PROPERTIES AND CHARACTERIZATIONS OF NORMALOID, CONVEXOID, TRANSLOID AND SPECTRALOID OPERATORS IN HILBERT SPACES

George Kariuki Maina

A research project submitted in partial fulfillment of the requirements for the award of the degree of Masters of Science (Pure Mathematics) in the School of Pure and Applied Sciences, Kenyatta University

July 2021

DECLARATION

Declaration by the Candidate

This project is my original work and has not been published or presented for a degree award in any other university or any other award.

Signature............................

Date....................................

George Kariuki Maina

I56/37475/2017.

Declaration by the supervisor

This project has been submitted for examination with my approval as a University supervisor.

Signature...............................

Date.......................................

Dr.Mutie Kavila

Department of Mathematics and Actuarial Science,

Kenyatta University, Kenya.

DEDICATION

This work is dedicated to the community of researchers in operator theory.

ACKNOWLEDGEMENT

I am grateful to the Almighty God for the opportunity, strength and sound health in each and every step of this journey.

Also, I would like to thank my supervisor, Dr. Mutie Kavila for his helpful discussion, patience and constant support during this research.

Furthermore, I wish to acknowledge all my lecturers in the department of Mathematics and Actuarial Science , Kenyatta University who played an important role by equipping us rightly.

I would also like to extend my gratitude to my colleagues Dennis, Waweru, Moses and Kiprop for their support and encouragement in the entire period of my study.

Finally, my sincere gratitude to my family for their spiritual, mental and moral support without which I would not have gone this far.

ABSTRACT

There are several well-developed theorems, lemmas and propositions relating to normaloid, convexoid, spectraloid and transloid operators. It has been demonstrated that every normaloid operator is spectraloid and that every convexoid operator is spectraloid. In addition, several researchers have extended interesting and significant properties of these operators. However, they do not constitute deeper and useful generalizations and there still does not exist a comprehensive investigation of the properties under which these operators relate to one another. In this project, we have established the essential properties and characterizations of these operators. Moreover, we have given generalized results in a Hilbert space using relevant theorems, propositions and corollaries. Lastly, we have investigated conditions under which normaloid and convexiod operators relate to one another.

TABLE OF CONTENTS

NOTATIONS

σ_{app}	-	Approximate point spectrum
$\langle x, y\rangle$	-	The inner product of x and y on a Hilbert space H
$B(H)$	-	The Banach algebra of all bounded linear operators
$\mathbb{C}$	-	Set of complex numbers
$co\sigma(T)$	-	The convex hull of the spectrum of T
$\mathbb{H}$	-	Hilbert space over the complex numbers $\mathbb{C}$
$\sigma(T)$	-	The spectrum of T
$r(T)$	-	The spectral radius of T
$r(T)D$	-	The spectral radius of disc D
SVEP	-	Single valued extentsion property
T	-	A bound linear operator on a complex Hilbert space
$\parallel T \parallel$	-	The norm of T
$W(T)$	-	The numerical range of T
$\overline{W(T)}$	-	The closure of the numerical range of T
$\omega(T)$	-	The numerical radius of T
$WLOG$	-	Without Loss of Generality .

CHAPTER ONE

INTRODUCTION

This chapter is divided into several sections. In the first section, we collect background information related to normaloid, convexoid, transloid and spectraloid operators. The second part we highlight important definitions and concepts of these operators, mostly without proof. In the other two sections, we outline the objectives and significance of this study respectively.

1.1 Background information

Operator theory studies transformations between vector spaces studied in Functional Analysis. The most thorough history of operator theory was given by Jean Dieudonne in 1982. Today, many branches of analysis are inseparable from operator theory.

In recent years, mathematicians have made considerable progress in operator theory. They have analyzed various non-normal operators like hyponormal, paranormal, normaloid, transloid, convexoid and spectraloid operators by making them satisfy certain known properties.The intercommunication between these classes of operators intensifies the beauty of the subject invariably, and opens up ways for detailed study which is no doubt beneficial.

In our context, there are several theorems, propositions and corollaries in regard to normaloid, convexoid, transloid and spectraloid operators. For example, it is known that the class of normaloid and convexoid operators are both contained in the class of spectraloid. Furthermore , they have been classified into a series of inclusions that we have given in details under Section 2.3

The underlying problems in these operators is to look at the conditions under which an operator becomes normaloid, convexoid, transloid and spectraloid and to show the correlation between them. Therefore, we intend to extend our contributions by giving more in-depth and useful generalizations. We will also investigate the conditions under which normaloid and convexoid operators relate to one another.

1.2 Definition of terms

Definition 1.2.1. *Let X be a vector space over the complex scalars $\mathbb{C}$. If there exists a complex number $\langle x, y\rangle$ for each pair of vectors $x, y \in X$ satisfying $(i), (ii), (iii)$ and (iv) then $\langle x, y\rangle$ is said to be the inner product of x and y;*

(i.) $\langle x, x\rangle \geq 0$ *for all* $x \in X$ *and* $\langle x, x\rangle = 0$ *if and only if* $x = 0$.

(ii.) $\langle y, x\rangle = \overline{\langle x, y\rangle}$ *for all* $x, y \in X$.

(iii.) $\langle x + y, z\rangle = \langle x, z\rangle + \langle y, z\rangle$ *for all* $x, y, z \in X$.

(iv.) $\langle \lambda x, y\rangle = \lambda\langle x, y\rangle$ *for all* $x, y \in X$ *and* $\lambda \in \mathbb{C}$.

A complex vector space X having an inner product is said to be an inner product space, or a pre-Hilbert space.

Example 1. *For an infinite dimensional complex vector space, the appropriate inner product is, with $x = (x_1, x_2, x_3, ...)$ and $y = (y_1, y_2, y_3, ...)$ then,*

$$\langle x, y\rangle = \Sigma_{j=1}^{\infty} x_j \overline{y}_j.$$

The inner product establishes a geometry on a linear space quite similar to that of the Euclidean spaces. Inner product spaces are a generalization of Euclidean spaces to infinite dimensional spaces.

We thus define the norm of a vector by the immediate definition .

Definition 1.2.2. *A Pre-Hilbert Space is a normed vector space with the norm.*

$$||x|| = \langle x, x\rangle^{\frac{1}{2}}.$$

Definition 1.2.3. *A function T which maps H_1 and H_2 is called a linear operator if for all $x, y \in H_1$ and $\alpha \in \mathbb{C}$;*

$$T(x + y) = T(x) + T(y) \text{ and}$$
$$T(\alpha x) = \alpha(T(x)).$$

Definition 1.2.4. *The linear operator $T : H_1 \to H_2$ is called **bounded** if there exists $c > 0$ such that $\| Tx \| \leq c||x||$ for all $x \in H$. Norm of T is defined by*

$$||T|| = inf\{c > 0 : ||Tx|| \leq c||x|| \ ||x \in H\}.$$

Definition 1.2.5. *If $T \in B(H)$ then its adjoint T^* is the unique operator in $B(H)$ that satisfies $\langle Tx, y\rangle = \langle x, T^*y\rangle$ for all $x, y \in H$.*
*An operator $T \in B(H)$ is called **self-adjoint** if $T^* = T$.*

Let $T, S \in B(H)$. Then, the following properties hold;

$(T + S)^* = T^* + S^*$

$(\alpha T)^* = \overline{\alpha}\ T^*$

$(TS)^* = S^*T^*$

$T^{**} = T$.

Definition 1.2.6. *The numerical range of an operator T is the subset of the complex number $\mathbb{C}$, given by*

$$W(T) = \{\langle Tx, x\rangle, x \in H, \ ||x|| = 1\}.$$

Definition 1.2.7. *The numerical radius $w(T)$ of an operator T on H is given by*

$$w(T) = sup\{|\lambda|, \lambda \in W(T)\} = sup\{|\langle Tx, x\rangle|, \ ||x|| = 1\}.$$

Definition 1.2.8. *Convex set X is a collection of points (vectors x) having the following property. If $P1$ and $P2$ are any points in X, then the entire line segment $P1 - P2$ is also in X.*

Theorem 1.2.1. *(Hubner, 1995)* ***(Toeplitz-Hausdorf)*** *The numerical range $W(T)$ of an operator T is a convex set in the complex plane.*

Definition 1.2.9. *A subspace M is invariant for T if $T(M) \subseteq M$.*

Definition 1.2.10. ***(Browder's theorem)*** *A monotone linear relation with closed graph is maximal monotone if and only if its adjoint is monotone.*

Definition 1.2.11. *Let $T : X \to Y$ be a bounded linear operator on a normed complex space X. The spectrum $\sigma(T)$ is the set of $\lambda \in \mathbb{C}$ such that the operator $T - \lambda I$ is not invertible.*

Definition 1.2.12. *The spectral radius $r(T)$ of any operator is defined by*

$$r(T) = sup\{|\lambda| : \lambda \in \sigma(T)\}.$$

Definition 1.2.13. *The convex hull of the spectrum of T is the intersection of all convex supersets of $\sigma(T)$.*

Definition 1.2.14. *An operator is said to be a condition G_1 operator if*

$$||(T - \mu)||^{-1} = \frac{1}{d(\mu, \sigma(T))}, \ \forall \ \mu \in \sigma(T).$$

Definition 1.2.15. *An operator T is normaloid if and only if the spectral radius is equal to $||T||$ i.e $r(T) = ||T||$.*

equivalently, $||T^n|| = ||T||^n$ for all positive integers $n = 1, 2, 3, ...$
Also if $w(T) = ||T||$.

Definition 1.2.16. *An operator T is hereditarily normaloid if every part of it is normaloid.*

Definition 1.2.17. *An operator T is Totally hereditarily normaloid if it is hereditary normaloid and every invertible part of it has normaloid inverse. The class of all hereditarily normaloid operators from B(H) is denoted by HN, and the class of all totally hereditarily normaloid operators from HN is denoted by THN.*

Definition 1.2.18. *A convexoid operator is a bounded linear operator T on a complex Hilbert space H such that the closure of the numerical range coincides with the convex hull of the spectrum. i.e.* $\overline{W(T)} = co\sigma(T)$.

Definition 1.2.19. *An operator* $T : H \to H$ *is said to be* ***Transloid*** *if* $T - \mu$ *is normaloid for all complex numbers* μ.

Definition 1.2.20. *A bounded linear operator T on a Hilbert space H is said to be paranormal if:* $||T^2x|| \geq ||Tx\,||^2$ *for every unit vector x in H.*
T is $K^* -$ *paranormal for every positive k, if for every x in H,* $||T^kx|| \geq ||T^*x||^k$.
Finally,
T is K-paranormal if $||T^kx|| \geq ||Tx||^k$ $(k \geqslant 2)$ *for* $x \in H$, $||x|| = 1$.

Definition 1.2.21. *An operator T defined on a Hilbert space H is said to be hyponormal if* $T^*T - TT^* \geq 0$ *or equivalently if* $T^*T \geq TT^*$.

Definition 1.2.22. *A linear operator* $T : H \to H$ *is said to be normal if T commutes with its adjoint. i.e.* $T^*T = TT^*$.

Definition 1.2.23. *An operator T which maps a Banach space X into itself has the single valued extension property if the only analytic function which satisfies* $(\lambda I - T)f(\lambda) = 0$ *is* $f = 0$.

Definition 1.2.24. *A bounded operator T on H is said to be subnormal if T has a normal extension. In other words, T is subnormal if there exists a Hilbert space K such that H can be embedded in K .*

Definition 1.2.25. *The Operator T is called quasinormal if:* $T(T^*T) = (T^*T)T$.

Definition 1.2.26. *A linear operator $T : H_1 \to H_2$ is said to be positive operator if*

i.e $$T \geq 0 \ ; \langle Tx, x\rangle \geq 0 \quad \forall \ \ x \in H_1.$$

Definition 1.2.27. *T is called spectraloid if $r(T) = w(T)$.*

Theorem 1.2.2. ***(Cauchy-Schwarz Inequality)*** *For all x, y in an inner product space H, we have that*

$|\langle x, y\rangle| \leq ||x|| \cdot ||y||$.

Definition 1.2.28. *Let X and Y be normed spaces. A linear transformation $T : X \longrightarrow Y$ is* ***compact*** *if it maps bounded sets into relatively compact subset of Y. That is, T is compact if $\overline{T(A)}$ is compact in Y whenever A is bounded in X.*

1.3 Statement of the problem

In mathematics, the property of an operator refers to the inherited quality of a thing that the operator owns by its virtue of nature while characteristics refer to the quality of a thing by which the operator distinguishes itself from another.

These properties and characteristics have been studied for decades. Several researchers have also shown the correlations between normaloid, spectraloid and convexoid operators. For instance, Wintner(1929) defined a few properties such as every normaloid operator is spectraloid. Also, he stated that every convexoid operator is spectraloid.

There are several pending questions on how these operators relate to each other. Consequently, in an effort to contribute to this important field of knowledge, we derive the correlation between normaloid, convexoid, transloid and spectraloid operators and give generalized results that can be used to classify them.

1.4 Objectives of the study

1.4.1 General objective

To study the fundamental properties of normaloid, convexoid, transloid and spectraloid operators, and their general results in the Hilbert space.

1.4.2 Specific objectives

i. To show if T is normaloid operator, then T^n is normaloid for $n \geq 1$ and equivalently for convexoid, transloid and spectraloid operators.

ii. To extend corollaries on the characterizations of normaloid, convexoid,transloid and spectraloid operators.

iii. To provide substitute proofs of results obtained by other researchers on properties of normaloid, convexoid, transloid and spectraloid operators.

iv. To highlight conditions under which normaloid and convexoid operators relate to one another.

1.5 Significance of the study

Properties and characterizations of operators is a field that generates a lot of interest among mathematicians, chemists and physicists across the world. For instance, Hilbert spaces are useful for creating a conceptual bridge between mathematicians and physicists.

The latter developed a more intuitive notion of Hilbert spaces via application in quantum mechanics. Mathematicians approach Hilbert Spaces the way they do other entities. This is by a rigorous formal axiomatic way without necessarily tying it to anything in the physical world. Mathematicians and Physicists can use Hilbert Spaces as sort of "common ground," which they can revert to when discussing certain mathematical notions.

Consequently, this study has generalized and positively criticized some of the existing results in the area of normaloid, convexoid, transloid and spectraloid operators. By so doing, new results and concepts have been realized.

CHAPTER TWO

2.1 Introduction

In this chapter, we outline the literature review of this project in details. Also, we have shown inclusions between normaloid, convexoid,transloid and spectraloid operators.

2.2 Literature review

Wintner (1929) began a strategic attack in operators theory and introduced the class of normaloid operator whereby; for an operator T, such that for all $T \in B(H)$ we have $r(T) = ||T||$. Normaloid operators will be of great use in this project because they contain other classes of operators. Normaloid and paranormal operators are intimately connected since a paranormal operator is an immediate class example of normaloid operators. Istratessu (1967) introduced the notion of paranormal operators, naming them ”operators of class N”. Later, Furuta (1967) introduced the term ”paranormal operator”.

The class of paranormal operators can be seen as a generalization of other important classes of operators such as; hyponormal operators, subnormal operators and normal operators. The basic nature of hyponormal operators was noticed for the first time in 1950s by Halmos (1954). Berberian (1965) raised a question as to whether there exists a hyponormal compact operator which is not normal. It was proved by Stampfli (1965) that it is impossible to construct such an operator. On the other hand, it is possible to construct a normaloid compact operator which is not normal. So there arises a problem as to whether this result is true for non-normal operators. This problem was studied by Stampfli (1965).

In the above references, hyponormality and paranormality appear to be an important property when studying various properties of normaloid operators and will therefore be useful in this project.

Furuta and Takeda (1967) analyzed characterizations of normaloid operators defined by the equality $||T^n|| = ||T||^n$ for every natural number n. They extended this result and gave analogous characterizations of spectraloid operator and its generalization and defined two families of new classes of non-normal operators broader than the class of normaloid operators associating with these characterizations. Some of the results were as follows; T is spectraloid if only $||T^n_N|| = ||T||^n_N$ where $||T||_N$ is the new operator norm. Thus,

correspond to $r(T)$, $||T||^n_N$ and $||T||$ satisfying $r(T) \leq ||T||^n_N \leq ||T||$. Another remarkable result from the analogy was that if T is a spectraloid operator, then T^n is also spectraloid for every positive integer n.

The original concept of convexoid was introduced by Halmos (1967). He showed that for a bounded linear operator T on a complex Hilbert space H, T is convexoid if the closure of the numerical range coincides with the convex hull of its spectrum i.e. $\overline{W(T)} = co\sigma(T)$. Also, he defined and proved another useful result such that for any operator T, T is spectraloid if the numerical radius coincides with the spectral radius i.e. $w(T) = r(T)$. This shows that convexoid operators are spectraloid operators. Consequently, from the fact that normaloid operators are spectraloid, we will be able to show if normaloid and convexoid operators are independent.

Furuta and Namakoto (1969) discussed the tensor product of bounded linear operators on a complex Hilbert space H and gave conditions under which classes of normaloid, convexoid and spectraloid operators satisfy tensor products. For instance, if T and S are normaloids, then $S \otimes T$ is also normaloid . However, $S \otimes T$ is not always convexoid even if T and S are both convexoid.

On the numerical range of an operator by Furuta and Namakoto (1971), they proved an interesting result that for any operator T, T is convexoid if and only if $T-\lambda$ is spectraloid for every complex number λ. Consequently, after this collaboration of the two researchers, Furuta (1971) on certain convexoid operators gave alternative proof of the same theorem.

Furuta (1977) extended research in this field and studied the relationship between generalized growth conditions (condition G_1 operator) and several classes of convexoid operators. He proved several characterizations of convexoid operators in relation to condition G_1 operator. Putnam (1979) studied in-details the class of condition G_1 and listed operators that satisfy a G_1 conditions. For instance, any operator T that satifies condition G_1 is convexoid.

Khasbarbardar and Thakare (1978) discussed on spectraloid operators and similarities.The main results were; the conditions on spectraloid operators implying normality or uniticity and finally the similarity of an operator involving the inverse or adjoint. These led to the generalizations of some of the results of Sheth (1969) and William(1970).

Recently, Uchiyama and Tanahashi (2011) discussed the notion of hereditary normaloid operators. They showed that every hereditary normaloid has the single valued extension property (SVEP) and hence satisfy Browders Theorem. Also, they gave an example of a hereditarily normaloid operator which is not totally hereditarily normaloid (hence, not paranormal) and does not satisfy Weyl's theorem.

2.3 Operators inclusions

Many researchers have extended the significant properties of normal operators to different classes of non-normal operators. Broadly speaking, the known classes of non-normal operators can be classified into several inclusions as follows:

$$Self{-}adjoint \subseteq Normal \subseteq Quasinormal \subseteq Subnormal \subseteq Hypernormal \subseteq Paranormal$$

$\subseteq Normaloid \subseteq Spectraloid$

$$Self - adjoint \subseteq Normal \subseteq Quasinormal \subseteq Subnormal \subseteq Hyponormal \subseteq Transloid$$
$$\subseteq Convexoid \subseteq Spectraloid.$$

$$Self - adjoint \subseteq Normal \subseteq Quasinormal \subseteq Subnormal \subseteq Hyponormal \subseteq Transloid$$
$$\subseteq Normaloid \subseteq Spectraloid$$

$$Self - adjoint \subseteq Normal \subseteq Quasinormal \subseteq Subnormal \subseteq Hyponormal \subseteq condition G_1$$
$$\subseteq Spectraloid.$$

Therefore, we seek to derive the correlation between normaloid, convexoid, transloid and spectraloid operators and give generalized results that can be used to classify them further. In particular, this project seeks to investigate the relationship between normaloid and convexoid operators.

CHAPTER THREE

CONVEXOID OPERATORS

2.1 Introduction

In this section, we study some basic properties of the numerical range of an operator as the numerical range and convexoid operator are intimately connected. We will draw more information about convexoid operators from the numerical range. We begin with the definition of a convexoid operator.

A convexoid operator is a bounded linear operator T on a complex Hilbert Space H such that the closure of the numerical range coincides with the convex hull of its spectrum.i.e, By Definition 1.2.18 T is convexoid if $\overline{W(T)} = co\sigma(T)$.

Also, an operator T is convexoid if and only if

$$||(T-\mu)^{(-1)}|| \leq \frac{1}{d(\mu, Co\sigma(T))} \text{ for all } \mu \notin co\sigma(T).$$

An example of such an operator is a normal operator(or some of its generalizations.)

A closely related operator is a spectraloid operator: An operator whose spectral radius coincides with the numerical radius. Infact, an operator T is a convexoid if and only if $T - \lambda$ is a spectraloid for every complex number λ.

Let us discuss the notion of the numerical range before we demonstrate some of the characterizations of convexoid operators.

2.2 Numerical range

We study the basic details of the numerical range of an operator. By Definition 1.2.6 the numerical range $W(T)$ of an operator T on a Hilbert Space H is defined by;

$W(T) = \{\langle Tx, x\rangle, x \in H, ||x|| = 1\}$

The following theorem provides well known properties of the numerical range.

Theorem 2.2.1. *(Shapiro, 2004)) For all operators T on Hilbert space H we have that*

(i) $W(\alpha I + \beta T) = \alpha + \beta B(T)$ *for all* $\alpha, \beta \in \mathbb{C}$

(ii) $W(T^*) = \{\overline{\lambda} : \lambda \in W(T)\}$

(iii) $W(T) \leqslant ||T||$

(iv) $|\langle Tx, X\rangle| \leqslant w(T)||x||^2$

(v) $W(U^*TU) = W(T)$ *where U is unitary operator.*

Definition 2.2.1. *A set X is convex if for any two points x,y* $\in$ *X, we have that*

$z = tx + (1 - t)\, y \ \in \ X$ *for all* $t \in [0, 1]$.

We note that changing the conditions t $\in [0, 1]$ to t $\in \mathbb{R}$ would result in z describing a straight line through the point x and y.

The empty set and the set containing a single point are regarded as convex. We also note that the intersection of any family of X_i (finite or infinite) of convex set is convex.

Example 2. *In* $\mathbb{R}^2$*, the set* $X = \{(x_1, x_2) : x_1, x_2 \geq 0\}$ *is an example of a convex set.*

Remark 1. *Note that from Theorem 1.2.1, convexity is the most fundamental property of numerical range.*

2.2.1 Numerical radius

Associated with the numerical range is the numerical radii. By Definition 1.2.12

$w(T) = \sup\{|\lambda|, \lambda \in W(T)\}$.

In other words, the numerical radius of an operator $T \in B(H)$ is the radius of the smallest circle in the complex plane centered at the origin that encloses the numerical range of an operator T. It follows that the numerical radius of an operator T is the greatest distance between any point in the numerical range and the origin.

Consequently, one of the very important application of the numerical range is to bound the spectrum. Therefore, we purpose to show that the numerical range bounds the spectrum.

We do this by looking at the boundary of the spectrum. We note that the boundary of the spectrum is contained in the approximate point spectrum.

The approximate point spectrum consists of all those $\lambda \in \mathbb{C}$ for which there exists a sequence of unit vector x_n such that $||(T - \lambda I)|| \longrightarrow 0$.

As a result, it is already clear that the numerical range provides a background for the proceeding development.

The theorem below is an immediate consequence of the definitions above.

Theorem 2.2.2. *(Berberian, 1964) The closure of the numerical range of an operator includes the spectrum .*

Proof. Let $\lambda \in \sigma_{ap}(T)$ and x_n be a sequence of unit vectors such that $||(T - \lambda I)|| \longrightarrow 0$.

By Theorem 1.2.2 we have that;

$|\langle (T - \lambda I)x_n, x_n \rangle| \leq ||(T - \lambda I)x_n|| \; ||x_n|| \longrightarrow 0$

$|\langle (T - \lambda I)x_n, x_n \rangle| \leq ||(T - \lambda I)x_n|| \longrightarrow 0$

Thus $\langle Tx_n, x_n \rangle \longrightarrow \lambda$ and so $\lambda \in \overline{W(T)}$.

Which completes the proof. □

The following theorem will be important for the proceeding developments.

Theorem 2.2.3. *(Lee, 1996)* ***Spectral mapping theorem***

Let $\sigma(T)$ be the spectrum of an operator T and $p(T)$ be any polynormial of an operator T. then,

$\sigma(p(T)) = p(\sigma(T))$.

Example 3. *Let T be defined as follows.*

if $M = \begin{pmatrix} 2 & 1 \\ 6 & 1 \end{pmatrix}$ *then* $\sigma(T) = \{4, -1\}$.

From Theorem 2.2.3, we have that ;

$\sigma(T^2) = (\sigma(T))^2 = \{4^2, (-1)^2\} = \{16, 1\}$.

Convexoid operators have been split into several pieces of operators. This includes the classes of self-adjoint, normal, hyponormal, conditon G_1 and such operators like seen in the inclusions above. Therefore, we attempt to tackle some of the results closely related to convexoid operators.

2.3 Characterization of convexoid operators

Lemma 2.3.1. *(Blumenson,1963) For any operator T, the following (i),(ii) and (ii) hold.*

(i) $\frac{1}{2}||T|| \leq w(T) \leq ||T||$

(ii) $r(T) \leq w(T) \leq ||T||$.

(iii) $||(T-\mu)^{-1}|| = \dfrac{1}{d(\mu, \sigma(T))} \leq \dfrac{1}{d(\mu, co\sigma(T))}$ *for all* $\mu \notin \sigma(T)$ *to the first inequality and for all* $\mu \notin \overline{W(T)}$ *to the second inequality.*

Proof. (i) By generalized polarization identity.

$(Tx, y) = \frac{1}{4}\langle T(x+y), (x+y)\rangle - \langle T(x-y), (x-y)\rangle + i\langle T(x+iy), (x+iy)\rangle - i\langle T(x-iy), (x-iy)\rangle$

$|(Tx, y)| \leq \frac{1}{4}||T||\ \{||x+y||^2 + ||x-y||^2 + ||x+iy||^2 + ||x-iy||^2\}$

by definition of w(T) that we obtain

$|(Tx, y)| \leq \frac{1}{4}w(T)\{||x+y||^2 + ||x-y||^2 + ||x+iy||^2 + ||x-iy||^2\}$

Now,

$||x+y||^2 + ||x-y||^2 = 2\langle x, y\rangle + 2\langle y, x\rangle$ and

$i||x+iy||^2 - ||x-iy||^2 = 2\langle x, y\rangle - 2\langle y, x\rangle$ i.e

$|(Tx, y)| \leq \frac{1}{4}w(T)\{2\langle x, y\rangle + 2\langle y, x\rangle + 2\langle x, y\rangle - 2\langle y, x\rangle\}$

$|(Tx, y)| \leq \frac{1}{4}w(T)\{2\langle x, y\rangle + 2\langle x, y\rangle\}$ i.e

$|(Tx, y)| \leq \frac{1}{4}w(T)4\{||x||^2 + ||y||^2\}$

$|(Tx, y)| \leq w(T)\{||x||^2 + ||y||^2\}$

Since $||T|| = Sup|(Tx, y)| : ||x|| = ||y|| = 1$,

then we have $||T|| \leq 2w(T) \implies \frac{1}{2}||T|| \leq w(T)$ and the first inequality holds.

The second inequality since $|\langle Tx, y\rangle| \leq ||T||\ ||x||^2$. Hence the (i) hold.

Proof (ii). From Theorem 2.2.2, we know that $\sigma(T) \subset \overline{W(T)}$ for any operator T.

This implies that $r(T) \leq w(T)$ ....(i)

Again, from (i) above $\frac{1}{2}||T|| \leq w(T) \leq ||T||$ ....(ii)

Applying above two results , we get

$\frac{1}{2}||T|| \leq r(T) \leq w(T) \leq ||T||$.

i.e $r(T) \leq w(T) \leq ||T||$.

Proof (iii)

$\frac{1}{d(\mu,\sigma(T))} = Sup\frac{1}{|\sigma(T)-\mu|}$ for any $\mu \notin \sigma(T)$ ......(2)

$= Sup\frac{1}{|\sigma(T-\mu)|}$ by Theorem 2.2.3 $(\sigma(p(T)) = p(\sigma(T)))$

$= sup|\sigma(T-\mu)^{-1}|$ by Theorem 2.2.3 $(\sigma(p(T^{-1})) = p(\sigma(T))^{-1})$

$= r(T-\mu)^{-1}$ So that first inequality of (iii) follows by (2)

and $r((T-\mu)^{(-1)}) \leq ||(T-\mu)^{(-1)}||$ by (ii) above. □

The immediate class of convexoid operator is the class that satisfies the growth condition (G_1). Luecke proves the following theorem which gives a method of construction of operators satisfying the condition (G_1).

Theorem 2.3.1. *(Luecke, 1972) If A is an operator acting on a Hilbert space H, then there is an operator B acting on a Hilbert space $\mathbb{R}$ such that their direct sum $T = A \oplus B$ acting on $H \oplus \mathbb{R}$ satisfies the condition (G_1).*

Here we give a more general method of the same.

Corollary 2.3.1. *If A is any operator and $B \in G_1$ with $\overline{W(A)} \subset \sigma(B)$, Then $T = A \oplus B \in G_1$*

Proof. We have $\sigma(T) = \sigma(A) \bigcup \sigma(B) = \sigma(B)$. For $\mu \notin \sigma(T)$

$$||(T - \mu I)^{-1}|| = max\{||(A - \mu I)^{-1}||, ||(B - \mu I)^{-1}||\}.$$

$$= max\left\{\frac{1}{d(\mu, W(A))}, \frac{1}{d(\mu, W(B))}\right\}$$

$$= \frac{1}{d(\mu, W(B))}$$

$$= \frac{1}{d(\mu, W(T))}. \quad \therefore \quad T \in G_1$$

□

Theorem 2.3.2. *(Putnam, 1979) If T is a condition G_1 then T is a convexoid.*

Proof

Let T be a condition G_1 operator then we have

$||(T - \mu)^{-1}|| = \dfrac{1}{d(\mu, \sigma(T))}$ *for $\mu \notin co\sigma(T)$. It follows that,*

$||(T - \mu)^{-1}|| = \dfrac{1}{d(\mu, \sigma(T))} \leq \dfrac{1}{d(\mu, co\sigma(T))}$ *by Lemma 2.3.1.*

i.e

$||(T - \mu)^{-1}|| \leq \dfrac{1}{d(\mu, co\sigma(T))}$

i.e T is a convexoid operator by definition of Convexoid operators.

Consequently, being inspired by Corollary 2.3.1 we give a method to construct convexoid operator as follows.

Corollary 2.3.2. *If A is any operator and B is a convexoid such as $\overline{W}(A) \subset \overline{W}(B)$, then $T = A \bigoplus B$ is convexoid.*

Proof. By definition of convexoid, $\overline{W}(B) = co\sigma(B)$ we have

$\overline{W(T)} = co\{\overline{W}(A) \bigcup \overline{W}(B)\} = co\overline{W}(B) = co\sigma(B)$

$= co\{\sigma(A) \bigcup \sigma(B)\} = co\sigma(T)$

so that T is convexoid. □

Theorem 2.3.3. *(Istrăţescu, 1967) Let T be a hyponormal operator. Then the following properties hold.*

(i.) $T - \mu$ *is also hyponormal for any* $\mu \in \mathbb{C}$

(ii.) T^{-1} *is also a hyponormal operator if* T^{-1} *exists*

(iii.) T *is a condition* G_1 *operator.*

Proof (i)

T is hyponormal $\Rightarrow T^*T \geq TT^*. \Rightarrow T^*T - TT^* \geq 0$

Now, $(T-\mu)^*(T-\mu) - (T-\mu)(T-\mu)^* \geq 0$

$\Rightarrow (T^*T - \overline{\mu}T - \mu T^* + |\mu|^2) - (TT^* - \overline{\mu}T^* - \mu T + |\mu|^2) \geq 0$

$\Rightarrow T^*T - TT^* - \overline{\mu}T + \overline{\mu}T - \overline{\mu}T^* + \overline{\mu}T^* + |\mu|^2 - |\mu|^2 \geq 0$

$\Rightarrow T^*T - TT^* \geq 0$

Therefore $T - \mu$ is also hyponormal for any $\mu \in \mathbb{C}$.

Proof (ii)

Since hyponormality is preserved under translation, This implies that

$T^*T - TT^* \geq 0$ and hence

$0 \leq T^{-1}(T^*T - TT^*)T^{*-1} = T^{-1}T^*TT^{*-1} - I.$

Now since $K \geq I$ implies that $K^{-1} \leq 1$ we have that;

$I - T^*T^{-1}T^{*-1}T \geq 0$ and hence

$T^{*-1}T^{-1} - T^{-1}T^{*-1} = T^{*-1}(I - T^*T^{-1}T^{*-1}T)T^{*-1} \geq 0.$

$\implies T^{*-1}T^{-1} - T^{-1}T^{*-1} \geq 0 \quad \implies T^{*-1}T^{-1} \geq T^{-1}T^{*-1}$

$\implies T^{-1}$ is hyponormal.

Proof (iii)

Let $\lambda \in \rho(T)$, $x \in H$ and $||x|| = 1$. Then,

$$||(T-\lambda I)^{-1}x|| \leq ||(T-\lambda I)^{-1}|| = max\,\{|\omega| :\ \omega \in \sigma((T-\lambda I)^{-1})\}$$
$$= \frac{1}{min\,\{|\omega| :\ \omega \in \sigma(T-\lambda I)\}}$$
$$= \frac{1}{min\,\{|\omega-\lambda| :\ \omega \in \sigma(T)\}}$$
$$= \frac{1}{d(\mu, \sigma(T)}$$

. Therefore, $||(T-\mu)^{-1}|| = \dfrac{1}{d(\mu, \sigma(T))}$.

where $d(\lambda - \sigma(T)) = min\,\{|\lambda - \omega| :\ \omega \in \sigma(T)\}$.

Hence T is a condition G_1. i.e the resolvent of T has exactly first order rate of growth with respect to the spectrum of T. Thus proved.

Corollary 2.3.3. *Every hyponormal operator is convexoid.*

Proof. Let T be hyponormal.

This implies that $T^*T - TT^* \geq 0$.

It follows that $T-\mu$ is hyponormal by Theorem 2.3.3 (i).

$\Rightarrow T-\mu$ is normaloid.(every hyponormal operator is normaloid).

$\Rightarrow T-\mu$ is spectraloid.(every normaloid operator is spectraloid).

(An operator T is convexoid if and only if $T-\lambda$ if spectraloid for all $\lambda \in \mathbb{C}$), T turns out to be convexoid. Hence the theorem hold.

□

Corollary 2.3.4. *Every normal operator is convexoid.*

Proof. Let T be a normal operator. Then, $TT^* = T^*T$.

From the fact that every normal operator is hyponormal, it follows that T is hyponormal.

By Corollary 2.3.3, T is convexoid.

□

We have shown that hyponormal operators are convexoid. Now we cite counter example to show that convexoid operator need not be hyponormal.

Example 4. *Let C and D be as* $\begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}$ *and* $\begin{pmatrix} 2 & 1 \\ 1 & 1 \end{pmatrix}$ *respectively , and give T, T^2 by the following infinite matrices.*

$$T = \begin{pmatrix} . & . & . & . & . & . & . & . & . & . \\ . & . & . & C^{\frac{1}{2}} & 0 & 0 & 0 & . & . & . \\ . & . & . & 0 & C^{\frac{1}{2}} & 0 & 0 & . & . & . \\ . & . & . & 0 & 0 & C^{\frac{1}{2}} & 0 & . & . & . \\ . & . & . & 0 & 0 & 0 & D^{\frac{1}{2}} & . & . & . \\ . & . & . & 0 & 0 & 0 & 0 & D^{\frac{1}{2}} & . & . \\ . & . & . & . & . & . & . & . & . & . \end{pmatrix}$$

Where 0 shows the place of the (0,0) matrix. Now, $det(D) = 1$ and $det(C) = 0$.
$\Longrightarrow$ *$D \geq C$ which confirm that T is hypornomal.*
But this does not hold for $D^2 \geq C^2$. Basing on this fact, we can ascertain that T is hyponormal but T^2 is not so. For example,

$$T^2 = \begin{pmatrix} . & . & . & . & . & . & . & . & . & . \\ . & . & . & C & 0 & 0 & 0 & 0 & . & . \\ . & . & . & 0 & C & 0 & 0 & 0 & . & . \\ . & . & . & 0 & 0 & C & 0 & 0 & . & . \\ . & . & . & 0 & 0 & 0 & D^{\frac{1}{2}} & C^{\frac{1}{2}} & . & . \\ . & . & . & 0 & 0 & 0 & 0 & D & . & . \\ . & . & . & . & . & . & . & . & . & . \end{pmatrix}$$

$$D^2 = \begin{pmatrix} 2 & 1 \\ 1 & 1 \end{pmatrix} \begin{pmatrix} 2 & 1 \\ 1 & 1 \end{pmatrix} = \begin{pmatrix} 5 & 3 \\ 3 & 2 \end{pmatrix}$$

$$C^2 = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}$$

$$D^2 - C^2 = \begin{pmatrix} 4 & 3 \\ 3 & 2 \end{pmatrix} \ngeq 0 \Longrightarrow D^2 \ngeq C^2 \text{ although } D \geq 0, C \geq 0$$

In our problem $D > C$, But $D^2 \ngeq C^2$. So that T is hyponormal, and T^2 is not hyponormal. Next we show that this T^2 is convexoid as follows

$D = \begin{pmatrix} 2 & 1 \\ 1 & 1 \end{pmatrix}$ is a positive operator on the two dimensional space E.

The proper values of D are, $\frac{3+\sqrt{5}}{2}, \frac{3-\sqrt{5}}{2}$

Put $\mu = \frac{3+\sqrt{5}}{2}$ then we have $1 < \mu$

$||T|| = \sqrt{\mu}$ *and* $||T^2|| = \mu$

Let $v = (v_1, v_2)$ be the eigenvectors of D. For the eigenvalue μ *and* $\Psi = (v_1, 0)$, 0=(0,0)

$$|D - \lambda I| = \begin{vmatrix} 2-\lambda & 1 \\ 1 & 1-\lambda \end{vmatrix} = 0$$

$(2-\lambda)(1-\lambda) - 1 = 0$

$2 - 3\lambda + \lambda^2 - 1 = 0$

$\lambda^2 - 3\lambda + 1 = 0$

$\lambda = \frac{3 \pm \sqrt{9-4}}{2} = \frac{3 \pm \sqrt{5}}{2}$

$\Psi = \frac{3-\sqrt{5}}{2}$, $\mu = \frac{3-\sqrt{5}}{2}$ $\quad ||T|| = \sqrt{\frac{3-\sqrt{5}}{2}}$ $\quad ||T^2|| = \frac{3-\sqrt{5}}{2}$

$(T - \lambda I)v = 0$

$$\begin{pmatrix} 2-\lambda & 1 \\ 1 & 1-\lambda \end{pmatrix} \begin{pmatrix} v_1 \\ v_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

when $\lambda = \mu$

$$\begin{pmatrix} 2-\mu & 1 \\ 1 & 1-\mu \end{pmatrix} \begin{pmatrix} v_1 \\ v_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

$(2-\mu)v_1 + v_2 = 0$

$v_1 + (1-\mu)v_2 = 0$

This guarantees that every complex number λ such that $1 < |\lambda| < \mu$ is in the spectrum $\sigma(T^2)$ and so the convex hull of the spectrum coincides with the disc $\{z : |z| \le \mu\}$. On the other hand since $||T^2|| = \mu$ the numerical range of T^2 is contained in this disc. Hence T^2 is convexoid.

Remark 2. *We shall omit the proofs of some results in this chapter, as they can be obtained from the corresponding results of the next chapters. However, lets us give a characterization of convexoid in terms of quadratic polynomial.*

Corollary 2.3.5. *Every quadratic polynomial in T and T^* is convexoid.*

Proof

Assume that every quadratic polynomial in T and T^* i.e. the operator of the form

$Q = aT^2 + bTT^* + cT^*T + dT^{*2} + eT + fT^* + gI(a, b, \cdots, g, complex)$.

being complex constants is spectraloid.Then for any complex λ.

$Q - \lambda = aT^2 + bTT^* + cT^*T + dT^{*2} + eT + fT^* + g^{'}I(a, b, \cdots, g, complex)$ turns out to be spectraloid in view of the assumption. By applying a well known and often used characterization due to Furuta-Nakamoto (1971)(An operator T is convexoid if and only if $T - \lambda$ is spectraloid for all complex number λ).

We conclude that Q is convexoid. i.e Every quadratic polynomial in T and T^* is convexoid.

CHAPTER FOUR

NORMALOID OPERATORS

3.1 Introduction

In this chapter, we investigate conditions for an operator to be normaloid. Also, we investigate the relationship and inclusion between some sub-classes in normaloid operators. A well known sub-class of normaloid operators is the paranormal operator. It is introduced as an intermediate class between hyponormal and normaloid operators.

3.2 Paranormal operators

Recall by Definition 1.2.20, a bounded linear operator T on a Hilbert space is paranormal if $||Tx||^2 \leq ||T^2x||; ||x|| = 1$ for all $x \in H$.

Note that there are operators that are normaloid but non-paranormal. See Example 5.

Theorem 3.2.1. *(Halmos, 1967)* $r(T) = ||T||$ *if and only if* $||T||^n = ||T^n||$ *for n=1,2,3,...*

Proof

Assume that T is normaloid. (i.e) $||T|| = r(T)$.

We have that $||T^n|| \leq ||T||^n$............(i)

For, since $||ST|| \leq ||S||\ ||T||$ for all S,T$\in$ B(H),

$\therefore$ by induction $||T^n|| \leq ||T||^n$.

Now, we show that $||T^n|| \geq ||T||^n$.

Since $||T|| = r(T)$, then $||T||^n = r(T)^n$. By Theorem 2.2.3, $r(T)^n = r(T^n)$.

Thus by Lemma 2.3.1, $||T||^n = r(T)^n = r(T^n) \leq w(T^n)$.

Again by Lemma 2.3.1,

$||T||^n = r(T)^n = r(T^n) \leq w(T^n) \leq ||T^n||$.........(ii)

$\therefore$ by (i) and (ii), $||T||^n = ||T^n||$.

Conversely,

Assume that $||T||^n = ||T^n||$.

$||T^n||^{\frac{1}{n}} = [||T||^n]^{\frac{1}{n}} \quad [By\ ||T||^n = ||T^n||]$

$\implies ||T^n||^{\frac{1}{n}} = ||T||.$

Then we have $r(T) = \lim_{n\to\infty} ||T^n||^{\frac{1}{n}} = ||T||.$

i.e T is normaloid.

Theorem 3.2.2. *(Furuta, 1967) Every paranormal operator is normaloid.*

Proof

Taking $||x|| = 1$, *we have* $||Tx||^2 \leq ||T^2x||||x||$; $\forall x \in H, \ ||x|| = 1$

Hence $||T||^2 \leq ||T^2||$ *Since we always have* $||T^2|| \leq ||T||^2$,

we conclude that; $||T||^2 = ||T^2||$

By an induction argument we have that $||T||^n = ||T^n||$.

By Theorem 3.2.1, we now conclude that T *is normaloid.*

We now give an example of non-convexoid, non-paranormal normaloid operator.

Example 5. *Let A be an infinite matrix of the form*

$$\begin{pmatrix} 1 & 0 & 0 & 0 & . & . & . \\ 0 & M & 0 & 0 & . & . & . \\ 0 & 0 & M & 0 & . & . & . \\ 0 & 0 & 0 & M & . & . & . \\ 0 & 0 & 0 & 0 & . & . & . \\ . & . & . & . & . & . & . \end{pmatrix}, \text{ where } M=\begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}.$$

Then A is non-convexoid, non-paranormal,normaloid because;

$$A^2 = A * A = \begin{pmatrix} 1 & 0 & 0 & 0 & . & . & . \\ 0 & M & 0 & 0 & . & . & . \\ 0 & 0 & M & 0 & . & . & . \\ 0 & 0 & 0 & M & . & . & . \\ 0 & 0 & 0 & 0 & . & . & . \\ . & . & . & . & . & . & . \end{pmatrix} * \begin{pmatrix} 1 & 0 & 0 & 0 & . & . & . \\ 0 & M & 0 & 0 & . & . & . \\ 0 & 0 & M & 0 & . & . & . \\ 0 & 0 & 0 & M & . & . & . \\ 0 & 0 & 0 & 0 & . & . & . \\ . & . & . & . & . & . & . \end{pmatrix}$$

$$= \begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 & . & . & . \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & . & . & . \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 & . & . & . \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & . & . & . \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & . & . & . \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & . & . & . \\ . & . & . & . & . & . & . & . & . & . \end{pmatrix}^2$$

$$= \begin{pmatrix} 1 & 0 & 0 & 0 & \dots \\ 0 & 0 & 0 & 0 & \dots \\ 0 & 0 & 0 & 0 & \dots \\ 0 & 0 & 0 & 0 & \dots \\ 0 & 0 & 0 & 0 & \dots \\ . & . & . & . & \dots \end{pmatrix}$$

and $||A||^n = ||A^n||$=1. i.e $||A||^2 = ||A^2||$=1. Impying that T is normaloid. However, the relation $||A^2x|| \geqslant ||Ax||^2$ does not hold for the unit vector

$e_2(0, 1, 0, 0, ...), e_4(0, 0, 0, 1, 0, 0...)$,.. For example, we verify

$||A^2x|| \geqslant ||Ax||^2$ does not hold.

$$A^2x = A^2e_2 = \begin{pmatrix} 1 & 0 & 0 & 0 & \dots \\ 0 & 0 & 0 & 0 & \dots \\ 0 & 0 & 0 & 0 & \dots \\ 0 & 0 & 0 & 0 & \dots \\ 0 & 0 & 0 & 0 & \dots \\ . & . & . & . & \dots \end{pmatrix} \begin{pmatrix} 0 \\ 1 \\ 0 \\ 0 \\ 0 \\ . \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ . \end{pmatrix} \therefore ||A^2x|| = ||A^2e_2|| = 0 \text{ . Again,}$$

$$Ax = Ae_2 = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 \end{pmatrix} \begin{pmatrix} 0 \\ 1 \\ 0 \\ 0 \\ 0 \\ 0 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 1 \\ 0 \\ 0 \\ 0 \end{pmatrix} \therefore ||Ax||^2 = ||Ae_2||^2 = 1$$

Clearly, $0 \ngeqslant 1 \implies ||A^2x|| \ngeqslant ||Ax||^2$. Hence A is non-paranormal.

Therefore, this example shows that there is normaloid operator that is non-paranormal.

A is non-convexoid. In fact $\overline{W(A)}$ is the closed convex set spanned by the disc

$\{\lambda : |\lambda| \leq \frac{1}{2}\}$ and one point 1,

$\sigma(A) = \{0\} \bigcup \{1\}$ so the convex hull of $\sigma(A)$ is the closed interval $[0, 1]$.

$\therefore \overline{W(A)} \neq Co\sigma(A)$.

Hence A is non-convexoid, non-paranormal, normaloid.

Remark 3. *It was seen earlier in Chapter two that hyponormal operators are convexoid operators. Now, since the class of paranormal operator contain the class of hyponormal operators, the following theorem demonstrate that hyponormal operators are normaloid operators.*

Theorem 3.2.3. *(Putnam, 1965) Every hyponormal operator is normaloid.*

Proof. Let $T \in B(H)$ be hyponormal operator in Hilbert space H.

Claim 1. $||T^n||^2 \leq ||T^{n+1}||||T^{n-1}||$ for every positive integer n.

First note that for any operator $T \in B(H)$

$||T^n x||^2 = \langle T^n x, T^n x\rangle = |\langle T^* T^n x, T^{n-1}x\rangle| \leq ||T^* T^n x||\ ||T^{n-1}x||$

for $n \geq 1$ and every $x \in H$(by Theorem 1.2.2). Now if T is hyponormal, then

$||T^* T^n x||\ ||T^{n-1}x|| \leq ||T^{n+1}x||\ ||T^{n-1}x|| \leq ||T^{n+1}||\ ||T^{n-1}||\ ||x||^2$

and hence for each $n \geq 1$

$||T^n x||^2 \leq ||T^{n+1}||\ ||x||^2$, which ensures the claim result.

Claim 2.

$||T^n|| = ||T||^n$ for every $n \geq 1$.

The above result holds trivially if T $= 0$ and it is also holds trivially for n=1.

Let $T \neq 0$ and suppose the above result holds for some $n \geq 1$. By claim 1 we get

$||T||^{2n} = (||T||^n)^2 = ||T^n||^2 \leq ||T^{n+1}||\ ||T^{n-1}|| \leq ||T^{n+1}||\ ||T||^{n-1}$.

Therefore, as $||T^n|| \leq ||T||^n$ *and since* $T \neq 0$,

$||T||^{n+1} = ||T||^{2n}\ (||T||^{n-1})^{-1} \leq ||T^{n+1}|| \leq ||T||^{n+1}$.

Hence $||T^{n+1}|| = ||T||^{n+1}$.

Then the claimed result holds for $n + 1$ whenever it holds for n, which concludes the proof of claim 2 by induction.

Therefore $||T^n|| = ||T||^n$ for each $n \geq 1$ by Claim 2, and so T is normaloid. □

3.3 Some conditions on an operator implying normality

A normal operator is known to satisfy a variety of conditions (normaloid, convexoid, hyponormal). Now, if $r(T)$ is the spectral radius of an operator T, defined by

$$r(T) = sup\{|\lambda| : \lambda \in \sigma(T)\},$$

$0 \leqslant r(T) \leqslant ||T||$; then the spectral mapping theorem implies that $r(T^n) = (r(T))^n$ for every positive integer n. It frequently turns out that the spectral radius of an operator is easy to compute even when it is hard to find the spectrum; the tool that makes it easy is the following definition.

Definition 3.3.1. *For each operator T,* $r(T) = \lim_{n\to\infty} ||T^n||^{\frac{1}{n}}$
in the sense that the indicated limit always exists and has the indicated value.

It is a consequence of this result that if A and B are commutative operators, then

$$r(AB) \leqslant r(A)r(B).$$

It is a somewhat less easy consequence, but still a matter of no more than a little fussy analysis with inequalities, that if A and B commute, then

$$r(A + B) \leqslant r(A) + r(B)$$

If no commutativity assumptions are made, then two-dimensional examples, such as

$$\text{A}=\begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}, \text{B}=\begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}$$

show that neither the submultiplicative nor the subadditive property persists.

Now, we give some of the results of this assertion.

Lemma 3.3.1. *(Furuta, 2001)*

$||T|| = \sup\{||Tx|| : ||x|| = 1\} = \sup\{||Tx|| : ||x|| \leqslant 1\}$

$\sup\{|\langle Tx, y\rangle| : ||x|| \leqslant 1, ||y|| \leqslant 1\}$ *See (Furuta, 2001).*

Note that, self adjoint operators are the building blocks of all operators. Infact, they are by far the most important subclass of all bounded operators on a Hilbert space. Therefore, self adjoint operators arise naturally in the discussion of normal operators. We now proceed to make some observations about self adjoint operators.

Theorem 3.3.1. *(Furuta, 2001)* $||T|| = ||T^*||$ *for all operators* T *on a Hilbert space* H.

$|\langle Tx, y\rangle| = |\langle x, T^*y\rangle| \leqslant ||x||||T^*y||.$

When the supremum on the left hand side is taken we obtain the result $||T|| \leqslant ||T^*||$*.*

From this result we have

$||T^*|| \leqslant ||(T^*)^*|| = ||T||$

hence $||T^*|| = ||T||$*. From Theorem 3.3.1, we give the following corollary.*

Corollary 3.3.1. *A bounded linear operator* T *is normal if and only if* $||T^*x|| = ||Tx||$*, for every* $x \in H$*.*

Proof

Now $||T^*x|| = ||Tx||$, for every $x \in H$

$||T^*x||^2 = ||Tx||^2$, for every $x \in H$

$\langle T^*x, T^*x\rangle = \langle Tx, Tx\rangle$, for every $x \in H$

$\langle TT^*x, x\rangle = \langle T^*Tx, x\rangle$, for every $x \in H$

$\langle (TT^* - T^*T)x, x\rangle = 0$, for every $x \in H$

Which implies that $TT^* - T^*T = 0 \Rightarrow TT^* = T^*T$

Corollary 3.3.2. *If* T *is any operator on a Hilbert space* H *then*

$||T^*T|| = ||TT^*|| = ||T||^2.$

Proof

$||T^*|| = ||T||$ *by Theorem 3.3.1*

$\therefore ||T^*T|| \leq ||T^*||\ ||T|| = ||T||\ ||T|| = ||T||^2$

i.e $||T^*T|| \leq ||T||^2$*...........(i)*

On the other hand

$||Tx||^2 = \langle Tx, Tx\rangle = \langle T^*Tx, x\rangle$ *by Theorem 1.2.2*

$\leq ||T^*T||\ ||x||^2$

i.e $||Tx||^2 \leq ||T^*T||\ ||x||^2$

$\implies ||T||^2 \leq ||T^*T||$*...........(ii)*

From (i) and (ii) $||T^*T|| = ||T||^2$

Theorem 3.3.2. *(Sheth, 1969) Let* $T \in B(H)$*. Then* T *is normaloid if it is self-adjoint.*

Proof

Since T is self-adjoint, $||T^*T|| = ||TT|| = ||T^2||$, $||T^*T|| = ||T||^2$.

We have $||T^2|| = ||T||^2$ and by induction $||T^{2n}|| = ||T||^{2n}$

Implying that $||T|| = \lim_{n\to\infty} ||T^{2n}||^{\frac{1}{2n}}$.

We have $||T|| = \lim_{n\to\infty} ||T^{2n}||^{\frac{1}{2n}} = r(T)$

$\implies$ T is normaloid

Alternative Proof of the above Theorem 3.3.2

Proof

Let T be self-adjoint. Then $T = T^* \implies ||T|| = ||T^*||$

Now,

$||T|| = sup\{|\langle x, Tx\rangle| \ : \ ||x|| = 1\}=$

$w(T)$.

And this is a complete proof.

Note that every self-adjoint operator is normal. We now give an example to show that a normal operator need not be self adjoint.

Example 6. *Let $I : H \longrightarrow H$ be the identity mapping. Let $T = 2iI$.*

Then $T^* = (2iI)^* = -2iI$ and also $T^{-1} = -\frac{1}{2}iI$

So $T^*T = TT^* = 4I$ which satisfies normal operator. But comparing T, T^* and T^{-1} we have that; $T^* \neq T$ and $T^* \neq T^{-1}$. Therefore, T is normal but neither self adjoint nor unitary.

Corollary 3.3.3. *Let $T \in B(H)$ then T is normaloid if it is normal.*

Proof

Given T is normal $T^*T = TT^*$.

T commutes with T^* and T^*T is self-adjoint.

(i.e) $T^*T = (T^*T)^* = TT^*$

$\therefore (T^*T)^n = (TT^*)^n = T^{*^n}T^n$

So that $||T|| = ||T^*T||^{\frac{1}{2}} = \lim_{n\to\infty} ||(T^*T)^n||^{\frac{1}{2n}}$ [*By Theorem* 3.3.2]

$= \lim_{n\to\infty} ||T^{*^n}T^n||^{\frac{1}{2n}}$

$= \lim_{n\to\infty} ||T||^{\frac{1}{n}}$

$= r(T). \implies$ *T is normaloid.*

Corollary 3.3.4. *Let $T \in B(H)$ then T is normaloid if it is positive.*

Proof

Note that if T is self-adjoint then $\langle Tx, x\rangle \in \mathbb{R}$.

Now, if T is positive $\langle Tx, x\rangle \geq 0$

$\implies$ Every positive operator is self-adjoint.

Hence by Theorem 3.3.2, T is normaloid.

3.4 Some of the characteristics of normaloid operators

Theorem 3.4.1. *(Halmos,1967) The following assertions are mutually equivalent*

(i.) T is normaloid operator, that is $||T|| = r(T)$

(ii.) $||T^n|| = ||T||^n$

(iii.) $||T|| = \omega(T)$.

Proof

(i.) $\Rightarrow$ (ii.) and (ii.) $\Rightarrow$ (i.)

This proof has already been captured. See Threorem 3.2.1

To prove : $(i) \Rightarrow (iii)$

Assume T is normaloid.(i.e) $||T|| = r(T)$

$||T|| = r(T)$ *ensures* $||T|| = w(T)[\therefore\ r(T) \leq w(T) \leq ||T||$ *always holds*]

For, $||T|| = r(T) \leq w(T) \leq ||T||$

$\therefore ||T|| = w(T)$ by sandwich theorem

$||T|| = r(T)$ *and* $||T|| = w(T)$

Hence $||T|| = w(T)$ holds.

$(iii.) \Rightarrow (i.)$

Assume that $||T|| = r(T)$.

Claim: T is a normaloid . (i.e) $||T|| = r(T)$ We have to prove that, $||T|| = w(T) = 1$.

By the homogeneity of $w(T)$ and $r(T)$. Assume there is a sequence of unit vectors $\{x_n\}$ such that ;

$|\langle Tx_n, x_n\rangle| \to 1$ WLOG by multiplying a suitable constant modulus 1 since,

$|\langle Tx_n, x_n\rangle| \leq ||Tx_n|| \leq 1$ and $|\langle Tx_n, x_n\rangle| \to 1$

$= ||Tx_n - x_n||^2 = \langle Tx_n, x_n\rangle - \langle Tx_n, x_n\rangle$

$= \langle Tx_n, Tx_n\rangle - 2Re\langle Tx_n, x_n\rangle + \langle x_n, x_n\rangle$

$= ||Tx_n||^2 - 2Re\langle Tx_n, x_n\rangle + 1 \to 0$ so that 1 is an appropriate point spectrum of T.

Therefore, we obtain $r(T) = 1$.

Now we want to demonstrate an example to show that if T is normaloid, $T - \lambda$ need not be normaloid.

Example 7. *Consider a finite Hilbert space operator. Let* $H = \mathbb{C}^3$ *and*

$$T = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix}$$

$$T^2 = T \cdot T = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

T is normaloid since $||T^n|| = ||T||^n$ *i.e* $||T^2|| = ||T||^2$*=1*

Now, pick any λ *say 2. Then,*

$$T - \lambda = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix} - \begin{pmatrix} 2 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 2 \end{pmatrix} = \begin{pmatrix} -1 & 0 & 0 \\ 0 & -2 & 0 \\ 0 & 1 & -2 \end{pmatrix}$$

$$||(T - \lambda)||^2 = 3^2 = 9$$

$$(T - \lambda)^2 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 4 & 0 \\ 0 & -4 & 4 \end{pmatrix}$$

$$||(T - \lambda)^2|| = 8$$

Clearly, $||(T - \lambda)^2|| \neq (T - \lambda)||^2$

From the previous discussion, we give the next immediate result about the power of normaloid operators.

Corollary 3.4.1. *If T is a normaloid operator, then* T^k *is also normaloid for every positive integer k.*

Proof. Suppose that T is normaloid $\implies$ that $r(T) = ||T||$.

claim $r(T^k) = ||T^k||$.

Now, let us assume $r(T^k) = ||T^k||$ holds for k=n.

Let $k = n + 1$ then we have

$r(T^{(n+1)}) = sup\{|\lambda| \ : \ \lambda \in \sigma(T^{(n+1)})\}$

By Lemma 2.3.1 $(r(T^n) = r(T)^n \leq ||T^n||)$,

$\leq \lim_{m\to\infty} ||T^{m(n+1)}||^{\frac{1}{m}}$.

$(i.e)$ $r(T^{(n+1)}) \leq ||T^{(n+1)}||$. The reverse inequality holds.Impying that

$r(T^{(n+1)}) = ||T^{(n+1)}||$. Hence the proof. □

Corollary 3.4.2. *An operator T is normaloid if and only if its adjoint T^* is normaloid.*

Proof

From the fact that $T^{**} = T$, we can say immediately that, if T is normal then its adjoint T^* is also normal.

Now $||T^{*n}|| = ||T^n||$ for each $n \geq 1$ and it follows that $r(T^*) = r(T)$.

Remark 4. *We have shown that the class of hyponormal is contained in both classes of normaloid and convexoid operators. However, normaloid and convexoid operators are independent classes of operators. Here, we now demonstrate with examples that normaloid operator need not be convexoid and vice versa.*

3.5 Examples relating normaloid and convexoid operaters

Example 8. *We first give an example of a convexoid which is not normaloid.*

Let $M = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}$

and N be a normal operator whose spectrum is the closed disc D with the center and radius $\frac{1}{2}$.

if $T = \begin{pmatrix} M & 0 \\ 0 & N \end{pmatrix}$

Then $\sigma(T) = \{0\} \cup D = D$ and $W(T) = conv(W(M) \cup W(N)) = D \Rightarrow T$ is convexoid.

Since $||T|| = 1$ (in fact $||M|| = 1$), T is not normaloid.

Example 9. *We now give example of normaloid operator which is not convexoid.*

Let M be as given above.

if $A = \begin{pmatrix} M & 0 \\ 0 & 1 \end{pmatrix}$

Since $||A|| = 1$ and W(A)=conv(D∪{1}) it follows that W(A)=1 and hence A is normaloid since $||A|| = W(A) = 1$. However, $\sigma(A) = \{0\} \cup \{1\}$

so that con$\sigma(A) = [0,1] \neq \overline{W(A)}$. Hence A is not convexoid.

Example 10. *A normaloid operator need not be convexoid. Let $H = \mathbb{C}^3$ with the Euclidean norm given by*

$\| f \| = \| (f_1, f_2, f_3) \| = |f_1|^2 + |f_2|^2 + |f_3|^2.$

Let

$$T = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix}.$$

Then

$$Tf = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} f_1, & f_2, & f_3 \end{pmatrix} = \begin{pmatrix} f_2, & 0, & f_3 \end{pmatrix}$$

and $\| T^2 \| = 1$.

On the other hand, $< Tf, f >= f_2\overline{f_1} + f_3\overline{f_3} = 1$ *and consequently*

$\omega(T) = \sup\limits_{||f||=1} \{|f_2\overline{f_1} + f_3\overline{f_3}|\} = 1$ *by taking* $f = (0, 0, 1)$.

Hence T is normaloid.

This shows that $\sigma(T) = \{0\} \cup \{1\}$ *and* $\overline{W(T)}$ *is a closed convex set spanned by the disc* $\lambda : |\lambda| \leq \frac{1}{2}$ *and one point 1. Hence T is not convexoid.*

Example 11. *A convexoid operator need not be normaloid. Let* $\{x_1, x_2, ...\}$ *be an orthornormal base for* $H = \iota_2$. *Define* $z_n = x_{2n+1}, n = 1, 2, 3, ...$ *and*

$z_{\sim} = x_{2n}\, n = 1, 2, 3, ...$ *Every x and H can be written as*

$$x = \sum_{k=-\infty}^{\infty} \alpha_k z_k = 1$$

Let us now define the operator S on H by

$$Sx = \frac{1}{2}\sum_{k=-\infty}^{\infty} \alpha_k z_{k+1} = 1$$

where

$$x = \sum_{k=-\infty}^{\infty} \alpha_k z_k = 1$$

we can check easily that

$W(T) = \{\lambda \in \mathbb{C} : |\lambda| \leq \frac{1}{2}\}.$

Let us define operator $L = \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}$ *on* $\mathbb{C}^2$.

The operator defined on $H \bigoplus \mathbb{C}^2$ *by*

$$T(f, g) = (Lf, Sg)$$

yields $W(T) = \{\lambda \in \mathbb{C} : |\lambda| \leq \frac{1}{2}\} = co\sigma(T)$.

T is not normaloid since

$||T|| = 1$ *and* $w(T) = \frac{1}{2}$.

Example 12. *A normaloid operator need not be convexoid. In $H = \mathbb{C}^3$, consider the operator*

$$T = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix}$$

We have as in Example 10 above, that $\sigma(T) = \{0, 1\}$ *and* $W(T) = co\{\sigma(T), S\}$, *where*

$$S = \left\{\lambda \in \mathbb{C} : |\lambda| \leq \tfrac{1}{2}\right\}.$$

Finally, the following result gives a further explanation of normaloid operator in terms of direct sum of operators.

Corollary 3.5.1. *Let A be any operator and B be normaloid operator such that $||A|| \leq ||B||$ then $T = A \bigoplus B$ is normaloid.*

Proof. Take an operator B such as $||B|| \geq ||A|$.

Then, $r(T) = max\{r(A), r(B)\} = r(B) = ||B|| = ||T||$

and T is normaloid. □

CHAPTER FIVE

TRANSLOID OPERATORS

4.1 Introduction

In this section, we will study the class of transloid operators. Also, we will investigate the relationship between transloid, normaloid and convexoid operators. From Definition 1.2.19 of transloid operators, we have the following immediate thereom.

Theorem 4.1.1. *(Fujii and Nakamoto, 1973) T is transloid if $T - \lambda$ is normaloid for every complex number μ.*

Proof. Suppose that T is normaloid and by Definition 1.2.12 we have

$$\begin{aligned} r(T-\lambda) &= sup\,\{|\lambda| : \lambda \in \sigma(T-\lambda)\} \\ &= sup\,\{|\lambda| : \lambda \in \sigma(T) - \lambda\} \\ &= sup\,\{|\lambda| : \lambda \in \sigma(T^*) - \lambda\} \\ &= sup\,\{|\lambda| : \lambda \in \sigma(T^* - \lambda)\} \\ &= r(T^* - \lambda) = ||T^* - \lambda|| \\ &= ||T - \lambda|| \text{ since } T^* = T. \end{aligned}$$

This shows that $T - \lambda$ is normaloid and hence T is transloid

□

Now, the question arises, if T is transloid, is T^n transloid ? We answer this question using the following result.

Corollary 4.1.1. *If T is transloid, then T^n is a transloid operator.*

Proof. It follows by hypothesis of Theorem 4.1.1 that;

T^n is transloid if $T^n - \lambda$ is normaloid for every complex number λ.

Now suppose $T^n - \lambda$ is normaloid.

Then by induction we assume that $n = k$ is true .ie

$r(T^k - \lambda) = sup\left\{|\lambda| : \lambda \in \sigma(T^k - \lambda)\right\}$

$r(T^k - \lambda) = ||T^k - \lambda||$

Now, let $n = k + 1$. we have that;

$r(T^{k+1} - \lambda) = sup\left\{|\lambda| : \lambda \in \sigma(T^{k+1} - \lambda)\right\}$

$= sup\left\{|\lambda| : \lambda \in \sigma(T^{(k+1)}) - \lambda\right\}$

$= sup\left\{|\lambda| : \lambda \in \sigma(T^{*(k+1)}) - \lambda\right\}$

$= sup\left\{|\lambda| : \lambda \in \sigma(T^{*(k+1)} - \lambda)\right\}$

$= r(T^{*(k+1)} - \lambda) = ||T^{*(k+1)} - \lambda||$

$= ||T^{(k+1)} - \lambda||$.

Therefore , $T^n - \mu$ is normaloid and hence T^n is transloid □

Transloid operators is a class of both normaloid and convexoid operators. Hence we list the inclusion that follows;

$Normal \subset Transloid \subset Convexoid \subset Spectraloid.$

$Normal \subset Transloid \subset Normaloid \subset Spectraloid.$

Remark 5. *From the previous discussion, we have given notable results relating to hyponomal with convexoid and normaloid operators. Now, we proceed by giving another important result relating hyponormal and transloid operators. This leads us to the next immediate result.*

Theorem 4.1.2. *(Fujii and Nakamoto, 1973) Let T be a hyponormal operator, then T is a transloid operator.*

Proof

From the fact that T is hyponormal, it follows that $T - \mu$ is hyponormal for any $\mu \in \mathbb{C}$ by Theorem 2.3.3 (i). Therefore, $T - \mu$ is normaloid for any $\mathbb{C}$, (every hyponormal operator is normaloid). From the definition of a transloid operator, we have that T is transloid operator.

Definition 4.1.1. *Spectral sets. A (closed) set S in the plane is a spectral set for an operator T if* $\sigma(T) \subset S$*.*
Aand $||f(T)|| \leqslant ||f||_s$ *for any rational function f with poles off S, where*
$||f||_s = sup\{|f(\lambda)| : \lambda \in S\}$.

4.2 Characterizations of transloid operators

The following theorems are fundamental;

Theorem 4.2.1. *(Schreiber, 1963)*$\{\lambda; |\lambda - \mu| \leqslant k\}$ *is the spectral set for any operator T if and only if* $||\lambda - \mu|| \leqslant k$*.*

Theorem 4.2.2. *(Schreiber, 1963) An operator T is a normaloid if and only if* $r(T)D$ *is a spectral set for T.*

By Theorem 4.2.2, we shall give a characterization of transaloids in the following theorem

Theorem 4.2.3. *(Fuji and Nakamoto, 1973) An operator T is a transloid if and only if any disk containing the spectrum* $\sigma(T)$ *is a spectral set for T.*

Proof. First we note that S is a spectral set for T if and only if $S - \lambda = \{\mu - \lambda; \mu \in S\}$ is a spectral set for $T - \lambda$. Hence $r(T - \lambda)D$ is a spectral set for $T - \lambda$ if and only if $r(T - \lambda)D + \lambda$ is a spectral set for T. By Theorem 4.2.2, it suffices to prove that $r(T - \lambda)D + \lambda$ is a spectral set for T for any λ if and only if any disk containing $\sigma(T)$ is a spectral set for T. Suppose that $r(T - \lambda)D + \lambda$ is a spectral set for T for any λ and D' which is a disk containing $\sigma(T)$ with the center λ'. Since $r(T - \lambda')D + \lambda' \subset D'$ we conclude that D' is a spectral set for T. Conversely, suppose that any disk containing $\sigma(T)$ is a spectral set for T. Since $r(T - \lambda)D + \lambda$ is a disk and contains $\sigma(T)$ for any λ, we have that $r(T - \lambda)D + \lambda$ is a spectral set for T. □

By Theorem 4.2.3 and Theorem 4.2.2, we have an another characterization of transaloids operators.

Theorem 4.2.4. *(Fuji and Nakamoto, 1973) An operator T is a transloid if and only if any disk containing $W(T)$ is a spectral set for T.*

We prove this theorem by completely analogous method to the proof of Theorem 4.2.3, replacing r and σ by ω and $\overline{W}$ respectively. This theorem is a consequence of Theorem 4.2.3 by the fact that a disk contains $\sigma(T)$ if and only if it contains $W(T)$.

Now, we show the correlation between convexoid and transloid operator.

Corollary 4.2.1. *If T is a transloid operator, then T is a convexoid.*

Proof. Let $T - \lambda$ be normaloid for any $\mu \in \mathbb{C}$. Then, $T - \mu$ is spectraloid for any $\mu \in \mathbb{C}$ since a normaloid operator is a spectraloid. Therefore, T is convexoid
by Theorem 5.3.1. □

Now, we conclude this chapter with the following result.

Corollary 4.2.2. *Let A be any operator and B be transloid operator such that*
$||(A - \lambda I)|| \leq |||(B - \lambda I)||$ *for all complex number λ.*
Then $T = A \bigoplus B$ is transloid.

Proof. Since $T - \lambda I = (A - \lambda I) \bigoplus (B - \lambda I)$ and
$(B - \lambda I)$ is normaloid by
$(A - \lambda I) \leq (B - \lambda I)$
then $T - \lambda$ is normaloid by Corollary 3.5.1. Hence T is transloid. □

We have earlier seen if T is a condition G_1 then T is a convexoid (see Theorem 2.3.2). Also, if T is transloid then it is convexoid. However, a convexoid operator need not be condition G_1. Therefore, we demonstrate with example that it does not hold.

Example 13. *There exists a transloid operator which is not G_1 and vice versa.*

Proof. Let $T = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}$, then $\sigma(T) = \{0\}, \overline{W(T)} = \dfrac{D}{2}$ where D is the unit disc.
Let B be a normal operator with $\sigma(B) = C$ where C is unit circle in complex plane.

For $T = A \bigoplus B$, we show $T \notin G_1$ but T is transloid.
Now, since $|A + \lambda| = \sqrt{\frac{2p+1+\sqrt{4p+1}}{2}}$ where $p = |\lambda|^2$ and

$|B + \lambda| = |\lambda| + 1$

Thus $|A + \lambda| \leq |B + \lambda|$. Hence T is transloid by Corollary 4.2.2

For the converse we can consider finite dimensional normal operator with more that one point in $\sigma(T)$ and $dim \leq 2$ □

CHAPTER SIX

SPECTRALOID OPERATORS

5.1 Introduction

In this chapter, we will study the class of spectraloid operators and the relationship between normaloid, convexoid and spectraloid operators. Throughout this chapter, let $||T||_N$ be the new norm equivalent to the operator norm $||T||$. Now, by power inequality:

$||T^n|| \leq ||T||^n, \quad ||T^n||_N \leq ||T||_N^n \quad , \quad r(T^n) \leq r(T)^n$

and $r(T^n) = r(T)^n$ by spectral mapping theorm (see Theorem 2.2.3).

Now, we notice that; An operator T is spectraloid if $||T||_N = r(T)$. This definitions is fundamental for the next immediate theorem.

Theorem 5.1.1. *(Furuta and Takeda, 1967) T is a spectraloid operator if and only if* $||T^n||_N \leq ||T||_N^n$.

Proof. If T is spectraloid, then $||T||_N^n = r(T^n) = r(T)^n \leq ||T^n||_N$, the reverse inequality is due to the power inequality, so $||T^n||_N = ||T||_N^n$.

If $||T^n||_N = ||T||_N^n$ holds, then

$||T^n||_N = ||T||_N^n \leq ||T^n||$

$\Rightarrow ||T||_N \leq ||T^n||^{\frac{1}{n}}$ so

$\Rightarrow ||T||_N \leq \lim_{n\to\infty} ||T^n||^{\frac{1}{n}} = r(T)$, thus

$||T||_N = r(T)$ because the reverse inequality is valid by $r(T) \leq ||T||_N \leq ||T||$. □

Corollary 5.1.1. *If T is a spectraloid operator, then T^n is also spectraloid for every positive integer n.*

Proof. Let T be spectraloid, then the following equality holds by the above Theorem, $||T^n||_N = ||T||_N^n = r(T^n) = r(T)^n$. Hence the proof. □

We now show that the class of spectraloid operators contain both the normaloid and convexoid operators.

Theorem 5.1.2. *(Halmos,1967) For any operator T, every normaloid operator is spectraloid.*

Proof. Let T be normaloid. i.e $r(T) = ||T||$.
From Theorem 3.4.1, for any operator T, $r(T) = ||T||$ if and only if $w(T) = ||T||$
i.e $r(T) = ||T|| = w(T)$
$\implies r(T) = w(T)$
Hence T is a spectraloid operator.

□

Theorem 5.1.3. *(Halmos,1967) For any operator T, every convexoid operator is spectraloid.*

Proof. For any operator T, we know that $\sigma(T) \subseteq \overline{W(T)}$.
Thus we have $r(T) \leq \sigma(T)$ for any T.
Now the closed disc with center 0 and radius $r(T)$ includes $\sigma(T^n)$ and is convex.
Hence if T is convexoid then that disc contains
$W(T) \Rightarrow \omega(T) \leq r(T)$.
Hence, $r(T) = \omega(T)$ i.e. T is spectraloid.

□

Remark 6. *We have shown that the class of normaloid and convexoid operators are both contained in the class of spectraloid operators. However, the converse implication is not true.*

Example 14. *It is important to note that a spectraloid operator is not necessarily normaloid.*

$$Let\ T = \begin{pmatrix} 1 & 0 & 0 \\ 1 & 0 & 0 \\ 0 & 2 & 0 \end{pmatrix}$$

We have $||T|| = 2$ and $\omega(T) = r(T) = 1 \Rightarrow ||T|| \neq r(T)$.

5.2 Characterizations of spectraloid operators

Theorem 5.2.1. *(Halmos,1967)The following statements are equivalent.*

(i.) T is a spectraloid operator

(ii.) $w(T^n) = w(T)^n$ for every natural number n.

Proof. **(i.)** $\Rightarrow$ **(ii.)** : Assume T is spectraloid.(i.e) $r(T) = w(T)$.

Claim : $w(T^n) = w(T)^n$ For this we have only to prove

$w(T^n) \geq w(T)^n$ since reserve inequality always holds. since $||ST|| \leq ||S||\ ||T||$ for all S,T$\in$ B(H).

$\therefore$ By induction $w(T^n) \leq w(T)^n$.

By hypothesis,

$w(T)^n = r(T)^n$ and by Theorem 2.2.3 $r(T)^n = r(T^n)$, which shows that;

$w(T)^n = r(T)^n = r(T^n)$.

$w(T)^n = r(T)^n = r(T^n) \leq w(T^n)$

(i.e) $w(T)^n \leq w(T^n)$.

(ii.) $\Rightarrow$ **(i.)** Assume $w(T^n) = w(T)^n$ for all natural number n.

By Lemma 2.3.1, $r(T) \leq w(T) \leq ||T||$ for any operator T. We have $r(T) \leq w(T) \leq ||T||$.

$r(T) = r(T^n)^{\frac{1}{n}} \leq w(T^n)^{\frac{1}{n}}$ and

$[w(T)^n]^{\frac{1}{n}} = w(T) \leq ||T^n||^{\frac{1}{n}}$.

$\implies r(T) \leq ||T^n||^{\frac{1}{n}}$ and by

$r(T) = \lim_{n\to\infty} ||T^n||^{\frac{1}{n}} = ||T||$

We have that $w(T) = r(T)$. Hence the proof.

□

Now, we have discussed several properties and characterizations of spectraloid operators. Furthermore, we want to show that if T is spectraloid, then $T - \lambda$ is also spectraloid for $\lambda \in \mathbb{C}$.

Corollary 5.2.1. *If T is a spectraloid operator, then $T - \lambda$ is also spectraloid for $\lambda \in \mathbb{C}$.*

Proof. Assume T is spectraloid.(i.e) $r(T) = w(T)$.

Claim : $w(T^n - \lambda) = w(T)^n - \lambda$. The proof is similar to Theorem 5.2. For this, we have only to prove

$w(T^n - \lambda) \geq w(T)^n - \lambda$ since reserve inequality always holds.

For, since $||ST|| \leq ||S||\ ||T||$ for all S,T$\in$ B(H).

$\therefore$ By induction $w(T^n - \lambda) \leq w(T)^n - \lambda$.

By hypothesis,

$w(T)^n - \lambda = r(T)^n - \lambda$ and by Theorem 2.2.3 $r(T)^n - \lambda = r(T^n - \lambda)$ which shows that;

$w(T)^n - \lambda = r(T)^n - \lambda = r(T^n - \lambda)$.

$w(T)^n - \lambda = r(T)^n - \lambda = r(T^n - \lambda) \leq w(T^n - \lambda)$

(i.e) $w(T)^n - \lambda \leq w(T^n - \lambda)$.

(i.e) $w(T)^n - \lambda = w(T^n - \lambda)$

□

Corollary 5.2.2. *Let A be any operator and B be a spectraloid operator such that; $w(A) \leq r(A)$, then $T = A \oplus B$ is spectraloid.*

Proof. Since B is spectraloid, $r(B) = w(B)$. Now,

$r(T) = max\{r(A), r(B)\} = r(B)$, since

$r(A) \leq w(A)$ and $r(B) \leq w(B)$ and

$w(T) = max\{w(A), w(B)\} = w(B) = r(B)$ because $w(A) \leq r(B) \leq w(B)$.

Thus $w(T) = r(T)$. (i.e) T is spectraloid. □

5.3 Relationship between convexoid, normaloid and spectraloid operators

Theorem 5.3.1. *(Furuta and Nakamoto, 1971) An operator T is convexoid if and only if $T - \lambda$ if spectraloid for all $\lambda \in \mathbb{C}$.*

Before proving Theorem 5.3.1, we have to prove the following lemma.

Lemma 5.3.1. *(Furuta, 1971)) If X is any bounded closed set in the complex plane, then;*
(i) coX=convex hull of X.

$= \{$ *The intersection of all circles which contain the set* $X\}$

$=\bigcap_{\mu} \{\lambda : \ |\lambda - \mu| \leq_{r \in X}^{sup} |x - \mu|\}$

(ii) $coX = \{$ *The intersection of all closed half planes which contain the set* $X\}$

$=\bigcap_{\theta} \{\lambda : Re\lambda^{i\theta} \geq_{\theta \in X}^{inf} ReSe^{i\theta}\}$.

Proof of Theorem 5.3.1

Taking X=$\overline{W(T)}$ and $\sigma(T)$ in (i) of Lemma 5.3.1, then the following (1) and (2) holds since $X = \overline{W(T)}$ is convex.

$\overline{W(T)} = \bigcap_{\mu} \{\lambda \ : \ |\lambda - \mu| \leq w|T - \mu|\}$..........(1)

$co\sigma(T) = \bigcap_{\mu} \{\lambda \ : \ |\lambda - \mu| \leq r|T - \mu|\}$..........(2)

From (1) and (2) , the proof follows from the relation

$co\sigma(T) - \lambda = co\sigma(T - \lambda)$ by spectral mapping theorem. (Theorem 2.2.3)

Remark 7. *From the above characterization theorem for convexoid operators, we have that if $T^n - \lambda$ is spectraloid for every λ, then T^n is a convexoid by induction. However, Luecke in 1972 gave a counter example as follows to show it does not hold.*

Example 15. *(Luecke, 1972) There is an operator T satisfying (G_1) such that $T^2 - \lambda$ is not a spectraloid for some λ.*

Proof. Let

$A = \begin{pmatrix} 1 & 0 \\ 1 & 0 \end{pmatrix}$

If N is a normal operator with $\sigma(N) = W(A)$,then $T = A \bigoplus N$ satisfies (G_1)

Clearly, $T^2 = A^2 \bigoplus N^2$ since $\sigma(N) \subset \frac{1}{2}D + 1$ and

$co\sigma(T^2) = co\sigma(T)^2$ we see that $co\sigma(T^2)$ is contained in the open right half plane. On the other hand, $W(A) = D + 1$, so that we have $0 \notin W(T^2) - co\sigma(T^2)$.Therefore, T^2 is not convexoid. Hence there is some such that $T^2 - \lambda$ is not a spectraloid □

It was seen earlier that normaloid operators need not be convexoid and vice versa. However, there are conditions under which these operators relate to one another. Therefore, we give one of the condition as follows.

Corollary 5.3.1. *An operator T is convexoid if* $T - \lambda I$ *is normaloid for all* $\lambda \in \mathbb{C}$.

Proof

Let $T - \lambda I$ be normaloid for all $\lambda \in \mathbb{C}$. By Theorem 5.1.2, it follows that $T - \lambda I$ is spectraloid.

i.e *normaloid* $\subset$ *spectraloid.*

Now, by Theorem 5.3.1, T is convexoid.

CHAPTER SEVEN

CONCLUSIONS AND RECOMMENDATIONS

6.1 Conclusions

In this research, we were to interpret theorems and propositions of normaloid, convexoid, transloid and spectraloid operators and give generalized results. For convexoid operators which we have discussed in Chapter three, we have given characterizations as well as the relationships between subclasses of convexoid operators and convexoid operators. Also, we have come up with proofs of Corollary 2.3.3 and 2.3.4. Moreover, we have presented alternative proofs and corollaries. Also, we have outlined several concrete examples to support our results.

In Chapter four, we have shown that a normaloid operator need not be convexoid and vice versa. We have supported this with several counter examples. We have also come up with proofs of Corollary 3.3.1 and 3.3.2

In Chapter five, we have given characterizations of transloid operators. For example, we have demonstrated that if any operator T is transloid, then T^n is also transloid. In addition, we have shown the relationship between transloid and convexoid operators.

Lastly, we have discussed and related normaloid, convexoid, transloid and spectraloid operators. This has been achieved through theorems and counter examples. For example, we have shown that spectraloid operators need not be normaloid. We have also shown that if T is spectraloid, then T^n is also spectraloid.

6.2 Recommendations

A relationship between normaloid and convexoid is unexplained area. Since we have investigated sub-classes and relationship between normaloid and convexoid operators, there is still a lot to be done especially investigating under which conditions is normaloid a subset of convexoid and vice-versa.

REFERENCES

Ando, T. (1972). *Operators with a norm conditions.* Acta Sci. Math. Szeged, 33, 169-178.

Aiena, P., **Aponte**, E., & **Balzan**, E. (2010). *Weyl type theorems for left and right polaroid operators.* Integral Equations and Operator Theory, 66(1), 1-20.

Berberian, S. K. (1964). T*he numerical range of a normal operator.* Duke Mathematical Journal, 31(3), 479-483.

Berberian, S. K. (1965). *An extension of Weyl's theorem to a class of not necessarily normal operator*s. The Michigan Mathematical Journal, 16(3), 273-279.

Berberian, S. K. (1970). *Some conditions on an operator implying normality.* Mathematische Annalen, 184(3), 188-192.

Blumenson, L. (1963). *A generalized polarization identity.* The American Mathematical Monthly, 70(3), 277-281.

Douglas, R. G., & **Rosenthal**, P. (1968). *A necessary and sufficient condition that an operator be normal.*

Dieudonné, J. (1982). *History of functional analysis.* Bull. Amer. Math. Soc, 7, 403-409.

Dragomir, S. S. (2006). Some inequalities for normal operators in Hilbert spaces. Acta Mathematica Vietnamica, 31(3), 291-300.

Furuta, T., & **Takeda**, Z. (1967). *A characterization of spectraloid operators and its generalization.* Proceedings of the Japan Academy, 43(7), 599-604.

Furuta, T., **Horie**, M., & **Nakamoto**, R. (1967). *A remark on a class of operators.* Proceedings of the Japan Academy, 43(7), 607-609.

Furuta, T. (1967). *On the class of paranormal operators.* Proceedings of the Japan Academy, 43(7), 594-598.

Furuta, T., & **Nakamoto**, R. (1969). *On tensor products of operators.* Proceedings of the Japan Academy, 45(8), 680-685.

Furuta, T. (1971). *Certain convexoid operators.* Proceedings of the Japan Academy, 47(SupplementI), 888-893.

Furuta, T., & **Nakamoto**, R. (1971). *On the numerical range of an operator.* Proceedings of the Japan Academy, 47(3), 279-284.

Furuta, T. (1973). *Some characterizations of convexoid operators.* Rev. Roum. Math. Pures Et Appl, 18(6), 892-900.

Fujii, M., & **Nakamoto**, R. (1973). *On some examples of non-normal operators.* Proceedings of the Japan Academy, 47(5), 458-463.

Furuta, T. (1977). *Relations between generalized growth conditions and several classes of convexoid operators.* Canadian Journal of Mathematics, 29(5), 1010-1030.

Furuta, T. (2001). *Invitation to linear operators: from matrices to bounded linear operators on a Hilbert space.* CRC Press.

Gustafson, K. E., & **Rao**, D. K. (1997). *Numerical range.* In Numerical Range (pp. 1-26). Springer, New York, NY.

Halmos, P. R. (1954). *Commutators of operators, II.* American Journal of Mathematics, 76(1), 191-198.

Hildebrandt, S. (1964). *The closure of the numerical range of an operator as spectral set.* Communications on Pure and Applied Mathematics, 17(4), 415-421.

Halmos, P. R. (1967). *A Hilbert space problem book, Prnceton.* K. J.

Hübner, M. (1995). *Spectrum where the boundary of the numerical range is not round.* The Rocky Mountain Journal of Mathematics, 1351-1355.

Istrăţescu, V. (1967). *On some hyponormal operators.* Pacific Journal of Mathematics, 22(3), 413-417.

Istrătescu, V. I. (1972). *On a class of normaloid operators.* Mathematische Zeitschrift, 124(3), 199-202.

Khasbardar, S. K., & **Thakare**, N. K. (1978). *Spectraloid operators, similarity and related results.* Indian J. pure appl. Marin, 9(5), 494-500.

Lee, W. Y., & **Lee**, S. H. (1996). *A spectral mapping theorem for the Weyl spectrum.* Glasgow Mathematical Journal, 38(1), 61-64.

Luecke, G. R. (1972). *Topological properties of paranormal operators on Hilbert space.* Transactions of the American Mathematical Society, 172, 35-43.

Orland, G. H. (1964). *On a class of operators.* Proceedings of the American Mathematical Society, 15(1), 75-79.

Okelo, B. N., & **Mogotu**, P. O. (2017). New conditions for contractivity of normaloid operators.

Putnam, C. R. (1965). *On the spectra of semi-normal operators.* Transactions of the American Mathematical Society, 119(3), 509-523.

Putnam, C. R. (1968). *The spectra of operators having resolvents of first-order growth.* Transactions of the American Mathematical Society, 133(2), 505-510.0.

Putnam, C. (1979). *Operators satisfying a G1 condition.* Pacific Journal of Mathematics, 84(2), 413-426.

Stampfli, J. G. (1962). Hyponormal operators. Pacific Journal of Mathematics, 12(4), 1453-1458.

Stampfli, J. G. (1965). *Hyponormal operators and spectral density.* Transactions of the American Mathematical Society, 117, 469-476.

Saitô, T., & **Yoshino**, T. (1965). *On a conjecture of Berberian.* Tohoku Mathematical Journal, Second Series, 17(2), 147-149.

Schreiber, M. (1963). *Numerical range and spectral set*s. The Michigan Mathematical Journal, 10(3), 283-288.

Sheth, I. H. (1969). *Normaloid operators.* Pacific j. Math, 28(3), 675-680.

Shapiro, J. H. (2004). *Notes on the numerical range.* Michigan state University, East Lansing, MI 48824-1027, USA.

Uchiyama A., & **Tanahashi**, K. (2011). *the single valued extension property for hereditarily normaloid operators.* Scientiae Mathematicae Japonicae, 74(2+ 3), 185-189.

Wintner, A. (1929). *Zur theorie der beschränkten bilinearformen.* Mathematische Zeitschrift, 30(1), 228-281.

Williams, J. P. (1970). *Finite operators.* Proceedings of the American Mathematical Society, 26(1), 129-136.