QUASIAFFINE INVERSE AND MOORE-PENROSE INVERSE OF OPERATORS IN HILBERT SPACES

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DECLARATION

This thesis is my original work and has not been presented for a degree in any other university or for any other award.

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To my parents, John Mwanzia and Celestine Mutinda, my beloved wife Victoria Mutile, my sons Benjamin Mutisya Mutuku and Lawrence Musau Mutuku.

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ACRONYMS AND NOTATIONS

The following notations have been used in this thesis:

- H – Hilbert space.
- $A\{1\} = \{A' \in B(H): AA'A = A\}$.
- $A\{2\} = \{A' \in B(H): A'AA' = A'\}$.
- $A\{3\} = \{A' \in B(H): (AA')^* = AA'\}$.
- $A\{4\} = \{A' \in B(H): (A'A)^* = A'A\}$.
- $A\{i,j\} = \{A' \in B(H): A' = A\{i\} \cap A\{j\}\}$ for $i, j = 1,2,3$ and 4.
- A' - The Generalised inverse of A where $A' \in A\{1\} = \{A' \in B(H): AA'A = A\}$.
- $A^+ -$ The Moore-Penrose inverse of A.
- A^*- Adjoint of A.
- $\overline{A^T}$ – Conjugate transpose of A.
- A^{-1} – Inverse of A.
- $\{A\}^c = \{B \in B(H): AB = BA\}$.
- A_D – Drazin inverse of A.
- $A_L^{(-1)} = P_L(AP_L + P_{L^\perp})^{-1}$- Bott-Duffin inverse of A.
- A_π – Spectral idempotent of A at 0.
- $A_\pi = I - A_D A$.
- $\|A\| -$ norm of A.
- A^T- The transpose of matrix A.
- $B(H)$ - Space of bounded linear operators in H.
- $B_c(H)$ - $B(H)$ with closed range.
- $D(A)$- Domain of operator A.
- Ker $A/N(A)$ - Kernel of A (Or the null space of A)

- LSS – Least Square Solution.
- $N(A)^{\perp}$ - Orthogonal complement of null space of operator A.
- $\mathbb{N}$ - a set of Positive integer/natural number.
- P_V - Orthogonal projection onto V.
- $\mathrm{Ran}(A) -$ Range of A.
- $r(A)$ - Rank of A.
- $R\,(T,S)\,X = TXS - X$ for $T, S \in B(H)$.
- SVD− Singular Value Decomposition.
- $\overline{V}$ is the closure of sub-space V.
- $< x, y > -$ Inner product of x and y.
- $\sigma(A)$ - Spectrum of A.

ABSTRACT

The study of inverses of operators by the concept of the Moore-Penrose Inverse and the quasiaffine inverse started in 1920 and 1985 respectively. Precisely, Moore (1920) and Penrose (1955) independently gave conditions satisfied by the MPI. Hongke and Chuan (1985) and Khalagai (1996) studied invertibility of normal (sub-normal) operators and one-sided invertible operators respectively, by the concept of the quasiaffine inverse. Since then, several researchers have contributed to these areas. Particularly, it has been shown that the Moore-Penrose inverse of an operator A with closed range satisfies the following conditions: $AA^{+}A = A$, $A^{+}AA^{+} = A^{+}$, $(AA^{+})^{*} = AA^{+}$ and $(A^{+}A)^{*} = A^{+}A$. If A is a quasiaffine inverse of B then both A and B are quasiaffinities and if A is an EP operator then $\mathrm{Ran}(A) = \mathrm{Ran}(A^{*})$. It is also known that the Fuglede-Putnam Theorems and Fuglede-Putnam type commutativity theorems hold for normal operators and EP operators under some conditions. On quasiaffine inverses, this thesis establishes the uniqueness of the quasiaffine inverse of A given $AXB = X$ and $BYA = Y$ as well as establishing that $B = A^{-1}$ under given conditions. The invertibility of quasinormal partial isometry with dense range as well as results on invertibility of operators A and B satisfying the equations $AX = XB$ or $BY = YA$ or both is shown under some given conditions. On MPI, the invertibility of an EP operator in terms of its Moore-Penrose inverse is established. In particular, the case where the Moore-Penrose inverse of an EP operator turns to be its usual inverse under some given conditions is shown. The Moore-Penrose inverse of a perturbed operator $A + B$ with closed range, where A is expressible as a product of two operators $P, Q \in B_C(H)$ with closed ranges and B a bounded operator satisfying some given conditions is exhibited as well as the relation between the ranges and null spaces of these operators. Moreover, this thesis establishes that Fuglede-Putnam-type results hold for EP operators, injective operators and operators with dense range satisfying some commutativity conditions involving operators AA^{*}, $A^{*}A$, $A^{*}A^{+}$, BB^{*}, $B^{*}B$ and $B^{*}B^{+}$.

CHAPTER ONE

1 INTRODUCTION

The mathematical concept of two dimensional and three dimensional vector spaces was generalized to Euclidean spaces. The inner product space was a generalization of the dot product in a Euclidean space, $\mathbb{R}^n$. A complete inner product space is called a Hilbert space, a concept introduced and developed in 20th century by David Hilbert in his study of integral function equations. Hilbert showed that the inner product of two integral function on an interval (ρ_1, ρ_2) is expressed as:

$$\langle A, B \rangle = \int_{\rho_1}^{\rho_2} A(x)\ \overline{B(x)}\, dx$$

Linear equations arise in mathematics, engineering, economics, planning and many other spheres. They are of the form:

$$\sum_{\delta=1}^{k} \omega_\delta v_\delta = \gamma$$

where $\omega_1, \omega_2, \omega_3, \dots, \omega_k$ and γ are constants and $v_1, v_2, v_3, \dots, v_k$ are variables. A system of linear equations is an assemblage of linear equations whose variables are common. In many fields where linear equations arise, the central concern is to use a suitable and systematic method to solve for the values of the unknowns. These systems of linear equations can be expressed in operator form as $Av = \gamma$ with v as the unknown vector, γ the known vector and A an operator. For example, researchers use operators to simplify difficult equations such as the Schrodinger wave equation. In this case the Hamiltonian operator is used. Also, when solving the Fredholm equations of the first kind such as:

$$\int_s^t G(x,z)H(z)\, dz = P(x)$$

with G as the continuous kernel function and P a given function both satisfying some conditions, to find the function H operator theory is applied. By use of expansion in terms of orthogonal functions or finite differences, the Fredholm equation can be substituted by approximating the matrix equation $\mathrm{Av} = \gamma$ and as a result, v is solved. If the operator is in $\mathbb{C}^{\mathrm{nxn}}$, then several methods such as crammers rule, reduced row echelon form or inverse of A may be used to determine the value of v. The inverse of A exists and it is unique if and only if A is bijective and can be used to find $\mathrm{v} = \mathrm{A}^{-1}\gamma$. If v is a particular solution, then the general solution can be given as $\mathrm{v} + \mathrm{v}_0$. Note that here we have $\mathrm{v_o} = \{\mathrm{u}: \mathrm{Au} = 0\}$.

If A is non-invertible, then researchers approximate the value of v which is used as an alternative to the original solution. If an operator is injective, then it has a left inverse and if its surjective then it has a right inverse. In case $\mathrm{Nul(A)} = \{0\}$, one solution exists. If $\mathrm{Nul(A)} \neq \{0\}$, infinitely many solutions are realized. In these cases, the system is said to be consistent. If $\gamma \notin \mathrm{Ran(A)}$, then $\mathrm{Av} = \gamma$ is said to be inconsistent and has no solution. For instance, consider$-\mathrm{x} + 2\mathrm{y} = 1$ and $\mathrm{x} - 2\mathrm{y} = 3$. These are parallel lines and do not intersect. This implies that they cannot be solved simultaneously. On the other hand, if $-\mathrm{x} + 2\mathrm{y} = 1$ and $\mathrm{x} - 2\mathrm{y} = -1$, then the equations have infinitely many solutions. Notably, A is a matrix $\mathrm{A} = \begin{pmatrix} -1 & 2 \\ 1 & -2 \end{pmatrix}, \gamma = (1,3)$ and $(1, -1)$ respectively. Precisely, the inverse of an operator is key in finding solutions to operator equations. In many fields such as oil exploration, linear programming and electrical networks, computers are used to solve millions of separate systems of linear equations. These computers compute the generalized inverses of an operator by use of computational algorithms that are developed by scholars.

A linear system that arises from experimental data frequently has no solution. In such a case, a substitute (approximate) solution (v') which give the smallest distance between Av' and γ is used. If the operator has no usual inverse, then the generalized inverses are used to approximate solutions to these operator equations. There are several generalised inverse with specific importance in analyzing operator equations. For instance, $A\{1\}$-inverse plays a very important role in solving linear systems while $A\{1,3\}$ and $A\{1,4\}$-inverses are important in the minimizing properties of least squares solution. There exists a unique generalised inverse referred to as the Moore-Penrose Inverse denoted by A^{+}. The Moore-Penrose inverse gives the best approximate solution v' of minimum norm. In case A is invertible, then A^{-1} turns out to be the Moore-Penrose inverse of A as it satisfies the Penrose conditions. There are various methods of finding the Moore-Penrose inverse of an operator in $\mathbb{C}^{mxn}$. If either A^*A or AA^* is invertible then, $A^{+} = (A^*A)^{-1}A^*$or $A^{+} = A^*(AA^*)^{-1}$. If A^*A and AA^* are singular, then the Tikhonov's regularization procedure is usually used where the singular matrices A^*A and AA^* are replaced by non-singular matrices $A^*A + \lambda I$ or $AA^* + \lambda I$, where λ is very small. The Moore-Penrose inverse of A is obtained as $\lambda \to 0$.

The Singular value decomposition theorem and the Householder QR decomposition are widely known methods for determining A^{+}. The matrix A is expressed as a product of three matrices $A = XSY$, where X and Y are matrices with orthonormal columns and the square roots of eigen values of AA^* are the diagonal elements of S. The Moore-Penrose inverse of A is realized as: $A^{+} = Y^{-1}S^{+}X^{-1}$ where

$$S^{+} = \begin{bmatrix} \frac{1}{\sigma_1} & \cdots & 0 \\ \vdots & \ddots & \vdots \\ 0 & \cdots & \frac{1}{\sigma_n} \end{bmatrix}$$

and σ_i's are square roots of eigen values of A^*A, $i = 1,2, \dots, n$.

The householder QR-method expresses $A \in \mathbb{C}^{mxn}$ with rank r, as $A = QR$, where $Q \in \mathbb{C}^{mxr}$ and $R \in \mathbb{C}^{rxn}$. In this case $A^{+} = R^{*}(RR^{*})^{-1}(Q^{*}Q)^{-1}Q^{*}$.

In analyzing the stability of the solution, the continuity of the Moore-Penrose inverse is important. The perturbation results show the extent to which an operator can be perturbed without losing the desired properties of the solution. Some operator equations are difficult to solve to get a particular solution hence the operator equations with exact solution are used to approximate solutions to these equations. That is, if $Av = \gamma$ has a particular solution and $Wv = \gamma$ is difficult to solve, then A can be perturbed so that $W = A + B$. In this case, some conditions are imposed so that A and W have same range where B is a perturbation of A satisfying some conditions. This thesis assumes that A is expressible as a product of two operators P and Q. That is $A = PQ$ and B is a perturbation of A satisfying some conditions to give an expression of $W^{+} = (A + B)^{+}$.

Also, there exists another emerging inverse called the quasiaffine inverse. Given the operator equation $AXB = X$ where X is a quasiaffinity, B is a right quasiaffine inverse of A and A is left quasiaffine inverse of B. Subsequently, A is a quasiaffine inverse of B if there is another quasiaffinity Y satisfying $AXB = X$ and $BYA = Y$. In case, $X = Y = I$, then $AB = BA = I$. That is $B = A^{-1}$. In this thesis, the uniqueness and invertibility of the operators A and B under certain conditions is investigated.

The idea of operators commuting is very significant in operator theory. Two linear operators A and B in $B(H)$ commute if $AB = BA$. If an operator commutes with its adjoint then it is said to be normal. The Fuglede-Putnam theorems gives results on normal operators commuting with bounded linear operators. These results can be generalized to EP operators and injective linear operators (and linear operators with dense range) satisfying some given commutativity conditions as illustrated in this thesis.

1.1 Historical background

Generalized inverses of linear operators were studied by Fredholm (1903) where the author looked at generalized inverses of integral functions referring to them as Pseudo inverse. Hurwitz (1912) came up with simple algebraic construction of generalized inverse by using the finite dimension of the null spaces of the Fredholm operators and characterized the classes of all pseudo inverses. Moore (1920) came up with a unique generalized inverse for any matrix whether square or rectangular referring it as the “general reciprocal”. Though this work was published in 1920 the results are speculated to have been obtained as early as 1906. After the death of Moore, the general reciprocal of matrix was sent to oblivion for quite some time until in 1955. For this time the generalized inverse for matrix was given by several authors such as Siegel (1937) and for operators by Tseng (1936) and others. In 1951, Bjerhammar discussed the least squares properties for generalized inverse and revived Moore’s work. In 1955, Penrose widened Bjerhammar’s results on linear systems and illustrated the uniqueness of the Moore’s inverse of a given matrix which satisfy four Penrose’s equations given in the definition of the Moore-Penrose inverse. The conditions of the Moore’s inverse are equivalent to the four conditions stated by Penrose (1955). Since Penrose work, many more properties and applications of generalized inverse have been studied by a number of authors.

Invertibility of linear operators with left (right) quasiaffine inverses is an emerging concept in mathematics. Inverses of normal and subnormal linear operators with left (right) quasiaffine inverse were first discussed by Hongke and Chuan (1985) and extended by Khalagai (1996) to left and right invertible linear operators. Khalagai and Otieno (2000) discussed quasiaffinity and quasiaffine inverses of partial isometry. Several other researchers have worked on this area.

On Fuglede-Putnam theorems, Fuglede (1950) showed that if a bounded operator commutes with normal operator, then it also commutes with the adjoint of the normal operator. This was extended to two normal operators by Putnam (1951). Since then several scholars have contributed to this area.

1.2 Statement of the problem

Several scholars have given results on Moore-Penrose inverse and quasiaffine inverse of operators. For instance, the uniqueness and reverse order of the Moore-Penrose inverse have been established. The expression of Moore-Penrose inverse of an operator A in terms of its adjoint, $A^{+} = A^{*}(AA^{*})^{+} = (A^{*}A)^{+}A^{*}$ has been given. The Moore-Penrose inverse of an operator A expressible as a product of two operators (matrices) $A = PQ$ has been discussed in terms of Moore-Penrose inverse of P and Moore-Penrose inverse of Q. Also, the Moore-Penrose inverse of a perturbed linear operator $W = A + B$, where B is a perturbation of A satisfying $B = AA^{+}B$, $BA^{+}A = B|_{D(A)}$ and $\|A^{+}B\| < 1$ has been established, where the Moore-Penrose inverse of W has been expressed in terms of Moore-Penrose inverse of A. However, conditions under which the Moore-Penrose inverse of an EP operator turns out to be the usual inverse of the operator has been overlooked as well as the expression of the Moore-Penrose inverse of $A + B$ in terms of Moore-Penrose inverse of P and Moore-Penrose inverse of Q when $A = PQ$. This thesis establishes the Moore-Penrose inverse of a perturbed linear operator $W = A + B$, where $A = PQ$ and B is such that $B = PP^{+}B$, $BQ^{+}Q = B|_{D(Q)}$ and $\|P^{+}BQ^{+}\| < 1$

On quasiaffine inverse, it has been shown that the quasiaffine inverse of an operator can be the usual inverse of the operator under some conditions and that the quasiaffine inverse is a quasiaffinity. The invertibility of a quasi-invertible partial isometry has been established. However, the uniqueness of a quasiaffine inverse of a bounded linear operator has been overpassed as well as the conditions under which the quasiaffine inverse of an operator turns out to be the usual inverse of the operator. Moreover, the invertibility of a quasinormal partial isometry with a dense range has not been established.

Finally, it has been established that the Fuglede –Putnam theorems hold true for normal operators and some other classes of operators. This study extends the Fuglede-Putnam theorems to EP operators, injective operators and operators with dense range satisfying some given conditions.

1.3 Objectives

1.3.1 General objective

To investigate properties of quasiaffine inverses and Moore-Penrose inverse of linear operators on Hilbert spaces.

1.3.2 Specific objectives

(i) To establish whether quasiaffine inverse of an operator can be the usual inverse of an operator under some given conditions.

(ii) To ascertain whether quasiaffine inverse of an operator on a Hilbert space is unique.

(iii) To establish whether the Moore-Penrose inverse of an EP operator can be its usual inverse under some conditions.

(iv) To ascertain whether the Moore-Penrose inverse of a perturbed linear operator $(A + B)$ can be expressed in terms of Moore-Penrose inverse of P and Moore-Penrose inverse of Q, when $A = PQ$ under some conditions.

(v) To establish if Fuglede-Putnam Theorems hold on linear operators and EP operators under some given conditions.

1.4 Significance of the study

Inverses of linear operators are applied in physics, engineering, partial differential equations and data analysis in finding solutions to operator equations. Moore-Penrose inverse is the favored generalised inverse to date. It is used to compute best fit solution for systems of linear equations $Av = \gamma$ incase A is not invertible. Without this inverse, many systems of linear equations would have no solution. The Moore-Penrose inverse is applicable in digital image restoration. This science is trending owing to the fact that restoration of original image is of importance so as to better study distorted or old images for medical, military and astronomical purposes just to name a few.

Computational algorithms for calculating Moore-Penrose inverse have been developed by many scholars. The race to have a fast and accurate digital imaging is always on. This is important so as to reduce computational load and reduce time load associated with these algorithms. Classical techniques such as Singular Value decomposition (SVD) are time intensive despite their accuracy. Modern methods such as Lagrange Multiples and others, see Spiros *et al. (*2009), are constantly being developed or improved. Thus, finding the inverse data within a short time is paramount and generalized inverses like the MPI and quasiaffine inverse contribute immensely to this as they form the backbone of the formulation of these algorithms.

Generalised inverses are valuable theoretical tools even when it is not usually necessary to compute the inverses as is the case in this thesis, see Ben-Israel (2002). The generalised inverse is an extension of the concept of inverses that applies to square singular matrices and rectangular matrices which would otherwise reduce to the usual inverse. This study hopes to improve the process and technique of finding these inverses by studying the different properties and results associated with quasiaffine inverse and Moore-Penrose inverse. This thesis will go a long way in improving computational algorithms for calculating generalized inverses.

Lastly, the Fuglede-Putnam theorems play a key role in determining whether quantities can be measured to precision. In quantum mechanics quantities are depicted using operators. The operators representing these quantities should commute for the quantities to be observed at the same time. For instance, time and energy as well as position and momentum. If the operators are non-commuting, then accuracy in measuring one quantity vis-à-vis the other is difficult. This thesis establishes the Fuglede –Putnam Theorem for Moore-Penrose Invertible operators in Hilbert spaces

1.5 Definitions

- The scalar valued function $<,>$ on a vector space V over the complex scalars $\mathbb{C}$ such that:

(i) $< x, x > \geq 0$ for all x in V and $< x, x >= 0$ if and only if $x = 0$.

(ii) $< \lambda x, y >= \lambda < x, y >$ for all x and y in V and all $\lambda \in \mathbb{C}$

(iii) $< x, y > = \overline{< y, x >}$ for all x and y in V.

(iv) $< x + y, z\rangle = < x, z > + < y, z >$ for $x, y, z \in V$ is referred to as an inner product function.

- A complex vector space V together with the inner product function is called inner product space or **Pre-Hilbert space.**
- If a sequence $\{u_i\}$ is such that $\|u_i - u_j\| < \varepsilon$, where i and j are natural numbers greater than a natural number depending on a positive ε, then $\{u_i\}$ is referred to as a Cauchy sequence.
- If every Cauchy sequence of points in V has a sequence that converges in V, then V is a complete space.
- Hilbert space is a complete inner product space.
- Let $B(H)$ denote the Banach algebra of bounded linear operators on an infinite dimensional complex Hilbert Space H on itself.
- The spectrum of $A \in B(H)$ denoted $\sigma(A)$ is the set $\{\lambda \in \mathbb{C}: A - \lambda I \text{ is non-invertible}\}$.
- Numerical range of $A \in B(H)$, denoted by $W(A)$, is the subset of the complex field given by $\{\langle Au, u\rangle: \|u\| = 1\}$ for $u \in H$.
- A linear operator A on a Hilbert space, H is said to be bounded if there exist constant $\lambda > 0$ such that $\|Au\| \leq \lambda\|u\|$ for all $u \in H\}$ where;

  $\|A\| = \inf\{\lambda > 0: \|Au\| \leq \lambda\|u\| \text{ for all } u \in H\}$
- A linear operator on a Hilbert space H is said to be bounded from below if there exist $\lambda > 0$ such that $\|Au\| \geq \lambda\|u\|$ for every $u \in H$ and $\lambda \in \mathbb{C}$.
- A bounded linear operator $A \in B(H)$ is said to be invertible if it is both injective (one-to-one) and surjective.
- A linear operator on Hilbert space that is both injective and surjective is said to be bijective.
- An injective operator $A \in B(H)$ whose range is dense in Hilbert space H (or equivalently if A and A^* are both injective.) is said to be quasi-invertible (or a quasiaffinity)

- A bounded linear operator $A \in B(H)$ is a left quasiaffine inverse of $B \in B(H)$ (equivalently, $B \in B(H)$ right quasiaffine inverse of $A \in B(H)$) if there exists a quasiaffinity X such that $AXB = X$. If there exists another quasiaffinity Y such that $AXB = X$ and $BYA = Y$, then A and B are quasiaffine inverses of each other.
- For a densely defined $A \in B(H)$ there is an unparalleled operator called the adjoint of A denoted by A^* such that $< Au, v > = < u, A^*v >$. If A is a matrix, then $A^* = \overline{A^T}$.
- In this work, a linear operator $A' \in B(H)$ such that $AA'A = A$ is the generalized inverse of $A \in B(H)$. That is $A' \in A\{1\} = \{ A' : AA'A = A \}$.
- A unique solution to the following equations is referred to as the MPI of $A \in B(H)$.

  (i) $A = AXA \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots . \ldots . \ldots E_1$

  (ii) $X = XAX \ldots \ldots \ldots \ldots . \ldots . \ldots . \ldots \ldots \ldots . . E_2$

  (iii) $(XA)^* = XA . . \ldots \ldots \ldots \ldots . \ldots . \ldots \ldots \ldots E_3$

  (iv) $(AX)^* = AX . . \ldots \ldots . \ldots \ldots . \ldots . \ldots \ldots . . E_4$
- An operator is said to have a weak Moore-Penrose inverse if it satisfies E_1, E_2 and either E_3 or E_4 above.
- Consider the operator equation $Av = y$ where $A \in B(H)$. Then $L(y) := \{v' \in D(A) : \|Av' - y\| \leq \|Av - y\| \; \forall \, v \in D(A)\}$. Here any $v \in L(y)$ is referred to as the least square solution (lss) of $Av = y$. The vector $v' = A^+y \in L(y)$ satisfying $\|v'\| = \|A^+y\| \leq \|v\| \; \forall \, v \in L(y)$ and is the lss to $Av = y$ of minimal norm.
- An operator A_d is said to be the Drazin inverse of a bounded linear operator A on a Hilbert space H if it satisfies the following. conditions: $A_d AA_d = A_d$, $A^{k+1}A_d = A^k$, $A_d A = AA_d$ for some nonnegative integer k (Drazin index of A).
- If L is a proper subspace of a Hilbert space, H then an operator denoted by

$A_L^{(-1)} = P_L(AP_L + P_{L^\perp})^{-1}$ is called the Bott-Duffin inverse of A constrained to L where P_L is the orthogonal projection on L and $P_{L^\perp}$ is the orthogonal projection on the orthogonal complement of L

- A subspace M of H where $M^\perp = \{u \in H: < u, v >= 0, \forall v \in M\}$ is the orthogonal complement of M . $H = M \oplus M^\perp$ (Orthogonal decomposition of H).
- The range of an operator $A \in B(H)$, denoted by, $\mathrm{Ran}(A)$ is given as the set
  $\mathrm{Ran}(A) = \{v: Au = v \text{ for } u \text{ in } H\}$.
- The kernel of $A \in B(H)$(or equivalently null space of A), denoted by Ker A (or Nul A) is the set Ker A/ Nul A $=\{u \in H: Au = 0\}$.
- The cokernel of $A \in B(H)$ [equivalently, Kernel of A^*) is the set:
  $\mathrm{coker}A = \ker A^* = \{v \in H: A^* v = 0\}$.
- A bounded operator A on a Hilbert space H is said to have a dense range if $BB^*A = A$ implies $BB^* = I$.
- A bounded linear operator C on Hilbert space H is said to be a commutator if there exist operators A and B on H such that $C = AB = BA$.
- The commutator of A and B is denoted by $[A, B]$ where$[A, B] = C = AB - BA$.
- If the Commutator $[A, B] = C = AB - BA = 0$, then A is said to be a commutant of B and B a commutant of A denoted as $B \in \{A\}^C$ and $A \in \{B\}^C$. This implies that the commutants of A will be denoted by $\{A\}^C = \{B: AB = BA \text{ for } B \in B(H)\}$.
- Two operators $A \in B(H)$ and $B \in B(H)$ are said to commute if $AB - BA = 0$ also written as $[A, B] = 0$.
- Suppose V and U are subspaces of H and $V \cap U = \{0\}$. Then $V \oplus U$ denotes the direct sum of V and U while $V \oplus^\perp U$ denotes the orthogonal direct sum of V and U whenever

$< v, u > = 0$ for each $v \in V$ and $u \in U$. In this case, $U = V^{\perp}$, hence it can be written as $H = V \bigoplus V^{\perp}$, where $< v, u > = 0$ for every $v \in V$ and $u \in V^{\perp}$.

- Singular Value Decomposition (SVD) of a matrix: The expression of a matrix $A \in \mathbb{C}^{mxn}$ as a composition of three matrices $A = XSY^t$ where X and Y are matrices with orthonormal columns and S is a diagonal matrix whose diagonal entries are the singular values of A.
- LU-factorization of a matrix is the expression of a matrix A as $A = LU$ with L being a lower trapezoidal matrix with 1's in the principal diagonal and U an upper triangular matrix .
- QR-decomposition of A is given as $A = QR$ with columns of Q being orthonormal and R is a upper triangular matrix.
- Similar operators: Given an invertible operator $X \in B(H)$, the equation $A = XBX^{-1}$ implies $A \in B(H)$ is similar to $B \in B(H)$ and they are unitarily equivalent if X is a unitary operator. In this case, $X^* = X^{-1}$.
- In this thesis, $C(A, B)X$ and $R(A, B)X$ are defined as follows: $C(A, B)X = AX - XB$ and $R(A, B)X = AXB - X$.
- Two linear operators $A, B \in B(H)$ are said to be quasi-similar if there exist two quasi-invertible operators $X, Y \in B(H)$ such that $AX = XB$ and $BY = YA$.
- An operator $A \in B(H)$ is said to be a partial isometry provided $A = AA^*A$ (or equivalently, $A^* = A^+$).
- An operator $A \in B(H)$ is said to be an isometry if $A^*A = I$.
- An operator $A \in B(H)$ is said to be co-isometry if $AA^* = I$.
- An operator $A \in B(H)$ unitary if A is both an isometry and co-isometry. That is $AA^* = A^*A = I$.
- An operator $A \in B(H)$ is self-adjoint or Hermitian if $A = A^*$ or if $< Au, u > \in \mathbb{R}$

for every u in H .

- If $A \in B(H)$ commutes with A^*, then A is said to be normal. That is $AA^* = A^*A$.

  (Or equivalently, $\|Au\| = \|A^*u\|$ for all $u \in H$).

- If $A \in B(H)$ commutes with A^*A, then A is said to be quasinormal. That is $AA^*A = A^*AA$.
- An operator $A \in B(H)$ is said to a contraction if $\|A\| \leq 1$.
- The operator A will be said to be an involution if $A^2 = 1$. In other words a projectivity of 2.
- $A \in B(H)$ is said to be dominant if for every $\lambda \in \mathbb{C}$ there is a non-negative number $M_\lambda \geq 1$ where $\|(A-\lambda I)^*u\| \leq M_\lambda \|(A-\lambda I)u\|$ for every u in H .
- An operator $A \in B(H)$ is said to be M-hyponormal if there is a constant $M \geq M_\lambda$ where $\|(A-\lambda I)^*u\| \leq M_\lambda \|(A-\lambda I)u\|$ for every u in H .
- Alternatively, $A \in B(H)$ is M-hyponormal operator if there exist a positive number M such that $M^2(A-\lambda)^*(A-\lambda) \geq (A-\lambda)(A-\lambda)^*$ for all $\lambda \in \mathbb{C}$,
- An operator $A \in B(H)$ is said to be hyponormal if $A^*A \geq AA^*$ or $\|A^*u\| \leq \|Au\|$ for all u in H.
- An operator $A \in B(H)$ is a p-hyponormal operator if $(A^*A)^p \geq (AA^*)^p$ where $0 < p \leq 1$,
- If for $A \in B(H)$, we have $\|A^2u\| \leq \|Au\|^2$ for all unit vectors u in H, then A is said to be paranormal.
- An operator $A \in B(H)$ is said to be quasinilpotent if $\sigma(A) = \{0\}$ and nilpotent if $A^m = 0$ for a positive integer m.
- An operator $A \in B(H)$ is an orthogonal projection if A is such that $A = A^*$ and $A^2 = A$.
- An operator $A \in B(H)$ is subnormal if B is a bounded normal operator in U and U is a subspace of Hilbert space, H.
- An operator $A \in B(H)$ is an EP operator provided $Ran(A) = Ran(A^*)$.

- If a bounded operator A on a Hilbert space H commute with AA^{+}, then A is a R-quasi EP operator.
- If a bounded operator A on a Hilbert space H commute with $A^{+}A$, then A is a L-quasi EP operator.
- An operator A is said to be Fredholm if its range is closed and both Ker A and Ker A^{*} are finite dimensional.
- An operator A is said to be Upper semi-Fredholm if its range is closed and Ker A is of finite dimension.

CHAPTER TWO

LITERATURE REVIEW

2 Introduction

The literature of past studies relevant to this work is given in this chapter. It is divided into three sections.

2.1 Quasiaffine Inverse

Hongke and Chuan (1985) showed that subnormal (normal) operators with a left (right) quasiaffine inverses are invertible. They also showed that given dominant operators A and B^* such that $AXB = X$, then $A^*XB^* = X$ for a quasi-invertible operator X.

Khalagai and Sheth (1987) established some conditions under which two operators can commute. That is R and S commute if $[R, S^2] = 0$ and either $\sigma(S) \cap \sigma(-S) = \emptyset$ or S is normal and $0 \notin W(S)$ or $\{S\}^C = \{S^{2n}\}^C$ for some non-negative integer n.

Khalagai (1996) showed that a left (right) invertible linear operator with right (left) quasiaffine inverse is invertible. The author exhibited that quasiaffine inverse of an operator can be its usual inverse. That is $S = R^{-1}$ provided R and S^* are dominant satisfying $RYS = Y$ where Y is self-adjoint and $\sigma(Y) \cap \sigma(-Y) = \emptyset$.

Khalagai and Nyamai (1998) demonstrated that B is injective whenever $R(A, B)Y = 0$ and Y is injective. They showed that A together with B are normal if Y is one -to - one or $\overline{R(Y)} = H$ whenever $AY = YB$ and $A^*Y = YB^*$. Also it was established that A and B are quasiaffinities whenever $AYB = Y$ implies $A^*YB^* = Y$ for a quasi-invertible Y.

Khalagai and Otieno (2000) showed that if two operators are quasiaffine inverses, then they are quasiaffinities. They also showed that an operator A is unitary whenever A is a partial isometry and either A or A^2 is a quasiaffinity.

2.2 Moore-Penrose Inverse

Generalised inverse of operators were first studied by Fredholm (1903) where the author worked with integral functions. The study shifted to using differential operators with Westfall (1909) and Bounitzky (1909). Hurwitz (1912) came up with simplified algebraic construction of generalized inverses by use of finite dimensionality null spaces of Fredholm operators. The concept of Moore-Penrose inverse was brought about by Moore (1920) and Penrose (1955).

Moore (1920) described the inverse B of a matrix A as a matrix satisfying the following properties:

$\mathrm{A}\{\mathrm{a}\}$ means columns of A are linear combinations of the conjugate rows of matrix B,

$\mathrm{A}\{\mathrm{b}\}$ means rows of A are linear combinations of the conjugate columns of B.

$\mathrm{A}\{\mathrm{c}\}$ implies $\mathrm{B} = \mathrm{BAB}$.

The invertibility of the matrix $\mathrm{A} - \mathrm{PQ}^*$ where A is mxm non-singular and P, Q are mxn matrices was investigated by Shermen and Morrison (1950) where they showed that if one element of the original matrix is substituted with another then it causes transformation of its inverse. This concept was generalized by Woodbury (1950) which resulted to the Shermen-Morrison-Woodbury formula (SMW). They coined out the now widely known fact that $\mathrm{A} - \mathrm{PQ}^*$ is invertible whenever $\mathrm{I} - \mathrm{Q}^*\mathrm{A}^{-1}\mathrm{P}$ is invertible and vice versa. The Shermen-Morrison-Woodbury formula is given as:

$$(\mathrm{A} - \mathrm{PQ}^*)^{-1} = \mathrm{A}^{-1} + \mathrm{A}^{-1}\mathrm{P}(\mathrm{I}_\mathrm{n} - \mathrm{Q}^*\mathrm{A}^{-1}\mathrm{P})^{-1}\mathrm{Q}^*\mathrm{A}^{-1}.$$

Bartlett (1951) established the equation of the inverse of $\mathrm{W} = \mathrm{A} + \mathrm{B}$ where $\mathrm{B} = \mathrm{pq}^\mathrm{T}$ with p and q being column vectors.

On generalized inverses, Bjerhammar (1951) studied least-squares therein and their relationship to solutions of linear systems. That is, the author gave characterization of generalized inverse and showed that if one type of GI exists then other type also exists under some conditions.

The conditions of Moore (1920) were rediscovered by Penrose (1955) by describing the generalised inverse of any matrix $T \in \mathbb{C}^{mxn}$ as the matrix $X \in \mathbb{C}^{nxm}$ upholding these conditions:

(i) $T = TXT$, (ii) $XTX = X$, (iii) $(XT)^* = XT$, (iv) $(TX)^* = TX$.

This came to be known as the Moore-Penrose inverse.

Noble (1966) discussed the methods for computing inverses based on factorizing the $m \times n$ matrix A. The matrix was given as $A = UV$ with U a mxk matrix and V a kxn matrix and all matrices being of rank k. The author used three different factorizations:

(i) $A = UV$ where U is a lower trapezoidal matrix containing $1's$ in the leading diagonal and V is an upper trapezoidal. MPI of A was given by $A^+ = V^+U^+$ where $V^+ = V^T(VV^T)^{-1}$ and $U^+ = (U^TU)^{-1}U^T$.

(ii) $A = UV$ where U is mxk matrix having orthonormal columns and V is upper triangular. Thus, $A^+ = V^T(VV^T)^{-1}U^T$.

(iii) $A = XSY$ where X and Y are matrices with orthonormal columns and the square roots of the eigen values of A^*A are the elements of the leading diagonal of S, so that $A^+ = X^TS^{-1}Y^T$.

Moreover, the author discussed results on perturbation theory where the author gave the expression of MPI of $W = A + B$ where A, B and W are mxn matrices.

Meyer (1973) gave an expression for $(A + PQ^*)^+$ using sum modifications of A^+.

Israel and Greville (1974) discussed the characterizations of generalized inverses. For instance, the situations where $A\{1\}$-inverse implies $A\{2\}$-inverse and vice versa.

The reverse order of Moore-Penrose inverse, $(AB)^+ = B^+A^+$ has been investigated by several authors among them, Israel and Greville (2003). Brock (1990) gave some characterization of EP operator. That is operator A such that $\mathrm{Ran}(A) = \mathrm{Ran}(A^*)$.

Koliha (2000) illustrated that an upper semi-Fredholm operator satisfying some properties is an EP. This was achieved by showing that the following statements are equivalent for an upper semi-Fredholm operator say A:

(i) $A^+A = AA^+$

(ii) $(A^*A)_\pi A = 0$

(iii) $A(AA^*)_\pi = 0$.

Gaoxiong (2001) investigated how the Moore's conditions and Penrose's conditions relate when studying matrices and their generalized inverses. The author gave characterization of two weak Moore-Penrose inverses which translated in ascertaining that the Moore (1920) conditions are the Penrose (1955) conditions.

Wei and Ding (2001) discussed sufficient conditions implying that $R(A + B)$ is closed in case $R(A)$ is closed as well as coming up with a formula for its generalized inverse.

Petryshyn (1967) showed that the Moore-Penrose inverse of A is the unique operator A^+ satisfying the following conditions:

(i) $A = AA^+A, A^+ = A^+AA^+, (AA^+)^* = AA^+, (A^+A)^* = A^+A$

(ii) $A^+A = P_{N(A)^\perp}, AA^+ = P_{\overline{(R(A)}}$

Steerneman and Kleij (2005) was able to extend the Shermen-Morrison-Woodbury formula to the case where A is singular and $I - Q^*A^{-1}P$ is invertible as:

$(A - PQ^*)^+ = A^+ + A^+P(I_n - Q^*A^+P)^+Q^*A^+$ provided $r(A, P) = r\,(A)$ and $r\begin{pmatrix} A \\ Q^* \end{pmatrix} = r(A)$ hold. The author also showed that $(A - PQ^*)^+ = (I_m - WW^+)A^{-1}(I_m - MM^+)$ where $W = A^{-1}P$ and $M = (A^{-1})^*Q$. In this case A is invertible and $Q^*A^{-1}P = I_n$.

Chen et al. (2009) showed that $(A - PQ^*)^+ = (I - WW^+)A^{-1}(I - MM^+)$ provided $M^*PM^* = M^*$, $PM^*P = P$ where $W = A^{-1}P$ and $M = (A^{-1})^*Q$ as long as $A - PQ^*, P, Q$ have closed ranges and A is invertible.

Du and Xue (2010) expressed the MPI of a perturbed linear operator $A - PQ^*$ as:

$(A - PQ^*)^+ = (I - (A^+P)(A^+P)^+)A^+(I - (Q^*A^+)^+ (Q^*A^+)$ if $PQ^*A^+P = P$ and $Q^*A^+PQ^* = Q^*$ and vice versa.

Barata and Hussein (2011) gave the definition of MPI and showed the existence of the MPI by producing algorithms for the computation of MPI. They showed that for any $m \times n$ matrix A, $(AA^*)^+ = (A^*)^+A^+$ and $A^+ = A^*(AA^*)^+ = (A^*A)^+A^*$.

Shani and Sivakumar (2013) gave the expression of MPI of $W = A + B$ with $R(W)$ closed, under some four given cases.

MacAusland (2014) showed how to obtain the MPI by full row rank decomposition of a matrix A as $A^{+} = A^{*}(AA^{*})^{-1}$ and full column rank decomposition as $A^{+} = (A^{*}A)^{-1}A^{*}$. The author discussed how to use the QR- decomposition of a matrix and the Singular Value decomposition to find Moore-Penrose inverse of a matrix.

Kulkarni and Ramesh (2015) discussed the perturbation of operators with closed range and the Moore-Penrose inverse of operators in Hilbert spaces. They gave conditions under which closed range of A implies range of $A + B$ is closed, the relation between A^{+} and $(A + B)^{+}$ as well as the relation between A^{+} and B^{+} where B is a bounded linear operator. The authors did this by restricting operator B to satisfy $AA^{+}B = B$ and $BA^{+}A = B|_{D(A)}$ and $\|A^{+}B\| < 1$.

2.3 Fuglede-Putnam Theorems

On Fuglede-Putnam Theorems, Fuglede (1950) came up with a result generally referred to as Fuglede theorem which was later generalized by Putnam (1951).

Fuglede (1950) showed that if $P \in B(H)$ and A a normal operator such that $AP = PA$, then $A^{*}P = PA^{*}$.

Putnam (1951) extended this to two normal operators

Mahmood (2017) went ahead to show that if $[A, A^{*}] = 0$ such that $PA = AQ$ and $QA = AP$, then $PA^{*} = A^{*}Q$ and $QA^{*} = A^{*}P$.

Johnson *et al*. (2021) extended the results of Fuglede (1950) and Putnam (1951) to EP operators. In this thesis, it has been established that Fuglede-Putnam theorems hold true for EP operator. That is $[P, A^*] = 0$ if $P \in B(H)$, A is an EP operator and $[P, A] = 0$, under some conditions different from the ones stated by Johnson et al.(2021).

CHAPTER THREE

ON QUASIAFFINE INVERSE OF LINEAR OPERATORS IN HILBERT SPACES

3 Introduction

Properties of quasiaffine inverses starting with left and right quasiaffine inverses of linear operators are established in this chapter. It is deduced that quasiaffine inverse of an operator is its usual inverse if some given conditions are upheld. The inverses of quasinormal partial isometries and operators with left (right) quasiaffine inverse under some conditions are exhibited. Since quasiaffine inverse is a quasiaffinity, the uniqueness of a quasiaffine inverse is established.

3.1 Results

Theorem 3.1.1

Suppose X and Y are quasiaffinities and $\mathrm{A}, \mathrm{B} \in \mathrm{B}(\mathrm{H})$ satisfiy $\mathrm{AXB} = \mathrm{X}$ and $\mathrm{BYA} = \mathrm{Y}$. Then:

(i) $\mathrm{XYA} = \mathrm{AXY}$.

(ii) $\mathrm{YXB} = \mathrm{BYX}$.

Proof

(i) From $\mathrm{AXB} = \mathrm{X}$ post multiplying each side by Y then $\mathrm{AXBY} = \mathrm{XY}$. This implies $\mathrm{AXBYA} = \mathrm{XYA}$. Since $\mathrm{BYA} = \mathrm{Y}$ then $\mathrm{XYA} = \mathrm{AXBYA} = \mathrm{AXY}$. Hence the result $\mathrm{XYA} = \mathrm{AXY}$.

(ii) Similarly, if $\mathrm{BYA} = \mathrm{Y}$ post multiplying each side by X then $\mathrm{BYAX} = \mathrm{YX}$. Thus, $\mathrm{BYAXB} = \mathrm{YXB}$. Again, $\mathrm{AXB} = \mathrm{X}$. Thus, $\mathrm{YXB} = \mathrm{BYAXB} = \mathrm{BYX}$. Thus, $\mathrm{YXB} = \mathrm{BYX}$.

Remark 3.1.2

From the Theorem 3.1.1 it is noted that if B is a quasiaffine inverse of A, then both operators commute with a product of the implementing quasiaffinities in a given order.

Corollary 3.1.3

Suppose $A, B \in B(H)$ satisfy $AXB = X$ and $A^*XB^* = X$. Then

(i) $AXX^* = XX^*A$.

(ii) $BX^*X = X^*XB$.

Proof

(i) By taking adjoint both sides of $A^*XB^* = X$ we have $BX^*A = X^*$. Multiplying $AXB = X$ by X^*A from the right we obtain $AXBX^*A = XX^*A$. Since $BX^*A = X^*$ then $XX^*A = AXBX^*A = AXX^*$. Thus, $AXX^* = XX^*A$.

(ii) Next, by multiplying $BX^*A = X^*$ by XB from the right we consequently have $BX^*AXB = X^*XB$. Since $AXB = X$, we have $X^*XB = BX^*AXB = BX^*X$. Thus, $BX^*X = X^*XB$.

Theorem 3.1.4 (Hongke and Chuan, 1985)

Suppose $A, B \in B(H)$ satisfy $AXB = X$ where X is a quasiaffinity with A and B^* dominant. Then $A^*XB^* = X$.

Corollary 3.1.5

Suppose $R, S \in B(H)$ satisfy $RYS = Y$ where Y is a quasiaffinity with R and S^* dominant. Then S is a quasiaffine inverse of R.

Proof

Since R and S^* are dominant, then $RYS = Y$ implies $R^*YS^* = Y$. Taking adjoint gives

$SY^*R = Y^*$.Now, if Y is a quasiaffinity then Y^* is also a quasiaffinity. Hence S is a quasiaffine inverse of R.

Remark 3.1.6

We note that in Theorem 3 (ii) of Khalagai (1996) the condition $\sigma(Y) \cap \sigma(-Y) = \emptyset$ can be replaced with a number of other conditions, which are less stringent by using the following results due to Khalagai and Sheth (1987).

Theorem 3.1.7 (Khalagai and Sheth, 1987)

Suppose $R, S \in B(H)$ satisfy $[R, S^2] = 0$. Then $[R, S] = 0$ if; $\sigma(S) \cap \sigma(-S) = \emptyset$ or $[S, S^*] = 0$ and $0 \notin W(S)$ or $\{S\}^C = \{S^{2n}\}^C$ for some non-negative integer n.

Theorem 3.1.8

Suppose $R, S \in B(H)$. If $RYS = Y$ implies $R^*Y\,S^* = Y$ where Y is self - adjoint. Then $S = R^{-1}$ in each of these conditions:

(i) $0 \notin W(Y)$.

(ii) $\{Y\}^C = \{Y^{2n}\}^C$ and $N(Y) = 0$ or $\overline{R(Y)} = H$, $n \in \mathbb{N}$.

Proof

Given that $RYS = Y$ implies $R^*YS^* = Y$, by Corollary 3.3 we have that $[R, YY^*] = 0$ and $[S, Y^*Y] = 0$. But Y is self-adjoint. Hence, $[R, Y^2] = 0 = [S, Y^2]$.

By Theorem 3.1.7, each of conditions (i) and (ii) implies $[R, Y] = 0$ and $[S, Y] = 0$. Consequently, $[R^*, Y] = 0 = [S^*, Y]$. Now, from $RYS = Y$, we have $RSY = Y$ implying $(RS - I)Y = 0$. But $0 \notin W(Y)$ which means that Y has a dense range. Thus, $RS - I = 0$. That is $RS = I$.

Also, from $R^*YS^* = Y$ taking adjoints gives $SYR = Y$. If $[R, Y] = 0$ then $SRY = Y$. That is $(SR - I)Y = 0$. Since Y has a dense range, then $SR - I = 0$ implying $SR = I$.

For part (ii), again $YRS = Y$ and $YSR = Y$. Since $YRS = Y$, $Y(RS - I) = 0$. Since $YSR = Y$, $Y(SR - I) = 0$. Now, $RS - I = 0$ and $SR - I = 0$ for Y is one to one. Hence $RS = I$ and $SR = I$. In both cases $RS = I$ and $SR = I$ meaning $S = R^{-1}$.

Remark 3.1.9

It follows that if Y is a quasiaffinity in Theorem 3.1.8 then the quasiaffine inverse S of R is the same as the usual inverse of R under any of the two conditions.

Corollary 3.1.10

Suppose $R, S \in B(H)$ satisfy $RYS = Y$ where Y is a self-adjoint and a quasiaffinity with R and S^* dominant. Then S is the quasi-affine inverse of R which is equal to R^{-1} under each of these statements:

(i) $0 \notin W(Y)$.

(ii) $\{Y\}^C = \{Y^{2n}\}^C$ for some non-negative integer n.

Proof

Given that $RYS = Y$ implies $R^*YS^* = Y$ by Corollary 3.1.3, we have that $[R, YY^*] = 0$ and $[S, Y^*Y] = 0$. But Y is self-adjoint. Hence $[R, Y^2] = 0 = [S, Y^2]$. By Theorem 3.1.7, each of the statements (i) and (ii) implies $RY = YR$ and $SY = YS$. Consequently, $R^*Y = YR^*$ and $S^*Y = YS^*$. Now, from $RYS = Y$ and $SYR = Y$ we have $RSY = Y$ and $SRY = Y$ or $YRS = Y$ and $YSR = Y$. Thus, $(RS - I)Y = 0$ and $(SR - I)Y = 0$ or $Y(RS - I) = 0$ and $Y(SR - I) = 0$. This implies, if $N(Y) = 0$ or $\overline{R(Y)} = H$, then $RS - I = 0$ and $SR - I = 0$. Hence $RS = I$ and $SR = I$ implying $S = R^{-1}$.

Corollary 3.1.11

Let $R, S \in B(H)$. If $RXS = X$ implies $R^{*}XS^{*} = X$ with X being a quasiaffinity then:

(i) R is co-isometric if S is isometric and conversely.

(ii) S is co-isometric if R is isometric and conversely.

Proof

To proof (i) If $RXS = X$ and $R^{*}XS^{*} = X$ then $X = RXS = RR^{*}XS^{*}S$. That is $X = RR^{*}XS^{*}S$. If S is isometric then $S^{*}S = I$ and $X = RR^{*}XS^{*}S = RR^{*}X$. This implies $X = RR^{*}X$ and $(I - RR^{*})X = 0$. Since X has a dense range then $I - RR^{*} = 0$ and $RR^{*} = I$. Similarly, if $X = RR^{*}XS^{*}S$, substituting $RR^{*} = I$, we have $X = XS^{*}S$ and $X - XS^{*}S = 0$. This implies $X(I - S^{*}S) = 0$. Since X is injective then $I - S^{*}S = 0$ implying $S^{*}S = I$.

To Proof (ii), We re-arrange the equations $RXS = X$ and $R^{*}XS^{*} = X$ so that $X = R^{*}XS^{*} = R^{*}RXSS^{*}$ implying $X = R^{*}RXSS^{*}$. If S is co-isometric then $SS^{*} = I$ and $X = R^{*}RXSS^{*} = R^{*}RX$.This implies $X = R^{*}RX$ which means $(I - R^{*}R)X = 0$. Since $\overline{R(X)} = H$, then $I - R^{*}R = 0 \Leftrightarrow R^{*}R = I$. Also, if $X = R^{*}RXSS^{*}$ and R is isometric, then substituting $R^{*}R = I$ we obtain $X = R^{*}RXSS^{*} = XSS^{*}$. So that $X = XSS^{*} \Leftrightarrow X - XSS^{*} = 0$ implying $X(I - SS^{*}) = 0$. Since X is injective, then $I - SS^{*} = 0$ implying $SS^{*} = I$.

Corollary 3.1.12

If $R, S \in B(H)$ such that $RYS = Y$ implies $R^{*}YS^{*} = Y$ with Y being a quasiaffinity, then R is unitary if S is unitary and conversely.

Proof

Let R be unitary. Then it is isometric and co-isometric. From Corollary 3.1.11, S is co-isometric and isometric thus unitary. Similarly if S is unitary, it is-isometric and co - isometric thus by Corollary 3.1.11, R is co-isometric and isometric operator, hence unitary.

Corollary 3.1.13

Suppose $R, S \in B(H)$ satisfy $RYS = Y$ where Y is a quasiaffinity with R and S^* dominant. Then R is unitary if and only if S is unitary.

Proof

If R and S^* are dominant, then $RYS = Y$ implies $R^*YS^* = Y$. From Corollary 3.1.12 above we have, if R is unitary, then $SS^* = I$ and $S^*S = I$. Thus S is unitary. Again if S is unitary, then $RR^* = I$ and $R^*R = I$ implying R is unitary. Thus R is unitary if and only if S is unitary.

Theorem 3.1.14

Suppose $P, Q \in B(H)$ satisfy $PYQ = Y$ or $QXP = X$. If there exisst an operator $Q_1 \in B(H\ )$ satisfying:

(i) $PYQ_1 = Y$ where P and Y are one –to-one then $Q_1 = Q$.

(ii) $Q_1XP = X$ where P and X have a dense range then $Q_1 = Q$.

Proof

(i) Since $PYQ = Y$ and $PYQ_1 = Y$, we have $PYQ = PYQ_1 = Y$. Thus, $PYQ - PYQ_1 = 0$. Since P and Y are one –to-one, then $P(YQ - YQ_1) =0$ implies $YQ - YQ_1 = 0$ and $Q - Q_1 = 0$. Thus $Q = Q_1$.

(ii) Since $QXP = X$ and $Q_1XP = X$ then $QXP = Q_1XP = X$. Thus $QXP - Q_1XP = 0$ and $(QX - Q_1X)P =0$. If P and X have a dense range then $(Q - Q_1)X = 0$ and $Q - Q_1 = 0$. Thus $Q = Q_1$.

Theorem 3.1.15 (Khalagai and Otieno, 2000)

Let $R, S \in B(H)$ where S is a quasiaffine inverse of R, then both R and S are quasiaffinities.

Corollary 3.1.16

Suppose $\mathrm{P}, \mathrm{Q} \in \mathrm{B(H)}$ where Q is a quasiaffine inverse of P with X and Y the implementing quasiaffinities. If there exists operator $\mathrm{Q}_1 \in \mathrm{B(H)}$ satisfying $\mathrm{PXQ}_1 = \mathrm{X}$ and $\mathrm{Q}_1\mathrm{YP} = \mathrm{Y}$, then $\mathrm{Q}_1 = \mathrm{Q}$.

Proof

From Theorem 3.1.15 both P and Q are quasiaffinities. This implies $kerA = \{0\}$ and $\overline{\mathrm{Ran(P)}} = \mathrm{H} = \overline{\mathrm{Ran(Q)}}$. If $\mathrm{PXQ}_1 = \mathrm{X}$ and $\mathrm{Q}_1\mathrm{YP} = \mathrm{Y}$, then by tracing Theorem 3.1.14, we have $\mathrm{PX}(\mathrm{Q}_1 - \mathrm{Q}) = 0$. Since PX is one-to-one, then $\mathrm{Q}_1 - \mathrm{Q} = 0$ implying $\mathrm{Q}_1 = \mathrm{Q}$. Again, $(\mathrm{Q}_1 - \mathrm{Q})\mathrm{YP} = 0$. If YP has a dense range, then $\mathrm{Q}_1 - \mathrm{Q} = 0$. Thus, $\mathrm{Q}_1 = \mathrm{Q}$.

Remark 3.1.17

Note that from Theorem 3.1.14 and Corollary 3.1.16, quasiaffine inverse of a bounded operator is indeed unique.

Theorem 3.1.18

Suppose $\mathrm{P}, \mathrm{Q}, \mathrm{Y} \in \mathrm{B(H)}$ satisfy $\mathrm{R(P, Q)Y} = 0$. If Y has dense range and P is quasinormal partial isometry, then P is unitary.

Proof

From definition, $\mathrm{P} = \mathrm{PP^*P} = \mathrm{P^*PP}$. Thus, $\mathrm{Y} = \mathrm{PYQ} = \mathrm{PP^*PYQ} = \mathrm{P^*PPYQ}$. That is $\mathrm{Y} = \mathrm{PP^*PYQ} = \mathrm{PP^*Y}$ implying $(\mathrm{I} - \mathrm{PP^*})\mathrm{Y} = 0$. If Y has a dense range then $\mathrm{I} - \mathrm{PP^*} = 0$ implying $\mathrm{PP^*} = \mathrm{I}$.

Similarly, $\mathrm{Y} = \mathrm{P^*PPYQ} = \mathrm{P^*PY}$ implying $(\mathrm{I} - \mathrm{P^*P})\mathrm{Y} = 0$. If Y has a dense range then $\mathrm{I} - \mathrm{P^*P} = 0$ and $\mathrm{P^*P} = \mathrm{I}$. Thus, $\mathrm{P^*P} = \mathrm{PP^*} = \mathrm{I}$.

Corollary 3.1.19

Suppose $P, Q, Y \in B(H)$ satisfy $R(P, Q)Y = 0$ where Y is a quasiaffinity and P a quasinormal partial isometry respectively. Then P is unitary.

Proof

If Y is a quasiaffinity then Y has a dense range. If P, Q and Y satisfy $PYQ = Y$, where P is a quasinormal partial isometry, then by tracing Theorem 3.1.18 , $P^*P - I = 0$ and $PP^* - I = 0$. Thus, $P^*P = I$ and $PP^* = I$, implying $P^* = P^{-1}$. Thus, P is unitary.

Corollary 3.1.20

Suppose $P, Q \in B(H)$. If a quasiaffine inverse Q of P is a quasinormal partial isometry, then it is unitary.

Proof

Since Q is a quasiaffine inverse of P we have $PXQ = X$ and $QYP = Y$ where X and Y are quasiaffinities. If Q is a quasinormal partial isometry and $QYP = Y$, then by Corollary 3.1.19, $Q^*Q - I = 0$ and $QQ^* - I = 0$. Thus, $Q^*Q = I$ and $QQ^* = I$, implying $Q^* = Q^{-1}$. Hence Q is unitary.

Remark 3.1.21

Khalagai and Otieno (2000) showed that if a partial isometry is a quasiaffinity, then it is unitary. In Corollary 3.1.22 , we extend the result to a quasinormal partial isometry with a dense range and notice that it is indeed invertible.

Corollary 3.1.22

Let $P \in B(H)$ be a quasinormal partial isometry. Then P is unitary provided it has a dense range.

Proof

From the definition, $P = PP^*P = P^*PP$. This means $P - PP^*P = 0$ and $(I - PP^*)P = 0$. Since $\overline{Ran(P)} = H$, $I - PP^* = 0$. That is $PP^* = I$.

On the other hand, $P - P^*PP = 0$ translating to $(I - P^*P)P = 0$. Since P has a dense range then $I - P^*P = 0$ and $P^*P = I$. Thus, $P^*P = PP^* = I$. This implies P is unitary.

Theorem 3.1.23 Suppose $P, Q \in B(H)$ satisfy $R(P, Q)Y = 0$. Then

(i) $C(P^*, Q)Y = 0$ if P is isometric.

(ii) P is co-isometric whenever $C(P^*, Q)Y = 0$ and Y has dense range.

Proof

(i) If $R(P, Q)Y = 0$ and given that P is isometric ($P^*P = I$), then

$YQ = P^*PYQ = P^*Y$. This implies $P^*Y = YQ$.

(ii) If $P^*Y = YQ$, then multiplying both sides by P from the left we have

$PP^*Y = PYQ = Y$. Thus, $PP^*Y - Y = 0 \Leftrightarrow (PP^* - I)Y = 0$. Since Y has a dense range then $PP^* - I = 0 \Leftrightarrow PP^* = I$.

Corollary 3.1.24

Suppose $P, Q \in B(H)$ satisfy $PYQ = Y$. Then

(i) $PY = YQ^*$ if Q is co-isometric.

(ii) Q is isometric whenever $PY = YQ^*$ and Y is one- to-one.

Proof

Following the steps of Theorem 3.1.23 , we multiply $PYQ = Y$ by Q^* from the right to have $PYQQ^* = YQ^*$. Since Q is co-isometric, then $QQ^* = I$. So that $YQ^* = PYQQ^* = PY$. Thus, $PY = YQ^*$.

Also, multiplying $PY = YQ^*$ by Q from the right we obtain $PYQ = YQ^*Q$. Since $PYQ = Y$, then $YQ^*Q = PYQ = Y$. Hence, $Y - YQ^*Q = 0$. Since Y is one –to- one then, $Y(I - Q^*Q) = 0$ implies $I - Q^*Q = 0$ and $Q^*Q = I$.

Theorem 3.1.25

Suppose $P, Q \in B(H)$ satisfy $PY = YQ$ for some operator Y. If:

(i) P is a one- to-one partial isometry then $P^*YQ = Y$.

(ii) Q is a partial isometry with a dense range then $PYQ^* = Y$.

Proof.

(i) By definition, $P = PP^*P$. If $C(P, Q)Y = 0$, then it follows that $PY = PP^*PY = PP^*YQ$. Thus, $PY = PP^*YQ$ and $PY - PP^*YQ = 0$. This implies $P(Y - P^*YQ) = 0$ and since P is one-to-one then, $Y - P^*YQ = 0$ and $P^*YQ = Y$. That is $R(P^*, Q)Y = 0$.

(ii) In the same manner, if Q is a partial isometry and $PY = YQ$, then $YQ = YQQ^*Q = PYQ^*Q$. Hence, $YQ = PYQ^*Q$ .This implies $(Y - PYQ^*)Q = 0$. Since Q has a dense range, then $Y - PYQ^* = 0$ and $PYQ^* = Y$. That is $R(P, Q^*)Y = 0$.

Remark 3.1.26

It is worth noting that a partial isometry which is one-to-one is an isometry and if it has a dense range then it is a co-isometry.

Theorem 3.1.27 (Duggal, 1986)

Suppose $P, Q \in B(H)$. If P and Q^* are dominant and M-hyponormal respectively, then $PY = YQ$ implies $P^*Y = YQ^*$

Corollary 3.1.28

Suppose $P, Q \in B(H)$ where P^* and Q are dominant and M-hyponormal respectively. Let $P^*Y = YQ^*$ for some operator Y. If:

(i) P is a one-to-one partial isometry then $P^*YQ = Y$.

(ii) Q is a partial isometry with a dense range then $PYQ^* = Y$.

Proof.

Since P^* and Q are dominant and M-hyponormal respectively, then $P^*Y = YQ^*$ imply $PY = YQ$. Thus, by Theorem 3.1.25, if P is a one- to-one partial isometry, then $P^*YQ = Y$ and if Q is a partial isometry with a dense range, then $PYQ^* = Y$.

Theorem 3.1.29 (Khalagai and Nyamai, 1998)

Suppose $P, Q \in B(H)$. If $PY = YQ$ implies $P^*Y = YQ^*$ for some $Y \in B(H)$, then P and Q are normal whenever Y is one-to-one or has a dense range.

Theorem 3.1.30

Suppose $P, Q \in B(H)$ are partial isometries. If $PY = YQ$ imply $P^*Y = YQ^*$ where Y is a quasiaffinity, then P and Q are unitary if any of these hold:

(i) P is a one-to-one.

(ii) Q has a dense range.

Proof

(i) If $PY = YQ$ and P is one-to-one partial isometry, then from Theorem 3.1.25, $Y = P^*YQ$. Again, if $P^*Y = YQ^*$, then $Y = P^*YQ = YQ^*Q$ implying $Y = YQ^*Q$ and $Y - YQ^*Q = 0$ so that $Y(I - Q^*Q) = 0$. Since Y is one-to-one, then $I - Q^*Q = 0$ implies $Q^*Q = I$. By Theorem 3.1.29, Q is normal. Hence, $I = Q^*Q = QQ^*$. Similarly, $Y = P^*YQ = P^*PY$ implying $Y = P^*PY$ and

$Y - P^*PY = 0$. Thus, $(I - P^*P)Y = 0$. Since Y has a dense range, then

$I - P^*P = 0$ and $P^*P = I$. Since $PY = YQ$ implies $P^*Y = YQ^*$ and Y has a dense

range, then P is normal from Theorem 3.1.29. Hence, $P^*P = PP^* = I$.

(ii) If Q is a partial isometry with a dense range and $PY = YQ$, then from Theorem 3.1.25 , $Y = PYQ^*$. Since $PY = YQ$ implies $P^*Y = YQ^*$, then: $Y = PYQ^* = YQQ^*$.

Thus, $Y - YQQ^* = 0$ and $Y(I - QQ^*) = 0$. Since Y is one-to-one then, $I - QQ^* = 0$ and $QQ^* = I$. By Theorem 3.1.29, Q is normal implying $I = QQ^* = Q^*Q$.

Similarly, $Y = PYQ^*$=PP^*Y implying $Y - PP^*Y = 0$. Thus, $(I - PP^*)Y = 0$. Since Y has a dense range, then $I - PP^* = 0$ and $PP^* = I$. Since P is normal, then $P^*P = PP^* = I$.

Theorem 3.1.31 (Duggal, 1996)

Suppose $P, Q \in B(H)$ where P and Q^* are p-hyponormal. Then $PY = YQ$ implies $P^*Y = YQ^*$ for some $Y \in B(H)$.

Remark 3.1.32

From Theorem 3.1.31 , note that parallel results on dominant operators and M-hyponormal operators can be obtained for p-hyponormal operators satisfying the same conditions.

Corollary 3.1.33

Suppose $P, Q \in B(H)$ are partial isometries with P and Q^* dominant and m-hyponormal respectively. If $PY = YQ$ where Y is a quasiaffinity, then P and Q are unitary operators if either of these hold:

(i) P is one-to-one.

(ii) $\overline{Rsn(Q)} = H$.

Proof

Since by hypothesis P and Q^* are dominant and m-hyponormal respectively, then

$PY = YQ \Rightarrow P^*Y = YQ^*$. By Theorem 3.1.30, if Y is a quasiaffinity and either P is one-to-one partial isometry or Q is a partial isometry with dense range, then P and Q are unitary.

Corollary 3.1.34

Let $P \in B(H)$ be a one-to-one partial isometry. Then P is unitary incase P^* is quasinormal.

Proof

If P is one-to-one and P^* quasinormal, then $\ker(P) = \{0\}$. Thus, $P - PPP^* = 0$ which gives $P(I - PP^*) = 0$. But P is one to one meaning $I - PP^* = 0$ implying $PP^* = I$.

On the other hand $P - PP^*P = 0$ which translates to $P(I - P^*P) = 0$. But $\ker(P) = \{0\}$.

Thus, $I - P^*P = 0$. Hence, $P^*P = I$. Thus, P is unitary.

Remark 3.1.35

From Corollary 3.1.34, it is deduced that P is normal under the stated conditions since unitary operators $\subseteq$ Normal operators.

Corollary 3.1.36

Suppose $P, Q \in B(H)$ where Q is a partial isometry satisfying $R(P, Q)Y = 0$. If Q^* is quasinormal and Y is one-to-one, then Q is unitary.

Proof

If $Q = QQ^*Q$ and $PYQ = Y$, then $Y = PYQQ^*Q = YQ^*Q$. Thus, $Y = YQ^*Q$ and $Y(I - Q^*Q) = 0$. Since Y is one-to-one, then $I - Q^*Q = 0$ implying $Q^*Q = I$.

Next, if $Q^*$is quasinormal, then $Y = PYQ = PYQQ^*Q = PYQQQ^* = YQQ^*$.

Thus, $Y = YQQ^*$and $Y - YQQ^* = 0$ implying $Y(I - QQ^*) = 0$. Since Y is one-to-one, then

$I - QQ^* = 0$ implying $QQ^* = I$. Thus, $QQ^* = I$ and $Q^*Q = I$ implying Q is unitary.

Theorem 3.1.37

Suppose $P, Q \in B(H)$ satisfy $PY = YQ$ for some operator Y. If:

(i) P is isometric, then $P^*YQ = Y$.

(ii) Q is co-isometric, then $PYQ^* = Y$.

Proof

(i) If P is isometric and $PY = YQ$, then $Y = P^*PY = P^*YQ$. Thus, $P^*YQ = Y$.

(ii) If Q is co-isometric, then $PYQ^* = YQQ^* = Y$. Thus $PYQ^* = Y$.

Corollary 3.1.38

Suppose $P, Q \in B(H)$ and $PY = YQ$ imply $P^*Y = YQ^*$. If P or Q is unitary, then $P^*YQ = Y$ and $PYQ^* = Y$.

Proof

If P is unitary, then $Y = PP^*Y = PYQ^*$. Also, $Y = P^*PY = P^*YQ$. Thus $P^*YQ = Y$ and $PYQ^* = Y$. If Q is unitary, then $Y = YQQ^* = PYQ^*$ and $Y = YQ^*Q = P^*YQ$.

Theorem 3.1.39

Suppose $P, Q \in B(H)$ and $PY = YQ$ imply $P^*Y = YQ^*$. If:

(i) P is unitary and Y is one-to-one, then Q is unitary.

(ii) Q is unitary then P is unitary as long as Y has dense range.

Proof

(i) If P is unitary then it's isometric and if $PY = YQ$, then $P^*YQ = Y$. Also if, $P^*Y = YQ^*$, then $PYQ^* = PP^*Y = Y$. Thus $PYQ^* = Y$. Next, we have

$Y = PYQ^* = PP^*YQQ^* = YQQ^*$. Thus $Y = YQQ^*$ and $Y - YQQ^* = 0$. Since Y is one-to-one, then $Y(I - QQ^*) = 0$ and $I - QQ^* = 0$ implying $QQ^* = I$. Again, $Y = P^*YQ = P^*PYQ^*Q = YQ^*Q$ and $Q^*Q = I$. Thus, Q is unitary.

(ii) Similarly from Theorem 3.1.38, $Y = PYQ^*$ and $P^*YQ = Y$. Thus, $Y = PP^*YQQ^* = PP^*Y$. Thus, $(PP^* - I)Y = 0$. Since $\overline{\mathrm{Ran}(Y)} = H$, then $PP^* - I = 0$ implying $PP^* = I$. Also, $Y = P^*YQ = P^*PYQ^*Q = P^*PY$. Thus, $(P^*P - I)Y = 0$. Since Y has a dense range, then $P^*P = I$. Hence, $PP^* = P^*P = I$. Thus, P is unitary.

Corollary 3.1.40

Suppose $P, Q \in B(H)$ and $PY = YQ$ imply $P^*Y = YQ^*$, Y being a quasiaffinity, then P is unitary if Q is unitary and conversely.

Proof

From Corollary 3.1.38 , if P is unitary and Y a quasiaffinity such that $PY = YQ$, then $P^*YQ = Y$ and if $P^*Y = YQ^*$, $Y = P^*YQ = YQ^*Q$. This implies $Y = YQ^*Q$ and $Y - YQ^*Q = 0$. Thus, $Y(I - Q^*Q) = 0$. Since Y is a quasiaffinity, then $\ker(Y) = \{0\}$ and $\overline{\mathrm{Ran}(Y)} = H$, hence $I - Q^*Q = 0$ implying $Q^*Q = I$. By Theorem 3.1.29, Q is normal, hence $QQ^* = Q^*Q = I$. Thus, Q is unitary.

Similarly, If Q is unitary and $PY = YQ$, then $PYQ^* = Y$ and if $P^*Y = YQ^*$, $Y = PYQ^* = PP^*Y$. Thus, $Y - PP^*Y = 0$ and $(I - PP^*)Y = 0$. Since Y has a dense range, then $PP^* = I$. Again, by Corollary 3.1.29, P is normal hence, $P^*P = PP^* = I$. Thus, P is unitary.

Corollary 3.1.41

Suppose $P, Q \in B(H)$ where P and Q^* are dominant and m-hyponormal respectively. If $PY = YQ$ where Y is a quasiaffinity, then P is unitary if and only if Q is unitary.

Proof

From Theorem 3.27, $PY = YQ$ implies $P^*Y = YQ^*$. By tracing Corollary 3.1.40 , we have $QQ^* = Q^*Q = I$ and $P^*P = PP^* = I$. Thus, both P and Q are unitary.

Theorem 3.1.42

Suppose $P, Q \in B(H)$ satisfy $PYQ = Y$. If Y has a dense range and P is isometric, then P is unitary.

Proof

If P is isometric, then $P^*P = I$. Thus if $PYQ = Y$, then $YQ = P^*PYQ = P^*Y$ implying $P^*Y = YQ$. Next, $Y = PYQ = PP^*Y$. This means $(PP^* - I)Y = 0$. Since $\overline{\mathrm{Ran}(Y)} = H$, then $PP^* = I$. If P is isometric satisfying the given conditions, then it's co-isometric. Hence, P is unitary.

Corollary 3.1.43

Suppose $P, Q \in B(H)$ satisfy $PYQ = Y$. If Y is one to one and Q co-isometric, then Q is unitary.

Proof

As in Theorem 3.1.42, if Q is co-isometric then $QQ^* = I$. If $PYQ = Y$, then $PY = PYQQ^* = YQ^*$ implying $PY = YQ^*$. Next, $Y = PYQ = YQ^*Q$. This means $Y - YQ^*Q = 0$ and $Y(I - Q^*Q) = 0$. If Y is one-to-one, then $Q^*Q = I$. Thus, if Q is co-isometric satisfying the stated conditions, then it is isometric implying Q is unitary.

CHAPTER FOUR

THE MOORE-PENROSE INVERSE OF LINEAR OPERATORS IN HILBERT SPACES

4 Introduction

This chapter investigates properties of the Moore-Penrose inverse of an EP operator as well as relation between the Moore-Penrose inverse of an EP operator and its usual inverse under some given conditions without assuming the invertibility of the EP operator in Hilbert spaces. It also establishes the Moore-Penrose inverse of $(A+B)$ where $A = PQ \in B_c(H)$ with closed ranges and $B \in B(H)$ under some given conditions distinct from the ones used by Kulkarni and Ramesh (2015) and other scholars. Corollaries relating to these theorems are also given as well as showing that if $\text{Ran}(P)$ is closed, then $\text{Ran}(A+B)$ is closed under some conditions.

4.1 Results

Remark 4.1.1

The generalised inverse of an operator exists if the range of the operator is closed and vice-versa. In our results we use A' as generalised inverse of A where $A' \in A\{1\}$. That is $A = AA'A$.

Theorem 4.1.2

Suppose $A \in B_C(H)$ and A' its generalised inverse. If:

(i) $\text{Ker}(A) = \{0\}$, then A' is the left inverse of A.

(ii) $\overline{\text{Ran}(A)} = H$, then A' is the right inverse of A.

Proof

(i) The generalised inverse of A satisfies $A = AA'A$. This implies $A - AA'A = 0$ and $A(I - A'A) = 0$. Since A is injective, then $I - A'A = 0$. Thus, $A'A = I$.

(ii) Since A' in $B(H)$ is a generalised inverse of A in $B_c(H)$ whose range is closed, then $Ran(A) = Ran(AA'A) \subseteq Ran(AA') \subseteq Ran(A)$. Thus, $Ran(A) = Ran(AA')$. If A has a closed range, that is dense in H, then A is onto. This implies that $Ran(A) = H$. Since $Ran(AA') = Ran(A)$, then $Ran(AA') = H$ .Thus, $AA': H \longrightarrow H$ if $A = AA'A$, then $A - AA'A = 0$. Hence,$(I - AA')A = 0$. Since $\overline{Ran(A)} = H$, then $A \neq 0$ implying $I - AA' = 0$. Thus, $AA' = I$ is defined on entire H.

Remark 4.1.3

It is important to observe that since $Ran(A) = Ran(AA') = H$, then AA' is a projection on the range of A ($AA' = P_{\overline{Ran(A)}}$). This means that $AA': H \rightarrow H$ is defined on entire H and $AA' = I$ on H. This is important because if $Ran(AA') \neq Ran(A) \neq H$, then $AA' = I$ is not defined on entire H but subspace $Ran(A)$. Recall that $Ix = x$ for all x in H.

Corollary 4.1.4

Suppose $A \in B_C(H)$ and A^+ be the Moore-Penrose inverse of A. If A is a quasiaffinity, then $A^+ = A^{-1}$.

Proof

An operator A being a quasiaffinity implies that it has a dense range and it is one-to-one. From Theorem 4.1.2, $A'A = AA' = I$. Since A^+ is the unique generalised inverse of A, then $A^+ = A^{-1}$.

Remark 4.1.5

In Corollary 4.1.4, it has been deduced that $A^{+} = A^{-1}$ for the case of a quasiaffinity. Theorem 4.1.6, relaxes the condition of quasiaffinity in Corollary 4.1.4, to operator with either dense range, meaning $\overline{\mathrm{Ran}(A)} = H$, or injective under the proviso that A is an EP operator.

Theorem 4.1.6

Let $A \in B_C(H)$ be EP operator and A^{+} its Moore-Penrose inverse. Then $A^{+} = A^{-1}$ in each of these cases:

(i) A is injective.

(ii) $\overline{\mathrm{Ran}(A)} = H$.

Proof

The Moore-Penrose inverse of A satisfies $A = AA^{+}A$ as one of its properties. Since A is an EP operator then it satisfies the equation $A^{+}A = AA^{+}$. This implies $A = AA^{+}A = AAA^{+}$ and $A = AA^{+}A = A^{+}AA$. Thus, $A(I - A^{+}A) = 0$ and $A\,(I - AA^{+}) = 0$. Also, $(I - AA^{+})A = 0$ and $(I - A^{+}A)A = 0$.

If A is injective, then $I - AA^{+} = 0$ and $I - A^{+}A = 0$. Thus, $A^{+}A = AA^{+} = I$, implying $A^{+} = A^{-1}$. Also, if $A \in B_C(H)$ and $\overline{\mathrm{Ran}(A)} = H$, then it is onto implying $A^{+}A = I$ and $AA^{+} = I$ is defined on entire H. Hence $A^{+} = A^{-1}$.

Remark 4.1.7

In Corollary 4.1.8, Brock (1990) gave a characterization of EP operator as follows.

Corollary 4.1.8 (Brock, 1990)

The Statements below imply each other for T in $B_c(H)$.

(i) $T^+T = TT^+$.

(ii) $H = Nul(T) \oplus^{\perp} Ran(T)$.

(iii) $Nul(T) = Nul(T^*)$.

(iv) $T^* = PT$ for an invertible operator P in H.

Corollary 4.1.9

Suppose $T \in B_C(H)$. If T is one-to-one or $\overline{Ran(T)} = H$, then

$T^+ = T^{-1}$ in each of the cases listed below.

(i) $H = Nul(T) \oplus^{\perp} Ran(T)$.

(ii) $Nul(T) = Nul(T^*)$.

(iii) $T^* = PT$ for some invertible operator P in H.

Proof

Since $T \in B_C(H)$, then $T = TT^+T$ so that $T(I - T^+T) = 0$ and $(I - TT^+)T = 0$.If T is injective, then $I - T^+T = 0$ so that $T^+T = I$. Again, if $\overline{R(T)} = H$, then $I - TT^+ = 0$ implying $TT^+ = I$. From Corollary 4.1.8, each of conditions (i) – (iii) implies T is an EP operator. Thus, $T^+T = TT^+$. Hence $T^+T = TT^+ = I$ implying $T^+ = T^{-1}$.

Corollary 4.1.10 (Koliha, 2000)

Let $T \in B_C(H)$ be upper semi-Fredholm. Then the statements below are equivalent.

i) $T^+T = TT^+$.

ii) $(T^*T)_\pi T = 0$.

iii) $T(TT^*)_\pi = 0$.

Remark 4.1.11

Corollary 4.1.12, uses Corollary 4.1.10 to introduce the Drazin inverse which capitalizes on the spectral idempotent of an operator to establish that under some conditions the Moore-Penrose inverse of an operator is its usual inverse.

Corollary 4.1.12

Let $A \in B_C(H)$ be an upper semi-Fredholm. If $(A^*A)_\pi A = 0$ or $A(AA^*)_\pi = 0$, then

$A^+ = A^{-1}$ provided A is injective or $\overline{Ran(A)} = H$.

Proof

By definition, null space of A has a finite dimension. As a result, $A = AA^+A$. This means $A(I - A^+A) = 0$ and $(I - AA^+)A = 0$. If $\ker(A) = 0$, then $I - A^+A = 0$ implying $A^+A = I$. Also, $\overline{R(A)} = H$ implies A has a dense range, so that $I - AA^+ = 0$. Consequently, $AA^+ = I$.

Thus, from Corollary 4.1.10, the imposed conditions imply $[A, A^+] = 0$.

Hence $A^+A = AA^+ = I$ indicating $A^+ = A^{-1}$.

Theorem 4.1.13 (Wong, 1986)

If the Moore-Penrose inverse of $A \in B_C(H)$ is a polynomial of A , then it is an EP-operator provided that H has a finite dimension and vice versa.

Remark 4.1.14

Notably, from Theorem 4.1.13, every subspace of a space that has finite dimension is closed. Since $A \in B(H)$ and H has a finite dimension, then it has a closed range and if

$\overline{Ran(A)} = H$ then A is surjective. Again, in case A is injective, then it is invertible. Hence,

$A^+ = A^{-1}$.

Corollary 4.1.15

If the Moore-Penrose inverse of $A \in B_C(H)$ is a polynomial of A and H is a finite dimension, then $A^+ = A^{-1}$ incase $\overline{Ran(A)} = H$.

Proof

Since Moore-Penrose inverse of A is a polynomial of A and H is of finite dimension, then A is EP operator and $A^+A = AA^+$. Thus, $A = AA^+A = A^+AA$. This means $A - AA^+A = 0$ implying

$(I - A^+A)A = 0$. Since $\overline{Ran(A)} = H$, then $I - AA^+ = 0$ and $AA^+ = I$. Similarly,

$A - A^+AA = 0$ and $(I - A^+A)A = 0$. Again, if A has a dense range, then $I - A^+A = 0$

and $A^+A = I$. Thus $A^+A = I$ and $AA^+ = I$ implying $A^+ = A^{-1}$.

Theorem 4.1.16 (Khalagai and Sheth, 1987)

Let $[A, B^2] = 0$, then $[A, B] = 0$ in each of the cases listed below for A and B bounded operators on a Hilbert space H.

(i) $\{B\}^C = \{B^{2m}\}^C$ for some positive integer m.

(ii) B is normal and either $\sigma(\text{Re } B) \cap \sigma(-\text{Re } B) = \emptyset$ or $\sigma(\text{Im } B) \cap \sigma(-\text{Im } B) = \emptyset$.

(iii) B is normal and either $0 \notin W(\text{Re } B)$ or $0 \notin W(\text{Im } B)$.

Corollary 4.1.17

Suppose $A \in B_C(H)$ with $[A^+, A^2] = 0$ and either $Nul(A) = 0$ or $\overline{Ran(A)} = H$, then $A^+ = A^{-1}$ in each of the statements listed below.

(i) $\{A\}^C = \{A^{2m}\}^C$ for some positive integer m.

(ii) A is normal and either $\sigma(\text{Re } A) \cap \sigma(-\text{Re } A) = \emptyset$ or $\sigma(\text{Im } A) \cap \sigma(-\text{Im } A) = \emptyset$.

(iii) A is normal and either $0 \notin W(\text{Re } A)$ or $0 \notin W(\text{Im } A)$.

Proof

From the given conditions A^+ exists implying $A = AA^+A$. Again $A(I - A^+A) = 0$ and $(I - AA^+)A = 0$. If A is injective, then $I - A^+A = 0$ implying $A^+A = I$. Also, Since $\overline{\text{Ran}(A)} = H$, then $I - AA^+ = 0$ and $AA^+ = I$. From Theorem 4.1.16, statements $(i) - (iii)$ imply $[A^+, A] = 0$. Thus, $AA^+ = A^+A = I$ implying $A^+ = A^{-1}$.

Remark 4.1.18

It is worth noting that if an operator is bounded from below, then the operator is one-to-one and its range is closed. Thus, if it has a dense range, then it is invertible. If an operator is normal and has a closed range, then it is an EP operator. Thus, conditions of Corollary 4.1.17 (ii) and (iii) imply $[A^+, A] = 0$.

Lemma 4.1.19 (Anderson, 2011)

Suppose $A \in B_C(H)$ is bounded from below. Then A is injective and its range is closed.

Corollary 4.1.20

Suppose $A \in B_C(H)$ where A is normal and bounded from below. Then $A^+ = A^{-1}$.

Proof

By Lemma 4.1.19, A is one-to-one with a closed range. Thus, $A(I - A^+A) = 0$ implies $I - A^+A = 0$. This results to $A^+A = I$. Again, since A has a range that is closed and it is normal, then A is EP operator and consequently $A^+A = AA^+ = I$. Therefore, $A^+ = A^{-1}$.

Proposition 4.1.21 (Israel and Greville, 2003)

The statements below are equivalent for $A \in B(H)$ with dense domain.

(i) A has a closed range.

(ii) A^* has a closed range.

(iii) A^+ is bounded.

(iv) A^*A has a closed range.

(v) AA^* has a closed range.

Corollary 4.1.22

Suppose $A \in B_C(H)$ and $[A^*, A] = 0$ with a dense range. If A^* or A^*A or AA^* have closed ranges or A^+ is bounded, then $A^+ = A^{-1}$

Proof

Each of the stated conditions imply that range of A is closed meaning A^+ exists and

$A = AA^+A$. Hence, $(I - AA^+)A = 0$ and since $\overline{Ran(A)} = H$, then $I - AA^+ = 0$.

Consequently, $AA^+ = I$.

Again if Ran (A) is closed and it is normal, then $[A^+, A] = 0$.

Thus, $A^+A = AA^+ = I$ implying $A^+ = A^{-1}$.

Theorem 4.1.23

If $A \in B_C(H)$ is R- quasi –EP operator, then $A^+ = A^{-1}$ if A is bounded from below.

Proof

By the hypothesis $[A, AA^+] = 0$ and $A = AA^+A = AAA^+$. This implies $A = AA^+A$ and

$A = AAA^+$. Hence, $A - AA^+A = 0$ and $A - AAA^+ = 0$. Since A is linear, then

$A(I - A^+A) = 0$ and $A(I - AA^+) = 0$ accordingly. Since A is bounded from below, from results of Lemma 4.1.19 , $Nul(A) = \{0\}$ implying $I - A^+A = 0$ and $I - AA^+ = 0$.

Consequently, $A^+A = I$ and $AA^+ = I$. Hence, $A^+ = A^{-1}$.

Corollary 4.1.24

Suppose $A \in B_C(H)$ is a partial isometry that is bounded from below. Then A is unitary provided A^* is quasinormal.

Proof

By definition of partial isometry, $A^* = A^+$. If A^* is quasinormal, we have

$AAA^* = AA^*A = A.$ Substituting A^* for A^+ we have $A = AA^+A = AAA^+$. Thus, from steps of Theorem 4.1.23, $I - A^+A = 0$ and $I - AA^+ = 0$. This implies, $A^+A = I$ and $AA^+ = I$. Thus, $A^+ = A^{-1}$.

Theorem 4.1.25

Let $A \in B_C(H)$ be L- quasi –EP operator. If $\overline{\mathrm{Ran}(A)} = H$, then $A^+ = A^{-1}$.

Proof

By hypothesis, $[A, A^+A] = 0$. Thus, $(AA^+)A = (A^+A)A$. Since A^+ exists, then

$A = AA^+A$. Thus, $A = AA^+A = A^+AA$. Hence, $A - AA^+A = 0$ and $A - A^+AA = 0$. This imply, $(I - AA^+)A = 0$ and $(I - A^+A)A = 0$. Since $\overline{\mathrm{Ran}(A)} = H$, then $I - A^+A = 0$ as well as $I - AA^+ = 0$. Thus, $A^+A = I$ and $AA^+ = I$ implying $A^+ = A^{-1}$. If A has a dense and closed range, then $\mathrm{Ran}(A) = \mathrm{Ran}(AA^+) = H$.

Remark 4.1.26

If in Theorem 4.1.25, A^+ is substituted for A^*, then Corollary 3.1 22 is rediscovered as established in Corollary 4.1.27 for partial isometries.

Corollary 4.1.27

Suppose $A \in B_C(H)$ is quasinormal partial isometry. If A has a dense range then it is unitary.

Proof

See Corollary 3.1.22.

Corollary 4.1.28

Suppose $A \in B_C(H)$ is quasinormal partial isometry. If A has dense range, then $A^+ = A^{-1}$.

Proof

$A^*AA = AA^*A = A$ since A is a quasinormal partial isometry. Since $\overline{\text{Ran}(A)} = H$ and is closed, then $A \neq 0$ and it is onto. From Theorem 4.1.25, $I - A^*A = 0$ as well as $I - AA^* = 0$. Hence, $A^*A = I$ and $AA^* = I$ implying $A^* = A^{-1}$. From definition $A^* = A^+$. Thus, $A^+ = A^{-1}$.

Remark 4.1.29

In the sequel, inclusions and equalities involving the range and null spaces of operators are proved and the expressions of the Moore-Penrose inverse of $(A + B)$where $A = PQ$, $PP^+B = B$, $BQ^+Q = B|_{D(Q)}$ and $\|P^+BQ^+\| < 1$ with $B \in B(H)$ being a perturbation of A are given. Frigyes and Bela (1955) gave the following result which helps in showing the relation between the ranges of operators in Hilbert spaces.

Proposition 4.1.30 (Frigyes and Bela, 1955)

If $T, P \in B(H)$ are densely defined and $S \in B(H)$, then $ST + SP = S(T + P)$.

Theorem 4.1.31

If $A = PQ$ is in $B_c(H)$ and densely defined with B in $B(H)$ where P and Q are densely defined with closed ranges then we have:

(i) $\mathrm{Ran}(A) \subseteq \mathrm{Ran}(P)$.

(ii) $\mathrm{Ran}(A+B) \subseteq \mathrm{Ran}(P)$ implies $PP^{+}B = B$ and conversely.

Proof

(i) $v \in \mathrm{Ran}(A)$ and u be in $D(A)$. This means $v = Au = PQu$.

Thus, $PQu = v$. Hence, v is in $\mathrm{Ran}(P)$.

Via implication $\mathrm{Ran}(A) \subseteq \mathrm{Ran}(P)$.

(ii) Let $\mathrm{Ran}(A+B) \subseteq \mathrm{Ran}(P)$. Then PP^{+} is in $P_{\mathrm{Ran}(A+B)}$. This means,

$A + B = PP^{+}(A+B) = PP^{+}PQ + PP^{+}B = PQ + PP^{+}B = A + PP^{+}B.$

Thus, $PP^{+}B = B$.

Conversely, let $PP^{+}B = B$, then $A + B = PQ + PP^{+}B$. By Proposition 4.30,

$A + B = P(Q + P^{+}B)$. Hence, $\mathrm{Ran}(A+B) \subseteq \mathrm{Ran}(P)$.

Remark 4.1.32

From results of Theorem 4.1.31, $\mathrm{Ran}(A) \subseteq \mathrm{Ran}(P)$ and $\mathrm{Ran}(A+B) \subseteq \mathrm{Ran}(P)$ means $PP^{+}A = A$ and $PP^{+}(A+B) = (A+B)$ under the stated conditions.

Corollary 4.1.33

Suppose $A = PQ \in B_c(H)$ is densely defined. If P is bounded from below, then $P^{+}PQ = Q$.

Proof

Since $A = PQ$, by Theorem 4.1.31, PP^{+} is in $P_{\mathrm{Ran}(A)}$. That is $PP^{+}A = A$. Thus, $PP^{+}PQ = PQ$ implying $PP^{+}PQ - PQ = 0$ and $P(P^{+}PQ - Q) = 0$. Since P is bounded from below, then by Lemma 4.1.19 , P has a closed range and it is injective.

Hence, $P^{+}PQ - Q = 0$ and $P^{+}PQ = Q$.

Proposition 4.1.34 (Kulkarni and Ramesh, 2015)

Suppose $A \in B_c(H)$ is densely defined, then

(i) $\mathrm{Nul}(A) = \mathrm{Ran}(A^*)^{\perp} = \mathrm{Ran}(A^+)^{\perp}$.

(ii) $\mathrm{Nul}(A^*) = \mathrm{Ran}(A)^{\perp} = \mathrm{Nul}(A^+)$.

Lemma 4.1.35 (Israel et al., 2003)

Suppose $A \in B(H)$ where $\|A\| < 1$, then $(I + A)^{-1}$ exists.

Corollary 4.1.36

Let $A = PQ \in B_c(H)$ with P and Q densely defined with closed ranges. If Q is surjective and $B \in B(H)$ satisfying $PP^+B = B$, $BQ^+Q = B|_{D(Q)}$ and $\|P^+BQ^+\| < 1$, then

$\mathrm{Ran}(A + B) = \mathrm{Ran}(P)$.

Proof

From Theorem 4.1.31, $\mathrm{Ran}(A + B) \subseteq \mathrm{Ran}(P)$ … … … (i)

Next, let $y \in \mathrm{Ran}(P)$ for x in D(P). Then $Px = y$. From Lemma 4.1.35, if $\|P^+BQ^+\| < 1$, then $(I + P^+BQ^+)^{-1} \in B(H)$. Since Q is surjective and $(I + P^+BQ^+)^{-1}$ exists, then $(I + P^+BQ^+)Q$ is surjective. Thus, there exists $u \in D(Q)$ such that $x = (I + P^+BQ^+)Qu$.

Thus, $y = Px = P(1 + P^+BQ^+)Qu = (P + PP^+BQ^+)Qu = (PQ + PP^+BQ^+Q)u$

$= (PQ + PP^+B)u = (A + B)u$.

Thus, $(A + B)u = y$ implying $y \in \mathrm{Ran}(A + B)$.

Hence, $\mathrm{Ran}(P) \subseteq \mathrm{Ran}(A + B)$ … … … … … (ii)

Consequently, $\mathrm{Ran}(P) = \mathrm{Ran}(A + B)$.

Theorem 4.1.37

Let $A = PQ \in B_c(H)$ where P is bounded from below with dense domain. If

$B \in B(H)$ satisfies $PP^+B = B$, $BQ^+Q = B|_{D(Q)}$ and $\|P^+BQ^+\| < 1$, then

$Nul(Q) = Nul(A + B)$.

Proof

For $x \in Nul(Q)$, then

$(A + B)x = (PQ + PP^+BQ^+Q)x = P(I + P^+BQ^+)Qx = P(I + P^+BQ^+)0.$

Since $\|P^+BQ^+\| < 1$, then $(I + P^+BQ^+)^{-1}$ exists and $P(I + P^+BQ^+)0 = P0$.

Since P is bounded from below then by Lemma 4.1.35, P is injective and thus,

$(A + B)x = (PQ + BQ^+Q)x = P(I + P^+BQ^+)Qx = P(I + P^+BQ^+)0 = P0 = 0.$

This means $x \in Nul(A + B)$ implying $Nul(Q) \subseteq Nul(A + B) \ldots\ldots\ldots.(*)$.

Next, let $x \in N(A + B)$, then $0 = (A + B)x = (PQ + PP^+BQ^+Q)x = P(I + P^+BQ^+)Qx$.

That is $P(I + P^+BQ^+)Qx = 0$. Since P is injective, then $(I + P^+BQ^+)Qx = 0$.

Also if $\|P^+BQ^+\| < 1$, then $(I + P^+BQ^+)^{-1}$ exists implying $I + P^+BQ^+$ is injective.

Thus, $(I + P^+BQ^+)Qx = 0$ implying $Qx = 0$.

Hence, $x \in Nul(Q)$ implying $Nul(A + B) \subseteq Nul(Q) \ldots\ldots\ldots\ldots\ldots(**)$

From $(*)$ and $(**)$, then $Nul(Q) = Nul(A + B)$.

Corollary 4.1.38

Let $A = PQ \in B_c(H)$ where P is bounded from below and densely defined, then

$Nul(Q) = Nul(A)$.

Proof

For $x \in Nul(Q)$ means $Ax = PQx = P0$. Since P is bounded from below, then Lemma 4.1.35,

P is injective implying $P0 = 0$. Hence $Ax = P0 = 0$ implying $Ax = 0$. Hence,

$Nul(Q) \subseteq Nul(A) \ldots\ldots\ldots\ldots\ldots(i)$

Next, let $x \in Nul(A)$. This means $PQx = Ax = 0$ implying $PQx = 0$. Since P is injective then

$Qx = 0$. Hence, $x \in Nul(Q)$.

By implication $\mathrm{Nu}(A) \subseteq \mathrm{Nul}(Q) \ldots \ldots \ldots (ii)$

From (i) and (ii) $\mathrm{Nul}(A) = \mathrm{Nul}(Q)$.

Theorem 4.1.39

Let A= $PQ \in B_c(H)$ where P and Q are densely defined operators with closed ranges. If Q is surjective and $B \in B(H)$ satisfies $\|P^+BQ^+\| < 1$, $PP^+B = B$ and $BQ^+Q = B|_{D(Q)}$.

Then

(i) $\mathrm{Ran}(A + B)$ is closed.

(ii) $(A + B)^+ = Q^+(I + P^+BQ^+)^{-1}P^+$.

Proof

(i) From Corollary 4.1.36, $\mathrm{Ran}(A + B) = \mathrm{Ran}(P)$. Thus, if range of P is closed, then $\mathrm{Ran}(A + B)$ is closed.

(ii) Since $Ran(A + B)$ is closed, then $(A + B)^+$ exists. Also

$A + B = (PQ + BQ^+Q) = (P + PP^+BQ^+)Q = P(I + P^+BQ^+)Q$. Thus,

$A + B = P(I + P^+BQ^+)Q$. If $\|P^+BQ^+\| < 1$, then by Lemma 4.35,

$1 + P^+BQ^+$ is bijective and $(1 + P^+BQ^+)^{-1} \in B(H)$. Thus,

$(A + B)^+ = Q^+(I + P^+BQ^+)^{-1}P^+$.

This equation satisfies the properties of Moore-Penrose inverse as illustrated below:

Let $y \in \mathrm{Ran}(A + B)^+ = \mathrm{Ran}\{Q^+(I + P^+BQ^+)^{-1}P^+\}$. This implies that for

$x \in D(P^+)$, $Q^+(I + P^+BQ^+)^{-1}P^+x = y$. Pre multiplying each side by Q gives,

$QQ^+(I + P^+BQ^+)^{-1}P^+x = Qy$. Since QQ^+ is in $P_{\mathrm{Ran}(Q)}$, it translates to

$(I + P^+BQ^+)^{-1}P^+x = Qy$. Thus, $P^+x = (I + P^+BQ^+)Qy$ implying,

$(I + P^+BQ^+)Qy \in \mathrm{Ran}(P^+)$. Again P^+P is in $P_{\mathrm{Ran}(P^+)}$. Thus,

$(A + B)^+(A + B)y = Q^+(I + P^+BQ^+)^{-1}P^+P(I + P^+BQ^+)Qy.$

$= Q^+(I + P^+BQ^+)^{-1}(I + P^+BQ^+)Qy.$

$= Q^+Qy.$

Also, $Q^+(I + P^+BQ^+)^{-1}P^+x = y$ implying $y \in Ran(Q^+)$. Thus,

$(A + B)^+(A + B)y = Q^+Qy = y.$

That is $(A + B)^+(A + B)y = P_{Nul(A+B)^{\perp}}y \;\; \forall \; y \in D(A + B)$ … … … … … … … . … (a)

Next**,** we let $y \in Ran(A + B) = Ran\{ P(I + P^+BQ^+)Q\}$. This implies $y \in Ran(P)$, thus there exists $x \in D(Q)$ such that $P(I + P^+BQ^+)Qx = y$. Pre-multiplying each side by P^+ yields: $P^+P(I + P^+BQ^+)Qx = P^+y$. Since P^+P is in $P_{Ran(P^+)}$ then it implies $(I + P^+BQ^+)Qx = P^+y$ and $Qx = (I + P^+BQ^+)^{-1}P^+y$. Thus, $(I + P^+BQ^+)^{-1}P^+y \in Ran(Q)$. Since QQ^+ is in $P_{Ran(Q)}$, then $(A + B)(A + B)^+y = P(I + P^+BQ^+)QQ^+(I + P^+BQ^+)^{-1}P^+y.$

$$= P(I + P^+BQ^+)(I + P^+BQ^+)^{-1}P^+y.$$

$$= PP^+y.$$

If $y \in Ran(A + B)$ and from Corollary 4.1.36, $Ran(A + B) = R(P)$.Thus $y \in Ran(P)$, implying $(A + B)(A + B)^+y = PP^+y = y$, Thus, $(A + B)(A + B)^+y = y$.

This implies$(A + B)(A + B)^+y = P_{\overline{R(A+B)}}\, y = y$ for every $y \in N(A + B)^+$ … … … (b).

Lastly, from results of Proposition 4.1.34, $(A + B) \in B(H)$ is densely defined.

Thus, $Nul(A + B)^+ = Ran(A + B)^{\perp}$ … … … … … … … … … … … … … … … … … . . . (c).

Equations (a), (b) and (c) justify the results.

Corollary 4.1.40

Let $A = PQ \in B_c(H)$where P and Q are densely defined operators with closed ranges. If Q is surjective and $B \in B(H)$ satisfies $BQ^+Q = B|_{D(Q)}$, $PP^+B = B$ and $\|P^+BQ^+\| < 1$, then

$P^+ = (1 + P^+BQ^+)Q(A + B)^+.$

Proof

From Theorem 4.1.39, we have $(A+B)^{+} = Q^{+}(I+P^{+}BQ^{+})^{-1}P^{+}$. Since Q is onto then, $QQ^{+} = I$ and via pre multiplication by $(1+P^{+}BQ^{+})Q$ both sides gives,

$P^{+} = (1+P^{+}BQ^{+})Q(A+B)^{+}$.

This satisfies the Properties of Moore-Penrose inverse as illustrated below. If,

$P^{+} = (1+P^{+}BQ^{+})Q(A+B)^{+}$and $P = (A+B)Q^{+}(I+P^{+}BQ^{+})^{-1}$, then to proof that $P^{+}P$ is in $P_{Nul(P)^{\perp}}$, we let $y \in Ran(P^{+})$.

Then there exists $x \in Nul(A+B)^{+}$ such that $(1+P^{+}BQ^{+})Q(A+B)^{+}x = y$.

Since $(1+P^{+}BQ^{+})$ is invertible, then $Q(A+B)^{+}x = (1+P^{+}BQ^{+})^{-1}y$.

This implies $(1+P^{+}BQ^{+})^{-1}y \in Ran(Q) \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots..(*)$

Pre-multiplying $Q(A+B)^{+}x = (1+P^{+}BQ^{+})^{-1}y$ by Q^{+} gives:

$Q^{+}Q(A+B)^{+}x = Q^{+}(1+P^{+}BQ^{+})^{-1}y$ implying $Q^{+}(1+P^{+}BQ^{+})^{-1}y \in R(Q^{+})$.

Thus, $(A+B)^{+}x = Q^{+}(1+P^{+}BQ^{+})^{-1}y$. Since $Q^{+}Q$ is in $P_{Ran(Q)^{+}}$.

This means $Q^{+}(1+P^{+}BQ^{+})^{-1}y \in Ran(A+B)^{+} \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots..(**)$

Now, $P^{+}Py = (1+P^{+}BQ^{+})Q(A+B)^{+}(A+B)Q^{+}(I+P^{+}BQ^{+})^{-1}y$.

From $(**)$ $Q^{+}(1+P^{+}BQ^{+})^{-1}y \in R(A+B)^{+}$. If $(A+B)^{+}(A+B)$ is in $P_{Ran(A+B)^{+}}$, then

$P^{+}Py = (1+P^{+}BQ^{+})QQ^{+}(I+P^{+}BQ^{+})^{-1}y$. Again, from $(*)$,

$(I+P^{+}BQ^{+})^{-1}\, y \in R(Q)$ and since QQ^{+} is in $P_{Ran(Q)}$, then

$P^{+}Py = (1+P^{+}BQ^{+})(I+P^{+}BQ^{+})^{-1}y = y$. Thus,

$P^{+}Py = P_{Nul(A)^{\perp}}y$ for all $y \in D(P) \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots..(a)$.

Next, to show that PP^{+} is in $P_{Ran(P)}$ let $y \in Ran(P)$ and $x \in D(I+P^{+}BQ^{+})^{-1}$ so that

$(A+B)Q^{+}(I+P^{+}BQ^{+})^{-1}x = y$. Pre-multiplying each side by $(A+B)^{+}$ gives:

$(A+B)^{+}(A+B)Q^{+}(I+P^{+}BQ^{+})^{-1}x = (A+B)^{+}y$.

Thus, $Q^{+}(I+P^{+}BQ^{+})^{-1}x = (A+B)^{+}y$. Hence $(A+B)^{+}y \in Ran(Q^{+})$ and

$PP^+y = (A+B)Q^+(I+P^+BQ^+)^{-1}(1+P^+BQ^+)Q(A+B)^+y$

$= (A+B)Q^+Q(A+B)^+y.$

Thus, $PP^+y = (A+B)Q^+Q(A+B)^+y$. Since Q^+Q is in $P_{Ran(Q^+)}$ and $(A+B)^+y \in Ran(Q^+)$, then $PP^+y = (A+B)Q^+Q(A+B)^+y = (A+B)(A+B)^+y$.

By Corollary 4.1.36, $Ran(A+B) = Ran(P)$ implying $PP^+y = (A+B)(A+B)^+y = y$.

Thus, $PP^+y = P_{\overline{Ran(P)}}y$ for every $y \in D(P^+)$ … … . . … … … … … … … … … … . . (b)

Lastly, since P is densely defined, then by Proposition 4.34,

$Ran(P)^{\perp} = Nul(P^+)$ … … … … … … … … … … … … … … … … … … … (c)

From equations a, b and c, $P^+ = (1+P^+BQ^+)Q(A+B)^+$ satisfies the properties of Moore-Penrose inverse.

Corollary 4.1.41

Let $A = PQ \in B_c(H)$where P and Q are densely defined operators with closed ranges. If P is bounded from below, Q is surjective and $B \in B(H)$ satisfies $PP^+B = B$, $BQ^+Q = B_{|D(Q)}$ and $\|P^+BQ^+\| < 1$, then $Q^+ = (A+B)^+P(\,I+P^+BQ^+)$.

Proof

From Theorem 4.1.39, $A+B = \ PQ + PP^+B = P(I+P^+BQ^+)Q$ and

$(A+B)^+ = Q^+(I+P^+BQ^+)^{-1}P^+$. Since P and $(I+P^+BQ^+)$ are injective, then post-multiplying the preceding equation by $P(I+P^+BQ^+)$ gives:

$Q^+ = (A+B)^+P(I+P^+BQ^+)$. This satisfies the properties of Moore-Penrose inverse as follows. Since, $Q = (I+P^+BQ^+)^{-1}P^+(A+B)$ and $Q^+ = (A+B)^+P(I+P^+BQ^+)$, then we let $y \in Ran(Q^+)$. This implies there is $x \in D(I+P^+BQ^+)$ so that $(A+B)^+P(I+P^+BQ^+)x = y$.

Pre-multiplying the equations on both sides by $(A+B)$, we obtain:

$(A+B)(A+B)^+P(I+P^+BQ^+)x = (A+B)y$. Since, $(A+B)(A+B)^+$ is in $P_{Ran(A+B)}$, then $P(I+P^+BQ^+)x = (A+B)y$. Thus, $(A+B)y \in Ran(P)$.

Thus, $PP^{+}(A+B)y = (A+B)y$.

Now, to show that $Q^{+}Q$ is in $P_{Nul(Q)^{\perp}}$ we have

$$Q^{+}Qy = (A+B)^{+}P(I+P^{+}BQ^{+})(I+P^{+}BQ^{+})^{-1}P^{+}(A+B)y$$
$$= (A+B)^{+}PP^{+}(A+B)y.$$
$$= (A+B)^{+}(A+B)y.$$

From Corollary 4.1.38, $Nul(Q) = Nul(A+B)$.

Thus, $Ran(Q^{+}) = Nul(Q)^{\perp} = Nul(A+B)^{\perp} = Ran(A+B)^{+}$.

This means $Q^{+}Qy = (A+B)^{+}(A+B)y = y$ implying $Q^{+}Qy = y$.

Thus $Q^{+}Qy = P_{\overline{Ran(Q^{+})}}y$ for every $x \in D(Q)$ … … … … … … … . … … . (a).

Next, to show QQ^{+} is in $P_{Ran(Q)}$, we have:

Let $y \in R(Q)$. This implies there exists a $y \in D(A+B)$ leading to

$(I+P^{+}BQ^{+})^{-1}P^{+}(A+B)x = y$. If $(I+P^{+}BQ^{+})^{-1}$ is injective, then

$P^{+}(A+B)x = (I+P^{+}BQ^{+})y$. This implies $(I+P^{+}BQ^{+})y \in R(P^{+})$ … … … … … … . . (∗)

Pre-multiplying $P^{+}(A+B)x = (I+P^{+}BQ^{+})y$ by P from both sides results to,

$PP^{+}(A+B)x = P(I+P^{+}BQ^{+})y$. Since PP^{+} is in $P_{R(P)}$ then,

$(A+B)x = P(I+P^{+}BQ^{+})y$ … … … … … … … … . . (∗∗)

This implies $P(I+P^{+}BQ^{+})y \in Ran(A+B)$.

Now, since $(A+B)(A+B)^{+}$ is in $P_{Ran(A+B)}$ from (∗) we have,

$$QQ^{+} = (I+P^{+}BQ^{+})^{-1}P^{+}(A+B)(A+B)^{+}P(I+P^{+}BQ^{+})y.$$
$$= (I+P^{+}BQ^{+})^{-1}P^{+}P(I+P^{+}BQ^{+})y.$$

Again, $P^{+}P$ is in $P_{Ran(P^{+})}$. Thus, from (∗)

$QQ^{+} = (I+P^{+}BQ^{+})^{-1}(I+P^{+}BQ^{+})y = y$. Thus, $QQ^{+}y = y$.

Thus, $QQ^{+}y = P_{\overline{Ran(Q)}}y$ for every $y \in D(Q^{+})$ … … … … … … … … … … … … … … . (b).

Since Q is densely defined, then from Proposition 4.1.34, we have

$\mathrm{Ran}(Q)^{\perp} = \mathrm{Nul}(Q^{+})$ … (c)

Thus, from equations a, b and c, $Q^{+} = (A + B)^{+}P(I + P^{+}BQ^{+})$ satisfies the properties of Moore-Penrose inverse.

Corollary 4.1.42

Suppose $P \in B(H)$with $Q \in B_{C}(H)$ having dense domain and $R(P^{*}) = H$ such that $R(Q) = H$ and $A = PQ$ is closed. If $B \in B(H)$ satisfies $PP^{+}B = B$ and $\|P^{+}B\| < \frac{1}{\|Q^{+}\|}$, then $A^{+} = (I{+}A^{+}B)(A + B)$.

Proof

If $A^{+} = Q^{+}P^{+}$, then:

$A + B = PQ + PP^{+}B = P(Q + P^{+}B) = PQ(I + Q^{+}P^{+}B) = A(I + A^{+}B)$. Thus,

$(A + B) = A(I + A^{+}B)$. If $\|P^{+}B\| < \frac{1}{\|Q^{+}\|}$ imply $||Q^{+}P^{+}B|| = \|A^{+}B\| < 1$ leading to

$(I + A^{+}B) \in B(H)$ is invertible. $(A + B)^{+} = (I + A^{+}B)^{-1}A^{+}$. Thus,

$A^{+} = (I{+}A^{+}B)(A + B)^{+}$.

CHAPTER FIVE

ON FUGLEDE-PUTNAM THEOREMS FOR MOORE-PENROSE INVERTIBLE OPERATORS

5 Introduction

The Fuglede-Putnam theorems will be established to hold true for EP operators under certain commutativity conditions involving the operators AA^*, A^*A^+, BB^* and B^*B^+as well as for injective operators or operators with dense range. Results on Fuglede-Putnam type commutativity theorems when the adjoint is replaced with the MPI and A being EP operator is replaced by either injective operator or an operator with dense range are also illustrated. These results are achieved under conditions different from the ones stated by Johnson *et al*. (2021) and other scholars.

5.1 Results

Remark 5.1.1

Johnson et al. (2021) gave results on a bounded operator commuting with Moore-Penrose inverse of an EP operator. Corollary 5.1.5, 5.1.7,and 5.1.8 below, show commutativity of a bounded operator $P \in B(H)$ and A^+A (or AA^+) as well as B^+B (or BB^+) under the proviso of having EP operators namely A and B.

Theorem 5.1.2 (Johnson *et al.,* 2021)

Suppose $A, B \in B_c(H)$ are EP operators and $P \in B(H)$. If

(i) $PA = AP$, then $PA^+ = A^+P$.

(ii) $BP = PA$, then $B^+P = PA^+$.

Theorem 5.1.3 (Johnson *et al.*, 2021)

Let $A, B \in B_c(H)$and EP operators and $P \in B(H)$, then:

(i) $PA = AQ$ and $QA = AP$, then $PA^+ = A^+Q$ and $QA^+ = A^+P$.

(ii) $PA = BQ$ and $QA = BP$, then $PA^+ = B^+Q$ and $QA^+ = B^+P$.

Remark 5.1.4

The following Corollaries 5.1.5, 5.1.7 and 5.1.8 are deduced from preceding theorems.

Corollary 5.1.5

Let $A \in B_c(H)$ be an EP operator and $P \in B(H)$. If $AP = PA$, then:

(i) $PA^+A = A^+AP$.

(ii) $PAA^+ = AA^+P$.

(iii) $PA^+A = AA^+P$

Proof

(i) Since $PA = AP$, then from Theorem 5.1.2 (i), $PA^+ = A^+P$. Thus,

$PA^+A = A^+PA = A^+AP$.

(ii) Since $PA = AP$, then $PA^+ = A^+P$. Hence, $AA^+P = APA^+ = PAA^+$.

(iii) Again, from 5.1.2 (i), $PA = AP$ implies $PA^+ = A^+P$. Thus,

$PA^+A = A^+AP = AA^+P$. Thus, $PA^+A = AA^+P$.

Remark 5.1.6

Note that for $A \in B(H)$, the range of A is orthogonal to null space of A^* but if

$\mathrm{Ran}(A) = \mathrm{Ran}(A^*)$, then $\mathrm{Ran}(A)$ is orthogonal to $\mathrm{Nul}(A)$.

Corollary 5.1.7

Suppose $A, B \in B(H)$ are EP operators and $P \in B(H)$. If $BP = PA$, then

$B^{+}BP = PA^{+}A$ and $BB^{+}P = PAA^{+}$.

Proof

From Theorem 5.1.2 (ii) if $BP = PA$ then, $B^{+}P = PA^{+}$. Thus, $PA^{+}A = B^{+}PA = B^{+}BP$.

Also, $PAA^{+} = BPA^{+} = BB^{+}P$. Thus, $PAA^{+} = BB^{+}P$.

Corollary 5.1.8

Suppose $A, B \in B_c(H)$ are EP operators and $P, Q \in B(H)$. If $PA = BQ$ and

$QA = BP$ then,

(i) $B^{+}BP = PA^{+}A$ and $BB^{+}P = PAA^{+}$.

(ii) $B^{+}BQ = Q\,A^{+}A$ and $BB^{+}Q = QAA^{+}$.

Proof

(i) If $PA = BQ$ and $QA = BP$, then by Theorem 5.1.3 (ii) $PA^{+} = B^{+}Q$ and

$QA^{+} = B^{+}P$ Thus, $PA^{+}A = B^{+}QA = B^{+}BP$. Hence, $PA^{+}A = B^{+}BP$.

Also, $PAA^{+} = BQA^{+} = BB^{+}P$. Thus, $PAA^{+} = BB^{+}P$.

(ii) Similarly, $QA^{+}A = B^{+}PA = B^{+}BQ$ and $QA^{+}A = B^{+}BQ$. Also,

$QAA^{+} = BPA^{+} = BB^{+}Q$. Hence $QAA^{+} = BB^{+}Q$.

Remark 5.1.9

The Fuglede-Putnam theorems for normal operators hold for EP-operators under some conditions as illustrated in the results that follow.

Theorem 5.1.10 (Fuglede, 1950)

Let $A, P \in B(H)$ with A normal. If $PA = AP$, then $PA^{*} = A^{*}P$.

Theorem 5.1.11 (Putnam, (1951)

Let A, B and $P \in \mathrm{B(H)}$ with A and B normal. If $PB = AP$, then $\mathrm{PB}^* = A^*\mathrm{P}$.

Remark 5.1.12

In the sequel, it is shown that results of Fuglede(1950) and Putnam (1951) are true for operators that are EP under conditions different from the ones of Johnson *et al.* (2021)) in Theorem 5.1.13 and Theorem 5.1.14 as well as for two bounded operators and two EP operators.

Theorem 5.1.13 (Johnson *et al.*, 2021)

Suppose $\mathrm{A} \in \mathrm{B_c(H)}$ is EP operators and $\mathrm{P} \in \mathrm{B(H)}$. If $AP = PA$ and $\mathrm{PA^*A} = A^*\mathrm{AP}$, then $A^*\mathrm{P} = \mathrm{PA}^*$.

Theorem 5.1.14 (Johnson *et al.*, 2021)

Suppose $\mathrm{A, B} \in \mathrm{B_c(H)}$ are EP operators on H and $\mathrm{P} \in \mathrm{B(H)}$. If $AP = PB$, then $A^*P = PB^*$ in each of these cases:

(i) $B^*BP = PA^*A$.

(ii) $B^+B^*P = PA^+A^*$

Theorem 5.1.15

Suppose $\mathrm{A} \in \mathrm{B_c(H)}$ is EP operator and $\mathrm{P} \in \mathrm{B(H)}$. If $AP = PA$ and $PAA^* = AA^*P$, then $A^*\mathrm{P} = \mathrm{PA}^*$.

Proof

The above conditions imply that $[\mathrm{P, A^+}] = 0$. Thus,

$\mathrm{PA}^* = \mathrm{P}\,(\mathrm{AA^+A})^* = \mathrm{P(A^+A)^*A^*} = \mathrm{PA^+AA^*} = \mathrm{A^+PAA^*}$. If $\mathrm{PAA^*} = \mathrm{A}A^*\mathrm{P}$, then

$PA^* = A^+PAA^* = A^+AA^*P = (A^+A)^*A^*P = (AA^+A)^*P = A^*P.$

Example 5.1.16

Consider the EP operator A such that:

$A(v_1, v_2, v_3, v_4, \dots) = (v_1 - v_2, v_1 + v_3, 2v_1 - v_2 + v_3, v_4, \dots)$ and $P \in B(H)$ defined by

$P(v_1, v_2, v_3, \dots) = (v_2, -v_1 + v_2 - v_3, -2v_1 + v_2, v_4, \dots)$ so that

$A^*(v_1, v_2, v_3, v_4, \dots) = (v_1 + v_2 + 2v_3, -v_1 - v_2, v_2 + v_3, v_4, \dots)$. so that

$PAA^*(v_1, v_2, v_3, v_4, \dots) = (v_1 + 2v_2 + 3v_3, -4v_1 - 2v_2 - 6v_3, -3v_1 - 3v_3, v_4, \dots)$.

$AA^*P(v_1, v_2, v_3, v_4, \dots) = (-7v_1 + 6v_2 - v_3, -8v_1 + 6v_2 - 2v_3, -15v_1 + 12v_2 - 3v_3, v_4, \dots)$. This implies $PAA^* \neq AA^*P$. By computing PA^* and A^*P,

we have $PA^*(v_1, v_2, v_3, v_4, \dots) = (-v_1 - v_3, -2v_1 - 2v_2 - 4v_3, -3v_1 - 2v_2 - 5v_3, v_4, \dots) \neq A^*P(v_1, v_2, v_3, v_4, \dots) = (-5v_1 + 4v_2 - v_3, 2v_1 - 2v_2, -3v_1 + 2v_2 - v_3, x_4, \dots)$.

This implies that the conditions cannot be overlooked for the conclusion to hold.

Corollary 5.1.17

Suppose $A \in B_c(H)$ is EP operator and $P, Q \in B(H)$ such that $PA = AQ$ and $QA = AP$. If:

(i) $QAA^* = AA^*P$, then $PA^* = A^*P$.

(ii) $PAA^* = AA^*Q$, then $QA^* = A^*Q$.

Proof

(i) If $PA = AQ$ and $QA = AP$, then $PA^+ = A^+Q$. So that

$PA^* = P(AA^+A)^* = P(A^+A)^*A^* = PA^+AA^* = A^+QAA^*$. If $QAA^* = AA^*P$, then

$PA^* = A^+QAA^* = A^+AA^*P = A^*P$.

(ii) From the stated conditions we have $C[Q, P]A^+ = 0$ and

$QA^* = Q(AA^+A)^* = Q(A^+A)^*A^* = QA^+AA^* = A^+PAA^*$. If

$PAA^* = AA^*Q$, then $QA^* = A^+PAA^* = A^+AA^*Q = A^*Q$.

Theorem 5.1.18

Let $A \in B_c(H)$ be EP operator and $P, Q \in B(H)$. Let $PA = AQ$ and $QA = AP$. If:

(i) $[Q, AA^*] = 0$, then $C[P, Q]A^* = 0$.

(ii) $[P, AA^*] = 0$, then $C[Q, P]A^* = 0$.

Proof

(i) Since $PA = AQ$ and $QA = AP$, then $PA^+ = A^+Q$ so that

$PA^* = P\,(AA^+A)^* = P(A^+A)^*A^* = PA^+AA^* = A^+QAA^*$.

If $[Q, AA^*] = 0$, then $PA^* = A^+QAA^* = A^+AA^*Q = A^*Q$.

Thus, $PA^* = A^*Q$ implying $C[P, Q]A^* = 0$.

(ii) Similarly, from the stated conditions $C[Q, P]A^+ = 0$. Thus,

$QA^* = Q(AA^+A)^* = Q(A^+A)^*A^* = QA^+AA^* = A^+PAA^*$. If $PAA^* = AA^*P$, then

$QA^* = A^+PAA^* = A^+AA^*P = A^*P$.

Hence $C[Q, P]A^* = 0$.

Remark 5.1.19

Normal operators whose range is closed are EP operators but the converse is not true. See Johnson *et al*. (2021). As a result, it is shown that Fuglede - Putnam results hold for operators whose null spaces equal to those of their adjoint operators (EP operators) under the given conditions.

Theorem 5.1.20

Suppose $A, B \in B_c(H)$ are EP operators and $P \in B(H)$. If $C[B, A]P = 0$ then

$C[BB^*, AA^*]P = 0 \Longrightarrow C[B^*, A^*]P = 0$.

Proof

The stated conditions imply $B^+P = PA^+$.

Thus, $PA^* = P(AA^+A)^* = P(A^+A)^*A^* = PA^+AA^* = B^+PAA^*$.

If $C[BB^*, AA^*]P = 0$, then $PA^* = B^+PAA^* = B^+BB^*P = B^*P$.

Thus, $C[B^*, A^*]P = 0$.

Example 5.1.21

Suppose $A, B \in B_c(H)$ are EP operators defined by:

$A(v_1, v_2, v_3, v_4, \dots) = (v_1 + v_3, 0, v_3, v_4, \dots)$,

$B(v_1, v_2, v_3, v_4, \dots) = (v_1 + v_2, v_2, 0, v_4, \dots)$.

Let $P \in B(H)$ be defined by $P(v_1, v_2, v_3, \dots) = (v_1 - v_3, v_3, 2v_2, v_4, \dots)$.

In this case, $PA(v_1, v_2, v_3, v_4, \dots) = (v_1, v_3, 0, v_4, \dots)$ and

$BP(v_1, v_2, v_3, v_4, \dots) = (v_1, v_3, 0, v_4, \dots)$. Thus $PA = BP$.

Also, $A^*(v_1, v_2, v_3, v_4, \dots) = (v_1, 0, v_1 + v_3, v_4, \dots)$. and

$B^*(v_1, v_2, v_3, v_4, \dots) = (v_1, v_1 + v_2, 0, v_4, \dots)$.

So that $PAA^*(v_1, v_2, v_3, v_4, \dots) = (v_1, v_1 + v_3, 0, v_4, \dots)$ and

$BB^*P(v_1, v_2, v_3, v_4, \dots) = (2v_1 - v_3, v_1, 0, v_4, \dots)$.

This implies $PAA^* \neq BB^*P$. By computing PA^* and B^*P we have, $PA^*(v_1, v_2, v_3, v_4, \dots) =$

$(-v_3, v_1 + v_3, 0, v_4, \dots)$ and

$B^*P(v_1, v_2, v_3, v_4, \dots) = (v_1 - v_3, v_1, 0, v_4, \dots)$. Thus, $PA^* \neq B^*P$.

Thus, the conditions in Theorem 5.20 are vital for the results to hold.

Theorem 5.1.22 (Mahmood, 2017)

Suppose A, B, P and $Q \in B(H)$ with A and B are normal. If:

(i) $PA = AQ$ and $QA = AP$, then $PA^* = A^*Q$ and $QA^* = A^*P$.

(ii) $PA = BQ$ and $QA = BP$, then $PA^* = B^*Q$ and $QA^* = B^*P$.

Theorem 5.1.23

Suppose A and $B \in B_c(H)$ are EP operators and $P, Q \in B(H)$ such that $PA = BQ$ and $QA = BP$. If $QAA^* = AA^*P$, then $C[B^*, A^*]P = 0$.

Proof

The stated conditions imply $PA^+ = B^+Q$. So that,

$PA^* = P(AA^+A)^* = P(A^+A)^*A^* = PA^+AA^* = B^+QAA^*$. If $QAA^* = BB^*P$, then

$PA^* = B^+QAA^* = B^+BB^*P = B^*P$. Thus $C[B^*, A^*]P = 0$

Theorem 5.1.24

Suppose $A, B \in B_c(H)$ are EP operators and $P, Q \in B(H)$ is such that $PA = BQ$ and $QA = BP$. If $PAA^* = BB^*Q$. then $C[B^*, A^*]\, Q = 0$.

Proof

From the stated conditions $QA^+ = B^+P$. So that,

$QA^* = Q(AA^+A)^* = Q(A^+A)^*A^* = QA^+AA^* = B^+PAA^*$. If $PAA^* = BB^*Q$, then

$QA^* = B^+PAA^* = B^+BB^*Q = B^*Q$. Thus $C[B^*, A^*]\, Q = 0$.

Theorem 5.1.25

Suppose A,B $\in B_c(H)$ and EP operators and $P, Q \in B(H)$ such that $PA = BQ$ and $QA = BP$. If:

(i) $C[BB^*, AA^*]P = 0$, then $QA^* = B^*P$.

(ii) $C[BB^*, AA^*]Q = 0$, then $PA^* = B^*Q$.

Proof

(i) The stated conditions imply, $PA^+ = B^+Q$ and $QA^+ = B^+P$ so that,

$QA^* = Q\,(AA^+A)^* = Q(A^+A)^*A^* = QA^+AA^* = B^+PAA^*$. If

$PAA^* = BB^*P$, then $QA^* = B^+PAA^* = B^+BB^*P = B^*P$.

(ii) Again, $PA^* = P\,(AA^+A)^* = P(A^+A)^*A^* = PA^+AA^* = B^+QAA^*$. If

$QAA^* = BB^*Q$, then $PA^* = B^+QAA^* = B^+BB^*Q = B^*Q$.

Theorem 5.1.26

Suppose $A \in B_c(H)$ is an EP operator and P in $B(H)$ is such that $PA = AP$.

If $PA^*A^+ = A^*A^+P$ then $PA^* = A^*P$.

Proof

We deduce that $A^* = (AA^+A\,)^* = (A^+A)^*A^* = A^+AA^*$. Thus,

$PA^* = PA^+AA^*$. Now, $[P, A^+] = 0$. Thus, $PA^* = PA^+AA^* = A^+PAA^*$. Now,

$PA^* = A^+PAA^* = A^+APA^* = A^+APA^*AA^+ = A^+APA^*A^+A$. Since $[P, A^*A^+] = 0$, then

$PA^* = A^+APA^*A^+A = A^+AA^*A^+PA = A^+AA^*A^+AP = A^+AA^*P = A^*P$.

Remark 5.1.27

It is worth noting that $A^* = A^+AA^* = A^*AA^+$. Since

$A^* = (AA^+A\,)^* = (A^+A\,)^*A^* = A^+AA^*$ or $A^* = (AA^+A\,)^* = A^*(AA^+\,)^* = A^*AA^+$.

Theorem 5.1.28

Suppose $A, B \in B_c(H)$ are EP operators and $P \in B(H)$such that $BP = PA$. If

$B^*B^+P = PA^*A^+$, then $B^*P = PA^*$.

Proof

If $BP = PA$, then the stated conditions imply $B^+P = PA^+$.

Thus, $PA^* = P(A^+A\,)^*A^* = PA^+AA^* = B^+PAA^* = B^+BPA^* = B^+BPA^*AA^+$

$= B^+BPA^*A^+A$. By assumption $B^*B^+P = PA^*A^+$. Hence,

$PA^* = B^+BPA^*A^+A = B^+BB^*B^+PA = B^+BB^*B^+BP = B^+BB^*P = B^*P$.

Theorem 5.1.29

Suppose A,B $\in B_c(H)$ are EP operators and $P, Q \in B(H)$ are such that $PA = BQ$ and

$QA = BP$. If:

(i) $B^*B^+P = PA^*A^+$, then $PA^* = B^*Q$.

(ii) $B^*B^+Q = QA^*A^+$, then $QA^* = B^*P$.

Proof

(i) From the given conditions we have $PA^+ = B^+Q$, then

$PA^* = PA^+AA^* = B^+QAA^* = B^+BPA^* = B^+BPA^*AA^+ = B^+BPA^*A^+A$.

Since, $B^*B^+P = PA^*A^+$, then:

$PA^* = B^+BPA^*A^+A = B^+BB^*B^+PA = B^+BB^*B^+BQ = B^+BB^*Q = B^*Q$.

(ii) Similarly, $QA^* = QA^*A^+A = B^*B^+QA = B^*B^+BP = B^*P$.

Corollary 5.1.30

Suppose $A, B \in B_c(H)$ and EP operators and $P, Q \in B(H)$ such that

$PA = BQ$ and $QA = BP$. If $PA^*A^+ = B^*B^+Q$, then $B^*P = PA^*$.

Proof

The above conditions imply $PA^+ = B^+Q$. So that

$PA^* = PA^+AA^* = B^+QAA^* = B^+BPA^* = B^+BPA^*AA^+ = B^+BPA^*A^+A$. Since

$PA^*A^+ = B^*B^+Q$, then

$PA^* = B^+BPA^*A^+A = B^+BB^*B^+QA = B^+BB^*B^+BP = B^+BB^*P = B^*P$.

Remark 5.1.31

In case the operators A, B, P and Q satisfy the conditions of Theorem 5.1.29 , then by interchanging the roles of P and Q gives $QA^* = B^*P$. Again, note in Corollary 5.1..30 that interchanging the roles of Q and P then $C[B^*, A^*]\, Q = 0$. In Theorem 5.32 extends the results Theorem 5.1.15 and Theorem 5.1.20 to operators with either dense range or injective without necessarily having A and B being EP operators.

Theorem 5.1.32

Suppose $A, P \in B(H)$. If $PA = AP$, then $PA^* = A^*P$ in each of these conditions:

(i) $PA^*A = A^*AP$ and $\mathrm{Ran}(A)$ is dense in H.

(ii) $PAA^* = AA^*P$ and A is one-one.

Proof

(i) If $PA^*A = A^*AP$ and $PA = AP$, then $PA^*A = A^*AP = A^*PA$. Thus, $PA^*A = A^*PA$. This implies that $PA^*A - A^*PA = 0$ and $(PA^* - A^*P)A = 0$. Since A has a dense range, then $PA^* - A^*P = 0$. Hence, $PA^* = A^*P$.

(ii) If $PAA^* = AA^*P$ and $AP = PA$, then $APA^* = PAA^* = AA^*P$. This implies $AA^*P - APA^* = 0$ and $A(A^*P - PA^*) = 0$. If $N(A) = 0$ then, $A^*P - PA^* = 0$. Hence, $PA^* = A^*P$.

Corollary 5.1.33

Suppose $A, P \in B_c(H)$ where A is a quasiaffinity and $PA = AP$. If $PA^*A = A^*AP$ or $PAA^* = AA^*P$, then $PA^* = A^*P$.

Proof

Operator A being a quasiaffinity implies its injective and has a dense range. Thus, by Theorem 5.1.32, $PA^* = A^*P$.

Corollary 5.1.34

Let $P, Q, A \in B(H)$ satisfy $PA = AQ$. If:

(i) $PA^*A = A^*AQ$ and A has a dense range, then $PA^* = A^*P$.

(ii) $PAA^* = AA^*Q$ and A is one-one, then $QA^* = A^*Q$.

Proof

(i) $PA^*A = A^*AQ = A^*PA$ implying $PA^*A - A^*PA = 0$. Thus, $(PA^* - A^*P)A = 0$.

If A has a dense range, then $PA^* - A^*P = 0$. Implying $PA^* = A^*P$.

(ii) $AA^*Q = PAA^* = AQA^*$. So that, $AQA^* - AA^*Q = 0$. This implies,

$A(QA^* - A^*Q) = 0$. Since A is one-one, then $QA^* - A^*Q = 0$ and $QA^* = A^*Q$.

Corollary 5.1.35

Let P, Q, B and $A \in B(H)$ where $PA = BQ$. If:

(i) $PA^*A = B^*BQ$ and $\overline{\mathrm{Ran}(A)} = H$ then $B^*P = PA^*$.

(ii) $PAA^* = BB^*Q$ and B is one $-$ one, then $B^*Q = QA^*$.

Proof

(i) If $PA^*A = B^*BQ$ and $PA = BQ$, then $PA^*A = B^*BQ = B^*PA$ implying

$PA^*A = B^*PA$ and $PA^*A - B^*PA = 0$. Thus, $(PA^* - B^*P)A = 0$.

Since $\overline{\mathrm{Ran}(A)} = H$, then $PA^* - B^*P = 0$. Thus, $PA^* = B^*P$.

(ii) If $PAA^* = BB^*Q$ and $PA = BQ$, then $BQA^* = PAA^* = BB^*Q$ and

$BQA^* = BB^*Q$. Hence, $BQA^* - BB^*Q = 0$. Thus, $B(QA^* - B^*Q) = 0$. If B

is one-one, then $QA^* - B^*Q = 0$. Thus, $QA^* = B^*Q$.

Remark 5.1.36

By replacing A^* with Moore-Penrose inverse of A in Theorem 5.1.32 and Corollary 5.1.34, the following Fuglede-Putnam type commutativity results are obtained as proved below.

Theorem 5.1.37

Let $A, P \in B(H)$ and $PA = AP$. Then $PA^+ = A^+P$ in each of these cases:

(i) $PA^+A = A^+AP$ as long as the range of A is dense in H.

(ii) $PAA^+ = PAA^+$ and A is one-one.

Proof

(i) If P commutes with A, then $PA^+A = A^+AP = A^+PA$. Thus, $PA^+A = A^+PA$.

This yields $(PA^+ - A^+P)A = 0$. Again A having a dense range implies

$PA^+ - A^+P = 0$. Hence $PA^+ = A^+P$.

(ii) Similarly, $AA^+P = PAA^+ = APA^+$. Thus $APA^+ = AA^+P$ and $A(PA^+ - A^+P) = 0$.

If $N(A) = \{0\}$, then $PA^+ - A^+P = 0$. Thus, $PA^+ = A^+P$.

Corollary 5.1.38

Let A , P and $Q \in B(H)$. Let $PA = AQ$. If :

(i) $PA^+A = A^+AQ$ and $\overline{\mathrm{Ran}(A)} = H$, then $PA^+ = A^+P$.

(ii) $PAA^+ = AA^+Q$ and $\mathrm{Nul}(A) = \{0\}$, then $QA^+ = A^+Q$.

Proof

(i) $PA^+A = A^+AQ = A^+PA$. This implies $(PA^+ - A^+P)A = 0$.

Since range of A is dense in H, then $PA^+ - A^+P = 0$. Hence, $PA^+ = A^+P$.

(ii) Again, $AA^+Q = PAA^+ = AQA^+$ implying $AA^+Q - AQA^+ = 0$.

Thus, $A(QA^+ - A^+Q) = 0$. If $\mathrm{Nul}(A) = \{0\}$, then $QA^+ - A^+Q = 0$.

Consequently, $QA^+ = A^+Q$.

Corollary 5.1.39

Let A, B, P and $Q \in B(H)$ and $PA = BQ$. If:

(i) $PA^+A = B^+BQ$ and A has a dense range, then $B^+P = PA^+$.

(ii) $PAA^{+} = BB^{+}Q$ and B is one-one, then $B^{+}Q = QA^{+}$.

Proof

(i) If $PA^{+}A = B^{+}BQ$ and $PA = BQ$, then $PA^{+}A = B^{+}BQ = B^{+}PA$. Thus,

$PA^{+}A = B^{+}PA$ and $(PA^{+} - B^{+}P)A = 0$. Since A has a dense range, then

$PA^{+} - B^{+}P = 0$ implying $PA^{+} = B^{+}P$.

(ii) Similarly, if $PAA^{+} = BB^{+}Q$ and $PA = BQ$, then $BB^{+}Q = PAA^{+} = BQA^{+}$ and

$B(QA^{+} - B^{+}Q) = 0$. Since B is one –to-one, then $QA^{+} - B^{+}Q = 0$.

Thus, $QA^{+} = B^{+}Q$.

CHAPTER SIX

CONCLUSIONS AND RECOMMENDATIONS

6.1 CONCLUSIONS

This thesis has made very important achievements in the development of properties and characterizations of quasiaffine inverse, Moore-Penrose inverse and the Fuglede-Putnam Theorems.

Precisely, it was shown that Q was a quasiaffine inverse of P provided that $PYQ = Y$ implied $P^*YQ^* = Y$ where Y was a quasiaffinity. Where the implementing quasiaffinity Y was self-adjoint satisfying the property $0 \notin W(Y)$ or $\{Y\}^C = \{Y^{2m}\}^C$ for some positive integer m or $\sigma(Y) \cap \sigma(-Y) = \emptyset$, it turned out that Q was the usual inverse of P.

In addition, it was established that the quasiaffine inverse of an operator is unique. That is, where there was another operator satisfying the same operator equations with the same implementing quasiaffinities, the two operators involved turned out to be equal.

It was also shown that if one of the operators P or Q is unitary, then the other is unitary provided P and Q satisfied the operator equation $PY = YQ$ and $P^*Y = YQ^*$. In this case Y was injective and/or had dense range. The inverses of linear operators with left(right) quasiaffine inverse were established and it turned out that they were their adjoint operators. To be specific, the adjoint of P was found to be the left quasiaffine inverse of Q and the adjoint of Q the right quasiaffine inverse of P. This worked for isometric, co-isometric operators, injective partial isometry and partial isometries with dense range. A partial isometry with dense range was found to be co-isometric and where the partial isometry P was one-to-one, then it became isometric. A quasinormal partial isometry with dense range was found to be invertible. In this case the adjoint was its inverse. That is implying it was unitary.

The Moore-Penrose inverse of a quasiaffinity turned to be the usual inverse of the operator with closed range. On the Moore-Penrose inverse of an EP operator, it turned out to be the usual inverse if it was injective or had dense range. The MPI of a perturbed linear operator $A + B$ was also derived where A was expressible as $A = PQ$ and B a perturbation of A satisfying the conditions; $B = PP^{+}B$, $BQ^{+}Q = B_{|D(Q)}$ and $||P^{+}BQ^{+}|| < 1$. That is, $(A + B)^{+}$ was given in terms of $P^{+}$and Q^{+} . Also P^{+} and Q^{+} was given comprising of $(A + B)^{+}$.

Lastly, this thesis derived the Fuglede and Fuglede-Putnam Theorems for EP operators involving operators AA^{*}, $A^{*}A^{+}, BB^{*}$ and $B^{*}B^{+}$ as well as with injective operators and operators with dense range. For instance if $[P, A] = 0$, the stated conditions implied $[P, A^{*}] = 0$. Also, if $PA = BP$, then it was established that $PA^{*} = B^{*}P$ under the same stated conditions in this paragraph. As a result, the Fuglede and Fuglede-Putnam theorems were henceforth proved.

6.1.1 CHAPTER WISE SUMMARY

In the first chapter, an introduction was given where various methods of solving matrix and operator equations such as Crammers rule and inverse of operators were mentioned and different methods of obtaining the Moore-Penrose inverse of linear operators such as Singular Value Decomposition (SVD) and QR-method were mentioned. The concept of least squares solutions of minimum norm was given which is obtained by use of the Moore-Penrose inverse. Several generalized inverses were highlighted and their significances in solving operator equations. The concept of quasiaffine inverse was introduced with the implementing operators being quasiaffinities. The Fuglede and Fuglede-Putnam Theorem and their importance in quantum mechanics and physical chemistry was mentioned where quantities are represented by commuting operators. Notations and terminologies used in this thesis as well as definitions were also given.

Chapter two constituted literature review which was given in three parts. The first part consisted of an emerging inverse called the quasiaffine inverse of operators in Hilbert space. Results given by other scholars on quasiaffine inverse were highlighted and how this thesis intended to produce parallel and new results. The second part was literature review on MPI where chronological development of Moore-Penrose inverse of perturbed linear was given and the importance of using Moore-Penrose inverse in solving operator equations involving non-invertible operators on a Hilbert space. Lastly, literature review on Fuglede-Putnam Theorems was given and how this thesis intended to contribute to their development.

Chapter three comprised of results on quasiaffine inverse where first objective was achieved. By Theorem 3.1.8 and Corollary 3.1.10, it was shown that quasiaffine inverse Q of an operator P in $\mathrm{PYQ} = \mathrm{Y}$ and $P^*YQ^* = Y$ is the usual inverse of P ($\mathrm{Q} = \mathrm{P}^{-1}$) if $0 \notin \mathrm{W(Y)}$ or $\{\mathrm{Y}\}^{\mathrm{C}} = \{\mathrm{Y}^{2\mathrm{m}}\}^{\mathrm{C}}$ where m was a positive integer and Y was self-adjoint, one –to-one or had dense range. The second specific objective was achieved in Theorem 3.1.14 and Corollary 3.1.16 where it was established that the quasiaffine inverse is unique. Here Q was a quasiaffine inverse of P with X and Y as the implementing quasiaffinities and there existed another operator Q_1 such that $\mathrm{PYQ}_1 = \mathrm{Y}$ and $\mathrm{Q}_1\mathrm{XP} = \mathrm{X}$ with X and Y being quasiaffinities. Uniqueness was established by having $\mathrm{Q}_1 = \mathrm{Q}$. Results on inverse of operators by the concept of left (right) quasiaffine inverse were given. Using Theorem 3.1.42 and Corollary 3.1.43 it was shown that the operators P and Q satisfying the operator equation $\mathrm{PYQ} = \mathrm{Y}$ are invertible under some conditions for some classes of operators. It was also shown that if $\overline{\mathrm{R(Y)}} = \mathrm{H}$ and P is isometric, then P is unitary. When Y was one-to-one and Q a co-isometry, it turned out that Q was unitary. The invertibility of operators satisfying $PY = YQ$ was also established. By Theorem 3.1.39 it was established that if P, Q and Y are in $B(H)$ where $PY = YQ$ imply $P^*Y = YQ^*$, then Q became unitary provided P was unitary and Y was one-to-one. On the other hand it was shown that P turned out to be unitary provided Q was unitary and the range of Y was dense.

Chapter four was on Moore-Penrose inverse where the generalized inverse of linear operators was shown to be the usual inverse of the operator under certain conditions. The third specific objective was achieved by the results of Theorem 4.1.6 and its subsequent corollaries where it was established that the Moore-Penrose inverse of an EP operator whose range is closed is the usual inverse of the operator under the conditions that the EP operator is either injective or has a dense range. Several characterizations of EP operators given by other scholars were used to give the parallel results that the Moore-Penrose inverse of an EP operator is the usual inverse of the EP operator under the same conditions. On the same chapter, the fourth specific objective was achieved with the help of results of Theorem 4.1.36 which showed that $R(A + B) = R(P)$ where Theorem 4.1.39 established the MPI of a perturbed operator $(A + B)$ where $A = PQ$, Q is surjective and $B \in B(H)$ satisfying $BQ^+Q = B|_{D(Q)}$, $PP^+B = B$ and $\|P^+BQ^+\| < 1$. The MPI of $A + B$ was realized as $(A + B)^+ = Q^+(I + P^+BQ^+)^{-1}P^+$. Also, in the subsequent Corollary 4.1.40 and Corollary 4.1.41, we expressed the Moore-Penrose inverse of P and Q (P^+ and Q^+) comprising of $(A + B)$ and $(A + B)^+$.

Lastly, the last objective was achieved in chapter five where results on Fuglede-Putnam theorems and Fuglede-Putnam type commutativity theorems of EP operators, injective operators and operators with dense range were given. To be more precise, Theorem 5.1.15 established the Fuglede theorem for EP operators where it was shown that incase A is EP operator on H with P a bounded operator such that P commutes with both A and AA^*, then it also commutes of A^*. The results were extended to two bounded linear operators and an EP operator. That is Corollary 5.1.17 established that if A is EP operator with P, Q in $B(H)$ where $PA = AQ$ and $QA = AP$, then $PA^* = A^*P$ provided $QAA^* = AA^*P$. Also we had $QA^* = A^*Q$ provided $PAA^* = AA^*Q$. Again in Theorem 5.1.20, Fuglede-Putnam results for EP operators were shown. That is if A, B are EP operators and P a bounded operator such that $PA = BP$ and $PAA^* = BB^*P$, then $PA^* = B^*P$. The results were extended to two EP operators and two bounded operators in Theorem 5.1.25. From Theorem 5.1.32 and its subsequent corollaries it was established that the Fuglede-Putnam theorems hold for injective operators and operators with dense range.

6.2 RECOMMENDATIONS

This thesis has established some properties and results of generalized inverses such as Moore-Penrose inverse and quasiaffine inverses. However, there are other generalized inverses like the Drazin inverse and Bott-Duffin inverse which are of interest and key in solving operator equations. Although the Drazin inverse in the classical sense is not a generalized inverse since $AA_dA \neq A$ in general, a few of the properties of the Moore-Penrose inverse are shared. It will be of great interest and importance to see if the same results can be achieved with DI as those of Moore-Penrose inverse. If any disparities are found, what developments do the new results or lack thereof bring to the applicability arena? The Moore-Penrose inverse is connected with the orthogonality of operators as documented in this thesis. However, the Drazin inverse touches on spectral properties of the operators involved which brings in a totally different aspect of study from what has been covered in this thesis.

Drazin inverse is applicable in population modelling (which is an important science currently owing to the challenges of global warming associated to targeted straining on parts of our universe), Markov chains, solving of singular systems of linear differential equations among other trending areas in the current set-up of our world. As a result, it is an area rich in current research with numerous open problems.

The Moore-Penrose inverse is applied in analyzing lumped linear electric networks by using the least square property of the generalized inverse together with a classical variation principle for electric networks. The resulting analysis is related to the Bott-Duffin network analysis. In particular, a relation between the Bott-Duffin constrained inverse and the MPI is established. See Charnes and Israel (1963). Bott-Duffin inverse is also important in optimization, linear statistical estimation and two dimensional interpolations. See Chen (1990). This inverse is used in solving the constrained system equations $Tu + v = b$ in which $u \in L,\ v \in L^{\perp}$ given

$T \in \mathbb{C}^{nxn}$, $b \in \mathbb{C}^{n}$ and $L \subseteq \mathbb{C}^{n}$. These equations occur in electrical network theory. The Bott-Duffin inverse of T, represented by $T_L^{(-1)}$ is given by $T_L^{(-1)} = P_L(TP_L + P_{L^{\perp}})^{-1}$. The Bott-Duffin inverse was introduced by Bott and Duffin and exists when $TP_L + P_{L^{\perp}}$ is invertible. See Bott and Duffin (1953). The equation $Tx = b$ is consistent if $TT^{+}b = b$. In this case TT^{+} is in $P_{R(T)}$. So if in the constrained system equations $Tu + v = b$ where $v = 0$, then the equation reduces to $Tu = b$. Thus, the Moore-Penrose inverse of T can be used to obtain the value of u. In this case, $u = T_L^{(-1)}b$. Comparing it with Moore-Penrose inverse, then $\|u'\| = \|T^{+}b\| \leq \|u\|$. Thus ,$R(T) = L$ assuming $L^{\perp} = R(T)^{\perp} = \{0\}$. This implies that the Bott-Duffin inverse coincides with the Moore-Penrose inverse under some conditions. These three inverses coincide at a point, hence it is of great interest to investigate whether these same results can be obtained for both Drazin inverse and Bott-Duffin inverse.

In summary, the properties and characterization of Drazin inverse and Bott-Duffin inverse on a Hilbert space vis-à-vis the properties and characterization of quasiaffine inverse and Moore-Penrose inverse studied in this thesis is a recommendable area of study stemming from our work herein.

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