C-Loops Obtained From Hypercomplex Numbers

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Abstract: **Hypercomplex numbers have played a notable and critical role in the study and exploration of Loop Theory. Researchers have made numerous studies in this area especially in the investigation and construction of different loops. This paper has extended the research to C-loops where we are investigating the formation of C-loops obtained from hypercomplex numbers of dimension 2^n ; $1 \geq n \leq 4$. We are specifically working with the 2^4- dimensional algebra, called the sedenions. In constructing the C-loops, we have used the frame multiplication of hypercomplex numbers using the Cayley-Dickson construction. We have tested the satisfaction of the left, (x x) (y z) = (x (x y)) z and right, x ((y z) z) = (x y) (z z) C-loop identities by the sedenions. We have also formed split extension of sedenions and equally tested the satisfaction of the C-loop identities on them. We have found that the sedenions satisfy the C-loop identities hence forming C-loops. However, the split extension of sedenions satisfies the right C-loop identity only.**

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I. INTRODUCTION

C-loops refer to one of the special algebraic structures in Loop Theory. They were introduced by Fenyves Ferenc in 1968. C-loops are loops satisfying (x x) (y z) = (x (x y)) z and x ((y z) z) = (x y) (z z), the left and right C-loop identities respectively. They behave analogously to Moufang loops and are closely related to the Steiner triple system.

We have used the Cayley-Dickson doubling process in this paper. It is a construction formula which when applied to an algebraic structure; it doubles the dimension of its predecessor. To begin with, Complex numbers is a 2-dimensional number system. They can be written as ordered pairs (a, b) where a, b ∈ ℝ. When this construction is applied to complex number system, a new algebraic structure called the quaternions is formed. It is of dimension 2^2. Multiplication of quaternions is defined as;

$$(a, b)\,(c, d) = (ac - \bar{d}b;\ da + b\bar{c}) \qquad (1)$$

A continuous application of this formula leads to the formation of octonions, sedenions and the general 2^n-ons.

Over the years, Mathematicians have progressively researched and given much useful results in Loop Theory. The formation of hypercomplex is highly credited to Rowan Hamilton who formed the quaternions in 1843. His discovery was such critical for it paved way for the formation of further related number systems including octonions and sedenions as formed by Arther Cayley in 1846. Njuguna (2012) investigated the properties of the split extension of sedenions with the loop being non-abelian. She found that for a non-abelian loop L, the split extension of sedenions does not form a group but satisfies other loop properties like the Jordan and flexible properties.

Some notable study on C-loops was made by Phillips and Vojtechovsky (2007) where they proved that:

i. A Steiner loop is a C-loop,
ii. C-loops module their nucleus are diassociative,
iii. The nucleus of a C-loop form a normal subgroup,
iv. C-loops are alternative, IP loops with squares in the nucleus,
v. C-loops are not diassociative. They are however power alternative and also power associative.

II. PRELIMINARIES

Definition 1. Quasigroup

A quasigroup is a groupoid G in which any two elements in the equation x · y = z defines the third one uniquely. Equivalently, a quasigroup (Q, ·) is a groupoid G such that ∀ a, b ∈ G, there exist unique x and y such that a · x = b and y · a = b with solutions x = a \b and y = b/a respectively. The operation \ is called the left division while / is the right division.

Definition 2. Loop

A loop L is quasigroup Q with a two sided identity **e** satisfying the equation e · x = x · e ∀ x ∈ Q. A left loop has the left identity satisfying x \ x = x \ (x · e) = e, ∀ x ∈ Q while a right loop has the right identity satisfying x/x = (e · x) / x = e, ∀ x ∈ Q.

Definition 3. Inverse properties

A loop L is said to have;

i. Left inverse property if ∀ x ∈ L, there exist x^l ∈ L such that x^l(x· y) = y, ∀ y ∈L.
ii. Right inverse property if ∀ x ∈ L, there exist x^r ∈ L such that(y · x) x^r = y, ∀ y ∈ L.
iii. Inverse property (I.P) if it has both the left and right inverse properties and x · x^r = x^l · x = e. ∀ x ∈L

Definition 4. Subloop

A subloop M is a non-empty subset of a loop L which is itself a loop.

Definition 5. Division algebra

This is an algebra in which given b and a ≠ 0; then;

i. There exist z such that $a \cdot z = b$.
ii. There exist q such that $q \cdot a = b$.

Definition 6. C-loop

A loop L is called a C-loop if the identities,

i. $(x\, x)\,(y\, z) = (x\,(x\, y))\, z$
ii. $x\,((y\, z)\, z) = (x\, y)\,(z\, z)$

are satisfied $\forall$ x, y, z $\in$ L. The above equations are referred to as the left C-loop and the right C-loop identities respectively. Therefore, a C-loop should be both a left C-loop and right C-loop. Alternatively, the C-loop can also be defined as a loop L which satisfies the identity $x\,(y\,(y\, z)) = ((x\, y)\, y)\, z$ $\forall$ x, y, z $\in$ L.

Definition 7. Moufang loop

This is a loop L in which the following identities hold, $\forall$ x, y, z $\in$ L,

i. $x\, y \cdot z\, x = (x \cdot y\, z)\, x$
ii. $x\,(y \cdot x\, z) = (x\, y \cdot x)\, z$
iii. $x\,(y \cdot z\, y) = (x\, y \cdot z)\, y$.

Definition 8. Bol-Moufang loops

These are the types of loops defined by a single identity that,

i. Involve three distinct variables on both sides,
ii. Contain variables in same order on both sides,
iii. Exactly one of the variables appears twice on both sides.

Multiplicative structures that arise from Cayley-Dickson formula

Jenya (2012), Define the Cayley-Dickson loop $(L_n, \cdot)$ over the $\mathbb{R}$ field as;

$L_0 = \{1, -1\}$, $L_n = \{(x, 0), (x, 1)$ such that $x \in L_{n-1}\}$.

Its split extension is the set $L_n \times S^o$ with multiplication defined as;

$$(x, 0)\,(y, 0) = (x\, y, 0)$$
$$(x, 0)\,(y, 1) = (y\, x, 1)$$
$$(x, 1)\,(y, 0) = (x\, y^*, 1)$$
$$(x, 1)(y, 1) = (-y^*\, x, 0) \qquad (2)$$

while conjugation is described as;

$$(x, 0)^* = (x^*, 0)$$
$$(x, 1)^* = (-x, 1) \qquad (3)$$

III. RESULTS AND DISCUSSION

Frame multiplication of sedenions

In this section we are going to construct the multiplication tables of the sedenions and determine whether the sedenions satisfy the C-loop identities. This will be achieved using equations 2 and 3.

Sedenions refer to a 16-dimensional number system with basis 1, $e_0, e_1, e_2, \ldots,$ e15. Let $b_0 = (e_0, 0)$, $b_1 = (e_1, 0)$, $b_2 = (e_2, 0)$, $b_3 = (e_3, 0)$, $b_4 = (e_4, 0)$, $b_5 = (e_5, 0)$, $b_6 = (e_6, 0)$, $b_7 = (e_7, 0)$, $b_8 = (e_0, 1)$, $b_9 = (e_1, 1)$, $b_{10} = (e_2, 1)$, $b_{11} = (e_3, 1)$, $b_{12} = (e_4, 1)$, $b_{13} = (e_5, 1)$, $b_{14} = (e_6, 1)$, $b_{15} = (e_7, 1)$ be the elements of a sedenion, (Njuguna, 2012). Then the loop $S = L_4 = \pm \{ b_0 = (e_0, 0), b_1 = (e_1, 0), b_2 = (e_2, 0), b_3 = (e_3, 0), b_4 = (e_4, 0), b_5 = (e_5, 0), b_6 = (e_6, 0), b_7 = (e_7, 0), b_8 = (e_0, 1), b_9 = (e_1, 1), b_{10} = (e_2, 1), b_{11} = (e_3, 1), b_{12} = (e_4, 1), b_{13} = (e_5, 1), b_{14} = (e_6, 1), b_{15} = (e_7, 1) \}$ is referred to as the standard sedenion loop. The multiplication table of the elements of the sedenions is as shown below.

Table 1: Multiplication of sedenions

$\cdot$	e_0	e_1	e_2	e_3	e_4	e_5	e_6	e_7
e_0	e_0	e_1	e_2	e_3	e_4	e_5	e_6	e_7
e_1	e_1	$-e_0$	e_3	$-e_2$	e_5	$-e_4$	$-e_7$	e_6
e_2	e_2	$-e_3$	$-e_0$	e_1	e_6	e_7	$-e_4$	$-e_5$
e_3	e_3	e_2	$-e_1$	$-e_0$	e_7	$-e_6$	e_5	$-e_4$
e_4	e_4	$-e_5$	$-e_6$	$-e_7$	$-e_0$	e_1	e_2	e_3
e_5	e_5	e_4	$-e_7$	e_6	$-e_1$	$-e_0$	$-e_3$	e_2
e_6	e_6	e_7	e_4	$-e_5$	$-e_2$	e_3	$-e_0$	$-e_1$
e_7	e_7	$-e_6$	e_5	e_4	$-e_3$	$-e_2$	e_1	$-e_0$
e_8	e_8	$-e_9$	$-e_{10}$	$-e_{11}$	$-e_{12}$	$-e_{13}$	$-e_{14}$	$-e_{15}$
e_9	e_9	e_8	$-e_{11}$	e_{10}	$-e_{13}$	e_{12}	e_{15}	$-e_{14}$
e_{10}	e_{10}	e_{11}	e_8	$-e_9$	$-e_{14}$	$-e_{15}$	e_{12}	e_{13}
e_{11}	e_{11}	$-e_{10}$	e_9	e_8	$-e_{15}$	e_{14}	$-e_{13}$	e_{12}
e_{12}	e_{12}	e_{13}	e_{14}	e_{15}	e_8	$-e_9$	$-e_{10}$	$-e_{11}$
e_{13}	e_{13}	$-e_{12}$	e_{15}	$-e_{14}$	e_9	e_8	e_{11}	$-e_{10}$
e_{14}	e_{14}	$-e_{15}$	$-e_{12}$	e_{13}	e_{10}	e_{11}	e_8	e_9
e_{15}	e_{15}	e_{14}	$-e_{13}$	$-e_{12}$	e_{11}	e_{10}	$-e_9$	e_8

·	e_8	e_9	e_{10}	e_{11}	e_{12}	e_{13}	e_{14}	e_{15}
e_0	e_8	e_9	e_{10}	e_{11}	e_{12}	e_{13}	e_{14}	e_{15}
e_1	e_9	$-e_8$	$-e_{11}$	e_{10}	$-e_{13}$	e_{12}	e_{15}	$-e_{14}$
e_2	e_{10}	e_{11}	$-e_8$	$-e_9$	$-e_{14}$	$-e_{15}$	e_{12}	e_{13}
e_3	e_{11}	$-e_{10}$	e_9	$-e_8$	$-e_{15}$	e_{14}	$-e_{13}$	e_{12}
e_4	e_{12}	e_{13}	e_{14}	e_{15}	$-e_8$	$-e_9$	$-e_{10}$	$-e_{11}$
e_5	e_{13}	$-e_{12}$	e_{15}	$-e_{14}$	e_9	$-e_8$	$-e_{11}$	$-e_{10}$
e_6	e_{14}	$-e_{15}$	$-e_{12}$	e_{13}	e_{10}	$-e_{11}$	$-e_8$	e_9
e_7	e_{15}	e_{14}	$-e_{13}$	$-e_{12}$	e_{11}	e_{10}	$-e_9$	$-e_8$
e_8	$-e_0$	e_1	e_2	e_3	e_4	e_5	e_6	e_7
e_9	$-e_{91}$	$-e_0$	$-e_3$	e_2	$-e_5$	e_4	e_7	$-e_6$
e_{10}	$-e_2$	e_3	$-e_0$	$-e_1$	$-e_6$	$-e_7$	e_4	e_5
e_{11}	$-e_3$	$-e_2$	e_1	$-e_0$	$-e_7$	e_6	$-e_5$	e_4
e_{12}	$-e_4$	e_5	e_6	e_7	$-e_0$	$-e_1$	$-e_2$	$-e_3$
e_{13}	$-e_5$	$-e_4$	e_7	$-e_6$	e_1	$-e_0$	e_3	$-e_2$
e_{14}	$-e_6$	$-e_7$	$-e_4$	e_5	e_2	$-e_3$	$-e_0$	e_1
e_{15}	$-e_7$	e_6	$-e_5$	$-e_4$	e_3	e_2	$-e_1$	$-e_0$

Observations:

- Each element in the loop appears once in each row and column.
- $b_m b_n = -b_n b_m, \forall$ m, n $\in L_n$, m $\neq$ n.
- $(b_n)^2 = -b_0 \; \forall$ n = 1, 2, …, 15.
- $-b_0$ commutes and associates with all the elements.

We next examine how C-loop identities are satisfied. We are also providing several exam-ples in different cases.

(a) Left C-loop identity, (xx) (y z) = (x (xy)) z

Case 1: (trivial)

i. [(x, 0) (x, 0)] [(y, 0) (z, 0)] = (xx, 0) (y z, 0) = (xxyz, 0)
ii. {(x, 0) [(x, 0) (y, 0)]} (z, 0) = [(x, 0) (xy, 0)] (z, 0) = (xxy, 0) (z, 0) = (xxyz, 0)

$\Rightarrow$ [(x, 0) (x, 0)] [(y, 0) (z, 0)] = {(x, 0) [(x, 0) (y, 0)]} (z, 0)

Case 2:

i. [(x, 0) (x, 0)] [(y, 1) (z, 0)] = (xx, 0) (yz*, 1) = (yz*xx, 1)
ii. {(x, 0) [(x, 0) (y, 1)] } (z, 0) = [(x, 0) (yx, 1)] (z; 0) = (yxx, 1) (z, 0) = (yxxz*, 1) = (yz*xx, 1)

$\Rightarrow$ [(x, 0) (x, 0)][(y, 1)(z, 0)] = { (x, 0) [(x, 0) (y, 1)] } (z, 0)

Example.

- $(b_5 b_5)(b_{10} b_3) = -b_0(-b_9) = b_9$
- $(b_5 (b_5 b_{10})) b_3 = (b_5(b_{15})) b_3 = -b_{10} (b_3) = b_9$

$\Rightarrow (b_5 b_5) (b_{10} b_3) = (b_5 (b_5 b_{10})) b_3$

Case 3:

i. [(x, 0)(x, 0)][(y, 0)(z, 1)] = (xx, 0) (zy, 1) = (zyxx, 1)
ii. {(x, 0)[(x, 0)(y, 0)]} (z, 1) = [(x, 0)(xy, 0)](z, 1) = (xxy, 0)(z, 1) = (zxxy, 1) = (zyxx, 1)

$\Rightarrow$ [(x, 0) (x, 0)] [(y, 0) (z, 1)] = {(x, 0) [(x, 0) (y, 0)] } (z, 1).

Example.

- $(b_6 b_6) (b_3 b_{11}) = -b_0 (-b_8) = b_8$
- $(b_6 (b_6 b_3)) b_{11} = (b_6 (-b_5)) b_{11} = -b_3(b_{11}) = b_8$

$\Rightarrow (b_6 b_6) (b_3 b_{11}) = (b_6 (b_6 b_3)) b_{11}$

Case 4:

i. [(x, 0) (x, 0)][(y, 1)(z, 1)] = (xx, 0) (-z*y, 0) = (-xxz*y, 0)
ii. {(x, 0) [(x, 0)(y, 1)] }(z, 1) = [(x, 0) (yx, 1)] (z, 1) = (yxx, 1) (z, 1) = (-z*yxx, 0) = (-xxz*y; 0)

$\Rightarrow$ [(x, 0)(x, 0)] [(y, 1)(z, 1)] = { (x, 0)[(x, 0)(y, 1)] }(z, 1)

Example.

- $(b_4 b_4)(b_{14} b_{11}) = -b_0(b_5) = -b_5$
- $(b_4 (b_4 b_{14})) b_{11} = (b_4(-b_{10})) b_{11} = -b_{14}(b_{11}) = -b_5$

$\Rightarrow (b_4 b_4) (b_{14} b_{11}) = (b_4 (b_4 b_{14})) b_{11}$

Case 5:

i. [(x, 1) (x, 1)] [(y, 0)(z, 0)] = (-x*x, 0) (yz, 0) = (-x*xyz, 0)
ii. {(x, 1) [(x, 1) (y, 0)]} (z, 0) = [(x, 1) (xy*, 1)] (z; 0) = (-yx*x, 0) (z; 0) = (-yx*xz, 0) = (-x*xyz, 0)

$\Rightarrow$ [(x, 1) (x, 1)][(y, 0)(z, 0)] = { (x, 1)[(x, 1)(y, 0)] }(z, 0)

Example.

- $(b_{13}b_{13})(b_4b_7) = -b_0(b_3) = -b_3$
- $(b_{13}(b_{14}b_4))\, b_7 = (b_{13}(b_9))b_7 = -b_4(b_7) = -b_3$

$\Rightarrow (b_{13}b_{13})\,(b_4b_7) = (b_{13}(b_{13}b_4))\, b_7$

Case 6:

i. [(x, 1) (x, 1)][(y, 1)(z, 0)] = (-x*x, 0) (yz*, 1) = (-yz*x*x, 1)

ii. { (x, 1)[(x, 1)(y, 1)]} (z, 0) = [(x, 1) (-yx, 0)](z, 0) = (-xx*y, 1)(z, 0) = (-xx*yz*, 1) =

(-yz*x*x, 1)

⇒ [(x, 1) (x, 1)][(y, 1)(z, 0)] = { (x, 1) [(x, 1)(y, 1)]} (z, 0)

Example.

- $(b_{11}b_{11})(b_{15}b_3) = -b_0(-b_{12}) = b_{12}$
- $(b_{11}(b_{11}b_{15}))b_3 = (b_{11}(b_4))b_3 = -b_{15}(b_3) = b_{12}$

$\Rightarrow (b_{11}b_{11})\,(b_{15}b_3) = (b_{11}(b_{11}b_{15}))\, b_3$

Case 7:

i. [(x, 1)(x, 1)][(y, 0)(z, 1)] = (-x*x, 0)(zy, 1) = (-zyx*x, 1)

ii. {(x, 1)[(x, 1)(y, 0)] }(z, 1) = [(x, 1)(xy*, 1)](z; 1) = (-yx*x, 0)(z, 1) = (-zyx*x, 1)

⇒ [(x, 1)(x, 1)][(y, 0)(z, 1)] = {(x, 1)[(x, 1)(y, 0)] }(z, 1)

Example.

- $(b_{10}b_{10})(b_2b_{14}) = -b_0(b_{12}) = -b_{12}$
- $(b_{10}(b_{10}b_2))b_{14} = (b_{10}(b_8))b_{14} = -b_2(b_{14}) = -b_{12}$

$\Rightarrow (b_{10}b_{10})\,(b_2b_{14}) = (b_{10}\,(b_{10}b_2))\, b_{14}$

Case 8:

i. [(x, 1)(x, 1)][(y, 1)(z, 1)] = (-x*x, 0)(-z*y, 0) = (x*xz*y, 0)

ii. {(x, 1)[(x, 1)(y, 1)] }(z, 1) = [(x, 1)(-y*x, 0)](z, 1) = (-xx*y, 1)(z, 1) = (z*xx*y, 0) =

(xx*z*y, 0)

⇒ [(x, 1)(x, 1)][(y, 1)(z, 1)] = {(x, 1)[(x, 1)(y, 1)] }(z, 1)

Example.

- $(b_{14}b_{14})(b_9b_{11}) = -b_0(b_2) = -b_2$
- $(b_{14}(b_{14}b_9))b_{11} = (b_{14}(-b_7))b_{11} = -b_9(b_{11}) = -b_2$

$\Rightarrow (b_{14}b_{14})\,(b_9b_{11}) = (b_{14}(b_{14}b_9))\, b_{11}$

Conclusion: The left C-loop identity holds.

(b) The right C-loop identity, x((yz)z) = (xy)(zz)

Case 1: (trivial)

i. (x, 0) { [(y, 0) (z, 0)](z, 0) }=(x, 0)[(yz, 0)(z, 0)] = (x, 0)(yzz, 0) = (xyzz, 0)

ii. [(x, 0) (y, 0)] [(z, 0)(z, 0)]=(xy, 0)(zz, 0) = (xyzz, 0)

⇒ (x, 0) { [(y, 0) (z, 0)](z, 0) } = [(x, 0)(y, 0)][(z, 0)(z, 0)]

Case 2:

i. (x, 0) { [(y, 1)(z, 0)](z, 0) }=(x, 0)[(yz*, 1)(z, 0)] = (x, 0)(yz*z*x, 1) = (yxz*z*, 1)

ii. [(x, 0) (y, 1)][(z, 0)(z, 0)]=(yx, 1)(zz, 0) = (yxz*z*, 1)

⇒ (x, 0) { [(y, 1) (z, 0)](z, 0) }= [(x, 0)(y, 1)][(z, 0)(z, 0)]

Example.

- $b_2((b_{13}b_6)b_6) = b_2(-b_{10}b_1) = b_2(-b_{12}) = b_{15}$
- $(b_2b_{13})(b_6b_6) = -b_{15}(-b_0) = b_{15}$

$\Rightarrow b_2\,((b_{13}b_6)\, b_6) = (b_2b_{13})\,(b_6b_6)$

Case 3:

i. (x, 1) { [(y, 0)(z, 0)](z, 0) }=(x, 1)[(yz, 0)(z, 0)] = (x, 1)(yzz, 0) = (xy*z*z*, 1)

ii. [(x, 1)(y, 0)][(z, 0)(z, 0)]=(xy*, 1)(zz, 0) = (xy*z*z*, 1)

⇒ (x, 1) { [(y, 0) (z, 0)](z, 0) }= [(x, 1)(y, 0)][(z, 0)(z, 0)]

Example.

- $b_{10}((b_4b_6)b_6) = b_{10}(b_2b_6) = b_{10}(-b_4) = b_{14}$
- $(b_{10}b_4)(b_6b_6) = -b_{14}(-b_0) = b_{14}$

$\Rightarrow b_{10}\,((b_4b_6)\, b_6) = (b_{10}b_4)\,(b_6b_6)$

Case 4:

i. (x, 1){ [(y, 1)(z, 0)](z, 0) }=(x, 1)[(yz*, 1)(z, 0)] = (x, 1)(yz*z*, 1) = (-zzy*x, 0) = (-y*xzz, 0)

ii. [(x, 1) (y, 1)][(z, 0)(z, 0)]=(-y*x, 0)(zz, 0) = (-y*xzz, 0)

⇒ (x, 1) { [(y, 1) (z, 0)](z, 0) }= [(x, 1)(y, 1)][(z, 0)(z, 0)]

Example.

- $b_8((b_{13}b_5)b_5) = b_8(b_8b_5) = b_8(-b_{13}) = -b_5$
- $(b_8b_{13})(b_5b_5) = b_5(-b_0) = -b_5$

$\Rightarrow b_8\,((b_{13}b_5)\, b_5) = (b_8b_{13})\,(b_5b_5)$

Case 5:

i. (x, 0) { [(y, 0)(z, 1)](z, 1) }=(x, 0)[(zy, 1)(z, 1)] = (x, 0)(-z*zy, 0) = (-xz*zy, 0) = (-xyz*z, 0)

ii. [(x, 0) (y, 0)][(z, 1)(z, 1)]=(xy, 0)(-z*z, 0) = (-xyz*z, 0)

⇒ (x, 0) { [(y, 0)(z, 1)](z, 1) }= [(x, 0)(y, 0)][(z, 1)(z, 1)]

Example.

- $b_7((b_3b_{15})b_{15}) = b_7(b_{12}b_{15}) = b_7(-b_3) = -b_4$
- $(b_7b_3)(b_{15}b_{15}) = b_4(-b_0) = -b_4$

$\Rightarrow b_7\,((b_3b_{15})\, b_{15}) = (b_7b_3)\,(b_{15}b_{15})$

Case 6:

i. (x, 1) { [(y, 0)(z, 1)](z, 1) }= (x, 1)[(zy, 1)(z, 1)] = (x, 1)(-z*zy, 0) = (-xy*z*z, 1)

ii. [(x, 1) (y, 0)][(z, 1)(z, 1)]=(xy*, 1)(z*z, 0) = (-xy*z*z, 1)

$\Rightarrow$ (x, 1) { [(y, 0)(z, 1)](z, 1) } = [(x, 1)(y, 0)][(z, 1)(z, 1)]

Example.

- $b_{12}((b_3b_{14})b_{14}) = b_{12}(-b_{13}b_{14}) = b_{12}(-b_3) = -b_{14}$
- $(b_{12}b_3)(b_{14}b_{14}) = b_{14}(-b_0) = -b_{14}$

$\Rightarrow b_{12}\ ((b_3b_{14})\ b_{14}) = (b_{12}b_3)\ (b_{14}b_{14})$

Case 7:

i. (x, 0) { [(y, 1)(z, 1)](z, 1) }= (x, 0)[(-z*y, 0)(z, 1)] = (x, 0)(-zz*y, 1) = (-yxzz*, 1)

ii. [(x, 0)(y, 1)][(z, 1)(z, 1)]=(yx, 1)(-z*z, 0) = (-yxzz*, 1)

$\Rightarrow$ (x, 0) { [(y, 1)(z, 1)](z, 1) } = [(x, 0)(y, 1)][(z, 1)(z, 1)]

Example.

- $b_4((b_{15}b_{10})b_{10}) = b_4(-b_5b_{10}) = b_4(-b_{15}) = b_{11}$
- $(b_4b_{15})(b_{10}b_{10}) = -b_{11}(-b_0) = b_{11}$

$\Rightarrow b_4\ ((b_{15}b_{10})\ b_{10}) = (b_4b_{15})\ (b_{10}b_{10})$

Case 8:

i. (x, 1) { [(y, 1)(z, 1)](z, 1) }=(x, 1)[(-z*y, 0)(z, 1)] = (x, 1)(-z*zy, 1) = (yz*zx, 0) = (y*xz*z, 0)

ii. [(x, 1)(y;,1)][(z, 1)(z, 1)]=(-y*x, 0)(-z*z, 0) = (y*xz*z, 0)

$\Rightarrow$ (x, 1) { [(y, 1)(z, 1)](z, 1) } = [(x, 1)(y;,1)][(z, 1)(z, 1)]

Example.

- $b_{14}((b_{13}b_{12})b_{12}) = b_{14}(b_1b_{12}) = b_{14}(-b_{13}) = b_3$
- $(b_{14}b_{13})(b_{12}b_{12}) = -b_3(-b_0) = b_3$

$\Rightarrow b_{14}\ ((b_{13}b_{12})\ b_{12}) = (b_{14}b_{13})\ (b_{12}b_{12})$

Conclusion: The right C-loop identity holds.

IV. SUMMARY

In this section, we were testing the satisfaction of C-loop identities by sedenions. We have observed that sedenions satisfy both the left and right C-loop identities.

Split extension of sedenions

In this section we are going to construct the multiplication tables of the split extension of sedenions and determine whether they satisfy the c-loop identities.

Let the basis elements in $\boldsymbol{S} \times S^0$ be as follows (Magero, 2007 and Jenya, 2012). We obtain the following structures as the split extension of sedenions.

$\delta_0 = (b_0, 0)$, $\delta_1 = (b_1, 0)$, $\delta_2 = (b_2, 0)$, $\delta_3 = (b_3, 0)$, $\delta_4 = (b_4, 0)$, $\delta_5 = (b_5, 0)$, $\delta_6 = (b_6, 0)$, $\delta_7 = (b_7, 0)$, $\delta_8 = (b_8, 0)$, $\delta_9 = (b_9, 0)$, $\delta_{10} = (b_{10}, 0)$, $\delta_{11} = (b_{11}, 0)$, $\delta_{12} = (b_{12}, 0)$, $\delta_{13} = (b_{13}, 0)$, $\delta_{14} = (b_{14}, 0)$, $\delta_{15} = (b_{15}, 0)$, $\delta_{16} = (-b_0, 1)$, $\delta_{17} = (-b_1, 1)$, $\delta_{18} = (-b_2, 1)$, $\delta_{19} = (-b_3, 1)$, $\delta_{20} = (-b_4, 1)$, $\delta_{21=} (b_5, 1)$, $\delta_{22} = (-b_6, 1)$, $\delta_{23} = (-b_7, 1)$, $\delta_{24} = (-b_8, 1)$, $\delta_{25} = (-b_9, 1)$, $\delta_{26} = (-b_{10}, 1)$, $\delta_{27} = (-b_{11}, 1)$, $\delta_{28} = (-b_{12}, 1)$, $\delta_{29} = (-b_{13}, 1)$, $\delta_{30} = (-b_{14}, 1)$, $\delta_{31} = (-b_{15}, 1)$, $\delta_{32} = (-b_0, 0)$, $\delta_{33} = (-b_1, 0)$, $\delta_{34} = (-b_2, 0)$; $\delta_{35} = (-b_3, 0)$, $\delta_{36} = (-b_4, 0)$, $\delta_{37} = (-b_5, 0)$, $\delta_{38} = (-b_6, 0)$, $\delta_{39} = (-b_7, 0)$, $\delta_{40} = (-b_8, 0)$, $\delta_{41} = (-b_9, 0)$, $\delta_{42} = (-b_{10}, 0)$, $\delta_{43} = (-b_{11}, 0)$, $\delta_{44} = (-b_{12}, 0)$, $\delta_{45} = (-b_{13}, 0)$, $\delta_{46} = (-b_{14}, 0)$, $\delta_{47} = (-b_{15}, 0)$, $\delta_{48} = (b_0, 1)$, $\delta_{49} = (b_1, 1)$, $\delta_{50} = (b_2, 1)$, $\delta_{51} = (b_3, 1)$, $\delta_{52} = (b_4, 1)$, $\delta_{53} = (b_5, 1)$, $\delta_{54} = (b_6, 1)$, $\delta_{55} = (b_7, 1)$, $\delta_{56} = (b_8, 1)$, $\delta_{57} = (b_9, 1)$, $\delta_{58} = (b_{10}, 1)$, $\delta_{59} = (b_{11}, 1)$, $\delta_{60} = (b_{12}, 1)$, $\delta_{61} = (b_{13}, 1)$, $\delta_{62} = (b_{14}, 1)$, $\delta_{63} = (b_{15}, 1)$. The frame multiplication table is as shown below.

Table2: Multiplication of the split extension of sedenions.

·	δ_0	δ_1	δ_2	δ_3	δ_4	δ_5	δ_6	δ_7	δ_8	δ_9	δ_{10}	δ_{11}	δ_{12}	δ_{13}	δ_{14}	δ_{15}
δ_0	δ_0	δ_1	δ_2	δ_3	δ_4	δ_5	δ_6	δ_7	δ_8	δ_9	δ_{10}	δ_{11}	δ_{12}	δ_{13}	δ_{14}	δ_{15}
δ_1	δ_1	δ_{32}	δ_3	δ_{34}	δ_5	δ_{36}	δ_{39}	δ_6	δ_9	δ_{40}	δ_{43}	δ_{10}	δ_{45}	δ_{12}	δ_{15}	δ_{46}
δ_2	δ_2	δ_{35}	δ_{32}	δ_1	δ_6	δ_7	δ_{36}	δ_{37}	δ_{10}	δ_{11}	δ_{40}	δ_{41}	δ_{46}	δ_{47}	δ_{12}	δ_{13}
δ_3	δ_3	δ_2	δ_{33}	δ_{32}	δ_7	δ_{38}	δ_5	δ_{36}	δ_{11}	δ_{42}	δ_9	δ_{40}	δ_{47}	δ_{14}	δ_{45}	δ_{12}
δ_4	δ_4	δ_{37}	δ_{38}	δ_{39}	δ_{32}	δ_1	δ_2	δ_3	δ_{12}	δ_{13}	δ_{14}	δ_{15}	δ_{40}	δ_{41}	δ_{42}	δ_{43}
δ_5	δ_5	δ_4	δ_{39}	δ_6	δ_{33}	δ_{32}	δ_{35}	δ_2	δ_{13}	δ_{44}	δ_{15}	δ_{46}	δ_9	δ_{40}	δ_{43}	δ_{42}
δ_6	δ_6	δ_7	δ_4	δ_{37}	δ_{34}	δ_3	δ_{32}	δ_{33}	δ_{14}	δ_{47}	δ_{44}	δ_{13}	δ_{10}	δ_{43}	δ_{40}	δ_9
δ_7	δ_7	δ_{38}	δ_5	δ_4	δ_{35}	δ_{34}	δ_1	δ_{32}	δ_{15}	δ_{14}	δ_{45}	δ_{44}	δ_{11}	δ_{10}	δ_{41}	δ_{40}
δ_8	δ_8	δ_{41}	δ_{42}	δ_{43}	δ_{44}	δ_{45}	δ_{46}	δ_{47}	δ_{32}	δ_1	δ_2	δ_3	δ_4	δ_5	δ_6	δ_7
δ_9	δ_9	δ_8	δ_{43}	δ_{10}	δ_{45}	δ_{12}	δ_{15}	δ_{46}	δ_{33}	δ_{32}	δ_{35}	δ_2	δ_{37}	δ_4	δ_7	δ_{38}
δ_{10}	δ_{10}	δ_{11}	δ_8	δ_{41}	δ_{46}	δ_{47}	δ_{12}	δ_{13}	δ_{34}	δ_3	δ_{32}	δ_{33}	δ_{38}	δ_{39}	δ_4	δ_5
δ_{11}	δ_{11}	δ_{42}	δ_9	δ_8	δ_{47}	δ_{14}	δ_{45}	δ_{12}	δ_{35}	δ_{34}	δ_1	δ_{32}	δ_{39}	δ_6	δ_{37}	δ_4
δ_{12}	δ_{12}	δ_{13}	δ_{14}	δ_{15}	δ_8	δ_{41}	δ_{42}	δ_{43}	δ_{36}	δ_5	δ_6	δ_7	δ_{32}	δ_{33}	δ_{34}	δ_{35}

δ_{13}	δ_{13}	δ_{44}	δ_{15}	δ_{46}	δ_{9}	δ_{8}	δ_{11}	δ_{42}	δ_{37}	δ_{36}	δ_{7}	δ_{38}	δ_{1}	δ_{32}	δ_{3}	δ_{34}
δ_{14}	δ_{14}	δ_{47}	δ_{44}	δ_{13}	δ_{10}	δ_{11}	δ_{8}	δ_{9}	δ_{38}	δ_{39}	δ_{36}	δ_{5}	δ_{2}	δ_{35}	δ_{32}	δ_{1}
δ_{15}	δ_{15}	δ_{14}	δ_{45}	δ_{44}	δ_{11}	δ_{10}	δ_{41}	δ_{8}	δ_{39}	δ_{6}	δ_{37}	δ_{36}	δ_{3}	δ_{2}	δ_{33}	δ_{32}
δ_{16}	δ_{16}	δ_{49}	δ_{50}	δ_{51}	δ_{52}	δ_{53}	δ_{54}	δ_{55}	δ_{56}	δ_{57}	δ_{58}	δ_{59}	δ_{60}	δ_{61}	δ_{62}	δ_{63}
δ_{17}	δ_{17}	δ_{16}	δ_{51}	δ_{18}	δ_{53}	δ_{20}	δ_{23}	δ_{54}	δ_{57}	δ_{24}	δ_{27}	δ_{58}	δ_{29}	δ_{60}	δ_{63}	δ_{30}
δ_{18}	δ_{18}	δ_{19}	δ_{16}	δ_{49}	δ_{54}	δ_{55}	δ_{20}	δ_{21}	δ_{58}	δ_{59}	δ_{24}	δ_{25}	δ_{30}	δ_{31}	δ_{60}	δ_{61}
δ_{19}	δ_{19}	δ_{50}	δ_{47}	δ_{16}	δ_{55}	δ_{22}	δ_{53}	δ_{20}	δ_{59}	δ_{26}	δ_{57}	δ_{24}	δ_{31}	δ_{62}	δ_{29}	δ_{60}
δ_{20}	δ_{20}	δ_{21}	δ_{22}	δ_{23}	δ_{16}	δ_{49}	δ_{50}	δ_{51}	δ_{60}	δ_{61}	δ_{62}	δ_{63}	δ_{24}	δ_{25}	δ_{26}	δ_{27}
δ_{21}	δ_{21}	δ_{52}	δ_{23}	δ_{54}	δ_{17}	δ_{16}	δ_{19}	δ_{50}	δ_{61}	δ_{28}	δ_{63}	δ_{30}	δ_{57}	δ_{24}	δ_{27}	δ_{26}
δ_{22}	δ_{22}	δ_{55}	δ_{52}	δ_{21}	δ_{18}	δ_{51}	δ_{16}	δ_{17}	δ_{62}	δ_{31}	δ_{28}	δ_{61}	δ_{58}	δ_{27}	δ_{24}	δ_{57}
δ_{23}	δ_{23}	δ_{22}	δ_{53}	δ_{52}	δ_{19}	δ_{18}	δ_{49}	δ_{16}	δ_{63}	δ_{62}	δ_{29}	δ_{28}	δ_{59}	δ_{58}	δ_{25}	δ_{24}
δ_{24}	δ_{24}	δ_{25}	δ_{26}	δ_{27}	δ_{28}	δ_{29}	δ_{30}	δ_{31}	δ_{16}	δ_{49}	δ_{50}	δ_{51}	δ_{52}	δ_{53}	δ_{54}	δ_{55}
δ_{25}	δ_{25}	δ_{56}	δ_{27}	δ_{58}	δ_{29}	δ_{60}	δ_{63}	δ_{30}	δ_{17}	δ_{16}	δ_{19}	δ_{50}	δ_{21}	δ_{52}	δ_{55}	δ_{22}
δ_{26}	δ_{26}	δ_{59}	δ_{56}	δ_{25}	δ_{30}	δ_{31}	δ_{60}	δ_{61}	δ_{18}	δ_{51}	δ_{16}	δ_{17}	δ_{22}	δ_{23}	δ_{52}	δ_{53}
δ_{27}	δ_{27}	δ_{26}	δ_{57}	δ_{56}	δ_{31}	δ_{62}	δ_{29}	δ_{60}	δ_{19}	δ_{18}	δ_{49}	δ_{16}	δ_{23}	δ_{54}	δ_{21}	δ_{52}
δ_{28}	δ_{28}	δ_{61}	δ_{62}	δ_{63}	δ_{56}	δ_{25}	δ_{26}	δ_{27}	δ_{20}	δ_{53}	δ_{54}	δ_{55}	δ_{16}	δ_{17}	δ_{18}	δ_{19}
δ_{29}	δ_{29}	δ_{28}	δ_{63}	δ_{30}	δ_{57}	δ_{56}	δ_{59}	δ_{26}	δ_{21}	δ_{20}	δ_{55}	δ_{22}	δ_{49}	δ_{16}	δ_{51}	δ_{18}
δ_{30}	δ_{30}	δ_{31}	δ_{28}	δ_{61}	δ_{58}	δ_{59}	δ_{56}	δ_{57}	δ_{22}	δ_{23}	δ_{20}	δ_{53}	δ_{50}	δ_{19}	δ_{16}	δ_{49}
δ_{31}	δ_{31}	δ_{62}	δ_{29}	δ_{28}	δ_{59}	δ_{58}	δ_{25}	δ_{26}	δ_{23}	δ_{54}	δ_{21}	δ_{20}	δ_{51}	δ_{50}	δ_{17}	δ_{16}

•	δ_{0}	δ_{1}	δ_{2}	δ_{3}	δ_{4}	δ_{5}	δ_{6}	δ_{7}	δ_{8}	δ_{9}	δ_{10}	δ_{11}	δ_{12}	δ_{13}	δ_{14}	δ_{15}
δ_{32}	δ_{32}	δ_{33}	δ_{34}	δ_{35}	δ_{36}	δ_{37}	δ_{38}	δ_{39}	δ_{40}	δ_{41}	δ_{42}	δ_{43}	δ_{44}	δ_{45}	δ_{46}	δ_{47}
δ_{33}	δ_{33}	δ_{0}	δ_{35}	δ_{2}	δ_{37}	δ_{4}	δ_{7}	δ_{38}	δ_{41}	δ_{8}	δ_{11}	δ_{42}	δ_{13}	δ_{44}	δ_{47}	δ_{14}
δ_{34}	δ_{34}	δ_{3}	δ_{0}	δ_{33}	δ_{38}	δ_{39}	δ_{4}	δ_{5}	δ_{42}	δ_{43}	δ_{8}	δ_{9}	δ_{14}	δ_{15}	δ_{44}	δ_{45}
δ_{35}	δ_{35}	δ_{34}	δ_{1}	δ_{0}	δ_{39}	δ_{6}	δ_{37}	δ_{4}	δ_{43}	δ_{10}	δ_{41}	δ_{8}	δ_{15}	δ_{46}	δ_{13}	δ_{44}
δ_{36}	δ_{36}	δ_{5}	δ_{6}	δ_{7}	δ_{0}	δ_{33}	δ_{34}	δ_{35}	δ_{44}	δ_{45}	δ_{46}	δ_{47}	δ_{8}	δ_{9}	δ_{10}	δ_{11}
δ_{37}	δ_{37}	δ_{36}	δ_{7}	δ_{38}	δ_{1}	δ_{0}	δ_{3}	δ_{34}	δ_{45}	δ_{12}	δ_{47}	δ_{14}	δ_{41}	δ_{8}	δ_{11}	δ_{10}
δ_{38}	δ_{38}	δ_{39}	δ_{36}	δ_{5}	δ_{2}	δ_{35}	δ_{0}	δ_{1}	δ_{46}	δ_{15}	δ_{12}	δ_{45}	δ_{42}	δ_{11}	δ_{8}	δ_{41}
δ_{39}	δ_{39}	δ_{6}	δ_{37}	δ_{36}	δ_{3}	δ_{2}	δ_{33}	δ_{0}	δ_{47}	δ_{46}	δ_{13}	δ_{12}	δ_{43}	δ_{42}	δ_{9}	δ_{8}
δ_{40}	δ_{40}	δ_{9}	δ_{10}	δ_{11}	δ_{12}	δ_{13}	δ_{14}	δ_{15}	δ_{0}	δ_{33}	δ_{34}	δ_{35}	δ_{36}	δ_{37}	δ_{38}	δ_{39}
δ_{41}	δ_{41}	δ_{40}	δ_{11}	δ_{42}	δ_{13}	δ_{44}	δ_{47}	δ_{14}	δ_{1}	δ_{0}	δ_{3}	δ_{34}	δ_{5}	δ_{36}	δ_{39}	δ_{6}
δ_{42}	δ_{42}	δ_{43}	δ_{40}	δ_{9}	δ_{14}	δ_{15}	δ_{44}	δ_{45}	δ_{2}	δ_{35}	δ_{0}	δ_{1}	δ_{6}	δ_{7}	δ_{36}	δ_{37}
δ_{43}	δ_{43}	δ_{10}	δ_{41}	δ_{40}	δ_{15}	δ_{46}	δ_{13}	δ_{44}	δ_{3}	δ_{2}	δ_{33}	δ_{0}	δ_{7}	δ_{38}	δ_{5}	δ_{36}
δ_{44}	δ_{44}	δ_{45}	δ_{46}	δ_{47}	δ_{40}	δ_{9}	δ_{10}	δ_{11}	δ_{4}	δ_{37}	δ_{38}	δ_{39}	δ_{0}	δ_{1}	δ_{2}	δ_{3}
δ_{45}	δ_{45}	δ_{12}	δ_{47}	δ_{14}	δ_{41}	δ_{40}	δ_{43}	δ_{10}	δ_{5}	δ_{4}	δ_{39}	δ_{6}	δ_{33}	δ_{0}	δ_{35}	δ_{2}
δ_{46}	δ_{46}	δ_{15}	δ_{12}	δ_{45}	δ_{42}	δ_{43}	δ_{40}	δ_{41}	δ_{6}	δ_{7}	δ_{4}	δ_{37}	δ_{34}	δ_{3}	δ_{0}	δ_{33}

δ_{47}	δ_{47}	δ_{46}	δ_{13}	δ_{12}	δ_{43}	δ_{42}	δ_{9}	δ_{40}	δ_{7}	δ_{38}	δ_{5}	δ_{4}	δ_{35}	δ_{34}	δ_{1}	δ_{0}
δ_{48}	δ_{48}	δ_{17}	δ_{18}	δ_{19}	δ_{20}	δ_{21}	δ_{22}	δ_{23}	δ_{24}	δ_{25}	δ_{26}	δ_{27}	δ_{28}	δ_{29}	δ_{30}	δ_{31}
δ_{49}	δ_{49}	δ_{48}	δ_{19}	δ_{50}	δ_{21}	δ_{52}	δ_{55}	δ_{22}	δ_{25}	δ_{56}	δ_{59}	δ_{26}	δ_{61}	δ_{28}	δ_{31}	δ_{62}
δ_{50}	δ_{50}	δ_{51}	δ_{48}	δ_{17}	δ_{22}	δ_{23}	δ_{52}	δ_{53}	δ_{26}	δ_{27}	δ_{56}	δ_{57}	δ_{62}	δ_{63}	δ_{28}	δ_{29}
δ_{51}	δ_{51}	δ_{18}	δ_{49}	δ_{48}	δ_{23}	δ_{54}	δ_{21}	δ_{52}	δ_{27}	δ_{58}	δ_{25}	δ_{56}	δ_{63}	δ_{30}	δ_{61}	δ_{28}
δ_{52}	δ_{52}	δ_{53}	δ_{54}	δ_{55}	δ_{48}	δ_{17}	δ_{18}	δ_{19}	δ_{28}	δ_{29}	δ_{30}	δ_{31}	δ_{56}	δ_{57}	δ_{58}	δ_{59}
δ_{53}	δ_{53}	δ_{20}	δ_{55}	δ_{22}	δ_{49}	δ_{48}	δ_{51}	δ_{18}	δ_{29}	δ_{60}	δ_{31}	δ_{62}	δ_{25}	δ_{56}	δ_{59}	δ_{58}
δ_{54}	δ_{54}	δ_{23}	δ_{20}	δ_{53}	δ_{50}	δ_{19}	δ_{48}	δ_{49}	δ_{30}	δ_{63}	δ_{60}	δ_{29}	δ_{26}	δ_{59}	δ_{56}	δ_{25}
δ_{55}	δ_{55}	δ_{54}	δ_{21}	δ_{20}	δ_{51}	δ_{50}	δ_{17}	δ_{48}	δ_{31}	δ_{30}	δ_{61}	δ_{60}	δ_{27}	δ_{26}	δ_{57}	δ_{56}
δ_{56}	δ_{56}	δ_{57}	δ_{58}	δ_{59}	δ_{60}	δ_{61}	δ_{62}	δ_{63}	δ_{48}	δ_{17}	δ_{18}	δ_{19}	δ_{20}	δ_{21}	δ_{22}	δ_{23}
δ_{57}	δ_{57}	δ_{24}	δ_{59}	δ_{26}	δ_{61}	δ_{28}	δ_{31}	δ_{62}	δ_{49}	δ_{48}	δ_{51}	δ_{18}	δ_{53}	δ_{20}	δ_{23}	δ_{54}
δ_{58}	δ_{58}	δ_{27}	δ_{24}	δ_{57}	δ_{62}	δ_{63}	δ_{28}	δ_{29}	δ_{50}	δ_{19}	δ_{48}	δ_{49}	δ_{54}	δ_{55}	δ_{20}	δ_{21}
δ_{59}	δ_{59}	δ_{58}	δ_{25}	δ_{24}	δ_{63}	δ_{30}	δ_{61}	δ_{28}	δ_{51}	δ_{50}	δ_{17}	δ_{48}	δ_{55}	δ_{22}	δ_{53}	δ_{20}
δ_{60}	δ_{60}	δ_{29}	δ_{30}	δ_{31}	δ_{24}	δ_{57}	δ_{58}	δ_{59}	δ_{52}	δ_{21}	δ_{22}	δ_{23}	δ_{48}	δ_{49}	δ_{50}	δ_{51}
δ_{61}	δ_{61}	δ_{60}	δ_{31}	δ_{62}	δ_{25}	δ_{24}	δ_{27}	δ_{58}	δ_{53}	δ_{52}	δ_{23}	δ_{54}	δ_{17}	δ_{48}	δ_{19}	δ_{50}
δ_{62}	δ_{62}	δ_{63}	δ_{60}	δ_{29}	δ_{26}	δ_{27}	δ_{24}	δ_{25}	δ_{54}	δ_{55}	δ_{52}	δ_{21}	δ_{18}	δ_{51}	δ_{48}	δ_{17}
δ_{63}	δ_{63}	δ_{30}	δ_{61}	δ_{60}	δ_{27}	δ_{26}	δ_{57}	δ_{24}	δ_{55}	δ_{22}	δ_{53}	δ_{52}	δ_{19}	δ_{18}	δ_{49}	δ_{48}

$\cdot$	δ_{16}	δ_{17}	δ_{18}	δ_{19}	δ_{20}	δ_{21}	δ_{22}	δ_{23}	δ_{24}	δ_{25}	δ_{26}	δ_{27}	δ_{28}	δ_{29}	δ_{30}	δ_{31}
δ_{0}	δ_{16}	δ_{17}	δ_{18}	δ_{19}	δ_{20}	δ_{21}	δ_{22}	δ_{23}	δ_{24}	δ_{25}	δ_{26}	δ_{27}	δ_{28}	δ_{29}	δ_{30}	δ_{31}
δ_{1}	δ_{17}	δ_{48}	δ_{51}	δ_{18}	δ_{53}	δ_{20}	δ_{23}	δ_{54}	δ_{57}	δ_{24}	δ_{27}	δ_{58}	δ_{29}	δ_{60}	δ_{63}	δ_{30}
δ_{2}	δ_{18}	δ_{19}	δ_{48}	δ_{49}	δ_{54}	δ_{55}	δ_{20}	δ_{21}	δ_{58}	δ_{59}	δ_{24}	δ_{25}	δ_{30}	δ_{31}	δ_{60}	δ_{61}
δ_{3}	δ_{19}	δ_{50}	δ_{17}	δ_{48}	δ_{55}	δ_{22}	δ_{53}	δ_{20}	δ_{59}	δ_{26}	δ_{57}	δ_{24}	δ_{31}	δ_{62}	δ_{29}	δ_{60}
δ_{4}	δ_{20}	δ_{21}	δ_{22}	δ_{23}	δ_{48}	δ_{49}	δ_{50}	δ_{51}	δ_{60}	δ_{61}	δ_{62}	δ_{63}	δ_{24}	δ_{25}	δ_{26}	δ_{27}
δ_{5}	δ_{21}	δ_{52}	δ_{23}	δ_{54}	δ_{17}	δ_{48}	δ_{19}	δ_{50}	δ_{61}	δ_{28}	δ_{63}	δ_{30}	δ_{57}	δ_{24}	δ_{27}	δ_{26}
δ_{6}	δ_{22}	δ_{55}	δ_{52}	δ_{21}	δ_{18}	δ_{51}	δ_{48}	δ_{17}	δ_{62}	δ_{31}	δ_{28}	δ_{61}	δ_{58}	δ_{27}	δ_{24}	δ_{57}
δ_{7}	δ_{23}	δ_{22}	δ_{53}	δ_{52}	δ_{19}	δ_{18}	δ_{49}	δ_{48}	δ_{63}	δ_{62}	δ_{29}	δ_{28}	δ_{59}	δ_{58}	δ_{25}	δ_{24}
δ_{8}	δ_{24}	δ_{25}	δ_{26}	δ_{27}	δ_{28}	δ_{29}	δ_{30}	δ_{31}	δ_{48}	δ_{49}	δ_{50}	δ_{51}	δ_{52}	δ_{53}	δ_{54}	δ_{55}
δ_{9}	δ_{25}	δ_{56}	δ_{27}	δ_{58}	δ_{29}	δ_{60}	δ_{63}	δ_{30}	δ_{17}	δ_{48}	δ_{19}	δ_{50}	δ_{21}	δ_{52}	δ_{55}	δ_{22}
δ_{10}	δ_{26}	δ_{59}	δ_{56}	δ_{25}	δ_{30}	δ_{31}	δ_{60}	δ_{61}	δ_{18}	δ_{51}	δ_{48}	δ_{17}	δ_{22}	δ_{23}	δ_{52}	δ_{53}
δ_{11}	δ_{27}	δ_{26}	δ_{57}	δ_{56}	δ_{31}	δ_{62}	δ_{29}	δ_{60}	δ_{19}	δ_{18}	δ_{49}	δ_{48}	δ_{23}	δ_{54}	δ_{21}	δ_{52}
δ_{12}	δ_{28}	δ_{61}	δ_{62}	δ_{63}	δ_{56}	δ_{25}	δ_{26}	δ_{27}	δ_{20}	δ_{53}	δ_{54}	δ_{55}	δ_{48}	δ_{17}	δ_{18}	δ_{19}
δ_{13}	δ_{29}	δ_{28}	δ_{63}	δ_{30}	δ_{57}	δ_{56}	δ_{59}	δ_{26}	δ_{21}	δ_{20}	δ_{55}	δ_{22}	δ_{49}	δ_{48}	δ_{51}	δ_{18}
δ_{14}	δ_{30}	δ_{31}	δ_{28}	δ_{61}	δ_{58}	δ_{59}	δ_{56}	δ_{57}	δ_{22}	δ_{23}	δ_{20}	δ_{53}	δ_{50}	δ_{19}	δ_{48}	δ_{49}
δ_{15}	δ_{31}	δ_{62}	δ_{29}	δ_{28}	δ_{59}	δ_{58}	δ_{25}	δ_{56}	δ_{23}	δ_{54}	δ_{21}	δ_{20}	δ_{51}	δ_{50}	δ_{17}	δ_{48}
δ_{16}	δ_{32}	δ_{1}	δ_{2}	δ_{3}	δ_{4}	δ_{5}	δ_{6}	δ_{7}	δ_{8}	δ_{9}	δ_{10}	δ_{11}	δ_{12}	δ_{13}	δ_{14}	δ_{15}

δ_{17}	δ_{33}	δ_{32}	δ_{35}	δ_{2}	δ_{37}	δ_{4}	δ_{7}	δ_{38}	δ_{41}	δ_{8}	δ_{11}	δ_{42}	δ_{13}	δ_{44}	δ_{47}	δ_{14}
δ_{18}	δ_{34}	δ_{3}	δ_{32}	δ_{33}	δ_{38}	δ_{39}	δ_{4}	δ_{5}	δ_{42}	δ_{43}	δ_{8}	δ_{9}	δ_{14}	δ_{15}	δ_{44}	δ_{45}
δ_{19}	δ_{35}	δ_{34}	δ_{1}	δ_{32}	δ_{39}	δ_{6}	δ_{37}	δ_{4}	δ_{43}	δ_{10}	δ_{41}	δ_{8}	δ_{15}	δ_{46}	δ_{13}	δ_{44}
δ_{20}	δ_{36}	δ_{5}	δ_{6}	δ_{7}	δ_{32}	δ_{33}	δ_{34}	δ_{35}	δ_{44}	δ_{45}	δ_{46}	δ_{47}	δ_{8}	δ_{9}	δ_{10}	δ_{11}
δ_{21}	δ_{37}	δ_{36}	δ_{7}	δ_{38}	δ_{1}	δ_{32}	δ_{3}	δ_{34}	δ_{45}	δ_{12}	δ_{47}	δ_{14}	δ_{41}	δ_{8}	δ_{11}	δ_{10}
δ_{22}	δ_{38}	δ_{39}	δ_{36}	δ_{5}	δ_{2}	δ_{35}	δ_{32}	δ_{1}	δ_{46}	δ_{15}	δ_{12}	δ_{45}	δ_{42}	δ_{11}	δ_{8}	δ_{41}
δ_{23}	δ_{39}	δ_{6}	δ_{37}	δ_{36}	δ_{3}	δ_{2}	δ_{33}	δ_{32}	δ_{47}	δ_{46}	δ_{13}	δ_{12}	δ_{43}	δ_{42}	δ_{9}	δ_{8}
δ_{24}	δ_{40}	δ_{9}	δ_{10}	δ_{11}	δ_{12}	δ_{13}	δ_{14}	δ_{15}	δ_{32}	δ_{33}	δ_{34}	δ_{35}	δ_{36}	δ_{37}	δ_{38}	δ_{39}
δ_{25}	δ_{41}	δ_{40}	δ_{11}	δ_{42}	δ_{13}	δ_{44}	δ_{47}	δ_{14}	δ_{1}	δ_{32}	δ_{3}	δ_{34}	δ_{5}	δ_{36}	δ_{39}	δ_{6}
δ_{26}	δ_{42}	δ_{43}	δ_{40}	δ_{9}	δ_{14}	δ_{15}	δ_{44}	δ_{45}	δ_{2}	δ_{35}	δ_{32}	δ_{1}	δ_{6}	δ_{7}	δ_{36}	δ_{37}
δ_{27}	δ_{43}	δ_{10}	δ_{41}	δ_{40}	δ_{15}	δ_{46}	δ_{13}	δ_{44}	δ_{3}	δ_{2}	δ_{33}	δ_{32}	δ_{7}	δ_{38}	δ_{5}	δ_{36}
δ_{28}	δ_{44}	δ_{45}	δ_{46}	δ_{47}	δ_{40}	δ_{9}	δ_{10}	δ_{11}	δ_{4}	δ_{37}	δ_{38}	δ_{39}	δ_{32}	δ_{1}	δ_{2}	δ_{3}
δ_{29}	δ_{45}	δ_{12}	δ_{47}	δ_{14}	δ_{41}	δ_{40}	δ_{43}	δ_{10}	δ_{5}	δ_{4}	δ_{39}	δ_{6}	δ_{33}	δ_{32}	δ_{35}	δ_{2}
δ_{30}	δ_{46}	δ_{15}	δ_{12}	δ_{45}	δ_{42}	δ_{43}	δ_{40}	δ_{41}	δ_{6}	δ_{7}	δ_{4}	δ_{37}	δ_{34}	δ_{3}	δ_{32}	δ_{33}
δ_{31}	δ_{47}	δ_{46}	δ_{13}	δ_{12}	δ_{43}	δ_{42}	δ_{9}	δ_{40}	δ_{7}	δ_{38}	δ_{5}	δ_{4}	δ_{35}	δ_{34}	δ_{1}	δ_{32}

•	δ_{16}	δ_{17}	δ_{18}	δ_{19}	δ_{20}	δ_{21}	δ_{22}	δ_{23}	δ_{24}	δ_{25}	δ_{26}	δ_{27}	δ_{28}	δ_{29}	δ_{30}	δ_{31}
δ_{32}	δ_{48}	δ_{49}	δ_{50}	δ_{51}	δ_{52}	δ_{53}	δ_{54}	δ_{55}	δ_{56}	δ_{57}	δ_{58}	δ_{59}	δ_{60}	δ_{61}	δ_{62}	δ_{63}
δ_{33}	δ_{49}	δ_{16}	δ_{19}	δ_{50}	δ_{21}	δ_{52}	δ_{55}	δ_{22}	δ_{25}	δ_{56}	δ_{59}	δ_{26}	δ_{61}	δ_{28}	δ_{31}	δ_{62}
δ_{34}	δ_{50}	δ_{51}	δ_{16}	δ_{17}	δ_{22}	δ_{23}	δ_{52}	δ_{53}	δ_{26}	δ_{27}	δ_{56}	δ_{57}	δ_{62}	δ_{63}	δ_{28}	δ_{29}
δ_{35}	δ_{51}	δ_{18}	δ_{49}	δ_{16}	δ_{23}	δ_{54}	δ_{21}	δ_{52}	δ_{27}	δ_{58}	δ_{25}	δ_{56}	δ_{63}	δ_{30}	δ_{61}	δ_{28}
δ_{36}	δ_{52}	δ_{53}	δ_{54}	δ_{55}	δ_{16}	δ_{17}	δ_{18}	δ_{19}	δ_{28}	δ_{29}	δ_{30}	δ_{31}	δ_{56}	δ_{57}	δ_{58}	δ_{59}
δ_{37}	δ_{53}	δ_{20}	δ_{55}	δ_{22}	δ_{49}	δ_{16}	δ_{51}	δ_{18}	δ_{29}	δ_{60}	δ_{31}	δ_{62}	δ_{25}	δ_{56}	δ_{59}	δ_{58}
δ_{38}	δ_{54}	δ_{23}	δ_{20}	δ_{53}	δ_{50}	δ_{19}	δ_{16}	δ_{49}	δ_{30}	δ_{63}	δ_{60}	δ_{29}	δ_{26}	δ_{59}	δ_{56}	δ_{25}
δ_{39}	δ_{55}	δ_{54}	δ_{21}	δ_{20}	δ_{51}	δ_{50}	δ_{17}	δ_{16}	δ_{31}	δ_{30}	δ_{61}	δ_{60}	δ_{27}	δ_{26}	δ_{57}	δ_{56}
δ_{40}	δ_{56}	δ_{57}	δ_{58}	δ_{59}	δ_{60}	δ_{61}	δ_{62}	δ_{63}	δ_{16}	δ_{17}	δ_{18}	δ_{19}	δ_{20}	δ_{21}	δ_{22}	δ_{23}
δ_{41}	δ_{57}	δ_{24}	δ_{59}	δ_{26}	δ_{61}	δ_{28}	δ_{31}	δ_{62}	δ_{49}	δ_{16}	δ_{51}	δ_{18}	δ_{53}	δ_{20}	δ_{23}	δ_{54}
δ_{42}	δ_{58}	δ_{27}	δ_{24}	δ_{57}	δ_{62}	δ_{63}	δ_{28}	δ_{29}	δ_{50}	δ_{19}	δ_{16}	δ_{49}	δ_{54}	δ_{55}	δ_{20}	δ_{21}
δ_{43}	δ_{59}	δ_{58}	δ_{25}	δ_{24}	δ_{63}	δ_{30}	δ_{61}	δ_{28}	δ_{51}	δ_{50}	δ_{17}	δ_{16}	δ_{55}	δ_{22}	δ_{53}	δ_{20}
δ_{44}	δ_{60}	δ_{29}	δ_{30}	δ_{31}	δ_{24}	δ_{57}	δ_{58}	δ_{59}	δ_{52}	δ_{21}	δ_{22}	δ_{23}	δ_{16}	δ_{49}	δ_{50}	δ_{51}
δ_{45}	δ_{61}	δ_{60}	δ_{31}	δ_{62}	δ_{25}	δ_{n4}	δ_{27}	δ_{58}	δ_{53}	δ_{52}	δ_{23}	δ_{54}	δ_{17}	δ_{16}	δ_{19}	δ_{50}
δ_{46}	δ_{62}	δ_{63}	δ_{60}	δ_{29}	δ_{26}	δ_{27}	δ_{24}	δ_{25}	δ_{54}	δ_{55}	δ_{52}	δ_{21}	δ_{18}	δ_{51}	δ_{16}	δ_{17}
δ_{47}	δ_{63}	δ_{30}	δ_{61}	δ_{60}	δ_{27}	δ_{26}	δ_{57}	δ_{24}	δ_{55}	δ_{22}	δ_{53}	δ_{52}	δ_{19}	δ_{18}	δ_{49}	δ_{16}
δ_{48}	δ_{0}	δ_{33}	δ_{34}	δ_{35}	δ_{36}	δ_{37}	δ_{38}	δ_{39}	δ_{40}	δ_{41}	δ_{42}	δ_{43}	δ_{44}	δ_{45}	δ_{46}	δ_{47}
δ_{49}	δ_{1}	δ_{0}	δ_{3}	δ_{34}	δ_{5}	δ_{36}	δ_{39}	δ_{6}	δ_{9}	δ_{40}	δ_{43}	δ_{10}	δ_{45}	δ_{12}	δ_{15}	δ_{46}
δ_{50}	δ_{2}	δ_{35}	δ_{0}	δ_{1}	δ_{6}	δ_{7}	δ_{36}	δ_{37}	δ_{10}	δ_{11}	δ_{40}	δ_{41}	δ_{46}	δ_{47}	δ_{12}	δ_{13}

δ_{51}	δ_{3}	δ_{2}	δ_{33}	δ_{0}	δ_{7}	δ_{38}	δ_{5}	δ_{36}	δ_{11}	δ_{42}	δ_{9}	δ_{40}	δ_{47}	δ_{14}	δ_{45}	δ_{12}
δ_{52}	δ_{4}	δ_{37}	δ_{38}	δ_{39}	δ_{0}	δ_{1}	δ_{2}	δ_{3}	δ_{12}	δ_{13}	δ_{14}	δ_{15}	δ_{40}	δ_{41}	δ_{42}	δ_{43}
δ_{53}	δ_{5}	δ_{4}	δ_{39}	δ_{6}	δ_{33}	δ_{0}	δ_{35}	δ_{2}	δ_{13}	δ_{44}	δ_{15}	δ_{46}	δ_{9}	δ_{40}	δ_{43}	δ_{42}
δ_{54}	δ_{6}	δ_{7}	δ_{4}	δ_{37}	δ_{34}	δ_{3}	δ_{0}	δ_{33}	δ_{14}	δ_{47}	δ_{44}	δ_{13}	δ_{10}	δ_{43}	δ_{40}	δ_{9}
δ_{55}	δ_{7}	δ_{38}	δ_{5}	δ_{4}	δ_{35}	δ_{34}	δ_{1}	δ_{0}	δ_{15}	δ_{14}	δ_{45}	δ_{44}	δ_{11}	δ_{10}	δ_{41}	δ_{40}
δ_{56}	δ_{8}	δ_{41}	δ_{42}	δ_{43}	δ_{44}	δ_{45}	δ_{46}	δ_{47}	δ_{0}	δ_{1}	δ_{2}	δ_{3}	δ_{4}	δ_{5}	δ_{6}	δ_{7}
δ_{57}	δ_{9}	δ_{8}	δ_{43}	δ_{10}	δ_{45}	δ_{12}	δ_{15}	δ_{46}	δ_{33}	δ_{0}	δ_{35}	δ_{2}	δ_{37}	δ_{4}	δ_{7}	δ_{38}
δ_{58}	δ_{10}	δ_{11}	δ_{8}	δ_{41}	δ_{46}	δ_{47}	δ_{12}	δ_{13}	δ_{34}	δ_{3}	δ_{0}	δ_{33}	δ_{38}	δ_{39}	δ_{4}	δ_{5}
δ_{59}	δ_{11}	δ_{42}	δ_{9}	δ_{8}	δ_{47}	δ_{14}	δ_{45}	δ_{12}	δ_{35}	δ_{34}	δ_{1}	δ_{0}	δ_{39}	δ_{6}	δ_{37}	δ_{4}
δ_{60}	δ_{12}	δ_{13}	δ_{14}	δ_{15}	δ_{8}	δ_{41}	δ_{42}	δ_{43}	δ_{36}	δ_{5}	δ_{6}	δ_{7}	δ_{0}	δ_{33}	δ_{34}	δ_{35}
δ_{61}	δ_{13}	δ_{44}	δ_{15}	δ_{46}	δ_{9}	δ_{8}	δ_{11}	δ_{42}	δ_{37}	δ_{36}	δ_{7}	δ_{38}	δ_{1}	δ_{0}	δ_{3}	δ_{34}
δ_{62}	δ_{14}	δ_{47}	δ_{44}	δ_{13}	δ_{10}	δ_{11}	δ_{8}	δ_{9}	δ_{38}	δ_{39}	δ_{36}	δ_{5}	δ_{2}	δ_{35}	δ_{0}	δ_{1}
δ_{63}	δ_{15}	δ_{14}	δ_{45}	δ_{44}	δ_{11}	δ_{10}	δ_{41}	δ_{8}	δ_{39}	δ_{6}	δ_{37}	δ_{36}	δ_{3}	δ_{2}	δ_{33}	δ_{0}

•	δ_{32}	δ_{33}	δ_{34}	δ_{35}	δ_{36}	δ_{37}	δ_{38}	δ_{39}	δ_{40}	δ_{41}	δ_{42}	δ_{43}	δ_{44}	δ_{45}	δ_{46}	δ_{47}
δ_{0}	δ_{32}	δ_{33}	δ_{34}	δ_{35}	δ_{36}	δ_{37}	δ_{38}	δ_{39}	δ_{40}	δ_{41}	δ_{42}	δ_{43}	δ_{44}	δ_{45}	δ_{46}	δ_{47}
δ_{1}	δ_{33}	δ_{0}	δ_{35}	δ_{2}	δ_{37}	δ_{4}	δ_{7}	δ_{38}	δ_{41}	δ_{8}	δ_{11}	δ_{42}	δ_{13}	δ_{44}	δ_{47}	δ_{14}
δ_{2}	δ_{34}	δ_{3}	δ_{0}	δ_{33}	δ_{38}	δ_{39}	δ_{4}	δ_{5}	δ_{42}	δ_{43}	δ_{8}	δ_{9}	δ_{14}	δ_{15}	δ_{44}	δ_{45}
δ_{3}	δ_{35}	δ_{34}	δ_{1}	δ_{0}	δ_{39}	δ_{6}	δ_{37}	δ_{4}	δ_{43}	δ_{10}	δ_{41}	δ_{8}	δ_{15}	δ_{46}	δ_{13}	δ_{44}
δ_{4}	δ_{36}	δ_{5}	δ_{6}	δ_{7}	δ_{0}	δ_{33}	δ_{34}	δ_{35}	δ_{44}	δ_{45}	δ_{46}	δ_{47}	δ_{8}	δ_{9}	δ_{10}	δ_{11}
δ_{5}	δ_{37}	δ_{36}	δ_{7}	δ_{38}	δ_{1}	δ_{0}	δ_{3}	δ_{34}	δ_{45}	δ_{12}	δ_{47}	δ_{14}	δ_{41}	δ_{8}	δ_{11}	δ_{10}
δ_{6}	δ_{38}	δ_{39}	δ_{36}	δ_{5}	δ_{2}	δ_{35}	δ_{0}	δ_{1}	δ_{46}	δ_{15}	δ_{12}	δ_{45}	δ_{42}	δ_{11}	δ_{8}	δ_{41}
δ_{7}	δ_{39}	δ_{6}	δ_{37}	δ_{36}	δ_{3}	δ_{2}	δ_{33}	δ_{0}	δ_{47}	δ_{46}	δ_{13}	δ_{12}	δ_{43}	δ_{42}	δ_{9}	δ_{8}
δ_{8}	δ_{40}	δ_{9}	δ_{10}	δ_{11}	δ_{12}	δ_{13}	δ_{14}	δ_{15}	δ_{0}	δ_{33}	δ_{34}	δ_{35}	δ_{36}	δ_{37}	δ_{38}	δ_{39}
δ_{9}	δ_{41}	δ_{40}	δ_{11}	δ_{42}	δ_{13}	δ_{44}	δ_{47}	δ_{14}	δ_{1}	δ_{0}	δ_{3}	δ_{34}	δ_{5}	δ_{36}	δ_{39}	δ_{6}
δ_{10}	δ_{42}	δ_{43}	δ_{40}	δ_{9}	δ_{14}	δ_{15}	δ_{44}	δ_{45}	δ_{2}	δ_{35}	δ_{0}	δ_{1}	δ_{6}	δ_{7}	δ_{36}	δ_{37}
δ_{11}	δ_{43}	δ_{10}	δ_{41}	δ_{40}	δ_{15}	δ_{46}	δ_{13}	δ_{44}	δ_{3}	δ_{2}	δ_{33}	δ_{0}	δ_{7}	δ_{38}	δ_{5}	δ_{36}
δ_{12}	δ_{44}	δ_{45}	δ_{46}	δ_{47}	δ_{40}	δ_{9}	δ_{10}	δ_{11}	δ_{4}	δ_{37}	δ_{38}	δ_{39}	δ_{0}	δ_{1}	δ_{2}	δ_{3}
δ_{13}	δ_{45}	δ_{12}	δ_{47}	δ_{14}	δ_{41}	δ_{40}	δ_{43}	δ_{10}	δ_{5}	δ_{4}	δ_{39}	δ_{6}	δ_{33}	δ_{0}	δ_{35}	δ_{2}
δ_{14}	δ_{46}	δ_{15}	δ_{12}	δ_{45}	δ_{42}	δ_{43}	δ_{40}	δ_{41}	δ_{6}	δ_{7}	δ_{4}	δ_{37}	δ_{34}	δ_{3}	δ_{0}	δ_{33}
δ_{15}	δ_{47}	δ_{46}	δ_{13}	δ_{12}	δ_{43}	δ_{42}	δ_{9}	δ_{40}	δ_{7}	δ_{38}	δ_{5}	δ_{4}	δ_{35}	δ_{34}	δ_{1}	δ_{0}
δ_{16}	δ_{48}	δ_{17}	δ_{18}	δ_{19}	δ_{20}	δ_{21}	δ_{22}	δ_{23}	δ_{24}	δ_{25}	δ_{26}	δ_{27}	δ_{28}	δ_{29}	δ_{30}	δ_{31}
δ_{17}	δ_{49}	δ_{48}	δ_{19}	δ_{50}	δ_{21}	δ_{52}	δ_{55}	δ_{22}	δ_{25}	δ_{56}	δ_{59}	δ_{26}	δ_{61}	δ_{28}	δ_{31}	δ_{62}
δ_{18}	δ_{50}	δ_{51}	δ_{48}	δ_{17}	δ_{22}	δ_{23}	δ_{52}	δ_{53}	δ_{26}	δ_{27}	δ_{56}	δ_{57}	δ_{62}	δ_{63}	δ_{28}	δ_{29}
δ_{19}	δ_{51}	δ_{18}	δ_{49}	δ_{48}	δ_{23}	δ_{54}	δ_{21}	δ_{52}	δ_{27}	δ_{58}	δ_{25}	δ_{56}	δ_{63}	δ_{30}	δ_{61}	δ_{28}
δ_{20}	δ_{52}	δ_{53}	δ_{54}	δ_{55}	δ_{48}	δ_{17}	δ_{18}	δ_{19}	δ_{28}	δ_{29}	δ_{30}	δ_{31}	δ_{56}	δ_{57}	δ_{58}	δ_{59}

δ_{21}	δ_{53}	δ_{20}	δ_{55}	δ_{22}	δ_{49}	δ_{48}	δ_{51}	δ_{18}	δ_{29}	δ_{60}	δ_{31}	δ_{62}	δ_{25}	δ_{56}	δ_{59}	δ_{58}
δ_{22}	δ_{54}	δ_{23}	δ_{20}	δ_{53}	δ_{50}	δ_{19}	δ_{48}	δ_{49}	δ_{30}	δ_{63}	δ_{60}	δ_{29}	δ_{26}	δ_{59}	δ_{56}	δ_{25}
δ_{23}	δ_{55}	δ_{54}	δ_{21}	δ_{20}	δ_{51}	δ_{50}	δ_{17}	δ_{48}	δ_{31}	δ_{30}	δ_{61}	δ_{60}	δ_{27}	δ_{26}	δ_{57}	δ_{56}
δ_{24}	δ_{56}	δ_{57}	δ_{58}	δ_{59}	δ_{60}	δ_{61}	δ_{62}	δ_{63}	δ_{48}	δ_{17}	δ_{18}	δ_{19}	δ_{20}	δ_{21}	δ_{22}	δ_{23}
δ_{25}	δ_{57}	δ_{24}	δ_{59}	δ_{26}	δ_{61}	δ_{28}	δ_{31}	δ_{62}	δ_{49}	δ_{48}	δ_{51}	δ_{18}	δ_{53}	δ_{20}	δ_{23}	δ_{54}
δ_{26}	δ_{58}	δ_{27}	δ_{24}	δ_{57}	δ_{62}	δ_{63}	δ_{28}	δ_{29}	δ_{50}	δ_{19}	δ_{48}	δ_{49}	δ_{54}	δ_{55}	δ_{20}	δ_{21}
δ_{27}	δ_{59}	δ_{58}	δ_{25}	δ_{24}	δ_{63}	δ_{30}	δ_{61}	δ_{28}	δ_{51}	δ_{50}	δ_{17}	δ_{48}	δ_{55}	δ_{22}	δ_{53}	δ_{20}
δ_{28}	δ_{60}	δ_{29}	δ_{30}	δ_{31}	δ_{24}	δ_{57}	δ_{58}	δ_{59}	δ_{52}	δ_{21}	δ_{22}	δ_{23}	δ_{48}	δ_{49}	δ_{50}	δ_{51}
δ_{29}	δ_{61}	δ_{60}	δ_{31}	δ_{62}	δ_{25}	δ_{24}	δ_{27}	δ_{58}	δ_{53}	δ_{52}	δ_{23}	δ_{54}	δ_{17}	δ_{48}	δ_{19}	δ_{50}
δ_{30}	δ_{62}	δ_{63}	δ_{60}	δ_{29}	δ_{26}	δ_{27}	δ_{24}	δ_{25}	δ_{54}	δ_{55}	δ_{52}	δ_{21}	δ_{18}	δ_{51}	δ_{48}	δ_{17}
δ_{31}	δ_{63}	δ_{30}	δ_{61}	δ_{60}	δ_{27}	δ_{26}	δ_{57}	δ_{24}	δ_{55}	δ_{22}	δ_{53}	δ_{52}	δ_{19}	δ_{18}	δ_{49}	δ_{48}

•	δ_{32}	δ_{33}	δ_{34}	δ_{35}	δ_{36}	δ_{37}	δ_{38}	δ_{39}	δ_{40}	δ_{41}	δ_{42}	δ_{43}	δ_{44}	δ_{45}	δ_{46}	δ_{47}
δ_{32}	δ_{0}	δ_{1}	δ_{2}	δ_{3}	δ_{4}	δ_{5}	δ_{6}	δ_{7}	δ_{8}	δ_{9}	δ_{10}	δ_{11}	δ_{12}	δ_{13}	δ_{14}	δ_{15}
δ_{33}	δ_{1}	δ_{32}	δ_{3}	δ_{34}	δ_{5}	δ_{36}	δ_{39}	δ_{6}	δ_{9}	δ_{40}	δ_{43}	δ_{10}	δ_{45}	δ_{12}	δ_{15}	δ_{46}
δ_{34}	δ_{2}	δ_{35}	δ_{32}	δ_{1}	δ_{6}	δ_{7}	δ_{36}	δ_{37}	δ_{10}	δ_{11}	δ_{40}	δ_{41}	δ_{46}	δ_{47}	δ_{12}	δ_{13}
δ_{35}	δ_{3}	δ_{2}	δ_{33}	δ_{32}	δ_{7}	δ_{38}	δ_{5}	δ_{36}	δ_{11}	δ_{42}	δ_{9}	δ_{40}	δ_{47}	δ_{14}	δ_{45}	δ_{12}
δ_{36}	δ_{4}	δ_{37}	δ_{38}	δ_{39}	δ_{32}	δ_{1}	δ_{2}	δ_{3}	δ_{12}	δ_{13}	δ_{14}	δ_{15}	δ_{40}	δ_{41}	δ_{42}	δ_{43}
δ_{37}	δ_{5}	δ_{4}	δ_{39}	δ_{6}	δ_{33}	δ_{32}	δ_{35}	δ_{2}	δ_{13}	δ_{44}	δ_{15}	δ_{46}	δ_{9}	δ_{40}	δ_{43}	δ_{42}
δ_{38}	δ_{6}	δ_{7}	δ_{4}	δ_{37}	δ_{34}	δ_{3}	δ_{32}	δ_{33}	δ_{14}	δ_{47}	δ_{44}	δ_{13}	δ_{10}	δ_{43}	δ_{40}	δ_{9}
δ_{39}	δ_{7}	δ_{38}	δ_{5}	δ_{4}	δ_{35}	δ_{34}	δ_{1}	δ_{32}	δ_{15}	δ_{14}	δ_{45}	δ_{44}	δ_{11}	δ_{10}	δ_{41}	δ_{40}
δ_{40}	δ_{8}	δ_{41}	δ_{42}	δ_{43}	δ_{44}	δ_{45}	δ_{46}	δ_{47}	δ_{32}	δ_{n1}	δ_{n2}	δ_{n3}	δ_{n4}	δ_{n5}	δ_{n6}	δ_{n7}
δ_{41}	δ_{9}	δ_{8}	δ_{43}	δ_{10}	δ_{45}	δ_{12}	δ_{15}	δ_{46}	δ_{33}	δ_{32}	δ_{35}	δ_{2}	δ_{37}	δ_{4}	δ_{7}	δ_{38}
δ_{42}	δ_{10}	δ_{11}	δ_{8}	δ_{41}	δ_{46}	δ_{47}	δ_{12}	δ_{13}	δ_{34}	δ_{3}	δ_{32}	δ_{33}	δ_{38}	δ_{39}	δ_{4}	δ_{5}
δ_{43}	δ_{11}	δ_{42}	δ_{9}	δ_{8}	δ_{47}	δ_{14}	δ_{45}	δ_{12}	δ_{35}	δ_{34}	δ_{1}	δ_{32}	δ_{39}	δ_{6}	δ_{37}	δ_{4}
δ_{44}	δ_{12}	δ_{13}	δ_{14}	δ_{15}	δ_{8}	δ_{41}	δ_{42}	δ_{43}	δ_{36}	δ_{5}	δ_{6}	δ_{7}	δ_{32}	δ_{33}	δ_{34}	δ_{35}
δ_{45}	δ_{13}	δ_{44}	δ_{15}	δ_{46}	δ_{9}	δ_{8}	δ_{11}	δ_{42}	δ_{37}	δ_{36}	δ_{7}	δ_{38}	δ_{1}	δ_{32}	δ_{3}	δ_{34}
δ_{46}	δ_{14}	δ_{47}	δ_{44}	δ_{13}	δ_{10}	δ_{11}	δ_{8}	δ_{9}	δ_{38}	δ_{39}	δ_{36}	δ_{5}	δ_{2}	δ_{35}	δ_{32}	δ_{1}
δ_{47}	δ_{15}	δ_{14}	δ_{45}	δ_{44}	δ_{11}	δ_{10}	δ_{41}	δ_{8}	δ_{39}	δ_{6}	δ_{37}	δ_{36}	δ_{3}	δ_{2}	δ_{33}	δ_{32}
δ_{48}	δ_{16}	δ_{49}	δ_{50}	δ_{51}	δ_{52}	δ_{53}	δ_{54}	δ_{55}	δ_{56}	δ_{57}	δ_{58}	δ_{59}	δ_{60}	δ_{61}	δ_{62}	δ_{63}
δ_{49}	δ_{17}	δ_{16}	δ_{51}	δ_{18}	δ_{53}	δ_{20}	δ_{23}	δ_{54}	δ_{57}	δ_{24}	δ_{27}	δ_{58}	δ_{29}	δ_{60}	δ_{63}	δ_{30}
δ_{50}	δ_{18}	δ_{19}	δ_{16}	δ_{49}	δ_{54}	δ_{55}	δ_{20}	δ_{21}	δ_{58}	δ_{59}	δ_{24}	δ_{25}	δ_{30}	δ_{31}	δ_{60}	δ_{61}
δ_{51}	δ_{19}	δ_{50}	δ_{17}	δ_{16}	δ_{55}	δ_{22}	δ_{53}	δ_{20}	δ_{59}	δ_{26}	δ_{57}	δ_{24}	δ_{31}	δ_{62}	δ_{29}	δ_{60}
δ_{52}	δ_{20}	δ_{21}	δ_{22}	δ_{23}	δ_{16}	δ_{49}	δ_{50}	δ_{51}	δ_{60}	δ_{61}	δ_{62}	δ_{63}	δ_{24}	δ_{25}	δ_{26}	δ_{27}
δ_{53}	δ_{21}	δ_{52}	δ_{23}	δ_{54}	δ_{17}	δ_{16}	δ_{19}	δ_{50}	δ_{61}	δ_{28}	δ_{63}	δ_{30}	δ_{57}	δ_{24}	δ_{27}	δ_{26}
δ_{54}	δ_{22}	δ_{55}	δ_{52}	δ_{21}	δ_{18}	δ_{51}	δ_{16}	δ_{17}	δ_{62}	δ_{31}	δ_{28}	δ_{61}	δ_{58}	δ_{27}	δ_{24}	δ_{57}

δ_{55}	δ_{23}	δ_{22}	δ_{53}	δ_{52}	δ_{19}	δ_{18}	δ_{49}	δ_{16}	δ_{63}	δ_{62}	δ_{29}	δ_{28}	δ_{59}	δ_{58}	δ_{25}	δ_{24}
δ_{56}	δ_{24}	δ_{25}	δ_{26}	δ_{27}	δ_{28}	δ_{29}	δ_{30}	δ_{31}	δ_{16}	δ_{49}	δ_{50}	δ_{51}	δ_{52}	δ_{53}	δ_{54}	δ_{55}
δ_{57}	δ_{25}	δ_{56}	δ_{27}	δ_{58}	δ_{29}	δ_{60}	δ_{63}	δ_{30}	δ_{17}	δ_{16}	δ_{19}	δ_{50}	δ_{21}	δ_{52}	δ_{55}	δ_{22}
δ_{58}	δ_{26}	δ_{59}	δ_{56}	δ_{25}	δ_{30}	δ_{31}	δ_{60}	δ_{61}	δ_{18}	δ_{51}	δ_{16}	δ_{17}	δ_{22}	δ_{23}	δ_{52}	δ_{53}
δ_{59}	δ_{27}	δ_{26}	δ_{57}	δ_{56}	δ_{31}	δ_{62}	δ_{29}	δ_{60}	δ_{19}	δ_{18}	δ_{49}	δ_{16}	δ_{23}	δ_{54}	δ_{21}	δ_{52}
δ_{60}	δ_{28}	δ_{61}	δ_{62}	δ_{63}	δ_{56}	δ_{25}	δ_{26}	δ_{27}	δ_{20}	δ_{53}	δ_{54}	δ_{55}	δ_{16}	δ_{17}	δ_{18}	δ_{19}
δ_{61}	δ_{29}	δ_{28}	δ_{63}	δ_{30}	δ_{57}	δ_{56}	δ_{59}	δ_{26}	δ_{21}	δ_{20}	δ_{55}	δ_{22}	δ_{49}	δ_{16}	δ_{51}	δ_{18}
δ_{62}	δ_{30}	δ_{31}	δ_{28}	δ_{61}	δ_{58}	δ_{59}	δ_{56}	δ_{57}	δ_{22}	δ_{23}	δ_{20}	δ_{53}	δ_{50}	δ_{19}	δ_{16}	δ_{49}
δ_{63}	δ_{31}	δ_{62}	δ_{29}	δ_{28}	δ_{59}	δ_{58}	δ_{25}	δ_{56}	δ_{23}	δ_{54}	δ_{21}	δ_{20}	δ_{51}	δ_{50}	δ_{17}	δ_{16}

$\cdot$	δ_{48}	δ_{49}	δ_{50}	δ_{51}	δ_{52}	δ_{53}	δ_{54}	δ_{55}	δ_{56}	δ_{57}	δ_{58}	δ_{59}	δ_{60}	δ_{61}	δ_{62}	δ_{63}
δ_{0}	δ_{48}	δ_{49}	δ_{50}	δ_{51}	δ_{52}	δ_{53}	δ_{54}	δ_{55}	δ_{56}	δ_{57}	δ_{58}	δ_{59}	δ_{60}	δ_{61}	δ_{62}	δ_{63}
δ_{1}	δ_{49}	δ_{16}	δ_{19}	δ_{50}	δ_{21}	δ_{52}	δ_{55}	δ_{22}	δ_{25}	δ_{56}	δ_{59}	δ_{26}	δ_{61}	δ_{28}	δ_{31}	δ_{62}
δ_{2}	δ_{50}	δ_{51}	δ_{16}	δ_{17}	δ_{22}	δ_{23}	δ_{52}	δ_{53}	δ_{26}	δ_{27}	δ_{56}	δ_{57}	δ_{62}	δ_{63}	δ_{28}	δ_{29}
δ_{3}	δ_{51}	δ_{18}	δ_{49}	δ_{16}	δ_{23}	δ_{54}	δ_{21}	δ_{52}	δ_{27}	δ_{58}	δ_{25}	δ_{56}	δ_{63}	δ_{30}	δ_{61}	δ_{28}
δ_{4}	δ_{52}	δ_{53}	δ_{54}	δ_{55}	δ_{16}	δ_{17}	δ_{18}	δ_{19}	δ_{28}	δ_{29}	δ_{30}	δ_{31}	δ_{56}	δ_{57}	δ_{58}	δ_{59}
δ_{5}	δ_{53}	δ_{20}	δ_{55}	δ_{22}	δ_{49}	δ_{16}	δ_{51}	δ_{18}	δ_{29}	δ_{60}	δ_{31}	δ_{62}	δ_{25}	δ_{56}	δ_{59}	δ_{58}
δ_{6}	δ_{54}	δ_{23}	δ_{20}	δ_{53}	δ_{50}	δ_{19}	δ_{16}	δ_{49}	δ_{30}	δ_{63}	δ_{60}	δ_{29}	δ_{26}	δ_{59}	δ_{56}	δ_{25}
δ_{7}	δ_{55}	δ_{54}	δ_{21}	δ_{20}	δ_{51}	δ_{50}	δ_{17}	δ_{16}	δ_{31}	δ_{30}	δ_{61}	δ_{60}	δ_{27}	δ_{26}	δ_{57}	δ_{56}
δ_{8}	δ_{56}	δ_{57}	δ_{58}	δ_{59}	δ_{60}	δ_{61}	δ_{62}	δ_{63}	δ_{16}	δ_{17}	δ_{18}	δ_{19}	δ_{20}	δ_{21}	δ_{22}	δ_{23}
δ_{9}	δ_{57}	δ_{24}	δ_{59}	δ_{26}	δ_{61}	δ_{28}	δ_{31}	δ_{62}	δ_{49}	δ_{16}	δ_{51}	δ_{18}	δ_{53}	δ_{20}	δ_{23}	δ_{54}
δ_{10}	δ_{58}	δ_{27}	δ_{24}	δ_{57}	δ_{62}	δ_{63}	δ_{28}	δ_{29}	δ_{50}	δ_{19}	δ_{16}	δ_{49}	δ_{54}	δ_{55}	δ_{20}	δ_{21}
δ_{11}	δ_{59}	δ_{58}	δ_{25}	δ_{24}	δ_{63}	δ_{30}	δ_{61}	δ_{28}	δ_{51}	δ_{50}	δ_{17}	δ_{16}	δ_{55}	δ_{22}	δ_{53}	δ_{20}
δ_{12}	δ_{60}	δ_{29}	δ_{30}	δ_{31}	δ_{24}	δ_{57}	δ_{58}	δ_{59}	δ_{52}	δ_{21}	δ_{22}	δ_{23}	δ_{16}	δ_{49}	δ_{50}	δ_{51}
δ_{13}	δ_{61}	δ_{60}	δ_{31}	δ_{62}	δ_{25}	δ_{24}	δ_{27}	δ_{58}	δ_{53}	δ_{52}	δ_{23}	δ_{54}	δ_{17}	δ_{16}	δ_{19}	δ_{50}
δ_{14}	δ_{62}	δ_{63}	δ_{60}	δ_{29}	δ_{26}	δ_{27}	δ_{24}	δ_{25}	δ_{54}	δ_{55}	δ_{52}	δ_{21}	δ_{18}	δ_{51}	δ_{16}	δ_{17}
δ_{15}	δ_{63}	δ_{30}	δ_{61}	δ_{60}	δ_{27}	δ_{26}	δ_{57}	δ_{24}	δ_{55}	δ_{22}	δ_{53}	δ_{52}	δ_{19}	δ_{18}	δ_{49}	δ_{16}
δ_{16}	δ_{32}	δ_{33}	δ_{34}	δ_{35}	δ_{36}	δ_{37}	δ_{38}	δ_{39}	δ_{40}	δ_{41}	δ_{42}	δ_{43}	δ_{44}	δ_{45}	δ_{46}	δ_{47}
δ_{17}	δ_{33}	δ_{0}	δ_{3}	δ_{34}	δ_{5}	δ_{36}	δ_{39}	δ_{6}	δ_{9}	δ_{40}	δ_{43}	δ_{10}	δ_{45}	δ_{12}	δ_{15}	δ_{46}
δ_{18}	δ_{34}	δ_{35}	δ_{0}	δ_{1}	δ_{6}	δ_{7}	δ_{36}	δ_{37}	δ_{10}	δ_{11}	δ_{40}	δ_{41}	δ_{46}	δ_{47}	δ_{12}	δ_{13}
δ_{19}	δ_{35}	δ_{2}	δ_{33}	δ_{0}	δ_{7}	δ_{38}	δ_{5}	δ_{36}	δ_{11}	δ_{42}	δ_{9}	δ_{40}	δ_{47}	δ_{14}	δ_{45}	δ_{12}
δ_{20}	δ_{36}	δ_{37}	δ_{38}	δ_{39}	δ_{0}	δ_{1}	δ_{2}	δ_{3}	δ_{12}	δ_{13}	δ_{14}	δ_{15}	δ_{40}	δ_{41}	δ_{42}	δ_{43}
δ_{21}	δ_{37}	δ_{4}	δ_{39}	δ_{6}	δ_{33}	δ_{0}	δ_{35}	δ_{2}	δ_{13}	δ_{44}	δ_{15}	δ_{46}	δ_{9}	δ_{40}	δ_{43}	δ_{42}
δ_{22}	δ_{38}	δ_{7}	δ_{4}	δ_{37}	δ_{34}	δ_{3}	δ_{0}	δ_{33}	δ_{14}	δ_{47}	δ_{44}	δ_{13}	δ_{10}	δ_{43}	δ_{40}	δ_{9}
δ_{23}	δ_{39}	δ_{38}	δ_{5}	δ_{4}	δ_{35}	δ_{34}	δ_{1}	δ_{0}	δ_{15}	δ_{14}	δ_{45}	δ_{44}	δ_{11}	δ_{10}	δ_{41}	δ_{40}
δ_{24}	δ_{40}	δ_{41}	δ_{42}	δ_{43}	δ_{44}	δ_{45}	δ_{46}	δ_{47}	δ_{0}	δ_{1}	δ_{2}	δ_{3}	δ_{4}	δ_{5}	δ_{6}	δ_{7}

δ_{25}	δ_{41}	δ_{8}	δ_{43}	δ_{10}	δ_{45}	δ_{12}	δ_{15}	δ_{46}	δ_{33}	δ_{0}	δ_{35}	δ_{2}	δ_{37}	δ_{4}	δ_{7}	δ_{38}
δ_{26}	δ_{42}	δ_{11}	δ_{8}	δ_{41}	δ_{46}	δ_{47}	δ_{12}	δ_{13}	δ_{34}	δ_{3}	δ_{0}	δ_{33}	δ_{38}	δ_{39}	δ_{4}	δ_{5}
δ_{27}	δ_{43}	δ_{42}	δ_{9}	δ_{8}	δ_{47}	δ_{14}	δ_{45}	δ_{12}	δ_{35}	δ_{34}	δ_{1}	δ_{0}	δ_{39}	δ_{6}	δ_{37}	δ_{4}
δ_{28}	δ_{44}	δ_{13}	δ_{14}	δ_{15}	δ_{8}	δ_{41}	δ_{42}	δ_{43}	δ_{36}	δ_{5}	δ_{6}	δ_{7}	δ_{0}	δ_{33}	δ_{34}	δ_{35}
δ_{29}	δ_{45}	δ_{44}	δ_{15}	δ_{46}	δ_{9}	δ_{8}	δ_{11}	δ_{42}	δ_{37}	δ_{36}	δ_{7}	δ_{38}	δ_{1}	δ_{0}	δ_{3}	δ_{34}
δ_{30}	δ_{46}	δ_{47}	δ_{44}	δ_{13}	δ_{10}	δ_{11}	δ_{8}	δ_{9}	δ_{38}	δ_{39}	δ_{36}	δ_{5}	δ_{2}	δ_{35}	δ_{0}	δ_{1}
δ_{31}	δ_{47}	δ_{14}	δ_{45}	δ_{44}	δ_{11}	δ_{10}	δ_{41}	δ_{8}	δ_{39}	δ_{6}	δ_{37}	δ_{36}	δ_{3}	δ_{2}	δ_{33}	δ_{0}

•	δ_{48}	δ_{49}	δ_{50}	δ_{51}	δ_{52}	δ_{53}	δ_{54}	δ_{55}	δ_{56}	δ_{57}	δ_{58}	δ_{59}	δ_{60}	δ_{61}	δ_{62}	δ_{63}
δ_{32}	δ_{16}	δ_{17}	δ_{18}	δ_{19}	δ_{20}	δ_{21}	δ_{22}	δ_{23}	δ_{24}	δ_{25}	δ_{26}	δ_{27}	δ_{28}	δ_{29}	δ_{30}	δ_{31}
δ_{33}	δ_{17}	δ_{48}	δ_{51}	δ_{18}	δ_{53}	δ_{20}	δ_{23}	δ_{54}	δ_{57}	δ_{24}	δ_{27}	δ_{58}	δ_{29}	δ_{60}	δ_{63}	δ_{30}
δ_{34}	δ_{18}	δ_{19}	δ_{48}	δ_{49}	δ_{54}	δ_{55}	δ_{20}	δ_{21}	δ_{58}	δ_{59}	δ_{24}	δ_{25}	δ_{30}	δ_{31}	δ_{60}	δ_{61}
δ_{35}	δ_{19}	δ_{50}	δ_{17}	δ_{48}	δ_{55}	δ_{22}	δ_{53}	δ_{20}	δ_{59}	δ_{26}	δ_{57}	δ_{24}	δ_{31}	δ_{62}	δ_{29}	δ_{60}
δ_{36}	δ_{20}	δ_{21}	δ_{22}	δ_{23}	δ_{48}	δ_{49}	δ_{50}	δ_{51}	δ_{60}	δ_{61}	δ_{62}	δ_{63}	δ_{24}	δ_{25}	δ_{26}	δ_{27}
δ_{37}	δ_{21}	δ_{52}	δ_{23}	δ_{54}	δ_{17}	δ_{48}	δ_{19}	δ_{50}	δ_{61}	δ_{28}	δ_{63}	δ_{30}	δ_{57}	δ_{24}	δ_{27}	δ_{26}
δ_{38}	δ_{22}	δ_{55}	δ_{52}	δ_{21}	δ_{18}	δ_{51}	δ_{48}	δ_{17}	δ_{62}	δ_{31}	δ_{28}	δ_{61}	δ_{58}	δ_{27}	δ_{24}	δ_{57}
δ_{39}	δ_{23}	δ_{22}	δ_{53}	δ_{52}	δ_{19}	δ_{18}	δ_{49}	δ_{48}	δ_{63}	δ_{62}	δ_{29}	δ_{28}	δ_{59}	δ_{58}	δ_{25}	δ_{24}
δ_{40}	δ_{24}	δ_{25}	δ_{26}	δ_{27}	δ_{28}	δ_{29}	δ_{30}	δ_{31}	δ_{48}	δ_{49}	δ_{50}	δ_{51}	δ_{52}	δ_{53}	δ_{54}	δ_{55}
δ_{41}	δ_{25}	δ_{56}	δ_{27}	δ_{58}	δ_{29}	δ_{60}	δ_{63}	δ_{30}	δ_{17}	δ_{48}	δ_{19}	δ_{50}	δ_{21}	δ_{52}	δ_{55}	δ_{22}
δ_{42}	δ_{26}	δ_{59}	δ_{56}	δ_{25}	δ_{30}	δ_{31}	δ_{60}	δ_{61}	δ_{18}	δ_{51}	δ_{48}	δ_{17}	δ_{22}	δ_{23}	δ_{52}	δ_{53}
δ_{43}	δ_{27}	δ_{26}	δ_{57}	δ_{56}	δ_{31}	δ_{62}	δ_{29}	δ_{60}	δ_{19}	δ_{18}	δ_{49}	δ_{48}	δ_{23}	δ_{54}	δ_{21}	δ_{52}
δ_{44}	δ_{28}	δ_{61}	δ_{62}	δ_{63}	δ_{56}	δ_{25}	δ_{26}	δ_{27}	δ_{20}	δ_{53}	δ_{54}	δ_{55}	δ_{48}	δ_{17}	δ_{18}	δ_{19}
δ_{45}	δ_{29}	δ_{28}	δ_{63}	δ_{30}	δ_{57}	δ_{56}	δ_{59}	δ_{26}	δ_{21}	δ_{20}	δ_{55}	δ_{22}	δ_{49}	δ_{48}	δ_{51}	δ_{18}
δ_{46}	δ_{30}	δ_{31}	δ_{28}	δ_{61}	δ_{58}	δ_{59}	δ_{56}	δ_{57}	δ_{22}	δ_{23}	δ_{20}	δ_{53}	δ_{50}	δ_{19}	δ_{48}	δ_{49}
δ_{47}	δ_{31}	δ_{62}	δ_{29}	δ_{28}	δ_{59}	δ_{58}	δ_{25}	δ_{56}	δ_{23}	δ_{54}	δ_{21}	δ_{20}	δ_{51}	δ_{50}	δ_{17}	δ_{48}
δ_{48}	δ_{0}	δ_{1}	δ_{2}	δ_{3}	δ_{4}	δ_{5}	δ_{6}	δ_{7}	δ_{8}	δ_{9}	δ_{10}	δ_{11}	δ_{12}	δ_{13}	δ_{14}	δ_{15}
δ_{49}	δ_{1}	δ_{32}	δ_{35}	δ_{2}	δ_{37}	δ_{4}	δ_{7}	δ_{38}	δ_{41}	δ_{8}	δ_{11}	δ_{42}	δ_{13}	δ_{44}	δ_{47}	δ_{14}
δ_{50}	δ_{2}	δ_{3}	δ_{32}	δ_{33}	δ_{38}	δ_{39}	δ_{4}	δ_{5}	δ_{42}	δ_{43}	δ_{8}	δ_{9}	δ_{14}	δ_{15}	δ_{44}	δ_{45}
δ_{51}	δ_{3}	δ_{34}	δ_{1}	δ_{32}	δ_{39}	δ_{6}	δ_{37}	δ_{4}	δ_{43}	δ_{10}	δ_{41}	δ_{8}	δ_{15}	δ_{46}	δ_{13}	δ_{44}
δ_{52}	δ_{4}	δ_{5}	δ_{6}	δ_{7}	δ_{32}	δ_{33}	δ_{34}	δ_{35}	δ_{44}	δ_{45}	δ_{46}	δ_{47}	δ_{8}	δ_{9}	δ_{10}	δ_{11}
δ_{53}	δ_{5}	δ_{36}	δ_{7}	δ_{38}	δ_{1}	δ_{32}	δ_{3}	δ_{34}	δ_{45}	δ_{12}	δ_{47}	δ_{14}	δ_{41}	δ_{8}	δ_{11}	δ_{10}
δ_{54}	δ_{6}	δ_{39}	δ_{36}	δ_{5}	δ_{2}	δ_{35}	δ_{32}	δ_{1}	δ_{46}	δ_{15}	δ_{12}	δ_{45}	δ_{42}	δ_{11}	δ_{8}	δ_{41}
δ_{55}	δ_{7}	δ_{6}	δ_{37}	δ_{36}	δ_{3}	δ_{2}	δ_{33}	δ_{32}	δ_{47}	δ_{46}	δ_{13}	δ_{12}	δ_{43}	δ_{42}	δ_{9}	δ_{8}
δ_{56}	δ_{8}	δ_{9}	δ_{10}	δ_{11}	δ_{12}	δ_{13}	δ_{14}	δ_{15}	δ_{32}	δ_{33}	δ_{34}	δ_{35}	δ_{36}	δ_{37}	δ_{38}	δ_{39}
δ_{57}	δ_{9}	δ_{40}	δ_{11}	δ_{42}	δ_{13}	δ_{44}	δ_{47}	δ_{14}	δ_{1}	δ_{32}	δ_{3}	δ_{34}	δ_{5}	δ_{36}	δ_{39}	δ_{6}
δ_{58}	δ_{10}	δ_{43}	δ_{40}	δ_{9}	δ_{14}	δ_{15}	δ_{44}	δ_{45}	δ_{2}	δ_{35}	δ_{32}	δ_{1}	δ_{6}	δ_{7}	δ_{36}	δ_{37}

δ_{59}	δ_{11}	δ_{10}	δ_{41}	δ_{40}	δ_{15}	δ_{46}	δ_{13}	δ_{44}	δ_{3}	δ_{2}	δ_{33}	δ_{32}	δ_{7}	δ_{38}	δ_{5}	δ_{36}
δ_{60}	δ_{12}	δ_{45}	δ_{46}	δ_{47}	δ_{40}	δ_{9}	δ_{10}	δ_{11}	δ_{4}	δ_{37}	δ_{38}	δ_{39}	δ_{32}	δ_{1}	δ_{2}	δ_{3}
δ_{61}	δ_{13}	δ_{12}	δ_{47}	δ_{14}	δ_{41}	δ_{40}	δ_{43}	δ_{10}	δ_{5}	δ_{4}	δ_{39}	δ_{6}	δ_{33}	δ_{32}	δ_{35}	δ_{2}
δ_{62}	δ_{14}	δ_{15}	δ_{12}	δ_{45}	δ_{42}	δ_{43}	δ_{40}	δ_{41}	δ_{6}	δ_{7}	δ_{4}	δ_{37}	δ_{34}	δ_{3}	δ_{32}	δ_{33}
δ_{63}	δ_{15}	δ_{46}	δ_{13}	δ_{12}	δ_{43}	δ_{42}	δ_{9}	δ_{40}	δ_{7}	δ_{38}	δ_{5}	δ_{4}	δ_{35}	δ_{34}	δ_{1}	δ_{32}

Observations:

- Any two distinct elements in the table define a third one uniquely. This imply that $\boldsymbol{S} \times S^0$ is a quasigroup.
- δ_0 is the two sided identity in the split extension of sedenions. It follows that $\boldsymbol{S} \times S^0$ is a loop.
- $\delta_n \delta_m \neq \delta_m \delta_n$, $m \neq n$.
- $(\delta_n)^2 = \delta_{32}$ when n = 1, 2, 3,… , 31, 33, …, 47, 49,…, 63 while $(\delta_n)^2 = \delta_0$ when n =

  0, 32 and 48.

We next determine whether C-loop identities are satisfied. We are also providing examples in different cases.

(a) *Left C-loop identity, (xx) (yz) = (x(xy))z*

Case 1: (trivial)

i. [(x, 0) (x, 0)][(y, 0)(z, 0)] = (xx, 0)(yz, 0) = (xxyz, 0)

ii. {(x, 0) [(x, 0) (y, 0)]} (z, 0) = [(x, 0) (xy, 0)](z, 0) = (xxy, 0)(z, 0) = (xxyz, 0)

⇒ [(x, 0)(x, 0)][(y, 0)(z, 0)] = { (x, 0)[(x, 0)(y, 0)] }(z, 0)

Case 2:

i. [(x, 0) (x, 0)] [(y, 1) (z, 0)] = (xx, 0)(yz*, 1) = (yz*xx, 1)

ii. {(x, 0) [(x, 0) (y, 1)] } (z, 0) = [(x, 0) (yx, 1)](z, 0) = (yxx, 1)(z; 0) = (yxxz*, 1) = (yz*xx, 1)

⇒ [(x, 0) (x, 0)][(y, 1)(z, 0)] = { (x, 0)[(x, 0)(y, 1)] } (z, 0)

Example.

- $(\delta_{13} \delta_{13})(\delta_{29} \delta_{35}) = \delta_{32}(\delta_{62}) = \delta_{30}$
- $(\delta_{13}(\delta_{13} \delta_{29})) \delta_{35} = (\delta_{13}(\delta_{48})) \delta_{35} = \delta_{61}(\delta_{35}) = \delta_{30}$

$\Rightarrow (\delta_{13} \delta_{13}) (\delta_{29} \delta_{35}) = (\delta_{13} (\delta_{13} \delta_{29})) \delta_{35}$

Case 3:

i. [(x, 0)(x, 0)][(y, 0)(z, 1)] = (xx, 0)(zy, 1) = (zyxx, 1)

ii. {(x, 0)[(x, 0)(y, 0)] } (z, 1) = [(x, 0)(xy, 0)](z, 1) = (xxy, 0)(z, 1) = (zxxy, 1)

⇒ [(x, 0)(x, 0)][(y, 0)(z, 1)] ≠ { (x, 0)[(x, 0)(y, 0)] } (z, 1)

Example.

- $(\delta_{37} \delta_{37})(\delta_{46} \delta_{60}) = \delta_{32}(\delta_{50}) = \delta_{18}$
- $(\delta_{37}(\delta_{37} \delta_{46})) \delta_{60} = (\delta_{37}(\delta_{43})) \delta_{60} = \delta_{46}(\delta_{60}) = \delta_{50}$

$(\delta_{37} \delta_{37})(\delta_{46} \delta_{60}) \delta_{60} \neq (\delta_{37}(\delta_{37} \delta_{46})) \delta_{60}$

Conclusion: The left C-loop identity fails.

(b) The right C-loop identity, x ((yz) z) = (xy) (zz)

Case 1:

i. (x, 0) { [(y, 0)(z, 0)](z, 0) }=(x, 0)[(yz, 0)(z, 0)] = (x, 0)(yzz, 0) = (xyzz, 0)

ii. [(x, 0) (y, 0)][(z, 0)(z, 0)]=(xy, 0)(zz, 0) = (xyzz, 0)

⇒ (x, 0) { [(y, 0)(z, 0)](z, 0) } = [(x, 0)(y, 0)][(z, 0)(z, 0)]

Case 2:

i. (x, 0) { [(y, 1) (z, 0)](z, 0) }=(x, 0)[(yz*, 1)(z, 0)] = (x, 0)(yz*z*x, 1) = (yxz*z*, 1)

ii. [(x, 0) (y, 1)][(z, 0)(z, 0)]=(yx, 1)(zz, 0) = (yxz*z*, 1)

⇒ (x, 0) { [(y, 1)(z, 0)](z, 0) } = [(x, 0)(y, 1)][(z, 0)(z, 0)]

Example.

- $\delta_{46}((\delta_{21} \delta_{11}) \delta_{11}) = \delta_{46}(\delta_{30} \delta_{11}) = \delta_{46}(\delta_{53}) = \delta_{59}$
- $(\delta_{46} \delta_{21})(\delta_{11} \delta_{11}) = \delta_{27}(\delta_{32}) = \delta_{59}$

  $\Rightarrow \delta_{46}((\delta_{21} \delta_{11}) \delta_{11}) = (\delta_{46} \delta_{21}) (\delta_{11} \delta_{11})$

Case 3:

i. (x, 1) { [(y, 0)(z, 0)](z, 0) } = (x, 1)[(yz, 0)(z, 0)] = (x, 1)(yzz, 0) = (xy*z*z*, 1)

ii. [(x, 1) (y, 0)][(z, 0)(z, 0)]=(xy*, 1)(zz, 0) = (xy*z*z*, 1)

⇒ (x, 1) { [(y, 0)(z, 0)](z, 0) } = [(x, 1)(y, 0)][(z, 0)(z, 0)]

Example.

- $\delta_{58}((\delta_{40} \delta_{43}) \delta_{43}) = \delta_{58}(\delta_{3} \delta_{43}) = \delta_{58}(\delta_{8}) = \delta_{50}$
- $(\delta_{58} \delta_{40})(\delta_{43} \delta_{43}) = \delta_{18}(\delta_{32}) = \delta_{50}$

$\Rightarrow \delta_{58} ((\delta_{40} \delta_{43}) \delta_{43}) = (\delta_{58} \delta_{40}) (\delta_{43} \delta_{43})$

Case 4;

i. (x, 1) { [(y, 1)(z, 0)](z, 0) }=(x, 1)[(yz*, 1)(z, 0)] = (x, 1)(yz*z*, 1) = (-zzy*x, 0) =(-y*xzz, 0)

ii. [(x, 1)(y, 1)][(z, 0)(z, 0)]=(-y*x, 0)(zz, 0) = (-y*xzz, 0)

⇒ (x, 1) { [(y, 1)(z, 0)](z, 0) }= [(x, 1)(y, 1)][(z, 0)(z, 0)]

Example.

- $\delta_{24}((\delta_{18} \delta_{11}) \delta_{11}) = \delta_{24}(\delta_{25} \delta_{11}) = \delta_{24}(\delta_{50}) = \delta_{42}$
- $(\delta_{24} \delta_{18})(\delta_{11} \delta_{11}) = \delta_{10}(\delta_{32}) = \delta_{42}$

$\Rightarrow \delta_{24}((\delta_{18}\,\delta_{11})\,\delta_{11}) = (\delta_{24}\,\delta_{18})\,(\delta_{11}\,\delta_{11})$

Case 5:

i. (x, 0) { [(y, 0) (z, 1)](z, 1) }=(x; 0)[(zy, 1)(z, 1)] = (x, 0)(-z*zy, 0) = (-xz*zy, 0) = (-xyz*z, 0)

ii. [(x, 0) (y, 0)] [(z, 1)(z, 1)]=(xy, 0)(-z*z, 0) = (-xyz*z, 0)

⇒ (x, 0) {[(y, 0) (z, 1)] (z, 1) } = [(x, 0)(y, 0)][(z, 1)(z, 1)]

Example.

- $\delta_{38}((\delta_{14}\,\delta_{60})\,\delta_{60}) = \delta_{38}(\delta_{18}\,\delta_{60}) = \delta_{38}(\delta_{46}) = \delta_{40}$
- $(\delta_{38}\,\delta_{14})(\delta_{60}\,\delta_{60}) = \delta_{8}(\delta_{32}) = \delta_{40}$

$\Rightarrow \delta_{38}((\delta_{14}\,\delta_{60})\,\delta_{60}) = (\delta_{38}\,\delta_{14})\,(\delta_{60}\,\delta_{60})$

Case 6:

i. (x, 1) { [(y, 0) (z, 1)](z, 1) }=(x, 1)[(zy, 1)(z, 1)] = (x, 1)(-z*z, 0) = (-xy*z*z, 1)

ii. [(x, 1) (y, 0)][(z, 1) (z, 1)]=(xy*, 1)(z*z, 0) = (-xy*z*z, 1)

⇒ (x, 1) { [(y, 0) (z, 1)](z, 1) } = [(x, 1)(y, 0)][(z, 1)(z, 1)]

Example.

- $\delta_{59}((\delta_{45}\,\delta_{52})\,\delta_{52}) = \delta_{59}(\delta_{57}\,\delta_{52}) = \delta_{59}(\delta_{13}) = \delta_{22}$
- $(\delta_{59}\,\delta_{45})(\delta_{52}\,\delta_{52}) = \delta_{54}(\delta_{32}) = \delta_{22}$

$\Rightarrow \delta_{59}((\delta_{45}\,\delta_{52})\,\delta_{52}) = (\delta_{59}\,\delta_{45})\,(\delta_{52}\,\delta_{52})$

Case 7:

i. (x, 0) { [(y, 1) (z, 1)](z, 1) }=(x, 0)[(-z*y, 0)(z, 1)] = (x, 0)(-zz*y, 1) = (zz*yx, 1) = (-yxzz*, 1)

ii. [(x, 0) (y, 1)][(z, 1)(z, 1)]=(yx, 1)(-z*z, 0) = (-yxzz*, 1)

⇒ (x, 0) { [(y, 1) (z, 1)] (z, 1)} = [(x, 0)(y, 1)][(z, 1)(z, 1)]

Example.

- $\delta_{13}((\delta_{27}\,\delta_{49})\,\delta_{49}) = \delta_{13}(\delta_{42}\,\delta_{49}) = \delta_{13}(\delta_{59}) = \delta_{54}$
- $(\delta_{13}\,\delta_{27})(\delta_{49}\,\delta_{49}) = \delta_{22}(\delta_{32}) = \delta_{54}$

$\Rightarrow \delta_{13}((\delta_{27}\,\delta_{49})\,\delta_{49}) = (\delta_{13}\,\delta_{27})\,(\delta_{49}\,\delta_{49})$

Case 8:

i. (x, 1) { [(y, 1) (z, 1)] (z, 1) }=(x, 1)[(-z*y, 0)(z, 1)] = (x, 1)(-z*zy, 1) = (yz*zx, 0) = (y*xz*z, 0)

ii. [(x, 1) (y, 1)] [(z, 1) (z, 1)] = (-y*x, 0) (-z*z, 0) = (y*xz*z, 0)

⇒ (x, 1) { [(y, 1) (z, 1)] (z, 1) } = [(x, 1) (y, 1)][(z, 1)(z, 1)]

Example.

- $\delta_{22}((\delta_{57}\,\delta_{49})\,\delta_{49}) = \delta_{22}(\delta_{40}\,\delta_{49}) = \delta_{22}(\delta_{25}) = \delta_{15}$
- $(\delta_{22}\,\delta_{57})(\delta_{49}\,\delta_{49}) = \delta_{47}(\delta_{32}) = \delta_{15}$

$\Rightarrow \delta_{46}((\delta_{21}\,\delta_{11})\,\delta_{11}) = (\delta_{46}\,\delta_{21})\,(\delta_{11}\,\delta_{11})$

Conclusion: The right C-loop identity holds.

In this section, we were testing the satisfaction of C-loop identities by split extension of sedenions. We have observed that they satisfy the right C-loop identity only.

V. CONCLUSION

The main objective in our paper was to determine if the sedenions form C-loops. We have also investigated whether the split extension of sedenions form C-loops. We have used the Cayley-Dickson doubling formula in the construction of the frame multiplication tables which we have used in testing the C-loop identities. We have found that the sedenions satisfy both the left and the right C-loop identities. The split extension of sedenions has only satisfied the right C-loop identity. As result of the mentioned observations, it is evident that the sedenions form C-loops. However, the split extension of sedenions does not form C-loops.

REFERENCES

[1] Cayley, A. (1945). On Jacobi's elliptic functions in reply to Rev.B.Brown; and on quaternions, Proc, Philos.Magazine 26,208-211.

[2] Fenyves, F. (1968). Extra loops1, Publ.Math.Debrecen,15,235-238.

[3] Graves, T. (1845). On a connection between the general theory of normal couples and the theory of complete quadratic functions of two variables, Phil.Mag.,26:315-320.

[4] Hamilton, W.R. (1844). On quaternions, or on a new system of imaginaries in algebra. Philosophical Magazine. Vol.25, n 3.p.489-495.

[5] Jenya, K.(2012). Cayley-Dickson Loops, a dissertation presented to the faculty of natural Sciences and Mathematics University of Denver.

[6] Magero, F. B. (2007). Split of Complex numbers quaternions and octonions basis. Kenyatta University, Nairobi, Kenya.

[7] Njuguna, L. N. (2012). Investigating Sedenion Extension loops and frames of hypercomplex numbers: Ph.D Thesis: Kenyatta University, Kenya

[8] Phillips, J. D. & Vojtechovsky, P. (2007). C-loops.An introduction. ArXiv preprint math/0701711 .