{ \left(g[1,2], g[2,1] \right) \mid g \in G,\ [2,1] \in \Delta_1 \right\}$. Therefore, $\mathscr{G}_1$, the corresponding suborbital graph has an edge from *V* to *W* if and only if the first coordinate of *V* is identical to the second of *W* and the second coordinate of *V* is identical to the first of *W*. The graph $\mathscr{G}_1$ has no cycles and hence has girth zero. It is also disconnected, regular of degree 1 and bipartite with one part corresponding to pairs $[a,b]$ with $a < b$, and the other to $a > b$.

Example 3.2.2.1

If $n = 4$, $\mathscr{G}_1$ is as shown in Figure 3.1 below;

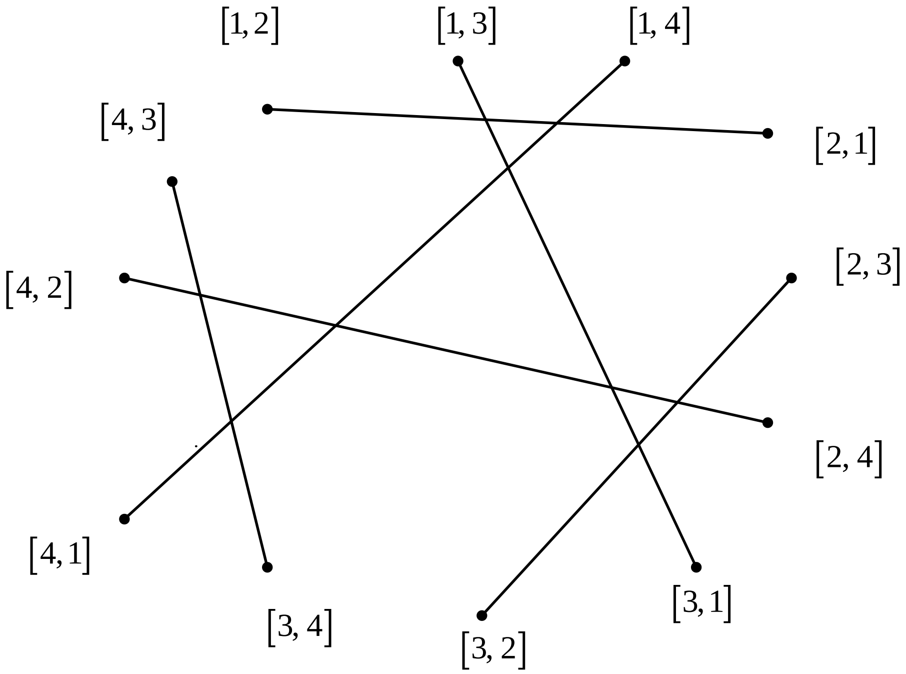

Figure 3.1: The suborbital graph $\mathcal{G}_1$ corresponding to the action of $G = S_4$ on $X^{[2]}$

b) The suborbital O_2 corresponding to suborbit Δ_2 is $O_2 = \left\{ \left(g[1,2], g[1,3] \right) \middle| \, g \in G, \; [1,3] \in \Delta_2 \right\}$. Therefore, the corresponding suborbital graph $\mathcal{G}_2$ has an edge from *V* to *W* if and only if the first coordinate of *V* is identical to the first of *W* and the second coordinate of *V* is not identical to the second of *W*. The graph $\mathcal{G}_2$ is disconnected, regular of degree 2 and has girth three.

Example 3.2.2.2

If $n = 4$, $\mathcal{G}_2$ is as shown in Figure 3.2 below;

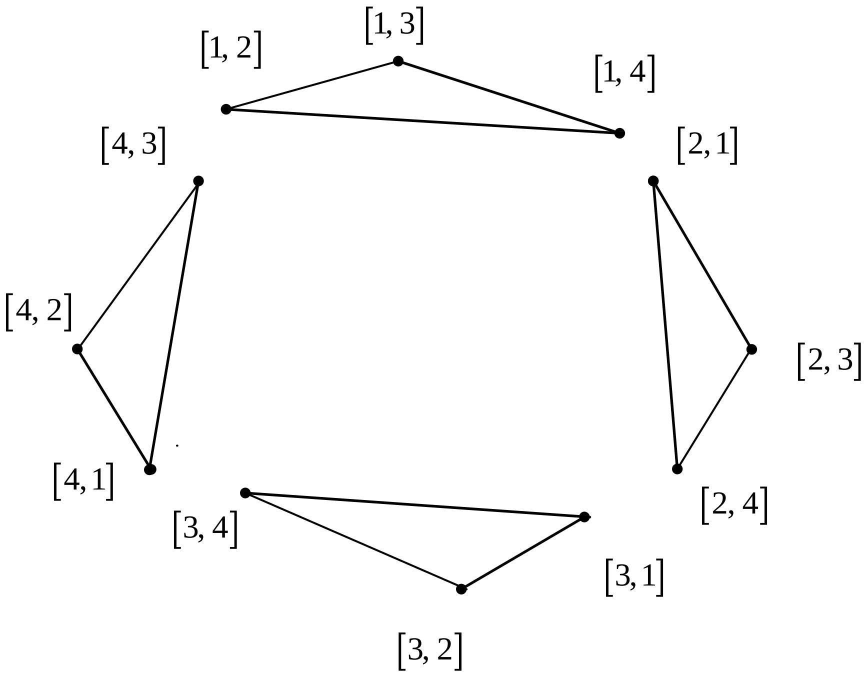

Figure 3.2: The suborbital graph $\mathscr{G}_2$ corresponding to the action of $G = S_4$ on $X^{[2]}$

c) The suborbital O_3 corresponding to suborbit Δ_3 is $O_3 = \{(g[1,2], g[3,1]) \mid g \in G,\ [3,1] \in \Delta_3\}$. Therefore, the corresponding suborbital graph $\mathscr{G}_3$ has a directed edge from V to W if and only if the first coordinate of V is identical to the second of W and the second coordinate of V is not identical to the first of W. The graph $\mathscr{G}_3$ is disconnected, has girth 3 and each of its vertices has indegree 2 and outdegree 2.

Example 3.2.2.3

If $n = 4$, $\mathscr{G}_3$ is as shown in Figure 3.3 below;

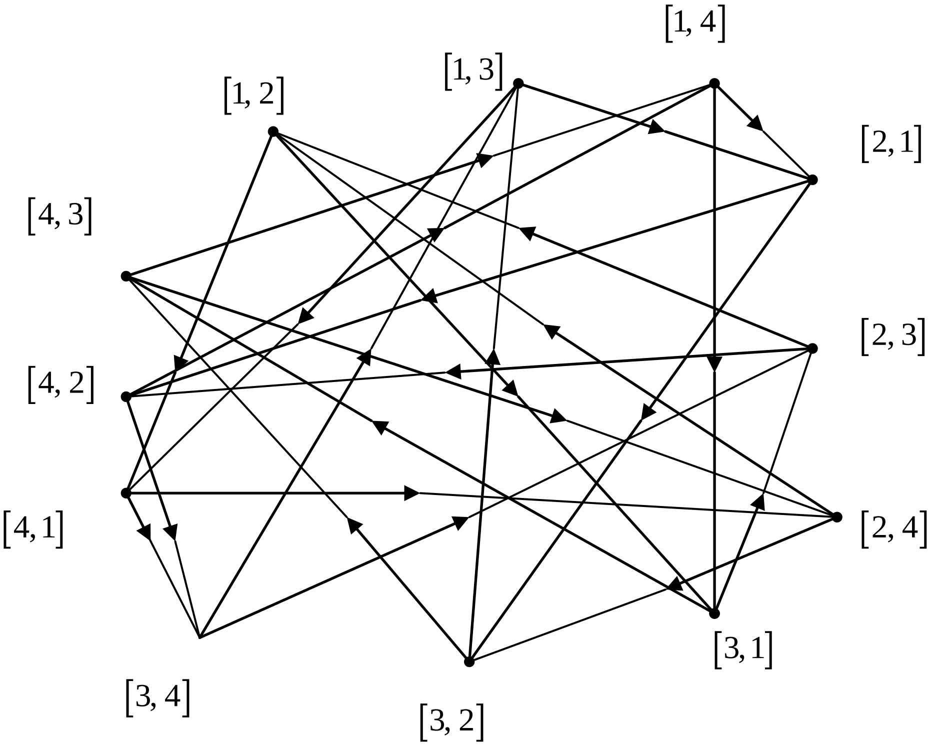

Figure 3.3: The suborbital graph $\mathscr{G}_3$ corresponding to the action of $G = S_4$ on $X^{[2]}$

d) The suborbital O_4 corresponding to suborbit Δ_4 is $\mathrm{O}_4 = \left\{\left(g[1, 2], g[2, 3]\right) \mid g \in G,\ [2, 3] \in \Delta_4\right\}$. Therefore, the corresponding suborbital graph $\mathscr{G}_4$ has a directed edge from *V* to *W* if and only if the first coordinate of *V* is not identical to the second of *W* and the second coordinate of *V* is identical to the first of *W*. The graph $\mathscr{G}_4$ is disconnected, has girth 3 and each of its vertices has indegree 2 and outdegree 2.

Example 3.2.2.4

If $n = 4$, $\mathscr{G}_4$ is as shown in Figure 3.4 below;

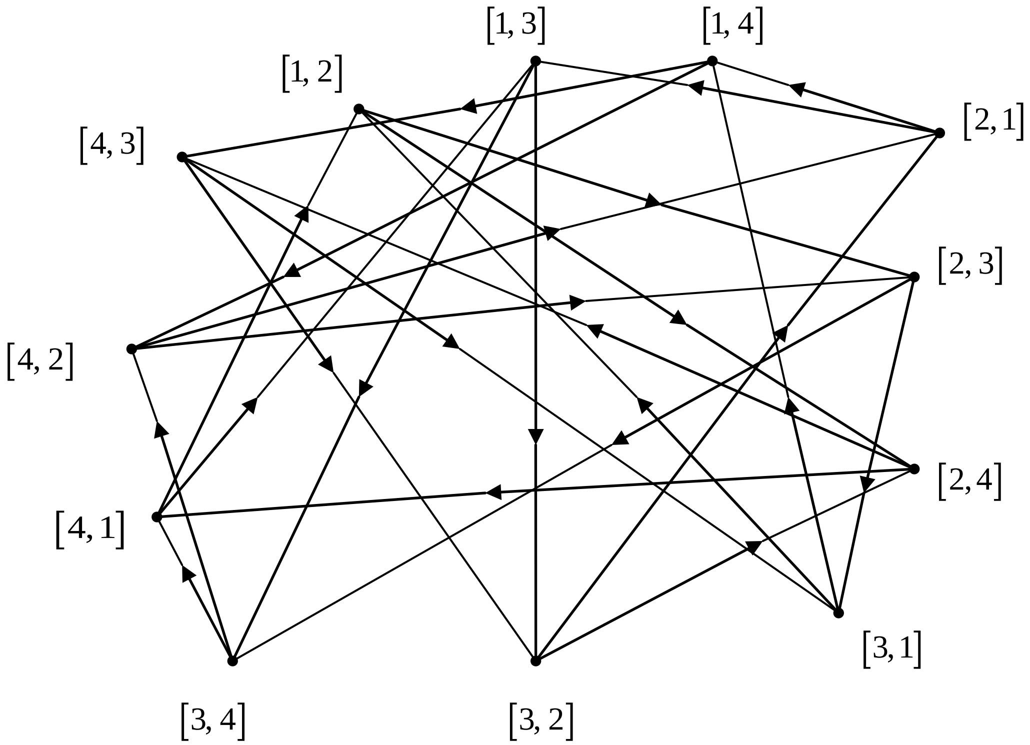

Figure 3.4: The suborbital graph $\mathcal{G}_4$ corresponding to the action of $G = S_4$ on $X^{[2]}$

e) The suborbital O_5 corresponding to suborbit Δ_5 is $O_5 = \{(g[1,2], g[3,2]) \mid g \in G, [3,2] \in \Delta_5\}$. Therefore, the corresponding suborbital graph $\mathcal{G}_5$ has an edge from V to W if and only if the first coordinate of V is not identical to the first of W and the second coordinate of V is identical to the second of W. The graph $\mathcal{G}_5$ is disconnected, regular of degree 2 and has girth three.

Example 3.2.2.5

If $n = 4$, $\mathcal{G}_5$ is as shown in Figure 3.5 below;

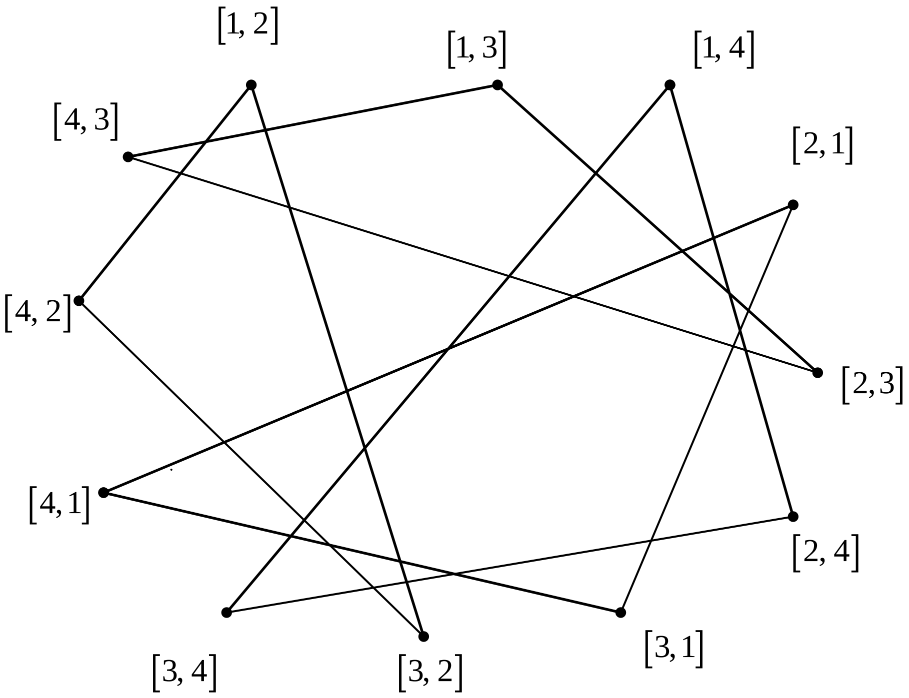

Figure 3.5: The suborbital graph $\mathscr{G}_5$ corresponding to the action of $G = S_4$ on $X^{[2]}$

f) Finally, the suborbital O_6 corresponding to suborbit Δ_6 is $O_6 = \{(g[1, 2], g[3, 4]) \mid g \in G, \ [3, 4] \in \Delta_6\}$. Thus, the corresponding suborbital graph $\mathscr{G}_6$ has an edge from *V* to *W* if and only if the coordinates of *V* are not identical to the coordinates of *W*. The graph $\mathscr{G}_6$ is disconnected, regular of degree 2 and has girth 4.

Example 3.2.2.6

If $n = 4$, $\mathscr{G}_6$ is as shown in Figure 3.6 below;

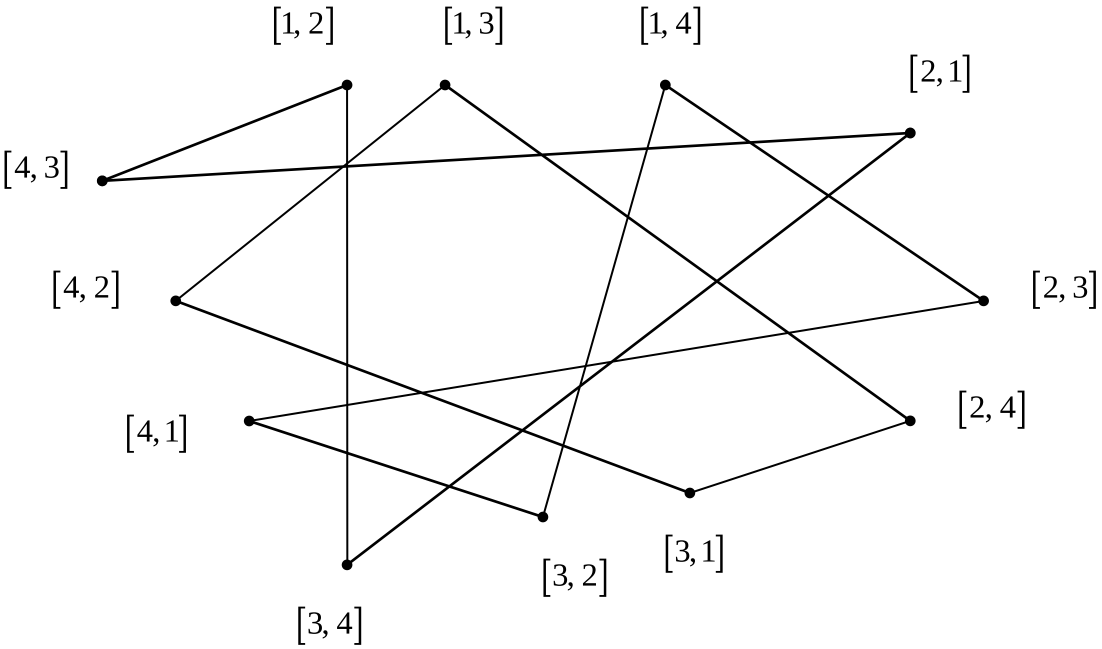

Figure 3.6: The suborbital graph $\mathscr{G}_6$ corresponding to the action of $G = S_4$ on $X^{[2]}$

Theorem 3.2.2.7

G acts imprimitively on $X^{[2]}$ provided $n > 3$.

Proof

Consider the orbits $\Delta_0 = [1, 2], \Delta_1, \Delta_2, \ldots, \Delta_6$ of $G_{[1,2]}$ on $X^{[2]}$. Suppose the lengths of these orbits are $n_0, n_1, n_2, \ldots, n_6$, where $n_0 \le n_1 \le n_2 \le \ldots \le n_6$. Then from Table 3.1, $n_1 = 1$ and $n_2 = (n-2)$. Now, let $j = 2$, then $j > 0$ and $n_j = (n-2) > (1)(1) = n_1 n_{j-1}$ if $n > 3$. Hence by Theorem 1.1.15, G acts imprimitively on $X^{[2]}$ provided $n > 3$. □

Corollary 3.2.2.8

Let G act on $X^{[2]}$. If $n > 3$, then all the corresponding suborbital graphs are disconnected.

Proof

From Theorem 3.2.2.7, *G* acts imprimitively on $X^{[2]}$ if $n > 3$ and so by Theorem 1.1.22 all the corresponding suborbital graphs are disconnected provided $n > 3$. □

3.3 Action of G on $X^{[3]}$

The cardinality of $X^{[3]}$ is ${}^{n}P_{3} = \frac{n!}{(n-3)!} = n(n-1)(n-2)$ and the action of *G* on $X^{[3]}$ is defined as follows;

$$g[x, y, z] = [g(x), g(y), g(z)],\ \forall\, g \in G,\ [x, y, z] \in X^{[3]}.$$

3.3.1 Properties of the action of $G = S_n$ on $X^{[3]}$

Theorem 3.3.1.1

G acts transitively on $X^{[3]}$.

Proof

Let $[x, y, z] \in X^{[3]}$, where $x, y, z \in \{1, 2, \ldots, n\}$. We need to show that the length of the orbit of $[x, y, z]$ is equal to the cardinality of $X^{[3]}$. We first determine $|H| = |Stab_G[x, y, z]|$. Now $g \in S_n$ fixes $[x, y, z]$ if and only if $g[x, y, z] = [x, y, z]$, so that $g(x) = x$, $g(y) = y$ and $g(z) = z$. This implies that *x*, *y* and *z* come from 1-cycles. Therefore *H* is isomorphic to the symmetric group S_{n-3} and so $|H| = (n-3)!$.

By Theorem 1.1.7,

$$|Orb_G[x, y, z]| = |G : Stab_G[x, y, z]|$$

$$= |G| \big/ \left|Stab_G[x, y, z]\right|$$

$$= \frac{n!}{(n-3)!}$$

$$= {}^{n}P_{3}$$

$$= \left|X^{[3]}\right|.$$

Hence *G* acts transitively on $X^{[3]}$. □

Theorem 3.3.1.2

If $n \geq 6$, then the rank of *G* on $X^{[3]}$ is equal to 34.

Proof

Let G act on $X^{[3]}$. Then $G_{[1,2,3]}$ has orbits with exactly 3, 2, 1 and no element from $A = \{1, 2, 3\}$. Now, there are $3! = 6$ suborbits with exactly 3 elements from *A*. Also there are ${}^{3}P_{2} = 6$ ways of selecting an ordered pair from *A* and each ordered pair can occupy the three positions in $\binom{3}{2} = 3$ ways. Hence there are 18 orbits of $G_{[1,2,3]}$ on $X^{[3]}$ containing exactly 2 elements from *A*. Similarly, there are ${}^{3}P_{1} \times {}^{3}C_{1} = 9$ suborbits with exactly 1 element from *A*. Finally, there is 1 suborbit with no element from *A*. Hence the rank of *G* on $X^{[3]}$ is equal to $R(G) = 6 + 18 + 9 + 1 = 34$. □

The suborbits $\Delta_0, \Delta_1, \ldots, \Delta_{33}$ of *G* on $X^{[3]}$ are as given in Appendix *A*. Finally, the subdegrees of *G* are as shown in Table 3.2 below:

Table 3.2: Subdegrees of *G* acting on $X^{[3]}$ for $n \geq 6$

Suborbit length	1	(*n*-3)	(*n*-3)(*n*-4)	(*n*-3)(*n*-4)(*n*-5)
Number of suborbits	6	18	9	1

3.3.2 Suborbital graphs corresponding to the action of $G = S_n \left(n \geq 6\right)$ on $X^{[3]}$

By Definition 1.1.10, the suborbits $\Delta_i \left(i = 1, 2, 5, 6, 9, 13, 16, 20, 23, 24, 28, 32, 33\right)$ are self-paired. Therefore, by Theorem 1.1.21 their corresponding suborbital graphs are undirected. On the other hand, the suborbits $\Delta_j \left(j = 3, 7, 8, 10, 11, 14, 17, 25, 26, 29\right)$ are respectively paired with suborbits $\Delta_k \left(k = 4, 12, 18, 15, 21, 19, 22, 27, 30, 31\right)$. Similarly by Theorem 1.1.21, for each *j* and *k*, the corresponding suborbital graphs are directed.

Theorem 3.3.2.1

G acts imprimitively on $X^{[3]}$ provided $n > 4$.

Proof

Consider the orbits $\Delta_0 = \left[1, 2, 3\right], \Delta_1, \Delta_2, \ldots, \Delta_{33}$ of $G_{[1, 2, 3]}$ on $X^{[3]}$. Suppose the lengths of these orbits are $n_0, n_1, n_2, \ldots, n_{33}$, where $n_0 \leq n_1 \leq n_2 \leq \ldots \leq n_{33}$. Then from Table 3.2, $n_1 = 1$, $n_5 = 1$ and $n_6 = \left(n-3\right)$. Now, let $j = 6$, then $j > 0$ and $n_j = \left(n-3\right) > \left(1\right)\left(1\right) = n_1 n_{j-1}$ if $n > 4$. Hence by Theorem 1.1.15, *G* acts imprimitively on $X^{[3]}$ provided $n > 4$. □

Corollary 3.3.2.2

Let G act on $X^{[3]}$. If $n > 4$, then all the corresponding suborbital graphs are disconnected.

Proof

From Theorem 3.3.2.1, G acts imprimitively on $X^{[3]}$ if $n>4$ and so by Theorem 1.1.22 all the corresponding suborbital graphs are disconnected provided $n>4$. □

3.4 Action of G on $X^{[4]}$

The set $X^{[4]}$ has ${}^{n}P_4 = \dfrac{n!}{(n-4)!} = n(n-1)(n-2)(n-3)$ elements and the action of G on $X^{[4]}$ is defined as follows;

$$g[x, y, z, w] = [g(x), g(y), g(z), g(w)],\ \forall\, g \in G,\ [x, y, z, w] \in X^{[4]}.$$

3.4.1 Properties of the action of G on $X^{[4]}$

Theorem 3.4.1.1

G acts transitively on $X^{[4]}$.

Proof

Let $[x, y, z, w] \in X^{[4]}$, where $x, y, z, w \in \{1, 2, \ldots, n\}$. We need to show that the length of the orbit of $[x, y, z, w]$ is equal to the cardinality of $X^{[4]}$. We first determine $|H| = |Stab_G[x, y, z, w]|$. Now $g \in S_n$ fixes $[x, y, z, w]$ if and only if $g[x, y, z, w] = [x, y, z, w]$, so that $g(x) = x,\ g(y) = y,\ g(z) = z$ and $g(w) = w$. This implies that *x*, *y*, *z* and *w* come from 1-cycles. Hence *H* is isomorphic to the symmetric group S_{n-4} and so $|H| = (n-4)!$.

By Theorem 1.1.7,

$$|Orb_G[x, y, z, w]| = |G : Stab_G[x, y, z, w]|$$

$$= {|G|}\Big/{|Stab_G[x, y, z, w]|}$$

$$= \frac{n!}{(n-4)!}$$

$$= {}^nP_4$$

$$= \left|X^{[4]}\right|.$$

Hence G acts transitively on $X^{[4]}$. □

Theorem 3.4.1.2

If $n \geq 8$, then the rank of G on $X^{[4]}$ is equal to 209.

Proof

Let G act on $X^{[4]}$. Then $G_{[1, 2, 3, 4]}$ has orbits with exactly 4, 3, 2, 1 and no element from $A = \{1, 2, 3, 4\}$. Now, there are $4! = 24$ suborbits with exactly 4 elements from *A*. Also there are ${}^4P_3 = 24$ ways of selecting an ordered triple from *A* and each ordered triple can occupy the four positions in $\binom{4}{3} = 4$ ways. Hence there are 96 orbits of $G_{[1, 2, 3, 4]}$ on $X^{[4]}$ containing exactly 3 elements from *A*. Similarly, there are ${}^4P_2 \times {}^4C_2 = 72$ and ${}^4P_1 \times {}^4C_1 = 16$ suborbits with exactly 2 and 1 element respectively from *A*. Finally, there is 1 suborbit with no element from *A*. Hence the rank of G on $X^{[4]}$ is equal to $R(G) = 24 + 96 + 72 + 16 + 1 = 209$. □

The suborbits $\Delta_0, \Delta_1, ..., \Delta_{208}$ of G on $X^{[4]}$ are as given in Appendix B. Finally, the subdedgrees of *G* are as shown in Table 3.3 below:

Table 3.3: Subdegrees of G acting on $X^{[4]}$ for $n \geq 8$

Suborbit Length	1	(*n*-4)	(*n*-4)(*n*-5)	(*n*-4)(*n*-5)(*n*-6)	(*n*-4)(*n*-5)(*n*-6)(*n*-7)
Number of Suborbits	24	96	72	16	1

3.4.2 Suborbital graphs corresponding to the action of $G = S_n \left(n \geq 8\right)$ on $X^{[4]}$

By Definition 1.1.10, the suborbits Δ_i ($i = 1$, 2, 5, 6, 7, 14, 16, 21, 23, 24, 29, 32, 38, 41, 46, 48, 53, 63,71, 78, 80, 87, 105, 114, 119, 120, 127, 134, 138, 149, 154, 157, 167, 171, 176, 184, 189, 192, 197, 202, 207, 208) are self-paired. Thus by Theorem 1.1.21 their corresponding suborbital graphs are undirected. On the other hand, the suborbits Δ_j ($j = 3$, 8, 9, 10, 11, 15, 17, 25, 26, 27, 30, 31, 33, 34, 35, 39, 42, 43, 47, 49, 50, 51, 54, 55, 56, 57, 58, 59, 64, 65, 66, 67, 74, 75, 79, 81, 82, 83, 89, 90, 91, 93, 94, 95, 106, 107, 115, 121, 122, 123, 124, 125, 128, 129, 130, 131, 135, 136, 137, 139, 140, 141, 142, 143, 147, 148, 153, 158, 159, 160, 161, 164, 165, 166 ,177, 178, 183, 193, 194, 195, 198, 199, 203) are paired respectively with suborbits Δ_k ($k = 4$, 12, 18, 13, 19, 20, 22, 28, 36, 60, 37, 61, 40, 44, 68, 62, 45, 69, 70, 52, 76, 84, 77, 85, 72, 96, 100, 108, 73, 97, 101, 109, 88, 92, 86, 104, 112, 116, 98, 102, 110, 99, 103, 111, 113, 117, 118, 126, 132, 168, 150, 144, 133, 169, 151, 145, 170, 152, 146. 156, 174, 186, 180, 162, 173, 155, 172, 175, 187, 181, 163, 179, 191, 185, 188, 182, 190, 196, 200, 204, 201, 205, 206). Similarly by Theorem 1.1.21, for each *j* and *k*, the corresponding suborbital graphs have directed edges.

Theorem 3.4.2.1

G acts imprimitively on $X^{[4]}$ provided $n > 5$.

Proof

Consider the orbits $\Delta_0 = [1, 2, 3, 4], \Delta_1, \Delta_2, \ldots, \Delta_{208}$ of $G_{[1, 2, 3, 4]}$ on $X^{[4]}$. Suppose the lengths of these orbits are $n_0, n_1, n_2, \ldots, n_{208}$ where, $n_0 \le n_1 \le n_2 \le \ldots \le n_{208}$. Then from Table 3.3, $n_1 = 1$, $n_{23} = 1$ and $n_{24} = (n-4)$. Now, let $j = 24$, then $j > 0$ and $n_j = (n-4) > (1)(1) = n_1 n_{j-1}$ if $n > 5$. Hence by Theorem 1.1.15, *G* acts imprimitively on $X^{[4]}$ provided $n > 5$. □

Corollary 3.4.2.2

Let *G* act on $X^{[4]}$. If $n > 5$, then all the corresponding suborbital graphs are disconnected.

Proof

From Theorem 3.4.2.1, *G* acts imprimitively on $X^{[4]}$ if $n > 5$ and so by Theorem 1.1.22 all the corresponding suborbital graphs are disconnected provided $n > 5$. □

CHAPTER FOUR

RANKS AND SUBDEGREES OF S_n ACTING ON $X^{[r]}$

4.1 Introduction

Let G act on $X=\{1,2,\ldots,n\}$. Then the action of *G* on *X* induces an action of *G* on $X^{[r]}$ that is defined as follows;

$$g[a_1,a_2,\ldots,a_r]=[g(a_1),g(a_2),\ldots,g(a_r)],\ \forall\, g\in G,\ [a_1,a_2,\ldots,a_r]\in X^{[r]}.$$

In this chapter we discuss in detail some properties of the above action. The chapter is presented in four sections.

In Section 4.2, we investigate some properties of the symmetric group S_n acting on $X^{[r]}$. In Section 4.3, we determine the ranks and subdegrees of S_n acting on $X^{[r]}$. Finally in Section 4.4 we investigate some properties of suborbits of S_n acting on $X^{[r]}$.

4.2 Some properties of *G* acting on $X^{[r]}$

Theorem 4.2.1

Let the cycle type of $g\in G$ be $(\alpha_1,\alpha_2,\ldots,\alpha_n)$. If $\alpha_1\geq r$, then the number of elements in $X^{[r]}$ fixed by *g* is given by the formula

$$|Fix(g)|=r!\binom{\alpha_1}{r}. \qquad 4.1)$$

Proof

Let $[a_1,a_2,\ldots,a_r]\in X^{[r]}$ and $g\in G$. Then g fixes $[a_1,a_2,\ldots,a_r]\in X^{[r]}$ if and only if $a_1,a_2,..,a_r$ are mapped onto themselves by *g*. That is

$g[a_1, a_2, \dots, a_r] = [g(a_1), g(a_2), \dots, g(a_r)] = [a_1, a_2, \dots, a_r]$ implying that $g(a_1) = a_1,\ g(a_2) = a_2, \dots, g(a_r) = a_r$. Therefore each of the elements $a_1, a_2, \dots, a_r$ comes from a 1-cycle. Hence the number of unordered *r*-element subsets fixed by $g \in S_n$ is $\binom{\alpha_1}{r}$. But an unordered *r*-element subset say, $\{a_1, a_2, \dots, a_r\}$ can be rearranged to give $r!$ distinct ordered *r*-element subsets. Hence

$$|Fix(g)| = r!\binom{\alpha_1}{r}. \qquad \square$$

Example 4.2.2

Consider the action of $G = S_7$ on $X^{[3]}$ and let $g = (1234) \in S_7$, then the cycle type of *g* is $(3, 0, 0, 1, 0, 0, 0)$. Now, by Theorem 4.2.1 we have,

$$|Fix(g)| = 3!\binom{3}{3} = 6$$

and

$$Fix(g) = \{[5, 6, 7],\ [5, 7, 6],\ [6, 5, 7],\ [6, 7, 5],\ [7, 5, 6], [7, 6, 5]\}.$$

Theorem 4.2.3

Suppose $g \in G$ has cycle type $(\alpha_1, \alpha_2, \dots, \alpha_n)$. Then the number of permutations fixing $[a_1, a_2, \dots, a_r] \in X^{[r]}$ and conjugate to *g* is given by

$$\frac{(n-r)!}{1^{\alpha_1 - r}(\alpha_1 - r)!\prod_{i=2}^{n} \alpha_i!\, i^{\alpha_i}}. \qquad 4.2)$$

Proof

Suppose $g \in G$ has cycle type $(\alpha_1, \alpha_2, \ldots, \alpha_n)$. Then *g* fixes an ordered *r*-element subset $[a_1, a_2, \ldots, a_r]$ if and only if each of the elements $a_1, a_2, \ldots, a_r$ comes from a 1- cycle. Hence the number of permutations in *G* fixing $[a_1, a_2, \ldots, a_r]$ and conjugate to *g* is equal to the number of permutations in S_{n-r} having cycle type $(\alpha_1 - r, \alpha_2, \ldots, \alpha_n)$. By Theorem 1.1.3, this number is

$$\frac{(n - r)!}{1^{\alpha_1 - r}(\alpha_1 - r)!\prod_{i=2}^{n}\alpha_i!\,i^{\alpha_i}}. \qquad \square$$

We can apply the above theorem to compute the order of the stabilizer of a point in a group action.

Example 4.2.4

Consider the action of $G = S_7$ on $X^{[3]}$. We want to compute $|Stab_G[1, 2, 3]|$. Now, by Theorem 4.2.3 we have the following results recorded in Table 4.1 below:

Table 4.1: Number of permutations in $Stab_G[1,2,3]$

Permutation Type	Number of permutations fixing $[1,2,3]$
1	1
(ab)	6
(abc)	8
$(abcd)$	6
$(abcde)$	0
$(abcdef)$	0
$(abcdefg)$	0
$(ab)(cd)$	3
$(ab)(cde)$	0
$(ab)(cdef)$	0
$(ab)(cdefg)$	0
$(ab)(cd)(ef)$	0
$(ab)(cd)(efg)$	0
$(abc)(def)$	0
$(abc)(defg)$	0

From the second column, we see that

$\left|Stab_G\left[1,2,3\right]\right| = 1+6+8+6+0+0+0+3+0+0+0+0+0+0+0 = 24$.

Theorem 4.2.5

Let G act on $X^{[r]}$. Then the order of the stabilizer of $\left[a_1, a_2, \ldots, a_r\right] \in X^{[r]}$ in G is equal to

$$\left|Stab_G\left[a_1, a_2, ..., a_r\right]\right| = (n-r)!. \tag*{4.3)}$$

Proof

Let $\left[a_1, a_2, \ldots, a_r\right] \in X^{[r]}$ and $g \in G$. Then g fixes $\left[a_1, a_2, ..., a_r\right]$ if and only if each of the elements $a_1, a_2, \ldots, a_r$ comes from a 1-cycle. Hence the order of $Stab_G\left[a_1, a_2, \ldots, a_r\right]$ is equal to the order of the group of all permutations of the set $\left\{a_{r+1}, a_{r+2}, \ldots, a_n\right\}$. But this group is isomorphic to S_{n-r}. Hence

$$\left|Stab_G\left[a_1, a_2, ..., a_r\right]\right| = (n-r)! \quad . \qquad \square$$

Example 4.2.6

Consider the action of $G = S_7$ on $X^{[3]}$, then by Theorem 4.2.5

$\left|Stab_G\left[1,2,3\right]\right| = (7-3)! = 24$.

This agrees with the results obtained earlier by applying Theorem 4.2.3 (See example 4.2.4).

Theorem 4.2.7

G acts transitively on $X^{[r]}$.

Proof

Let $[a_1, a_2, \ldots, a_r] \in X^{[r]}$, we only need to show that the length of the orbit of $[a_1, a_2, \ldots, a_r]$ is equal to $\left|X^{[r]}\right|$. Now, from Theorem 4.2.5 we have $\left|Stab_G[a_1,\ a_2, \ldots, a_r]\right| = (n-r)!$. Also $|G| = n!$; hence by Theorem 1.1.7 we have

$$\begin{aligned}\left|Orb_G[a_1, a_2, \ldots, a_r]\right| &= \left|G : Stab_G[a_1, a_2, \ldots, a_r]\right| \\ &= |G| \big/ \left|Stab_G[a_1, a_2, \ldots, a_r]\right| \\ &= \frac{n!}{(n-r)!} \\ &= {}^nP_r \\ &= \left|X^{[r]}\right|.\end{aligned}$$

Hence *G* acts transitively on $X^{[r]}$. □

4.3 Ranks and Subdegrees of *G* acting on $X^{[r]}$

4.3.1 Ranks of *G* acting on $X^{[r]}$

From the results obtained in Sections 3.2, 3.3 and 3.4, we can generalize the ranks of *G* on $X^{[r]}$ as follows;

Case 1

If $n = r$, then $G_{[1, 2, \ldots, r]}$ has $r!$ orbits with exactly *r* elements from $A = \{1, 2, \ldots, r\}$. Hence the rank of *G* on $X^{[r]}$ in this case is equal to

$$R(G) = r!. \qquad 4.4)$$

Case 2

If $n = r+1$, then $G_{[1,2,\ldots,r]}$ has orbits with exactly r and $(r-1)$ elements from $A = \{1, 2, \ldots, r\}$. Now, there are $r!$ suborbits with exactly r elements from A. Also there are ${}^{r}P_{r-1}$ ways of choosing an ordered $(r-1)$-tuple from A and each ordered $(r-1)$-tuple can occupy the r positions in ${}^{r}C_{r-1}$ ways. Hence there are ${}^{r}P_{r-1} \times {}^{r}C_{r-1} = r!r = r^2(r-1)!$ suborbits with exactly $(r-1)$ elements from A. Therefore the rank of G on $X^{[r]}$ in this case is equal to

$$R(G) = r! + r^2(r-1)!. \qquad 4.5)$$

Case 3

If $n = r+2$, then $G_{[1,2,\ldots,r]}$ has orbits with exactly r, $(r-1)$ and $(r-2)$ elements from $A = \{1, 2, \ldots, r\}$. Now, there are $r!$ suborbits with exactly r elements from A. Also there are ${}^{r}P_{r-1} \times {}^{r}C_{r-1} = r^2(r-1)!$ and ${}^{r}P_{r-2} \times {}^{r}C_{r-2} = \frac{r^2(r-1)^2(r-2)!}{2!2!}$ suborbits with exactly $(r-1)$ and $(r-2)$ elements from A respectively. Hence the rank of G on $X^{[r]}$ in this case is equal to

$$R(G) = r! + r^2(r-1)! + \frac{r^2(r-1)^2(r-2)!}{2!2!}. \qquad 4.6)$$

……………………………………………………………………………………

Case r

If $n = 2r - 1$, then $G_{[1,\,2,\,\ldots,\,r]}$ has orbits with exactly r, $(r\ -\ 1)$, $(r\ -\ 2)$, $(r\ -\ 3)$, …, 2 and 1 element from $A = \{1, 2, \ldots, r\}$. The number of each of these orbits is as shown in Table 4.2 below:

Table 4.2: Suborbits of G acting on $X^{[r]}$, where $X = \{1, 2, \ldots, r, r+1, \ldots, 2r-2,\ 2r-1\}$

Number of elements m from $A = \{1, 2, \ldots, r\}$	Number of orbits of $G_{[1,\,2,\,\ldots,\,r]}$ on $X^{[r]}$ containing exactly m elements from A
r	$r!$
$(r-1)$	$r!r = r^2(r-1)!$
$(r-2)$	$\frac{r!r(r-1)}{2!2!} = \frac{r^2(r-1)^2(r-2)!}{2!2!}$
$(r-3)$	$\frac{r!r(r-1)(r-2)}{3!3!} = \frac{r^2(r-1)^2(r-2)^2(r-3)!}{3!3!}$
$\cdots$	$\cdots$
$\cdots$	$\cdots$
2	$\frac{r^2(r-1)^2(r-2)^2 \ldots 4^2 3^2 2!}{(r-2)!(r-2)!} = \frac{r^2(r-1)^2}{2!}$
1	$\frac{r^2(r-1)^2(r-2)^2 \ldots 4^2 3^2 2^2 1!}{(r-1)!(r-1)!} = r^2$

Hence the rank of G on $X^{[r]}$ in this case is given by

$$R(G)=r!+r^2(r-1)!+\frac{r^2(r-1)^2(r-2)!}{2!2!}+\frac{r^2(r-1)^2(r-2)^2(r-3)!}{3!3!}+\ldots+\frac{r^2(r-1)^2}{2!}+r^2.$$

4.7)

Case *r + 1*

If $n=2r$, then $G_{[1,2,\ldots,r]}$ has orbits with exactly *r*, $(r-1)$, $(r-2)$, $(r-3)$, $\ldots$, 2, 1 and no element from $A=\{1,2,\ldots,r\}$. The number of each of these orbits is as shown in Table 4.3 below:

Table 4.3: Suborbits of *G* acting on $X^{[r]}$, where $X=\{1,2,\ldots,r,r+1,\ldots,2r-1,\ 2r\}$

Number of elements *m* from $A=\{1,2,\ldots,r\}$	Number of orbits of $G_{[1,2,\ldots,r]}$ on $X^{[r]}$ containing exactly *m* elements from *A*
r	$r!$
$(r-1)$	$r!r=r^2(r-1)!$
$(r-2)$	$\frac{r!r(r-1)}{2!2!}=\frac{r^2(r-1)^2(r-2)!}{2!2!}$
$(r-3)$	$\frac{r!r(r-1)(r-2)}{3!3!}=\frac{r^2(r-1)^2(r-2)^2(r-3)!}{3!3!}$
$\cdots$	$\cdots$
$\cdots$	$\cdots$
2	$\frac{r^2(r-1)^2(r-2)^2\ldots4^23^22!}{(r-2)!(r-2)!}=\frac{r^2(r-1)^2}{2!}$
1	$\frac{r^2(r-1)^2(r-2)^2\ldots4^23^22^21!}{(r-1)!(r-1)!}=r^2$
0	$\frac{r^2(r-1)^2(r-2)^2\ldots4^23^22^21^20!}{r!r!}=1$

Hence the rank of G on $X^{[r]}$ in this case is given by

$$R(G)=r!+r^2(r-1)!+\frac{r^2(r-1)^2(r-2)!}{2!2!}+\frac{r^2(r-1)^2(r-2)^2(r-3)!}{3!3!}+\ldots+\frac{r^2(r-1)^2}{2!}+r^2+1.$$

4.8)

For the values of $n \geq 2r$, we have the following theorem:

Theorem 4.3.1.1

If $n \geq 2r$, then the rank of G on $X^{[r]}$ is equal to

$$R(G)=r!+r^2(r-1)!+\frac{r^2(r-1)^2(r-2)!}{2!2!}+\frac{r^2(r-1)^2(r-2)^2(r-3)!}{3!3!}+\ldots+\frac{r^2(r-1)^2}{2!}+r^2+1.$$

Proof

We prove by induction. If $n=2r$, then the rank of G on $X^{[r]}$ is given by equation 4.8 (See case $r+1$). Hence the theorem holds for $n=2r$. Next, we assume that the theorem holds for $n=2r+k$, where $k \in \square$ and show that it holds for $n=2r+(k+1)$. Now, suppose we add an extra element to the set $X=\{1, 2, \ldots, r, r+1, \ldots, 2r, 2r+1, \ldots, 2r+k\}$. Then the extra element changes the lengths of suborbits of G with exactly $(r-1)$, $(r-2)$, $(r-3)$, $\ldots$, 2, 1 and no elements from $A=\{1, 2, \ldots, r\}$ and not the number of suborbits. Therefore if $n=2r+(k+1)$, then the rank of G on $X^{[r]}$ is the same as that when $n=2r+k$ and so the theorem holds for $n=2r+(k+1)$. Hence by the principle of mathematical induction the theorem holds for all $n \geq 2r$. □

Example 4.3.1.1

a) If $r=2$ and $n \geq 4$, then the rank of G on $X^{[2]}$ is equal to

$$R(G) = 2! + 2^2 1! + \frac{2^2 1^2 0!}{2!2!}$$

$$= 2 + 4 + 1 = 7.$$

b) If $r = 3$ and $n \geq 6$, then the rank of G on $X^{[3]}$ is equal to

$$R(G) = 3! + 3^2 2! + \frac{3^2 2^2 1!}{2!2!} + \frac{3^2 2^2 1^2 0!}{3!3!}$$

$$= 6 + 18 + 9 + 1 \ = 34.$$

c) If $r = 4$ and $n \geq 8$, then the rank of G on $X^{[4]}$ is equal to

$$R(G) = 4! + 4^2 3! + \frac{4^2 3^2 2!}{2!2!} + \frac{4^2 3^2 2^2 1!}{3!3!} + \frac{4^2 3^2 2^2 1^2 0!}{4!4!}$$

$$= 24 + 96 + 72 + 16 + 1 = 209.$$

This agrees with the results obtained in Sections 3.2, 3.3 and 3.4.

4.3.2 Subdegrees of *G* acting on $X^{[r]}$

From the results obtained in Sections 3.2, 3.3 and 3.4, the subdegrees of G on $X^{[r]}$ can be generalized as shown in Table 4.4 below:

Table 4.4: Subdegrees of S_n acting on $X^{[r]}$ for $n \geq 2r$

Suborbit length	Number of suborbits
1	$r!$
$(n-r)$	$r^2(r-1)!$
$(n-r)(n-r-1)$	$\frac{r^2(r-1)^2(r-2)!}{2!2!}$
$(n-r)(n-r-1)(n-r-2)$	$\frac{r^2(r-1)^2(r-2)^2(r-3)!}{3!3!}$
$\cdots$	$\cdots$
$\cdots$	$\cdots$
$(n-r)(n-r-1)(n-r-2)\ldots(n-2r+3)$	$\frac{r^2(r-1)^2}{2!}$
$(n-r)(n-r-1)(n-r-2)\ldots(n-2r+3)(n-2r+2)$	r^2
$(n-r)(n-r-1)(n-r-2)\ldots(n-2r+3)(n-2r+2)(n-2r+1)$	1

Example 4.3.2.1

If $r=4$ and $n \geq 8$, then the subdegrees of *G* on $X^{[4]}$ are as shown in Table 4.5 below:

Table 4.5: Subdegrees of *G* acting on $X^{[4]}$ for $n \geq 8$

Suborbit Length	1	(*n*-4)	(*n*-4)(*n*-5)	(*n*-4)(*n*-5)(*n*-6)	(*n*-4)(*n*-5)(*n*-6)(*n*-7)
Number of Suborbits	24	96	72	16	1

This also agrees with the results obtained in Section 3.4.

4.4 Properties of suborbits of *G* acting on $X^{[r]}$

Theorem 4.4.1

Let G act on $X^{[r]}$. Suppose $\Delta = \{[a_1, a_2, \dots, a_r]\}$ is an orbit of $G_{[1, 2, \dots, r]}$ on $X^{[r]}$ of length 1, where $a_i \in \{1, 2, \dots, r\}$ $\forall\ i = 1, 2, \dots, r$. Then Δ is self-paired if and only if the permutation $\sigma = \begin{pmatrix} 1 \ \ 2 \ \dots \ r \\ a_1 \ a_2 \dots a_r \end{pmatrix}$ is such that $\sigma^2 = 1$.

Proof

Let Δ be self-paired. Then there exists $g \in G$ such that

$g[a_1, a_2, \dots, a_r] = [1, 2, \dots, r]$, that is

$[g(a_1), g(a_2), \dots, g(a_r)] = [1, 2, \dots, r]$.

$\Rightarrow g(a_1) = 1,\ g(a_2) = 2, \dots, g(a_r) = r$.

Since Δ is self-paired, then by Definition 1.1.10

$g(1) = a_1,\ g(2) = a_2, \dots, g(r) = a_r$.

$\Rightarrow$ g exchanges a_i and i if $a_i \neq i$ or fixes i. Thus the permutation $\sigma = \begin{pmatrix} 1 \ \ 2 \ \dots \ r \\ a_1 \ a_2 \dots a_r \end{pmatrix}$ is such that $\sigma^2 = 1$. Conversely, let $\sigma^2 = 1$, then $\sigma = \sigma^{-1}$. Now, $g \in G$ such that $g = \begin{pmatrix} 1 \ \ 2 \ \dots \ r \ \dots \ n \\ a_1 \ a_2 \dots a_r \dots a_n \end{pmatrix}$ takes $[a_1, a_2, \dots, a_r]$ to $[1, 2, \dots, r]$ and $[1, 2, \dots, r]$ to $[a_1, a_2, \dots, a_r]$. Therefore Δ is self-paired. □

Example 4.4.2

Consider the following suborbit of $G = S_n$ on $X^{[4]}$ of length 1 (See Appendix B);

$$\Delta_5 = Orb_{G_{[1,2,3,4]}}[1,4,3,2] = \{[1,4,3,2]\}.$$

Now,

$$\sigma = \begin{pmatrix} 1\,2\,3\,4 \\ 1\,4\,3\,2 \end{pmatrix} = (24) \text{ and } \sigma^2 = (24)(24) = 1.$$

Hence Δ_5 is self-paired by Theorem 4.4.1. This agrees with the results in Section 3.4.

Corollary 4.4.3

Let G act on $X^{[r]}$ and Δ be an orbit of $G_{[1,2,\ldots,r]}$ on $X^{[r]}$ of length l with $l \neq 1$. Suppose $[a_1, a_2, \ldots, a_r] \in \Delta$, where $a_i \in \{1, 2, \ldots, n\}$ $\forall\ i = 1, 2, \ldots, r$. Then Δ is self-paired if and only if the permutation $\sigma = \begin{pmatrix} 1 & 2 \ldots r \\ a_1 & a_2 \ldots a_r \end{pmatrix}$ is such that $\sigma^2 = 1$.

Proof

Let Δ be a self-paired suborbit of G on $X^{[r]}$ of length $l \neq 1$ and suppose that $[a_1, a_2, \ldots, a_r] \in \Delta$. Then there exists $g \in G$ such that $g[a_1, a_2, \ldots, a_r] = [1, 2, \ldots, r]$. Proceeding with arguments similar to those of Theorem 4.4.1, we see that Δ is self-paired if and only if the permutation $\sigma = \begin{pmatrix} 1 & 2 \ldots r \\ a_1 & a_2 \ldots a_r \end{pmatrix}$ is such that $\sigma^2 = 1$. □

Theorem 4.4.4

Let G act on $X^{[r]}$ and suppose $\Delta_i = \{[a_1, a_2, \ldots, a_r]\}$ and $\Delta_j = \{[b_1, b_2, \ldots, b_r]\}$ are orbits of $G_{[1,2,\ldots,r]}$ on $X^{[r]}$ of length 1, where $a_i, b_i \in \{1, 2, \ldots, r\}$ $\forall\ i = 1, 2, \ldots, r$. Then Δ_i and

Δ_j are paired if and only if the permutations $\sigma_i = \begin{pmatrix} 1 & 2 \ldots r \\ b_1 & b_2 \ldots b_r \end{pmatrix}$ and $\sigma_j = \begin{pmatrix} 1 & 2 \ldots r \\ a_1 & a_2 \ldots a_r \end{pmatrix}$ are inverses of each other.

Proof

Suppose Δ_i and Δ_j are paired. Then there exist $g_i, g_j \in G$ such that

$g_i\left[a_1, a_2, \ldots, a_r\right] = \left[1, 2, \ldots, r\right]$ and $g_j\left[b_1, b_2, \ldots, b_r\right] = \left[1, 2, \ldots, r\right]$.

That is,

$\left[g_i\left(a_1\right), g_i\left(a_2\right), \ldots, g_i\left(a_r\right)\right] = \left[1, 2, \ldots, r\right]$ and $\left[g_j\left(b_1\right), g_j\left(b_2\right), \ldots, g_j\left(b_r\right)\right] = \left[1, 2, \ldots, r\right]$.

$\Rightarrow g_i\left(a_1\right) = 1, g_i\left(a_2\right) = 2, \ldots, g_i\left(a_r\right) = r$ and $g_j\left(b_1\right) = 1, g_j\left(b_2\right) = 2, \ldots, g_j\left(b_r\right) = r$.

Since Δ_i and Δ_j are paired, then by Definition 1.1.10

$g_i\left(1\right) = b_1, g_i\left(2\right) = b_2, \ldots, g_i\left(r\right) = b_r$ and $g_j\left(1\right) = a_1, g_j\left(2\right) = a_2, \ldots, g_j\left(r\right) = a_r$.

$\Rightarrow \left(g_i g_j\right)\left(1\right) = 1, \left(g_i g_j\right)\left(2\right) = 2, \ldots, \left(g_i g_j\right)\left(r\right) = r$.

Similarly,

$\left(g_j g_i\right)\left(1\right) = 1, \left(g_j g_i\right)\left(2\right) = 2, \ldots, \left(g_j g_i\right)\left(r\right) = r$.

Hence the permutations $\sigma_i = \begin{pmatrix} 1 & 2 \ldots r \\ b_1 & b_2 \ldots b_r \end{pmatrix}$ and $\sigma_j = \begin{pmatrix} 1 & 2 \ldots r \\ a_1 & a_2 \ldots a_r \end{pmatrix}$ are inverses of each other. Conversely, suppose $\sigma_i = \begin{pmatrix} 1 & 2 \ldots r \\ b_1 & b_2 \ldots b_r \end{pmatrix}$ and $\sigma_j = \begin{pmatrix} 1 & 2 \ldots r \\ a_1 & a_2 \ldots a_r \end{pmatrix}$ are inverses of each other. Now, if $g_i, g_j \in G$ where $g_i = \begin{pmatrix} 1 & 2 \ldots r \ldots n \\ b_1 & b_2 \ldots b_r \ldots b_n \end{pmatrix}$ and $g_j = \begin{pmatrix} 1 & 2 \ldots r \ldots n \\ a_1 & a_2 \ldots a_r \ldots a_n \end{pmatrix}$, then g_i takes $\left[a_1, a_2, \ldots, a_r\right]$ to $\left[1, 2, \ldots, r\right]$ and $\left[1, 2, \ldots, r\right]$ to $\left[b_1, b_2, \ldots, b_r\right]$. Similarly, g_j takes

$\left[b_1, b_2, \dots, b_r\right]$ to $\left[1, 2, \dots, r\right]$ and $\left[1, 2, \dots, r\right]$ to $\left[a_1, a_2, \dots, a_r\right]$. Hence Δ_i and Δ_j are paired. □

Example 4.4.5

Consider the following suborbits of $G = S_n$ on $X^{[4]}$ of length 1 (See Appendix B):

$$\Delta_3 = Orb_{G_{[1,2,3,4]}}\left[1,3,4,2\right] = \left\{\left[1,3,4,2\right]\right\}$$

and

$$\Delta_4 = Orb_{G_{[1,2,3,4]}}\left[1,4,2,3\right] = \left\{\left[1,4,2,3\right]\right\}.$$

Now,

$$\sigma_3 = \begin{pmatrix} 1 & 2 & 3 & 4 \\ 1 & 4 & 2 & 3 \end{pmatrix} = (324) \text{ and } \sigma_4 = \begin{pmatrix} 1 & 2 & 3 & 4 \\ 1 & 3 & 4 & 2 \end{pmatrix} = (234), \text{ so}$$

$$\sigma_3\sigma_4 = (324)(234) = 1$$

and

$$\sigma_4\sigma_3 = (234)(324) = 1.$$

$\Rightarrow \sigma_3$ and σ_4 are inverses of each other, hence Δ_3 and Δ_4 are paired by Theorem 4.4.4. Similarly, this agrees with the results in Section 3.4.

Corollary 4.4.6

Let G act on $X^{[r]}$ and let Δ_i and Δ_j be orbits of $G_{[1,2,\dots,r]}$ on $X^{[r]}$ of length l with $l \neq 1$. Suppose $\left[a_1, a_2, \dots, a_r\right] \in \Delta_i$ where $a_i \in \{1, 2, \dots, n\}$ $\forall\ i = 1, 2, \dots, r$. Then Δ_i is paired with Δ_j if there exists an element $\left[b_1, b_2, \dots, b_r\right] \in \Delta_j$ with $b_i \in \{1, 2, \dots, n\}$

$\forall\ i = 1, 2, \ldots, r$ such that the permutations $\sigma_i = \begin{pmatrix} 1 & 2 \ldots r \\ b_1 & b_2 \ldots b_r \end{pmatrix}$ and $\sigma_j = \begin{pmatrix} 1 & 2 \ldots r \\ a_1 & a_2 \ldots a_r \end{pmatrix}$ are inverses of each other.

Proof

Let Δ_i and Δ_j be paired suborbit of G on $X^{[r]}$ of length $l \neq 1$ and suppose that $[a_1, a_2, \ldots, a_r] \in \Delta_i$ and $[b_1, b_2, \ldots, b_r] \in \Delta_j$. Then there exists $g_i, g_j \in G$ such that $g_i[a_1, a_2, \ldots, a_r] = [1, 2, \ldots, r]$ and $g_j[b_1, b_2, \ldots, b_r] = [1, 2, \ldots, r]$. Proceeding with arguments similar to those of Theorem 4.4.4, we see that Δ_i and Δ_j are paired if and only if the permutations $\sigma_i = \begin{pmatrix} 1 & 2 \ldots r \\ b_1 & b_2 \ldots b_r \end{pmatrix}$ and $\sigma_j = \begin{pmatrix} 1 & 2 \ldots r \\ a_1 & a_2 \ldots a_r \end{pmatrix}$ are inverses of each other. □

Theorem 4.4.7

Let G act on $X^{[r]}$ and suppose $g \in G$ has cycle type $(\alpha_1, \alpha_2, \ldots, \alpha_n)$, then the number of self-paired suborbits of G on $X^{[r]}$ is given by

$$n_\pi = \frac{r!}{n!} \sum_g \binom{\alpha_1 + 2\alpha_2}{r}. \qquad 4.9)$$

Proof

Let the cycle type of $g \in G$ be $(\alpha_1, \alpha_2, \ldots, \alpha_n)$, then g^2 has $(\alpha_1 + 2\alpha_2)$ cycles of length 1. Hence by Theorem 4.2.1, the number of elements in $X^{[r]}$ fixed by g^2 is given by

$$\left|Fix\left(g^2\right)\right| = r! \binom{\alpha_1 + 2\alpha_2}{r}.$$

By using this together with Theorem 1.1.13 we see that the number of self-paired suborbits of G on $X^{[r]}$ is equal to

$$n_\pi = \frac{1}{|G|}\sum_{g \in G} \pi\left(g^2\right)$$

$$= \frac{1}{n!}\sum_{g} r!\binom{\alpha_1 + 2\alpha_2}{r}$$

$$= \frac{r!}{n!}\sum_{g} \binom{\alpha_1 + 2\alpha_2}{r}. \qquad \square$$

Example 4.4.8

Consider the action of $G = S_7$ on $X^{[3]}$. Let the cycle type of $g \in S_7$ be $\left(\alpha_1, \alpha_2, \ldots, \alpha_7\right)$, then the number of permutations in S_7 conjugate to g is given by Theorem 1.1.3. Hence we have the following results recorded in Table 4.6 below:

Table 4.6: Permutations in G and number of points fixed by g^2

Permutation Type	Cycle type	Number of permutations	$\binom{\alpha_1+2\alpha_2}{3}$
1	(7, 0, 0, 0, 0, 0, 0)	1	35
(ab)	(5, 1, 0, 0, 0, 0, 0)	21	35
(abc)	(4, 0, 1, 0, 0, 0, 0)	70	4
$(abcd)$	(3, 0, 0, 1, 0, 0, 0)	210	1
$(abcde)$	(2, 0, 0, 0, 1, 0, 0)	504	0
$(abcdef)$	(1, 0, 0, 0, 0, 1, 0)	840	0
$(abcdefg)$	(0, 0, 0, 0, 0, 0, 1)	720	0
$(ab)(cd)$	(3, 2, 0, 0, 0, 0, 0)	105	35
$(ab)(cde)$	(2, 1, 1, 0, 0, 0, 0)	420	4
$(ab)(cdef)$	(1, 1, 0, 1, 0, 0, 0)	630	1
$(ab)(cdefg)$	(0, 1, 0, 0, 1, 0, 0)	504	0
$(ab)(cd)(ef)$	(1, 3, 0, 0, 0 ,0 ,0)	105	35
$(ab)(cd)(efg)$	(0, 2, 1, 0, 0, 0, 0)	210	4
$(abc)(def)$	(1, 0, 2, 0, 0, 0, 0)	280	0
$(abc)(defg)$	(0, 0, 1, 1, 0, 0, 0)	420	0

Now,

$$n_\pi = \frac{r!}{n!}\sum_{g}\binom{\alpha_1 + 2\alpha_2}{r}$$

$$= \frac{3!}{7!}\sum_{g}\binom{\alpha_1 + 2\alpha_2}{3}$$

$$= \frac{3!}{7!}\begin{bmatrix} 1\times 35 + 21\times 35 + 70\times 4 + 210\times 1 + 504\times 0 + 840\times 0 + 720\times 0 + 105\times 35 + \\ 420\times 4 + 630\times 1 + 504\times 0 + 105\times 35 + 210\times 4 + 280\times 0 + 420\times 0 \end{bmatrix}$$

$$= \frac{6}{5040}\left[35 + 735 + 280 + 210 + 3675 + 1680 + 630 + 3675 + 840\right]$$

$$= \frac{6}{5040}\left[11{,}760\right]$$

$$= 14\,.$$

Hence S_7 has 14 self-paired suborbits on $X^{[3]}$. Alternatively, by Theorem 4.4.1 and Corollary 4.4.3, the orbits $\Delta_i \left(i = 0, 1, 2, 5, 6, 9, 13, 16, 20, 23, 24, 28, 32, 33\right)$ of $G_{[1,\, 2,\, \ldots,\, 7]}$ on $X^{[3]}$ are self-paired (See Appendix C). Thus S_7 on $X^{[3]}$ has 14 self-paired suborbits. This agrees with the results obtained earlier using Theorem 4.4.7.

CHAPTER FIVE

SUBORBITAL GRAPHS OF S_n ACTING ON $X^{[r]}$

5.1 Introduction

In this chapter we study the suborbital graphs corresponding to the action of G on $X^{[r]}$. In Section 5.2, we give a construction of suborbital graphs corresponding to the action of G on $X^{[r]}$. Finally in Section 5.3, we investigate some properties of the suborbital graphs constructed in Section 5.2.

5.2 Construction of Suborbital Graphs

Let G act on $X^{[r]}$ and let Δ be an orbit of $G_{[1,2,\ldots,r]}$ on $X^{[r]}$. Suppose $[a_1, a_2, \ldots, a_r] \in \Delta$, where $a_i \in \{1, 2, \ldots, n\}$ $\forall$ $i = 1, 2, \ldots, r$. Then the suborbital O corresponding to Δ is given by

$$\mathrm{O} = \left\{ \left(g[1, 2, \ldots, r],\ g[a_1, a_2, \ldots, a_r] \right) \middle|\ g \in G,\ [a_1, a_2, \ldots, a_r] \in \Delta \right\}. \qquad 5.1)$$

We form the suborbital graph $\mathscr{G}$ corresponding to suborbital O by taking $X^{[r]}$ as the vertex set and by including an edge from $[b_1, b_2, \ldots, b_r]$ to $[c_1, c_2, \ldots, c_r]$ if and only if $([b_1, b_2, \ldots, b_r], [c_1, c_2, \ldots, c_r]) \in \mathrm{O}$. Now, if the coordinates of $[1, 2, \ldots, r]$ in positions $i, j, k, \ldots$ are respectively identical to the coordinates of $[a_1, a_2, \ldots, a_r]$ in positions $x, y, z, \ldots$, then $([b_1, b_2, \ldots, b_r], [c_1, c_2, \ldots, c_r]) \in \mathrm{O}$ if and only if the coordinates of $[b_1, b_2, \ldots, b_r]$ in positions $i, j, k, \ldots$ are respectively identical to the coordinates of

$[c_1, c_2, \dots, c_r]$ in positions $x, y, z, \dots$. Consequently we have an edge from $[b_1, b_2, \dots, b_r]$ to $[c_1, c_2, \dots, c_r]$ in $\mathscr{G}$.

Example 5.2.1

Consider the action of $G = S_n$ on $X^{[4]}$ and the following suborbit of G (See Appendix B);

$$\Delta_{10} = Orb_{G_{[1,2,3,4]}}[2,4,1,3] = \{[2,4,1,3]\}.$$

Then the suborbital O_{10} corresponding to suborbit Δ_{10} is given by $\mathrm{O}_{10} = \{(g[1,2,3,4], g[2,4,1,3]) \mid g \in G,\ [2,4,1,3] \in \Delta_{10}\}$. Thus the suborbital graph $\mathscr{G}_{10}$ corresponding to O_{10} has $X^{[4]}$ as the vertex set and there exists an edge from $[b_1, b_2, b_3, b_4]$ to $[c_1, c_2, c_3, c_4]$ in $\mathscr{G}_{10}$ if and only if the coordinates of $[b_1, b_2, b_3, b_4]$ in positions 1, 2, 3 and 4 are respectively identical to the coordinates of $[c_1, c_2, c_3, c_4]$ in positions 3, 1, 4 and 2. That is, b_1, b_2, b_3 and b_4 are respectively identical to c_3, c_1, c_4 and c_2. By Theorem 4.4.4, Δ_{10} is paired with Δ_{13}, hence by Theorem 1.1.21 $\mathscr{G}_{10}$ has directed edges.

5.3 Properties of suborbital graphs corresponding to the action of *G* on $X^{[r]}$

Theorem 5.3.1

Let G act on $X^{[r]}$. If $n > r+1$, then G acts imprimitively on $X^{[r]}$.

Proof

Consider the orbits $\Delta_0 = [1, 2, \dots, r], \Delta_1, \Delta_2, \dots, \Delta_{k-1}$ of $G_{[1,2,\dots,r]}$ on $X^{[r]}$. Suppose the lengths of these orbits are $n_0, n_1, n_2, \dots, n_{k-1}$ where, $n_0 \le n_1 \le n_2 \le \dots \le n_{k-1}$. Then from

Table 4.4, $G_{[1,2,\ldots,r]}$ has r! orbits of length 1. Thus $n_1 = 1$ and $n_{(r!-1)} = 1$. Also $n_{r!} = (n-r)$. Now, let $j = r!$, then $j > 0$ and $n_j = (n-r) > (1)(1) = n_1 n_{j-1}$ if $n > r+1$. Hence by Theorem 1.1.15, *G* acts imprimitively on $X^{[r]}$ provided $n > r+1$. □

Corollary 5.3.2

If $n > r+1$, then all the suborbital graphs corresponding to the action of *G* on $X^{[r]}$ are disconnected.

Proof

By Theorem 5.3.1, *G* acts imprimitively on $X^{[r]}$ if $n > r+1$ hence by Theorem 1.1.22 all the corresponding suborbital graphs are disconnected provided $n > r+1$. □

Next, we consider the other two cases:

Case 1

If $n = r$, then each orbit of $G_{[1,2,\ldots,r]}$ on $X^{[r]}$ is of length 1. Thus the suborbital graphs corresponding to self-paired suborbits are regular of degree 1. Also the graphs corresponding to the paired suborbits have vertices each of which has indegree 1 and outdegree 1. Hence these graphs must be disconnected implying that *G* acts imprimitively on $X^{[r]}$ if $n = r$.

Case 2

If $n = r+1$, then $G_{[1,2,\ldots,r]}$ has orbits with exactly r and $(r-1)$ elements from $A = \{1, 2, \ldots, r\}$. Now, the former orbits have length 1 while the latter have length $n - r = (r+1) - r = 1$ (See Table 4.4). Thus the corresponding suborbital graphs have vertices each of which has degree 1 or indegree 1 and outdegree 1. Hence these graphs must be disconnected implying that *G* is imprimitive on $X^{[r]}$ if $n=r+1$.

Theorem 5.3.3

Let G act on $X^{[r]}$. Then the number of connected components in the suborbital graph $\mathscr{G}_i$ corresponding to a self-paired orbit of $G_{[1,2,\ldots,r]}$ on $X^{[r]}$ with exactly *r* elements from $A=\{1,2,\ldots,r\}$ is equal to

$$n(\mathscr{G}_i)=\frac{n!}{2(n-r)!}. \tag*{5.2)}$$

Proof

Let $\mathscr{G}_i$ be the suborbital graph corresponding to a self-paired orbit of $G_{[1,2,\ldots,r]}$ on $X^{[r]}$ with exactly *r* elements from $A=\{1,2,\ldots,r\}$. Since each vertex of $\mathscr{G}_i$ has degree 1, then the connected components in $\mathscr{G}_i$ are trees with two vertices and one edge (See Figure. 5.1). Hence the number of connected components in $\mathscr{G}_i$ is equal to

$$\begin{aligned}
\frac{\text{Number of vertices in } \mathscr{G}_i}{2} &= \left|X^{[r]}\right| \Big/ 2 \\
&= \binom{n}{r} r! \Big/ 2 \\
&= \frac{n!}{2(n-r)!}.
\end{aligned}$$

Figure 5.1: A connected component in $\mathscr{G}_i$

□

Theorem 5.3.4

Let G act on $X^{[r]}$. Then the number of connected components in the suborbital graph $\mathscr{G}_j$ corresponding to a paired orbit of $G_{[1,2,\ldots,r]}$ on $X^{[r]}$ with exactly r elements from $A=\{1,2,\ldots,r\}$ is equal to

$$n\left(\mathscr{G}_j\right) = \frac{n!}{3(n-r)!}. \qquad 5.3)$$

Proof

Let $\mathscr{G}_j$ be the suborbital graph corresponding to a paired orbit of $G_{[1,2,\ldots,r]}$ on $X^{[r]}$ with exactly r elements from $A=\{1,2,\ldots,r\}$. Then each vertex of $\mathscr{G}_j$ has indegree 1 and outdegree 1. Moreover, construction shows that the connected components of $\mathscr{G}_j$ are directed triangles (See Figure 5.2). Hence the number of connected components in $\mathscr{G}_j$ is equal to

$$\frac{\text{Number of vertices in } \mathscr{G}_j}{3} = \left|X^{[r]}\right|\big/3$$

$$= \binom{n}{r} r! \Big/ 3$$

$$= \frac{n!}{3(n-r)!}.$$

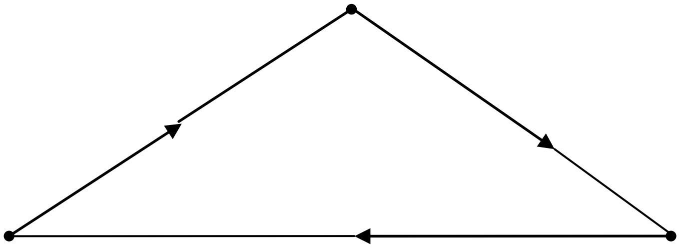

Figure 5.2: A connected component in $\mathscr{G}_j$

□

Theorem 5.3.5

Let G act on $X^{[r]}$ and let $\mathscr{G}_i$ be the suborbital graph corresponding to a self-paired orbit of $G_{[1,2,\ldots,r]}$ on $X^{[r]}$ with exactly r elements from $A=\{1,2,\ldots,r\}$. Then $\mathscr{G}_i$ has girth equal to zero.

Proof

By Theorem 5.3.3, the connected components in $\mathscr{G}_i$ are trees with two vertices and one edge. Hence $\mathscr{G}_i$ cannot have a cycle which implies that its girth is equal to zero. □

Theorem 5.3.6

Let G act on $X^{[r]}$ and let $\mathscr{G}_j$ be the suborbital graph corresponding to a paired orbit of $G_{[1,2,\ldots,r]}$ on $X^{[r]}$ with exactly r elements from $A=\{1,2,\ldots,r\}$. Then $\mathscr{G}_j$ has girth 3.

Proof

By Theorem 5.3.4, the connected components in $\mathscr{G}_j$ are directed triangles, that is, directed cycles of length 3. Hence the girth of $\mathscr{G}_j$ is equal to 3. □

Theorem 5.3.7

Let G act on $X^{[r]}$ and let $\mathscr{G}$ be the suborbital graph corresponding to the orbit of $G_{[1, 2, \ldots, r]}$ on $X^{[r]}$ with no element from $A = \{1, 2, ..., r\}$. Then $\mathscr{G}$ has girth 3 provided $n \geq 3r$.

Proof

Let Δ be the orbit of $G_{[1, 2, \ldots, r]}$ on $X^{[r]}$ with no element from $A = \{1, 2, \ldots, r\}$ and suppose $[d_1, d_2, \ldots, d_r] \in \Delta$. Then the suborbital O corresponding to Δ is given by

$$\mathrm{O} = \left\{ \left(g[1, 2, \ldots, r], g[d_1, d_2, \ldots, d_r] \right) \middle| \, g \in G, \ [d_1, d_2, \ldots, d_r] \in \Delta \right\}. \qquad 5.4)$$

Therefore the corresponding suborbital graph $\mathscr{G}$ has $X^{[r]}$ as the vertex set and has an edge from $[e_1, e_2, \ldots, e_r]$ to $[f_1, f_2, \ldots, f_r]$ if and only if $\{e_1, e_2, \ldots, e_r\} \cap \{f_1, f_2, \ldots, f_r\} = \phi$. Hence the cycle in Figure 5.3 below exists in $\mathscr{G}$ if and only if the sets $\{d_1, d_2, \ldots, d_r\}$, $\{e_1, e_2, \ldots, e_r\}$ and $\{f_1, f_2, \ldots, f_r\}$ are mutually disjoint. But clearly this is possible if $n \geq 3r$.

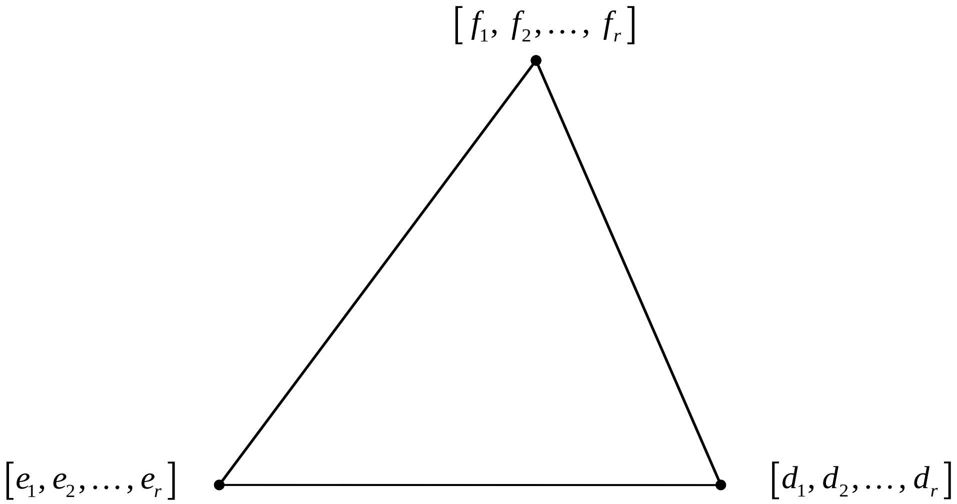

Figure 5.3: A cycle in $\mathscr{G}$

□

CHAPTER SIX

CONCLUSION AND RECOMMENDATIONS FOR FURTHER RESEARCH

6.1 Introduction

This chapter is divided into three sections. Section 6.2 gives the conclusion of this research while Section 6.3 gives recommendations for further research.

6.2 Conclusion

The purpose of this research was to study the action of the symmetric group S_n on the set of ordered *r*-element subsets, $X^{[r]}$. In particular, the ultimate goal was to determine the ranks and subdegrees of S_n acting on $X^{[r]}$ and consequently construct and investigate theoretic properties of the corresponding suborbital graphs.

To do this, we set out specific objectives which were achieved as follows:

i) For the ranks and subdegrees of S_n on $X^{[r]}$, particular cases when $r = 2, 3$ and 4 were considered first and then a generalization was made for any value of *r* and *n*. The formulas for computing the ranks of S_n on $X^{[r]}$ are as given in Section 4.3 while the subdegrees are as displayed in Table 4.4. This study shows that if $n \geq 2r$, then for a fixed value of *r*, the rank of S_n on $X^{[r]}$ is a constant while the subdegrees vary with *n*. This action was shown to be both transitive and imprimitive.

ii) The suborbital graphs for the action of S_n on $X^{[r]}$ were constructed and their theoretic properties studied. These graphs were all found to be disconnected. The girth of the graphs corresponding to self-paired orbits of $G_{[1,2,\dots,r]}$ on $X^{[r]}$ with exactly r elements from $A=\{1,2,\dots,r\}$ was shown to be zero while that of paired orbits with precisely r elements from A was found to be three. For the suborbital graph corresponding to the orbit of $G_{[1,2,\dots,r]}$ on $X^{[r]}$ with no element from A, the girth was found to be equal to three provided $n \geq 3r$. The formulas for computing the number of connected components were also derived and are as given by equations 5.2 and 5.3.

6.3 Recommendations for further research

We have described various properties of the symmetric group S_n acting on $X^{[r]}$. One can extend this work by considering the action of the alternating group A_n on both $X^{(r)}$ and $X^{[r]}$, that is, the set of all unordered and ordered *r*-element subsets from $X=\{1,2,\dots,n\}$ respectively.

REFERENCES

Akbas, M. (2001). Suborbital graphs for the modular group, *Bulletin of the London Mathematical Society* **33**: 647 – 652.

Besenk, M. (2010). Conditions to be a forest for normalizer. *Int. Journal of Math. Analysis* **4**: 1635-1643.

Burnside, W. (1911). *Theory of groups of finite order*. Cambridge University Press, Cambridge.

Cameron, P. J. (1972). Bounding the rank of certain permutation groups. *Math. Z.* **124**: 343-352.

Cameron, P. J. (1973). Primitive groups with most suborbits doubly transitive . *Geometriae Dedicata* **1**: 434-446.

Cameron, P. J. (1974). Suborbits in transitive permutation groups. *Proceedings of the NATO Advanced Study Institute on Combinatorics,* Breukelen, Netherlands.

Coxeter, H. S. (1983). My Graph. *Proc. London Math. Soc*. **46**: 117-136.

Evans, D. (2001). Suborbits in infinite primitive permutation groups. *Bulletin of the Mathematical Society* **33**: 583-590.

Faradzev, I. A. and Ivanov, A. A. (1990). Distance-transitive representations of groups G with $PSL(2,q) \trianglelefteq G < P\Gamma L(2,q)$, *European Journal of Combinatorics* **11**: 347-356.

Faradzev, I. A., Ivanov, A. A., Klin, M. H. and Woldar, A. J. (1994*). Investigations in algebraic theory of combinatorial objects*. Kluwer Academic Publishers, Moscow.

Guler, B., Kader, S. and Besenk, M. (2008). On suborbital graphs of the congruence subgroup $\Gamma_0(N)$. *International Journal of Computational and Mathematical Science* **2/3**: 153-156.

Heng, C., Ping, Z. and Marusic, D. (2004). On primitive permutation groups with small suborbits and their suborbital graphs. *Journal of Algebra* **279**: 749-770.

Higman, D. G. (1970). Characterization of families of rank 3 permutation groups by subdegrees I. *Arch. Math*. **21**: 151 – 156.

Hubaut, X. L (1975). Strongly regular graphs. *Discrete Math*. **13**: 357-381.

Ivanov, A. A., Klin, M. H., Tsaranov, S. V. and Shpektorov, S. V. (1983). On the problem of computing subdegrees of transitive permutation groups. *Soviet Mathematical Survey* **38**: 123 – 124.

Ivanov A. A. (1984). *Combinatoric-algebraic methods for the investigation of distance-regular graphs*, Ph. D. Thesis, University of Amsterdam, The Netherlands.

Jones, G., Singerman, D. and Wicks, K. (I991). The modular group and generalized Farey graph. *London Math. Soc. Lecture Note Series*. **160**: 316-338.

Kamuti, I. N. (1992). *Combinatorial formulas, invariants and structures associated with primitive permutation representations of PSL (2, q) and PGL (2, q)*, Ph. D. Thesis, Southampton University, U.K.

Kamuti, I. N. (2006). Subdegrees of primitive permutation representations of $PGL(2,q)$, *East African Journal of Physical Sciences* $\mathbf{7}\left(\boldsymbol{1/2}\right)$: 25 – 41.

Keskin, R. (2006). Suborbital graphs for the normalizer of $\Gamma_0\left(m\right)$. *European Journal of Combinatorics* **27**: 193-206.

Keskin, R and Bahar, D. (2009). On suborbital graphs for the normalizer of $\Gamma_0\left(N\right)$. *The Electronic Journal of Combinatorics* **16**: 116-134.

Krishnamurthy, V. (1985). *Combinatorics, theory and applications*, Affiliated East-West Press Private Limited, New Delhi.

Marusic, D. and Nedela, R. (1998). On point stabilizer of transitive groups with non-self-paired suborbits of length 2. *Preprint Series* ***36***.

Neumann, P. M. (1977). Finite permutation groups, edge-coloured graphs and matrices. *Topics in Group Theory and Computation*, Academic Press, London.

Neumann, P. M. (1992). Oligomorphic permutation groups, *London Math. Soc. Lecture Notes Series*, **152**, Cambridge Press, Cambridge.

Numata, M. (1978). Generalization of a theorem of Peter J. Cameron, *Osaka J. Math*. **15**: 311-342.

Petersen, J. (1898). Sur le Theore’me de Tait, *Intermed. Math*. **5**: 225-227.

Praeger, C. and Saxl, J. (1979). On fixed points in paired suborbits. *Archiv Der Mathematik* **33**: 325.

Quirin, L. (1971). Primitive permutation groups with small suborbits. *Math. Z.* **122**: 267-274.

Rose, J. S. (1978). *A course on group theory*. Cambridge University Press, Cambridge.

Sims, C. C. (1967). Graphs and finite permutation groups. *Math. Zeitschrift* **95**: 76 – 86.

Wang, J. (1992). The primitive permutation groups with an orbital of length 4. *Comm. Algebra* **20**: 889-921.

Wielandt, H. (1964). *Finite permutation groups*. Academic Press, New York.

APPENDIX A

A.1 Suborbits of $G = S_n$ acting on $X^{[3]}$

$$Orb_{G_{[1,2,3]}}[1,2,3] = \{[1,2,3]\} = \Delta_0$$

$$Orb_{G_{[1,2,3]}}[1,3,2] = \{[1,3,2]\} = \Delta_1$$

$$Orb_{G_{[1,2,3]}}[2,1,3] = \{[2,1,3]\} = \Delta_2$$

$$Orb_{G_{[1,2,3]}}[2,3,1] = \{[2,3,1]\} = \Delta_3$$

$$Orb_{G_{[1,2,3]}}[3,1,2] = \{[3,1,2]\} = \Delta_4$$

$$Orb_{G_{[1,2,3]}}[3,2,1] = \{[3,2,1]\} = \Delta_5$$

$$Orb_{G_{[1,2,3]}}[1,2,4] = \{[1,2,4],[1,2,5],\ldots,[1,2,n]\} = \Delta_6$$

$$Orb_{G_{[1,2,3]}}[1,4,2] = \{[1,4,2],[1,5,2],\ldots,[1,n,2]\} = \Delta_7$$

$$Orb_{G_{[1,2,3]}}[4,1,2] = \{[4,1,2],[5,1,2],\ldots,[n,1,2]\} = \Delta_8$$

$$Orb_{G_{[1,2,3]}}[2,1,4] = \{[2,1,4],[2,1,5],\ldots,[2,1,n]\} = \Delta_9$$

$$Orb_{G_{[1,2,3]}}[2,4,1] = \{[2,4,1],[2,5,1],\ldots,[2,n,1]\} = \Delta_{10}$$

$$Orb_{G_{[1,2,3]}}[4,2,1] = \{[4,2,1],[5,2,1],\ldots,[n,2,1]\} = \Delta_{11}$$

$$Orb_{G_{[1,2,3]}}[1,3,4] = \{[1,3,4],[1,3,5],\ldots,[1,3,n]\} = \Delta_{12}$$

$$Orb_{G_{[1,2,3]}}[1,4,3] = \{[1,4,3],[1,5,3],\ldots,[1,n,3]\} = \Delta_{13}$$

$$Orb_{G_{[1,2,3]}}[4,1,3] = \{[4,1,3],[5,1,3],\ldots,[n,1,3]\} = \Delta_{14}$$

$$Orb_{G_{[1,2,3]}}[3,1,4]=\{[3,1,4],[3,1,5],\ldots,[3,1,n]\}=\Delta_{15}$$

$$Orb_{G_{[1,2,3]}}[3,4,1]=\{[3,4,1],[3,5,1],\ldots,[3,n,1]\}=\Delta_{16}$$

$$Orb_{G_{[1,2,3]}}[4,3,1]=\{[4,3,1],[5,3,1],\ldots,[n,3,1]\}=\Delta_{17}$$

$$Orb_{G_{[1,2,3]}}[2,3,4]=\{[2,3,4],[2,3,5],\ldots,[2,3,n]\}=\Delta_{18}$$

$$Orb_{G_{[1,2,3]}}[2,4,3]=\{[2,4,3],[2,5,3],\ldots,[2,n,3]\}=\Delta_{19}$$

$$Orb_{G_{[1,2,3]}}[4,2,3]=\{[4,2,3],[5,2,3],\ldots,[n,2,3]\}=\Delta_{20}$$

$$Orb_{G_{[1,2,3]}}[3,2,4]=\{[3,2,4],[3,2,5],\ldots,[3,2,n]\}=\Delta_{21}$$

$$Orb_{G_{[1,2,3]}}[3,4,2]=\{[3,4,2],[3,5,2],\ldots,[3,n,2]\}=\Delta_{22}$$

$$Orb_{G_{[1,2,3]}}[4,3,2]=\{[4,3,2],[5,3,2],\ldots,[n,3,2]\}=\Delta_{23}$$

$$Orb_{G_{[1,2,3]}}[1,4,5]=\{[1,4,5],[1,4,6],\ldots,[1,4,n],[1,5,4],[1,5,6],\ldots,[1,n,n-1]\}=\Delta_{24}$$

$$Orb_{G_{[1,2,3]}}[4,1,5]=\{[4,1,5],[4,1,6],\ldots,[4,1,n],[5,1,4],[5,1,6],\ldots,[n,1,n-1]\}=\Delta_{25}$$

$$Orb_{G_{[1,2,3]}}[4,5,1]=\{[4,5,1],[4,6,1],\ldots,[4,n,1],[5,4,1],[5,6,1],\ldots,[n,n-1,1]\}=\Delta_{26}$$

$$Orb_{G_{[1,2,3]}}[2,4,5]=\{[2,4,5],[2,4,6],\ldots,[2,4,n],[2,5,4],[2,5,6],\ldots,[2,n,n-1]\}=\Delta_{27}$$

$$Orb_{G_{[1,2,3]}}[4,2,5]=\{[4,2,5],[4,2,6],\ldots,[4,2,n],[5,2,4],[5,2,6],\ldots,[n,2,n-1]\}=\Delta_{28}$$

$$Orb_{G_{[1,2,3]}}[4,5,2]=\{[4,5,2],[4,6,2],\ldots,[4,n,2],[5,4,2],[5,6,2],\ldots,[n,n-1,2]\}=\Delta_{29}$$

$$Orb_{G_{[1,2,3]}}[3,4,5]=\{[3,4,5],[3,4,6],\ldots,[3,4,n],[3,5,4],[3,5,6],\ldots,[3,n,n-1]\}=\Delta_{30}$$

$$Orb_{G_{[1,2,3]}}[4,3,5]=\{[4,3,5],[4,3,6],\ldots,[4,3,n],[5,3,4],[5,3,6],\ldots,[n,3,n-1]\}=\Delta_{31}$$

$$Orb_{G_{[1,2,3]}}[4,5,3]=\{[4,5,3],[4,6,3],\ldots,[4,n,3],[5,4,3],[5,6,3],\ldots,[n,n-1,3]\}=\Delta_{32}$$

$$Orb_{G_{[1,2,3]}}[4,5,6]=\{[4,5,6],[4,5,7],\ldots,[4,5,n],[4,6,5],[4,6,7],\ldots,[n,n-1,n-2]\}=\Delta_{33}$$

APPENDIX B

B.1 Suborbits of $G = S_n$ acting on $X^{[4]}$

$$Orb_{G_{[1,2,3,4]}}[1,2,3,4] = \{[1,2,3,4]\} = \Delta_0$$

$$Orb_{G_{[1,2,3,4]}}[1,2,4,3] = \{[1,2,4,3]\} = \Delta_1$$

$$Orb_{G_{[1,2,3,4]}}[1,3,2,4] = \{[1,3,2,4]\} = \Delta_2$$

$$Orb_{G_{[1,2,3,4]}}[1,3,4,2] = \{[1,3,4,2]\} = \Delta_3$$

$$Orb_{G_{[1,2,3,4]}}[1,4,2,3] = \{[1,4,2,3]\} = \Delta_4$$

$$Orb_{G_{[1,2,3,4]}}[1,4,3,2] = \{[1,4,3,2]\} = \Delta_5$$

$$Orb_{G_{[1,2,3,4]}}[2,1,3,4] = \{[2,1,3,4]\} = \Delta_6$$

$$Orb_{G_{[1,2,3,4]}}[2,1,4,3] = \{[2,1,4,3]\} = \Delta_7$$

$$Orb_{G_{[1,2,3,4]}}[2,3,1,4] = \{[2,3,1,4]\} = \Delta_8$$

$$Orb_{G_{[1,2,3,4]}}[2,3,4,1] = \{[2,3,4,1]\} = \Delta_9$$

$$Orb_{G_{[1,2,3,4]}}[2,4,1,3] = \{[2,4,1,3]\} = \Delta_{10}$$

$$Orb_{G_{[1,2,3,4]}}[2,4,3,1] = \{[2,4,3,1]\} = \Delta_{11}$$

$$Orb_{G_{[1,2,3,4]}}[3,1,2,4] = \{[3,1,2,4]\} = \Delta_{12}$$

$$Orb_{G_{[1,2,3,4]}}[3,1,4,2] = \{[3,1,4,2]\} = \Delta_{13}$$

$$Orb_{G_{[1,2,3,4]}}[3,2,1,4] = \{[3,2,1,4]\} = \Delta_{14}$$

$$Orb_{G_{[1,2,3,4]}}[3,2,4,1]=\{[3,2,4,1]\}=\Delta_{15}$$

$$Orb_{G_{[1,2,3,4]}}[3,4,1,2]=\{[3,4,1,2]\}=\Delta_{16}$$

$$Orb_{G_{[1,2,3,4]}}[3,4,2,1]=\{[3,4,2,1]\}=\Delta_{17}$$

$$Orb_{G_{[1,2,3,4]}}[4,1,2,3]=\{[4,1,2,3]\}=\Delta_{18}$$

$$Orb_{G_{[1,2,3,4]}}[4,1,3,2]=\{[4,1,3,2]\}=\Delta_{19}$$

$$Orb_{G_{[1,2,3,4]}}[4,2,1,3]=\{[4,2,1,3]\}=\Delta_{20}$$

$$Orb_{G_{[1,2,3,4]}}[4,2,3,1]=\{[4,2,3,1]\}=\Delta_{21}$$

$$Orb_{G_{[1,2,3,4]}}[4,3,1,2]=\{[4,3,1,2]\}=\Delta_{22}$$

$$Orb_{G_{[1,2,3,4]}}[4,3,2,1]=\{[4,3,2,1]\}=\Delta_{23}$$

$$Orb_{G_{[1,2,3,4]}}[1,2,3,5]=\{[1,2,3,5],[1,2,3,6],...,[1,2,3,n]\}=\Delta_{24}$$

$$Orb_{G_{[1,2,3,4]}}[1,2,5,3]=\{[1,2,5,3],[1,2,6,3],...,[1,2,n,3]\}=\Delta_{25}$$

$$Orb_{G_{[1,2,3,4]}}[1,5,2,3]=\{[1,5,2,3],[1,6,2,3],...,[1,n,2,3]\}=\Delta_{26}$$

$$Orb_{G_{[1,2,3,4]}}[5,1,2,3]=\{[5,1,2,3],[6,1,2,3],...,[n,1,2,3]\}=\Delta_{27}$$

$$Orb_{G_{[1,2,3,4]}}[1,2,4,5]=\{[1,2,4,5],[1,2,4,6],...,[1,2,4,n]\}=\Delta_{28}$$

$$Orb_{G_{[1,2,3,4]}}[1,2,5,4]=\{[1,2,5,4],[1,2,6,4],...,[1,2,n,4]\}=\Delta_{29}$$

$$Orb_{G_{[1,2,3,4]}}[1,5,2,4]=\{[1,5,2,4],[1,6,2,4],...,[1,n,2,4]\}=\Delta_{30}$$

$$Orb_{G_{[1,2,3,4]}}[5,1,2,4]=\{[5,1,2,4],[6,1,2,4],...,[n,1,2,4]\}=\Delta_{31}$$

$$Orb_{G_{[1,2,3,4]}}[1,3,2,5]=\{[1,3,2,5],[1,3,2,6],\ldots,[1,3,2,n]\}=\Delta_{32}$$

$$Orb_{G_{[1,2,3,4]}}[1,3,5,2]=\{[1,3,5,2],[1,3,6,2],\ldots,[1,3,n,2]\}=\Delta_{33}$$

$$Orb_{G_{[1,2,3,4]}}[1,5,3,2]=\{[1,5,3,2],[1,6,3,2],\ldots,[1,n,3,2]\}=\Delta_{34}$$

$$Orb_{G_{[1,2,3,4]}}[5,1,3,2]=\{[5,1,3,2],[6,1,3,2],\ldots,[n,1,3,2]\}=\Delta_{35}$$

$$Orb_{G_{[1,2,3,4]}}[1,3,4,5]=\{[1,3,4,5],[1,3,4,6],\ldots,[1,3,4,n]\}=\Delta_{36}$$

$$Orb_{G_{[1,2,3,4]}}[1,3,5,4]=\{[1,3,5,4],[1,3,6,4],\ldots,[1,3,n,4]\}=\Delta_{37}$$

$$Orb_{G_{[1,2,3,4]}}[1,5,3,\vdots 4]=\{[1,5,3,4],[1,6,3,4],\ldots,[1,n,3,4]\}=\Delta_{38}$$

$$Orb_{G_{[1,2,3,4]}}[5,1,3,4]=\{[5,1,3,4],[6,1,3,4],\ldots,[n,1,3,4]\}=\Delta_{39}$$

$$Orb_{G_{[1,2,3,4]}}[1,4,2,5]=\{[1,4,2,5],[1,4,2,6],\ldots,[1,4,2,n]\}=\Delta_{40}$$

$$Orb_{G_{[1,2,3,4]}}[1,4,5,2]=\{[1,4,5,2],[1,4,6,2],\ldots,[1,4,n,2]\}=\Delta_{41}$$

$$Orb_{G_{[1,2,3,4]}}[1,5,4,2]=\{[1,5,4,2],[1,6,4,2],\ldots,[1,n,4,2]\}=\Delta_{42}$$

$$Orb_{G_{[1,2,3,4]}}[5,1,4,2]=\{[5,1,4,2],[6,1,4,2],\ldots,[n,1,4,2]\}=\Delta_{43}$$

$$Orb_{G_{[1,2,3,4]}}[1,4,3,5]=\{[1,4,3,5],[1,4,3,6],\ldots,[1,4,3,n]\}=\Delta_{44}$$

$$Orb_{G_{[1,2,3,4]}}[1,4,5,3]=\{[1,4,5,3],[1,4,6,3],\ldots,[1,4,n,3]\}=\Delta_{45}$$

$$Orb_{G_{[1,2,3,4]}}[1,5,4,3]=\{[1,5,4,3],[1,6,4,3],\ldots,[1,n,4,3]\}=\Delta_{46}$$

$$Orb_{G_{[1,2,3,4]}}[5,1,4,3]=\{[5,1,4,3],[6,1,4,3],\ldots,[n,1,4,3]\}=\Delta_{47}$$

$$Orb_{G_{[1,2,3,4]}}[2,1,3,5]=\{[2,1,3,5],[2,1,3,6],\ldots,[2,1,3,n]\}=\Delta_{48}$$

$$Orb_{G_{[1,2,3,4]}}[2,1,5,3]=\{[2,1,5,3],[2,1,6,3],\ldots,[2,1,n,3]\}=\Delta_{49}$$

$$Orb_{G_{[1,2,3,4]}}[2,5,1,3]=\{[2,5,1,3],[2,6,1,3],\ldots,[2,n,1,3]\}=\Delta_{50}$$

$$Orb_{G_{[1,2,3,4]}}[5,2,1,3]=\{[5,2,1,3],[6,2,1,3],\ldots,[n,2,1,3]\}=\Delta_{51}$$

$$Orb_{G_{[1,2,3,4]}}[2,1,4,5]=\{[2,1,4,5],[2,1,4,6],\ldots,[2,1,4,n]\}=\Delta_{52}$$

$$Orb_{G_{[1,2,3,4]}}[2,1,5,4]=\{[2,1,5,4],[2,1,6,4],\ldots,[2,1,n,4]\}=\Delta_{53}$$

$$Orb_{G_{[1,2,3,4]}}[2,5,1,4]=\{[2,5,1,4],[2,6,1,4],\ldots,[2,n,1,4]\}=\Delta_{54}$$

$$Orb_{G_{[1,2,3,4]}}[5,2,1,4]=\{[5,2,1,4],[6,2,1,4],\ldots,[n,2,1,4]\}=\Delta_{55}$$

$$Orb_{G_{[1,2,3,4]}}[2,3,1,5]=\{[2,3,1,5],[2,3,1,6],\ldots,[2,3,1,n]\}=\Delta_{56}$$

$$Orb_{G_{[1,2,3,4]}}[2,3,5,1]=\{[2,3,5,1],[2,3,6,1],\ldots,[2,3,n,1]\}=\Delta_{57}$$

$$Orb_{G_{[1,2,3,4]}}[2,5,3,1]=\{[2,5,3,1],[2,6,3,1],\ldots,[2,n,3,1]\}=\Delta_{58}$$

$$Orb_{G_{[1,2,3,4]}}[5,2,3,1]=\{[5,2,3,1],[6,2,3,1],\ldots,[n,2,3,1]\}=\Delta_{59}$$

$$Orb_{G_{[1,2,3,4]}}[2,3,4,5]=\{[2,3,4,5],[2,3,4,6],\ldots,[2,3,4,n]\}=\Delta_{60}$$

$$Orb_{G_{[1,2,3,4]}}[2,3,5,4]=\{[2,3,5,4],[2,3,6,4],\ldots,[2,3,n,4]\}=\Delta_{61}$$

$$Orb_{G_{[1,2,3,4]}}[2,5,3,4]=\{[2,5,3,4],[2,6,3,4],\ldots,[2,n,3,4]\}=\Delta_{62}$$

$$Orb_{G_{[1,2,3,4]}}[5,2,3,4]=\{[5,2,3,4],[6,2,3,4],\ldots,[n,2,3,4]\}=\Delta_{63}$$

$$Orb_{G_{[1,2,3,4]}}[2,4,1,5]=\{[2,4,1,5],[2,4,1,6],\ldots,[2,4,1,n]\}=\Delta_{64}$$

$$Orb_{G_{[1,2,3,4]}}[2,4,5,1]=\{[2,4,5,1],[2,4,6,1],\ldots,[2,4,n,1]\}=\Delta_{65}$$

$$Orb_{G_{[1,2,3,4]}}[2,5,4,1]=\{[2,5,4,1],\ [2,6,4,1],\dots,[2,n,4,1]\}=\Delta_{66}$$

$$Orb_{G_{[1,2,3,4]}}[5,2,4,1]=\{[5,2,4,1],[6,2,4,1],\dots,[n,2,4,1]\}=\Delta_{67}$$

$$Orb_{G_{[1,2,3,4]}}[2,4,3,5]=\{[2,4,3,5],[2,4,3,6],\dots,[2,4,3,n]\}=\Delta_{68}$$

$$Orb_{G_{[1,2,3,4]}}[2,4,5,3]=\{[2,4,5,3],[2,4,6,3],\dots,[2,4,n,3]\}=\Delta_{69}$$

$$Orb_{G_{[1,2,3,4]}}[2,5,4,3]=\{[2,5,4,3],[2,6,4,3],\dots,[2,n,4,3]\}=\Delta_{70}$$

$$Orb_{G_{[1,2,3,4]}}[5,2,4,3]=\{[5,2,4,3],[6,2,4,3],\dots,[n,2,4,3]\}=\Delta_{71}$$

$$Orb_{G_{[1,2,3,4]}}[3,1,2,5]=\{[3,1,2,5],[3,1,2,6],\dots,[3,1,2,n]\}=\Delta_{72}$$

$$Orb_{G_{[1,2,3,4]}}[3,1,5,2]=\{[3,1,5,2],[3,1,6,2],\dots,[3,1,n,2]\}=\Delta_{73}$$

$$Orb_{G_{[1,2,3,4]}}[3,5,1,2]=\{[3,5,1,2],[3,6,1,2],\dots,[3,n,1,2]\}=\Delta_{74}$$

$$Orb_{G_{[1,2,3,4]}}[5,3,1,2]=\{[5,3,1,2],[6,3,1,2],\dots,[n,3,1,2]\}=\Delta_{75}$$

$$Orb_{G_{[1,2,3,4]}}[3,1,4,5]=\{[3,1,4,5],[3,1,4,6],\dots,[3,1,4,n]\}=\Delta_{76}$$

$$Orb_{G_{[1,2,3,4]}}[3,1,5,4]=\{[3,1,5,4],[3,1,6,4],\dots,[3,1,n,4]\}=\Delta_{77}$$

$$Orb_{G_{[1,2,3,4]}}[3,5,1,4]=\{[3,5,1,4],[3,6,1,4],\dots,[3,n,1,4]\}=\Delta_{78}$$

$$Orb_{G_{[1,2,3,4]}}[5,3,1,4]=\{[5,3,1,4],[6,3,1,4],\dots,[n,3,1,4]\}=\Delta_{79}$$

$$Orb_{G_{[1,2,3,4]}}[3,2,1,5]=\{[3,2,1,5],[3,2,1,6],\dots,[3,2,1,n]\}=\Delta_{80}$$

$$Orb_{G_{[1,2,3,4]}}[3,2,5,1]=\{[3,2,5,1],[3,2,6,1],\dots,[3,2,n,1]\}=\Delta_{81}$$

$$Orb_{G_{[1,2,3,4]}}[3,5,2,1]=\{[3,5,2,1],[3,6,2,1],\dots,[3,n,2,1]\}=\Delta_{82}$$

$$Orb_{G_{[1,2,3,4]}}\left[5,3,2,1\right]=\left\{\left[5,3,2,1\right],\left[6,3,2,1\right],\ldots,\left[n,3,2,1\right]\right\}=\Delta_{83}$$

$$Orb_{G_{[1,2,3,4]}}\left[3,2,4,5\right]=\left\{\left[3,2,4,5\right],\left[3,2,4,6\right],\ldots,\left[3,2,4,n\right]\right\}=\Delta_{84}$$

$$Orb_{G_{[1,2,3,4]}}\left[3,2,5,4\right]=\left\{\left[3,2,5,4\right],\left[3,2,6,4\right],\ldots,\left[3,2,n,4\right]\right\}=\Delta_{85}$$

$$Orb_{G_{[1,2,3,4]}}\left[3,5,2,4\right]=\left\{\left[3,5,2,4\right],\left[3,6,2,4\right],\ldots,\left[3,n,2,4\right]\right\}=\Delta_{86}$$

$$Orb_{G_{[1,2,3,4]}}\left[5,3,2,4\right]=\left\{\left[5,3,2,4\right],\left[6,3,2,4\right],\ldots,\left[n,3,2,4\right]\right\}=\Delta_{87}$$

$$Orb_{G_{[1,2,3,4]}}\left[3,4,1,5\right]=\left\{\left[3,4,1,5\right],\left[3,4,1,6\right],\ldots,\left[3,4,1,n\right]\right\}=\Delta_{88}$$

$$Orb_{G_{[1,2,3,4]}}\left[3,4,5,1\right]=\left\{\left[3,4,5,1\right],\left[3,4,6,1\right],\ldots,\left[3,4,n,1\right]\right\}=\Delta_{89}$$

$$Orb_{G_{[1,2,3,4]}}\left[3,5,4,1\right]=\left\{\left[3,5,4,1\right],\left[3,6,4,1\right],\ldots,\left[3,n,4,1\right]\right\}=\Delta_{90}$$

$$Orb_{G_{[1,2,3,4]}}\left[5,3,4,1\right]=\left\{\left[5,3,4,1\right],\left[6,3,4,1\right],\ldots,\left[n,3,4,1\right]\right\}=\Delta_{91}$$

$$Orb_{G_{[1,2,3,4]}}\left[3,4,2,5\right]=\left\{\left[3,4,2,5\right],\left[3,4,2,6\right],\ldots,\left[3,4,2,n\right]\right\}=\Delta_{92}$$

$$Orb_{G_{[1,2,3,4]}}\left[3,4,5,2\right]=\left\{\left[3,4,5,2\right],\left[3,4,6,2\right],\ldots,\left[3,4,n,2\right]\right\}=\Delta_{93}$$

$$Orb_{G_{[1,2,3,4]}}\left[3,5,4,2\right]=\left\{\left[3,5,4,2\right],\left[3,6,4,2\right],\ldots,\left[3,n,4,2\right]\right\}=\Delta_{94}$$

$$Orb_{G_{[1,2,3,4]}}\left[5,3,4,2\right]=\left\{\left[5,3,4,2\right],\left[6,3,4,2\right],\ldots,\left[n,3,4,2\right]\right\}=\Delta_{95}$$

$$Orb_{G_{[1,2,3,4]}}\left[4,1,2,5\right]=\left\{\left[4,1,2,5\right],\left[4,1,2,6\right],\ldots,\left[4,1,2,n\right]\right\}=\Delta_{96}$$

$$Orb_{G_{[1,2,3,4]}}\left[4,1,5,2\right]=\left\{\left[4,1,5,2\right],\left[4,1,6,2\right],\ldots,\left[4,1,n,2\right]\right\}=\Delta_{97}$$

$$Orb_{G_{[1,2,3,4]}}\left[4,5,1,2\right]=\left\{\left[4,5,1,2\right],\left[4,6,1,2\right],\ldots,\left[4,n,1,2\right]\right\}=\Delta_{98}$$

$$Orb_{G_{[1,2,3,4]}}\left[5,4,1,2\right]=\left\{\left[5,4,1,2\right],\left[6,4,1,2\right],\ldots,\left[n,4,1,2\right]\right\}=\Delta_{99}$$

$$Orb_{G_{[1,2,3,4]}}[4,1,3,5]=\left\{[4,1,3,5],[4,1,3,6],\ldots,[4,1,3,n]\right\}=\Delta_{100}$$

$$Orb_{G_{[1,2,3,4]}}[4,1,5,3]=\left\{[4,1,5,3],[4,1,6,3],\ldots,[4,1,n,3]\right\}=\Delta_{101}$$

$$Orb_{G_{[1,2,3,4]}}[4,5,1,3]=\left\{[4,5,1,3],[4,6,1,3],\ldots,[4,n,1,3]\right\}=\Delta_{102}$$

$$Orb_{G_{[1,2,3,4]}}[5,4,1,3]=\left\{[5,4,1,3],[6,4,1,3],\ldots,[n,4,1,3]\right\}=\Delta_{103}$$

$$Orb_{G_{[1,2,3,4]}}[4,2,1,5]=\left\{[4,2,1,5],[4,2,1,6],\ldots,[4,2,1,n]\right\}=\Delta_{104}$$

$$Orb_{G_{[1,2,3,4]}}[4,2,5,1]=\left\{[4,2,5,1],[4,2,6,1],\ldots,[4,2,n,1]\right\}=\Delta_{105}$$

$$Orb_{G_{[1,2,3,4]}}[4,5,2,1]=\left\{[4,5,2,1],[4,6,2,1],\ldots,[4,n,2,1]\right\}=\Delta_{106}$$

$$Orb_{G_{[1,2,3,4]}}[5,4,2,1]=\left\{[5,4,2,1],[6,4,2,1],\ldots,[n,4,2,1]\right\}=\Delta_{107}$$

$$Orb_{G_{[1,2,3,4]}}[4,2,3,5]=\left\{[4,2,3,5],[4,2,3,6],\ldots,[4,2,3,n]\right\}=\Delta_{108}$$

$$Orb_{G_{[1,2,3,4]}}[4,2,5,3]=\left\{[4,2,5,3],[4,2,6,3],\ldots,[4,2,n,3]\right\}=\Delta_{109}$$

$$Orb_{G_{[1,2,3,4]}}[4,5,2,3]=\left\{[4,5,2,3],[4,6,2,3],\ldots,[4,n,2,3]\right\}=\Delta_{110}$$

$$Orb_{G_{[1,2,3,4]}}[5,4,2,3]=\left\{[5,4,2,3],[6,4,2,3],\ldots,[n,4,2,3]\right\}=\Delta_{111}$$

$$Orb_{G_{[1,2,3,4]}}[4,3,1,5]=\left\{[4,3,1,5],[4,3,1,6],\ldots,[4,3,1,n]\right\}=\Delta_{112}$$

$$Orb_{G_{[1,2,3,4]}}[4,3,5,1]=\left\{[4,3,5,1],[4,3,6,1],\ldots,[4,3,n,1]\right\}=\Delta_{113}$$

$$Orb_{G_{[1,2,3,4]}}[4,5,3,1]=\left\{[4,5,3,1],[4,6,3,1],\ldots,[4,n,3,1]\right\}=\Delta_{114}$$

$$Orb_{G_{[1,2,3,4]}}[5,4,3,1]=\left\{[5,4,3,1],[6,4,3,1],\ldots,[n,4,3,1]\right\}=\Delta_{115}$$

$$Orb_{G_{[1,2,3,4]}}[4,3,2,5]=\left\{[4,3,2,5],[4,3,2,6],\ldots,[4,3,2,n]\right\}=\Delta_{116}$$

$$Orb_{G_{[1,2,3,4]}}[4,3,5,2]=\{[4,3,5,2],[4,3,6,2],\ldots,[4,3,n,2]\}=\Delta_{117}$$

$$Orb_{G_{[1,2,3,4]}}[4,5,3,2]=\{[4,5,3,2],[4,6,3,2],\ldots,[4,n,3,2]\}=\Delta_{118}$$

$$Orb_{G_{[1,2,3,4]}}[5,4,3,2]=\{[5,4,3,2],[6,4,3,2],\ldots,[n,4,3,2]\}=\Delta_{119}$$

$$Orb_{G_{[1,2,3,4]}}[1,2,5,6]=\{[1,2,5,6],[1,2,5,7],\ldots,[1,2,5,n],[1,2,6,5],\ldots,[1,2,n,n-1]\}=\Delta_{120}$$

$$Orb_{G_{[1,2,3,4]}}[1,5,2,6]=\{[1,5,2,6],[1,5,2,7],\ldots,[1,5,2,n],[1,6,2,5],\ldots,[1,n,2,n-1]\}=\Delta_{121}$$

$$Orb_{G_{[1,2,3,4]}}[1,5,6,2]=\{[1,5,6,2],[1,5,7,2],\ldots,[1,5,n,2],[1,6,5,2],\ldots,[1,n,n-1,2]\}=\Delta_{122}$$

$$Orb_{G_{[1,2,3,4]}}[5,6,1,2]=\{[5,6,1,2],[5,7,1,2],\ldots,[5,n,1,2],[6,5,1,2],\ldots,[n,n-1,1,2]\}=\Delta_{123}$$

$$Orb_{G_{[1,2,3,4]}}[5,1,6,2]=\{[5,1,6,2],[5,1,7,2],\ldots,[5,1,n,2],[6,1,5,2],\ldots,[n,1,n-1,2]\}=\Delta_{124}$$

$$Orb_{G_{[1,2,3,4]}}[5,1,2,6]=\{[5,1,2,6],[5,1,2,7],\ldots,[5,1,2,n],[6,1,2,5],\ldots,[n,1,2,n-1]\}=\Delta_{125}$$

$$Orb_{G_{[1,2,3,4]}}[1,3,5,6]=\{[1,3,5,6],[1,3,5,7],\ldots,[1,3,5,n],[1,3,6,5],\ldots,[1,3,n,n-1]\}=\Delta_{126}$$

$$Orb_{G_{[1,2,3,4]}}[1,5,3,6]=\{[1,5,3,6],[1,5,3,7],\ldots,[1,5,3,n],[1,6,3,5],\ldots,[1,n,3,n-1]\}=\Delta_{127}$$

$$Orb_{G_{[1,2,3,4]}}[1,5,6,3]=\{[1,5,6,3],[1,5,7,3],\ldots,[1,5,n,3],[1,6,5,3],\ldots,[1,n,n-1,3]\}=\Delta_{128}$$

$$Orb_{G_{[1,2,3,4]}}[5,6,1,3]=\{[5,6,1,3],[5,7,1,3],\ldots,[5,n,1,3],[6,5,1,3],\ldots,[n,n-1,1,3]\}=\Delta_{129}$$

$$Orb_{G_{[1,2,3,4]}}[5,1,6,3]=\{[5,1,6,3],[5,1,7,3],\ldots,[5,1,n,3],[6,1,5,3],\ldots,[n,1,n-1,3]\}=\Delta_{130}$$

$$Orb_{G_{[1,2,3,4]}}[5,1,3,6]=\{[5,1,3,6],[5,1,3,7],\ldots,[5,1,3,n],[6,1,3,5],\ldots,[n,1,3,n-1]\}=\Delta_{131}$$

$$Orb_{G_{[1,2,3,4]}}[1,4,5,6]=\{[1,4,5,6],[1,4,5,7],...,[1,4,5,n],[1,4,6,5],...,[1,4,n,n-1]\}=\Delta_{132}$$

$$Orb_{G_{[1,2,3,4]}}[1,5,4,6]=\{[1,5,4,6],[1,5,4,7],...,[1,5,4,n],[1,6,4,5],...,[1,n,4,n-1]\}=\Delta_{133}$$

$$Orb_{G_{[1,2,3,4]}}[1,5,6,4]=\{[1,5,6,4],[1,5,7,4],...,[1,5,n,4],[1,6,5,4],...,[1,n,n-1,4]\}=\Delta_{134}$$

$$Orb_{G_{[1,2,3,4]}}[5,6,1,4]=\{[5,6,1,4],[5,7,1,4],...,[5,n,1,4],[6,5,1,4],...,[n,n-1,1,4]\}=\Delta_{135}$$

$$Orb_{G_{[1,2,3,4]}}[5,1,6,4]=\{[5,1,6,4],[5,1,7,4],...,[5,1,n,4],[6,1,5,4],...,[n,1,n-1,4]\}=\Delta_{136}$$

$$Orb_{G_{[1,2,3,4]}}[5,1,4,6]=\{[5,1,4,6],[5,1,4,7],...,[5,1,4,n],[6,1,4,5],...,[n,1,4,n-1]\}=\Delta_{137}$$

$$Orb_{G_{[1,2,3,4]}}[2,1,5,6]=\{[2,1,5,6],[2,1,5,7],...,[2,1,5,n],[2,1,6,5],...,[2,1,n,n-1]\}=\Delta_{138}$$

$$Orb_{G_{[1,2,3,4]}}[2,5,1,6]=\{[2,5,1,6],[2,5,1,7],...,[2,5,1,n],[2,6,1,5],...,[2,n,1,n-1]\}=\Delta_{139}$$

$$Orb_{G_{[1,2,3,4]}}[2,5,6,1]=\{[2,5,6,1],[2,5,7,1],...,[2,5,n,1],[2,6,5,1],...,[2,n,n-1,1]\}=\Delta_{140}$$

$$Orb_{G_{[1,2,3,4]}}[5,6,2,1]=\{[5,6,2,1],[5,7,2,1],...,[5,n,2,1],[6,5,2,1],...,[n,n-1,2,1]\}=\Delta_{141}$$

$$Orb_{G_{[1,2,3,4]}}[5,2,6,1]=\{[5,2,6,1],[5,2,7,1],...,[5,2,n,1],[6,2,5,1],...,[n,2,n-1,1]\}=\Delta_{142}$$

$$Orb_{G_{[1,2,3,4]}}[5,2,1,6]=\{[5,2,1,6],[5,2,1,7],...,[5,2,1,n],[6,2,1,5],...,[n,2,1,n-1]\}=\Delta_{143}$$

$$Orb_{G_{[1,2,3,4]}}[2,3,5,6]=\{[2,3,5,6],[2,3,5,7],...,[2,3,5,n],[2,3,6,5],...,[2,3,n,n-1]\}=\Delta_{144}$$

$$Orb_{G_{[1,2,3,4]}}[2,5,3,6]=\{[2,5,3,6],[2,5,3,7],...,[2,5,3,n],[2,6,3,5],...,[2,n,3,n-1]\}=\Delta_{145}$$

$$Orb_{G_{[1,2,3,4]}}\left[2,5,6,3\right]=\left\{\left[2,5,6,3\right],\left[2,5,7,3\right],...,\left[2,5,n,3\right],\left[2,6,5,3\right],...,\left[2,n,n-1,3\right]\right\}=\Delta_{146}$$

$$Orb_{G_{[1,2,3,4]}}\left[5,6,2,3\right]=\left\{\left[5,6,2,3\right],\left[5,7,2,3\right],...,\left[5,n,2,3\right],\left[6,5,2,3\right],...,\left[n,n-1,2,3\right]\right\}=\Delta_{147}$$

$$Orb_{G_{[1,2,3,4]}}\left[5,2,6,3\right]=\left\{\left[5,2,6,3\right],\left[5,2,7,3\right],...,\left[5,2,n,3\right],\left[6,2,5,3\right],...,\left[n,2,n-1,3\right]\right\}=\Delta_{148}$$

$$Orb_{G_{[1,2,3,4]}}\left[5,2,3,6\right]=\left\{\left[5,2,3,6\right],\left[5,2,3,7\right],...,\left[5,2,3,n\right],\left[6,2,3,5\right],...,\left[n,2,3,n-1\right]\right\}=\Delta_{149}$$

$$Orb_{G_{[1,2,3,4]}}\left[2,4,5,6\right]=\left\{\left[2,4,5,6\right],\left[2,4,5,7\right],...,\left[2,4,5,n\right],\left[2,4,6,5\right],...,\left[2,4,n,n-1\right]\right\}=\Delta_{150}$$

$$Orb_{G_{[1,2,3,4]}}\left[2,5,4,6\right]=\left\{\left[2,5,4,6\right],\left[2,5,4,7\right],...,\left[2,5,4,n\right],\left[2,6,4,5\right],...,\left[2,n,4,n-1\right]\right\}=\Delta_{151}$$

$$Orb_{G_{[1,2,3,4]}}\left[2,5,6,4\right]=\left\{\left[2,5,6,4\right],\left[2,5,7,4\right],...,\left[2,5,n,4\right],\left[2,6,5,4\right],...,\left[2,n,n-1,4\right]\right\}=\Delta_{152}$$

$$Orb_{G_{[1,2,3,4]}}\left[5,6,2,4\right]=\left\{\left[5,6,2,4\right],\left[5,7,2,4\right],...,\left[5,n,2,4\right],\left[6,5,2,4\right],...,\left[n,n-1,2,4\right]\right\}=\Delta_{153}$$

$$Orb_{G_{[1,2,3,4]}}\left[5,2,6,4\right]=\left\{\left[5,2,6,4\right],\left[5,2,7,4\right],...,\left[5,2,n,4\right],\left[6,2,5,4\right],...,\left[n,2,n-1,4\right]\right\}=\Delta_{154}$$

$$Orb_{G_{[1,2,3,4]}}\left[5,2,4,6\right]=\left\{\left[5,2,4,6\right],\left[5,2,4,7\right],...,\left[5,2,4,n\right],\left[6,2,4,5\right],...,\left[n,2,4,n-1\right]\right\}=\Delta_{155}$$

$$Orb_{G_{[1,2,3,4]}}\left[3,1,5,6\right]=\left\{\left[3,1,5,6\right],\left[3,1,5,7\right],...,\left[3,1,5,n\right],\left[3,1,6,5\right],...,\left[3,1,n,n-1\right]\right\}=\Delta_{156}$$

$$Orb_{G_{[1,2,3,4]}}\left[3,5,1,6\right]=\left\{\left[3,5,1,6\right],\left[3,5,1,7\right],...,\left[3,5,1,n\right],\left[3,6,1,5\right],...,\left[3,n,1,n-1\right]\right\}=\Delta_{157}$$

$$Orb_{G_{[1,2,3,4]}}\left[3,5,6,1\right]=\left\{\left[3,5,6,1\right],\left[3,5,7,1\right],...,\left[3,5,n,1\right],\left[3,6,5,1\right],...,\left[3,n,n-1,1\right]\right\}=\Delta_{158}$$

$$Orb_{G_{[1,2,3,4]}}\left[5,6,3,1\right]=\left\{\left[5,6,3,1\right],\left[5,7,3,1\right],...,\left[5,n,3,1\right],\left[6,5,3,1\right],...,\left[n,n-1,3,1\right]\right\}=\Delta_{159}$$

$$Orb_{G_{[1,2,3,4]}}[5,3,6,1]=\{[5,3,6,1],[5,3,7,1],...,[5,3,n,1],[6,3,5,1],...,[n,3,n-1,1]\}=\Delta_{160}$$

$$Orb_{G_{[1,2,3,4]}}[5,3,1,6]=\{[5,3,1,6],[5,3,1,7],...,[5,3,1,n],[6,3,1,5],...,[n,3,1,n-1]\}=\Delta_{161}$$

$$Orb_{G_{[1,2,3,4]}}[3,2,5,6]=\{[3,2,5,6],[3,2,5,7],...,[3,2,5,n],[3,2,6,5],...,[3,2,n,n-1]\}=\Delta_{162}$$

$$Orb_{G_{[1,2,3,4]}}[3,5,2,6]=\{[3,5,2,6],[3,5,2,7],...,[3,5,2,n],[3,6,2,5],...,[3,n,2,n-1]\}=\Delta_{163}$$

$$Orb_{G_{[1,2,3,4]}}[3,5,6,2]=\{[3,5,6,2],[3,5,7,2],...,[3,5,n,2],[3,6,5,2],...,[3,n,n-1,2]\}=\Delta_{164}$$

$$Orb_{G_{[1,2,3,4]}}[5,6,3,2]=\{[5,6,3,2],[5,7,3,2],...,[5,n,3,2],[6,5,3,2],...,[n,n-1,3,2]\}=\Delta_{165}$$

$$Orb_{G_{[1,2,3,4]}}[5,3,6,2]=\{[5,3,6,2],[5,3,7,2],...,[5,3,n,2],[6,3,5,2],...,[n,3,n-1,2]\}=\Delta_{166}$$

$$Orb_{G_{[1,2,3,4]}}[5,3,2,6]=\{[5,3,2,6],[5,3,2,7],...,[5,3,2,n],[6,3,2,5],...,[n,3,2,n-1]\}=\Delta_{167}$$

$$Orb_{G_{[1,2,3,4]}}[3,4,5,6]=\{[3,4,5,6],[3,4,5,7],...,[3,4,5,n],[3,4,6,5],...,[3,4,n,n-1]\}=\Delta_{168}$$

$$Orb_{G_{[1,2,3,4]}}[3,5,4,6]=\{[3,5,4,6],[3,5,4,7],...,[3,5,4,n],[3,6,4,5],...,[3,n,4,n-1]\}=\Delta_{169}$$

$$Orb_{G_{[1,2,3,4]}}[3,5,6,4]=\{[3,5,6,4],[3,5,7,4],...,[3,5,n,4],[3,6,5,4],...,[3,n,n-1,4]\}=\Delta_{170}$$

$$Orb_{G_{[1,2,3,4]}}[5,6,3,4]=\{[5,6,3,4],[5,7,3,4],...,[5,n,3,4],[6,5,3,4],...,[n,n-1,3,4]\}=\Delta_{171}$$

$$Orb_{G_{[1,2,3,4]}}[5,3,6,4]=\{[5,3,6,4],[5,3,7,4],...,[5,3,n,4],[6,3,5,4],...,[n,3,n-1,4]\}=\Delta_{172}$$

$$Orb_{G_{[1,2,3,4]}}[5,3,4,6]=\{[5,3,4,6],[5,3,4,7],...,[5,3,4,n],[6,3,4,5],...,[n,3,4,n-1]\}=\Delta_{173}$$

$$Orb_{G_{[1,2,3,4]}}\left[4,1,5,6\right]=\left\{\left[4,1,5,6\right],\left[4,1,5,7\right],\dots,\left[4,1,5,n\right],\left[4,1,6,5\right],\dots,\left[4,1,n,n-1\right]\right\}=\Delta_{174}$$

$$Orb_{G_{[1,2,3,4]}}\left[4,5,1,6\right]=\left\{\left[4,5,1,6\right],\left[4,5,1,7\right],\dots,\left[4,5,1,n\right],\left[4,6,1,5\right],\dots,\left[4,n,1,n-1\right]\right\}=\Delta_{175}$$

$$Orb_{G_{[1,2,3,4]}}\left[4,5,6,1\right]=\left\{\left[4,5,6,1\right],\left[4,5,7,1\right],\dots,\left[4,5,n,1\right],\left[4,6,5,1\right],\dots,\left[4,n,n-1,1\right]\right\}=\Delta_{176}$$

$$Orb_{G_{[1,2,3,4]}}\left[5,6,4,1\right]=\left\{\left[5,6,4,1\right],\left[5,7,4,1\right],\dots,\left[5,n,4,1\right],\left[6,5,4,1\right],\dots,\left[n,n-1,4,1\right]\right\}=\Delta_{177}$$

$$Orb_{G_{[1,2,3,4]}}\left[5,4,6,1\right]=\left\{\left[5,4,6,1\right],\left[5,4,7,1\right],\dots,\left[5,4,n,1\right],\left[6,4,5,1\right],\dots,\left[n,4,n-1,1\right]\right\}=\Delta_{178}$$

$$Orb_{G_{[1,2,3,4]}}\left[5,4,1,6\right]=\left\{\left[5,4,1,6\right],\left[5,4,1,7\right],\dots,\left[5,4,1,n\right],\left[6,4,1,5\right],\dots,\left[n,4,1,n-1\right]\right\}=\Delta_{179}$$

$$Orb_{G_{[1,2,3,4]}}\left[4,2,5,6\right]=\left\{\left[4,2,5,6\right],\left[4,2,5,7\right],\dots,\left[4,2,5,n\right],\left[4,2,6,5\right],\dots,\left[4,2,n,n-1\right]\right\}=\Delta_{180}$$

$$Orb_{G_{[1,2,3,4]}}\left[4,5,2,6\right]=\left\{\left[4,5,2,6\right],\left[4,5,2,7\right],\dots,\left[4,5,2,n\right],\left[4,6,2,5\right],\dots,\left[4,n,2,n-1\right]\right\}=\Delta_{181}$$

$$Orb_{G_{[1,2,3,4]}}\left[4,5,6,2\right]=\left\{\left[4,5,6,2\right],\left[4,5,7,2\right],\dots,\left[4,5,n,2\right],\left[4,6,5,2\right],\dots,\left[4,n,n-1,2\right]\right\}=\Delta_{182}$$

$$Orb_{G_{[1,2,3,4]}}\left[5,6,4,2\right]=\left\{\left[5,6,4,2\right],\left[5,7,4,2\right],\dots,\left[5,n,4,2\right],\left[6,5,4,2\right],\dots,\left[n,n-1,4,2\right]\right\}=\Delta_{183}$$

$$Orb_{G_{[1,2,3,4]}}\left[5,4,6,2\right]=\left\{\left[5,4,6,2\right],\left[5,4,7,2\right],\dots,\left[5,4,n,2\right],\left[6,4,5,2\right],\dots,\left[n,4,n-1,2\right]\right\}=\Delta_{184}$$

$$Orb_{G_{[1,2,3,4]}}\left[5,4,2,6\right]=\left\{\left[5,4,2,6\right],\left[5,4,2,7\right],\dots,\left[5,4,2,n\right],\left[6,4,2,5\right],\dots,\left[n,4,2,n-1\right]\right\}=\Delta_{185}$$

$$Orb_{G_{[1,2,3,4]}}\left[4,3,5,6\right]=\left\{\left[4,3,5,6\right],\left[4,3,5,7\right],\dots,\left[4,3,5,n\right],\left[4,3,6,5\right],\dots,\left[4,3,n,n-1\right]\right\}=\Delta_{186}$$

$$Orb_{G_{[1,2,3,4]}}\left[4,5,3,6\right]=\left\{\left[4,5,3,6\right],\left[4,5,3,7\right],\dots,\left[4,5,3,n\right],\left[4,6,3,5\right],\dots,\left[4,n,3,n-1\right]\right\}=\Delta_{187}$$

$$Orb_{G_{[1,2,3,4]}}[4,5,6,3]=\{[4,5,6,3],[4,5,7,3],\ldots,[4,5,n,3],[4,6,5,3],\ldots,[4,n,n-1,3]\}=\Delta_{188}$$

$$Orb_{G_{[1,2,3,4]}}[5,6,4,3]=\{[5,6,4,3],[5,7,4,3],\ldots,[5,n,4,3],[6,5,4,3],\ldots,[n,n-1,4,3]\}=\Delta_{189}$$

$$Orb_{G_{[1,2,3,4]}}[5,4,6,3]=\{[5,4,6,3],[5,4,7,3],\ldots,[5,4,n,3],[6,4,5,3],\ldots,[n,4,n-1,3]\}=\Delta_{190}$$

$$Orb_{G_{[1,2,3,4]}}[5,4,3,6]=\{[5,4,3,6],[5,4,3,7],\ldots,[5,4,3,n],[6,4,3,5],\ldots,[n,4,3,n-1]\}=\Delta_{191}$$

$$Orb_{G_{[1,2,3,4]}}[1,5,6,7]=\{[1,5,6,7],[1,5,6,8],\ldots,[1,5,6,n],[1,5,7,6],[1,5,7,8],\ldots,[1,n,n-1,n-2]\}=\Delta_{192}$$

$$Orb_{G_{[1,2,3,4]}}[5,1,6,7]=\{[5,1,6,7],[5,1,6,8],\ldots,[5,1,6,n],[5,1,7,6],[5,1,7,8],\ldots,[n,1,n-1,n-2]\}=\Delta_{193}$$

$$Orb_{G_{[1,2,3,4]}}[5,6,1,7]=\{[5,6,1,7],[5,6,1,8],\ldots,[5,6,1,n],[5,7,1,6],[5,7,1,8],\ldots,[n,n-1,1,n-2]\}=\Delta_{194}$$

$$Orb_{G_{[1,2,3,4]}}[5,6,7,1]=\{[5,6,7,1],[5,6,8,1],\ldots,[5,6,n,1],[5,7,6,1],[5,7,8,1],\ldots,[n,n-1,n-2,1]\}=\Delta_{195}$$

$$Orb_{G_{[1,2,3,4]}}[2,5,6,7]=\{[2,5,6,7],[2,5,6,8],\ldots,[2,5,6,n],[2,5,7,6],[2,5,7,8],\ldots,[2,n,n-1,n-2]\}=\Delta_{196}$$

$$Orb_{G_{[1,2,3,4]}}[5,2,6,7]=\{[5,2,6,7],[5,2,6,8],\ldots,[5,2,6,n],[5,2,7,6],[5,2,7,8],\ldots,[n,2,n-1,n-2]\}=\Delta_{197}$$

$$Orb_{G_{[1,2,3,4]}}[5,6,2,7]=\{[5,6,2,7],[5,6,2,8],\ldots,[5,6,2,n],[5,7,2,6],[5,7,2,8],\ldots,[n,n-1,2,n-2]\}=\Delta_{198}$$

$$Orb_{G_{[1,2,3,4]}}[5,6,7,2]=\{[5,6,7,2],[5,6,8,2],\ldots,[5,6,n,2],[5,7,6,2],[5,7,8,2],\ldots,[n,n-1,n-2,2]\}=\Delta_{199}$$

$$Orb_{G_{[1,2,3,4]}}[3,5,6,7]=\{[3,5,6,7],[3,5,6,8],\ldots,[3,5,6,n],[3,5,7,6],[3,5,7,8],\ldots,[3,n,n-1,n-2]\}=\Delta_{200}$$

$$Orb_{G_{[1,2,3,4]}}[5,3,6,7]=\{[5,3,6,7],[5,3,6,8],\ldots,[5,3,6,n],[5,3,7,6],[5,3,7,8],\ldots,[n,3,n-1,n-2]\}=\Delta_{201}$$

$$Orb_{G_{[1,2,3,4]}}[5,6,3,7]=\{[5,6,3,7],[5,6,3,8],\ldots,[5,6,3,n],[5,7,3,6],[5,7,3,8],\ldots,[n,n-1,3,n-2]\}=\Delta_{202}$$

$$Orb_{G_{[1,2,3,4]}}\left[5,6,7,3\right]=\left\{\left[5,6,7,3\right],\left[5,6,8,3\right],\ldots,\left[5,6,n,3\right],\left[5,7,6,3\right],\left[5,7,8,3\right],\ldots,\left[n,n-1,n-2,3\right]\right\}=\Delta_{203}$$

$$Orb_{G_{[1,2,3,4]}}\left[4,5,6,7\right]=\left\{\left[4,5,6,7\right],\left[4,5,6,8\right],\ldots,\left[4,5,6,n\right],\left[4,5,7,6\right],\left[4,5,7,8\right],\ldots,\left[4,n,n-1,n-2\right]\right\}=\Delta_{204}$$

$$Orb_{G_{[1,2,3,4]}}\left[5,4,6,7\right]=\left\{\left[5,4,6,7\right],\left[5,4,6,8\right],\ldots,\left[5,4,6,n\right],\left[5,4,7,6\right],\left[5,4,7,8\right],\ldots,\left[n,4,n-1,n-2\right]\right\}=\Delta_{205}$$

$$Orb_{G_{[1,2,3,4]}}\left[5,6,4,7\right]=\left\{\left[5,6,4,7\right],\left[5,6,4,8\right],\ldots,\left[5,6,4,n\right],\left[5,7,4,6\right],\left[5,7,4,8\right],\ldots,\left[n,n-1,4,n-2\right]\right\}=\Delta_{206}$$

$$Orb_{G_{[1,2,3,4]}}\left[5,6,7,4\right]=\left\{\left[5,6,7,4\right],\left[5,6,8,4\right],\ldots,\left[5,6,n,4\right],\left[5,7,6,4\right],\left[5,7,8,4\right],\ldots,\left[n,n-1,n-2,4\right]\right\}=\Delta_{207}$$

$$Orb_{G_{[1,2,3,4]}}\left[5,6,7,8\right]=\left\{\left[5,6,7,8\right],\left[5,6,7,9\right],\ldots,\left[5,6,7,n\right],\left[5,7,6,8\right],\left[5,7,6,9\right],\ldots,\left[n,n-1,n-2,n-3\right]\right\}=\Delta_{208}$$

APPENDIX C

C.1 Suborbits of $G = S_7$ acting on $X^{[3]}$

$$Orb_{G_{[1,2,3]}}[1,2,3]=\{[1,2,3]\}=\Delta_0$$

$$Orb_{G_{[1,2,3]}}[1,3,2]=\{[1,3,2]\}=\Delta_1$$

$$Orb_{G_{[1,2,3]}}[2,1,3]=\{[2,1,3]\}=\Delta_2$$

$$Orb_{G_{[1,2,3]}}[2,3,1]=\{[2,3,1]\}=\Delta_3$$

$$Orb_{G_{[1,2,3]}}[3,1,2]=\{[3,1,2]\}=\Delta_4$$

$$Orb_{G_{[1,2,3]}}[3,2,1]=\{[3,2,1]\}=\Delta_5$$

$$Orb_{G_{[1,2,3]}}[1,2,4]=\{[1,2,4],[1,2,5],[1,2,6],[1,2,7]\}=\Delta_6$$

$$Orb_{G_{[1,2,3]}}[1,4,2]=\{[1,4,2],[1,5,2],[1,6,2],[1,7,2]\}=\Delta_7$$

$$Orb_{G_{[1,2,3]}}[4,1,2]=\{[4,1,2],[5,1,2],[6,1,2],[7,1,2]\}=\Delta_8$$

$$Orb_{G_{[1,2,3]}}[2,1,4]=\{[2,1,4],[2,1,5],[2,1,6],[2,1,7]\}=\Delta_9$$

$$Orb_{G_{[1,2,3]}}[2,4,1]=\{[2,4,1],[2,5,1],[2,6,1],[2,7,1]\}=\Delta_{10}$$

$$Orb_{G_{[1,2,3]}}[4,2,1]=\{[4,2,1],[5,2,1],[6,2,1],[7,2,1]\}=\Delta_{11}$$

$$Orb_{G_{[1,2,3]}}[1,3,4]=\{[1,3,4],[1,3,5],[1,3,6],[1,3,7]\}=\Delta_{12}$$

$$Orb_{G_{[1,2,3]}}[1,4,3]=\{[1,4,3],[1,5,3],[1,6,3],[1,7,3]\}=\Delta_{13}$$

$$Orb_{G_{[1,2,3]}}[4,1,3]=\{[4,1,3],[5,1,3],[6,1,3],[7,1,3]\}=\Delta_{14}$$

$$Orb_{G_{[1,2,3]}}[3,1,4]=\{[3,1,4],[3,1,5],[3,1,6],[3,1,7]\}=\Delta_{15}$$

$$Orb_{G_{[1,2,3]}}[3,4,1]=\{[3,4,1],[3,5,1],[3,6,1],[3,7,1]\}=\Delta_{16}$$

$$Orb_{G_{[1,2,3]}}[4,3,1]=\{[4,3,1],[5,3,1],[6,3,1],[7,3,1]\}=\Delta_{17}$$

$$Orb_{G_{[1,2,3]}}[2,3,4]=\{[2,3,4],[2,3,5],[2,3,6],[2,3,7]\}=\Delta_{18}$$

$$Orb_{G_{[1,2,3]}}[2,4,3]=\{[2,4,3],[2,5,3],[2,6,3],[2,7,3]\}=\Delta_{19}$$

$$Orb_{G_{[1,2,3]}}[4,2,3]=\{[4,2,3],[5,2,3],[6,2,3],[7,2,3]\}=\Delta_{20}$$

$$Orb_{G_{[1,2,3]}}[3,2,4]=\{[3,2,4],[3,2,5],[3,2,6],[3,2,7]\}=\Delta_{21}$$

$$Orb_{G_{[1,2,3]}}[3,4,2]=\{[3,4,2],[3,5,2],[3,6,2],[3,7,2]\}=\Delta_{22}$$

$$Orb_{G_{[1,2,3]}}[4,3,2]=\{[4,3,2],[5,3,2],[6,3,2],[7,3,2]\}=\Delta_{23}$$

$$\begin{aligned}Orb_{G_{[1,2,3]}}[1,4,5] &=\{[1,4,5],[1,4,6],[1,4,7],[1,5,4],[1,5,6],[1,5,7],[1,6,4],[1,6,5],\\ &\quad [1,6,7],[1,7,4],[1,7,5],[1,7,6]\}=\Delta_{24}\end{aligned}$$

$$\begin{aligned}Orb_{G_{[1,2,3]}}[4,1,5] &=\{[4,1,5],[4,1,6],[4,1,7],[5,1,4],[5,1,6],[5,1,7],[6,1,4],[6,1,5],\\ &\quad [6,1,7],[7,1,4],[7,1,5],[7,1,6]\}=\Delta_{25}\end{aligned}$$

$$\begin{aligned}Orb_{G_{[1,2,3]}}[4,5,1] &=\{[4,5,1],[4,6,1],[4,7,1],[5,4,1],[5,6,1],[5,7,1],[6,4,1],[6,5,1],\\ &\quad [6,7,1],[7,4,1],[7,5,1],[7,6,1]\}=\Delta_{26}\end{aligned}$$

$$\begin{aligned}Orb_{G_{[1,2,3]}}[2,4,5] &=\{[2,4,5],[2,4,6],[2,4,7],[2,5,4],[2,5,6],[2,5,7],[2,6,4],[2,6,5],\\ &\quad [2,6,7],[2,7,4],[2,7,5],[2,7,6]\}=\Delta_{27}\end{aligned}$$

$$\begin{aligned}Orb_{G_{[1,2,3]}}[4,2,5] &=\{[4,2,5],[4,2,6],[4,2,7],[5,2,4],[5,2,6],[5,2,7],[6,2,4],[6,2,5],\\ &\quad [6,2,7],[7,2,4],[7,2,5],[7,2,6]\}=\Delta_{28}\end{aligned}$$

$$\begin{aligned} Orb_{G_{[1,2,3]}}[4,5,2] = \{&[4,5,2],[4,6,2],[4,7,2],[5,4,2],[5,6,2],[5,7,2],[6,4,2],[6,5,2],\\ &[6,7,2],[7,4,2],[7,5,2],[7,6,2]\} = \Delta_{29} \end{aligned}$$

$$\begin{aligned} Orb_{G_{[1,2,3]}}[3,4,5] = \{&[3,4,5],[3,4,6],[3,4,7],[3,5,4],[3,5,6],[3,5,7],[3,6,4],[3,6,5],\\ &[3,6,7],[3,7,4],[3,7,5],[3,7,6]\} = \Delta_{30} \end{aligned}$$

$$\begin{aligned} Orb_{G_{[1,2,3]}}[4,3,5] = \{&[4,3,5],[4,3,6],[4,3,7],[5,3,4],[5,3,6],\ [5,3,7],[6,3,4],[6,3,5],\\ &[6,3,7],[7,3,4],[7,3,5],[7,3,6]\ \} = \Delta_{31} \end{aligned}$$

$$\begin{aligned} Orb_{G_{[1,2,3]}}[4,5,3] = \{&[4,5,3],[4,6,3],[4,7,3],[5,4,3],[5,6,3],[5,7,3],[6,4,3],[6,5,3],\\ &[6,7,3],[7,4,3],[7,5,3],[7,6,3]\} = \Delta_{32} \end{aligned}$$

$$\begin{aligned} Orb_{G_{[1,2,3]}}[4,5,6] = \{&[4,5,6],[4,5,7],[4,6,5],[4,6,7],[4,7,5],[4,7,6],[5,4,6],[5,4,7],\\ &[5,6,4,],[5,6,7],[5,7,4],[5,7,6],[6,4,5],[6,4,7],[6,5,4],[6,5,7],\\ &[6,7,4],[6,7,5],[7,4,5],\ [7,4,6],[7,5,4],[7,5,6],[7,6,4],[7,6,5]\} = \Delta_{33} \end{aligned}$$