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Modeling the Effect of Inpatient Rehabilitation of Tobacco Smokers on Smoking Dynamics

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Authors' contributions

This work was carried out in collaboration between both authors. Both authors read and approved the final manuscript.

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Abstract

Aims:

1. Develop and analyze a mathematical model of the effect of inpatient rehabilitation of tobacco smokers on tobacco smoking using Kenya as a case study.
2. Perform stability analysis on the smoking free equilibrium point and endemic equilibrium point of the model.
3. Use numerical simulation to investigate the impact of inpatient rehabilitation of tobacco smokers on smoking.

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Tobacco smoking is a serious burden in Kenya and the world at large. Smoking harms nearly every organ of the body and affects the overall health of a person. Despite the overwhelming facts about the consequences of tobacco smoking, it remains a bad wont which is socially accepted and widely spread. In this research we numerically analyze the dynamics of smoking incorporating the impact of inpatient rehabilitation to curb the smoking habit. We first present a three-compartment model incorporating inpatient rehabilitation, then develop the system of ordinary differential equations governing the smoking dynamics. The basic reproduction number is determined using next generation matrix method. The model equilibria were

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computed and the stability analysis carried out. The results of stability analysis indicate that the disease-free equilibrium (DFE) is both locally and globally asymptotically stable for $R_S < 1$ while the endemic equilibrium is both locally and globally asymptotically stable for $R_S > 1$. Numerical simulations of model carried out with the help of MATLAB shows that, when rehabilitation is implemented effectively, it helps in minimization of smoking in the community.

Keywords: Tobacco smoking; inpatient rehabilitation; reproduction number; stability; numerical simulation.

1. Introduction

Tobacco smoking epidemic is among the leading health threats in the world today. It kills more than 7 million people in a year according to WHO [1]. About 6 million of these deaths are as a result of direct tobacco intake while about 1 million of the deaths are due to exposure of non-smokers to second hand smoke. According to CDC [2] report, 14.0% of all adults globally are cigarette smokers (about thirty-four million people), 15.8% of these being men and 12.2% being women. In Kenya, according to GATS [3] report, 2.5 million adults smoke tobacco and according to GYTS [4] report, 12.5% of boys and 6.7% of girls aged 13-15 years smoke tobacco. Users of tobacco die prematurely dispossessing family's income, raising health care cost and also impeding economic development.

According to CDC [2], of the more than 7000 tobacco chemicals, about 250 are harmful and about 70 of these chemicals cause cancer. Exposure to second hand smoke causes serious respiratory diseases, coronary heart diseases, lung cancer, sudden deaths in infants and low birth weights. No level of exposure to second hand tobacco smoke can be said to be safe. In 2004, 28% of second-hand smoke deaths were children. Every person should be able to breathe air free from tobacco smoke.

Tobacco smoking is still a burden to Kenya despite the overwhelming facts about the consequences of smoking. Hence there is need to take a strong action in order to minimize tobacco smoking. Setting up of inpatient rehabilitation centers for tobacco smokers is greatly believed to promote tobacco smoking cessation. The current research examines the impact of inpatient rehabilitation of tobacco smokers on the dynamics of tobacco smoking.

2. Model Formulation

We denote the total population by N(t) and divide it into three classes; potential smokers P(t), smokers S(t) and rehabilitated smokers R(t), as shown in Fig. 1.

Thus, the size of the population is given by

$$N(t) = P(t) + S(t) + R(t) \tag{2.1}$$

We assume that individuals in all classes die naturally at the rate μ and in addition, those in S(t) and R(t) classes die as a result of smoking at the rate σ. Individuals are recruited to potential smokers' class at the rate Λ, which is the birth rate. Potential smokers progress to the smoker's class at the rate βS, where β is the rate of contact between potential smokers and smokers. The function $\tau\beta$ ($0 < \tau < 1$) denotes the reduced rate of contact between potential smokers and smokers as result of inpatient rehabilitation of smokers, where τ is the rate at which smokers move to rehabilitation class. On the other hand, individuals in the inpatient rehabilitation class move to potential smokers' class upon recovery at the rate γ.

2.1 Assumptions of the model

i. The population birth and death rates are different.
ii. All the newly born individuals join only potential smokers' class.
iii. Smoking is due to contact between potential smokers and smokers
iv. One must be rehabilitated if and only if he/she is a smoker in order to quit.
v. Individuals in the rehabilitation center remain there until they stop smoking.

vi. There is no immunity upon recovery.

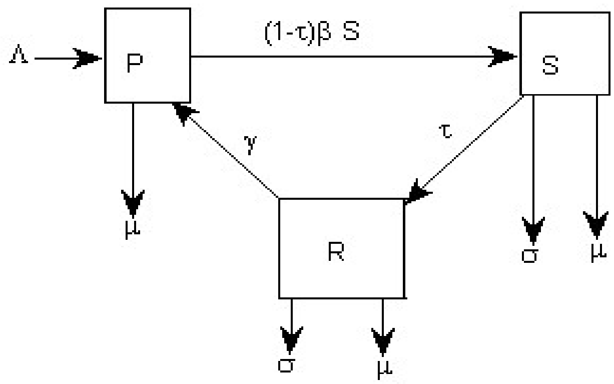

Fig.1. Flow chart

Based on the flow chart the equations governing the smoking dynamics are:

$$\frac{dP}{dt} = \Lambda - (1-\tau)\beta PS - \mu P + \gamma R \tag{2.2}$$

$$\frac{dS}{dt} = (1-\tau)\beta PS - (\tau + \sigma + \mu)S \tag{2.3}$$

$$\frac{dR}{dt} = \tau S - (\gamma + \sigma + \mu)R \tag{2.4}$$

2.2 Model analysis

In this section, we discuss positivity and boundedness of solutions, equilibrium points of the model and basic reproduction number.

2.2.1 Positivity and Boundedness of solutions

Here, we show that the solutions of the system of equations (2.2 – 2.4) are positive with non-negative initial values and bounded using the following theorems:

Theorem 1. *The solutions (P(t), S(t) and R(t)) of the system of equations* $(2.2 - 2.4)$ *are non-negative* ∀ *t* > 0 *with non-negative initial values.*

Proof. Equation (2.2), can be written as;

$$\frac{dP}{dt} > -\mu P \tag{2.5}$$

Solving equation (2.5) by separation of variables yields,

$$P(t) \geq P(0)e^{-\mu t}$$

as $t \longrightarrow \infty$, we have;

$P(t) \geq 0;\ \forall\ t \geq 0.$

Equation of (2.3) can be expressed as

$$\frac{dS}{dt} \geq -(\tau + \sigma + \mu)S \tag{2.6}$$

Solving equation (2.6) by separation of variables, we get,

$$S(t) \geq S(0)e^{-(\tau+\sigma+\mu)t}$$

as $t \to \infty$, we have;

$$S(t) \geq 0; \forall t \geq 0$$

Finally, equation (2.4), can be written as,

$$\frac{dR}{dt} \geq -(\gamma + \sigma + \mu)R \tag{2.7}$$

Solving equation (2.7) by separation of variables, we get,

$$R(t) \geq R(0)e^{-(\gamma+\sigma+\mu)t}$$

as $t \to \infty$, we have;

$$R(t) \geq 0; \forall t \geq 0$$

It is evident that the solutions P(t), S(t) and R(t) of the system of equations (2.2 – 2.4) are non-negative $\forall\, t > 0$.

Theorem 2. The solution N(t) of the system (2.1) is bounded.

Proof. Let $N = P + S + R$. Differentiating N with respect to t we get

$$\frac{dN}{dt} = \frac{dP}{dt} + \frac{dS}{dt} + \frac{dR}{dt}$$

Therefore, from system (2.1),

$$\frac{dN}{dt} = \Lambda - \mu(\mathrm{P} + \mathrm{S} + \mathrm{R}) - \sigma\mathrm{S} - \sigma\mathrm{R}$$

Which can be expressed as

$$\frac{dN}{dt} \leq -\mu(\mathrm{P} + \mathrm{S} + \mathrm{R}) \tag{2.8}$$

since $P + S + R = N$, equation (2.8) assumes the form

$$\frac{dN}{dt} + \mu N \leq \Lambda \tag{2.9}$$

Clearly equation (2.9) cannot be solved by separation of variable method. In order to solve by separation of variables method, we multiply both sides of the equation by the integrating factor $e^{\int \mu dt} = e^{\mu t}$ to get,

$$e^{\mu t}\frac{dN}{dt} + \mu e^{\mu t} N \leq \Lambda e^{\mu t}$$

Which is the same as

$$\frac{d(e^{\mu t}N)}{dt} \leq \Lambda e^{\mu t} \tag{2.10}$$

By integration of equation (2.10) we have

$$d(e^{\mu t}N) \leq \Lambda e^{\mu t}dt + c$$

which gives

$$e^{\mu t} N \leq \frac{\Lambda}{\mu} e^{\mu t} + c \quad (2.11)$$

Clearly

$$c = N(0) - \frac{\Lambda}{\mu} \quad \text{at } t = 0$$

Therefore, substituting for c in equation (2.11), we obtain

$$N(t) \leq \frac{\Lambda}{\mu} + \frac{N(0) - \frac{\Lambda}{\mu}}{e^{\mu t}}$$

which implies that

$$N(t) \leq \frac{\Lambda}{\mu} \quad (2.12)$$

as $t \to \infty$ Hence $N(t)$ is bounded

From Theorem 1 and Theorem 2, we analyze system (2.1) in the feasible region

$$\Omega = \{(P, S, R)\epsilon\mathbb{R}_+^3 ; P > 0; S, R \geq 0; N(t) \leq N(t) \leq \frac{\Lambda}{\mu}\}$$

Ω is positively invariant. That is, every solution of system of equations $(2.2 - 2.4)$, with initial conditions in Ω remains there $\forall\ t > 0$.

2.2.2 Existence of equilibrium points

In this section we determine the existence of Smoking Free Equilibrium (S.F.E) point denoted by $E^0(P^0, S^0, R^0)$ and Endemic Equilibrium (E.E) point denoted by $E^*(P^*, S^*, R^*)$. At the equilibrium point, the right hand side of the system of equations $(2.2 - 2.4)$ is equal to zero. In that

$$\Lambda - (1 - \tau)\beta PS - \mu P + \gamma R = 0$$

$$(1 - \tau)\beta PS - (\tau + \sigma + \mu)S + \alpha R = 0$$

$$\tau S - (\gamma + \alpha + \sigma + \mu)R = 0 \quad (2.13)$$

S.F.E point is a steady state solution of the system of equations (2.2 – 2.4) in the absence of Smoking. Therefore, in order to determine S.F.E we substitute $E^0(P^0, S^0, R^0)$ for (*P, S, R*) in system (2.13) and let

$S^0, R^0 = 0$ to remain with

$$\Lambda - \mu P^0 = 0 \quad (2.14)$$

Solving for P^0 in equation (2.14), we get

$$P^0 = \frac{\Lambda}{\mu}$$

Therefore S.F.E is

$$E^0(P^0, S^0, R^0) = E^0(\frac{\Lambda}{\mu}, 0, 0)$$

Next, E.E point is a steady state solution of the system of equations (2.2 – 2.4) in the presence of smoking. In order to determine E.E, we substitute $E^*(P^*, S^*, R^*)$ for (*P,S,R*) in system (2.13) then solve for $P^*, S^*, and\ R^*$). By doing so we obtain

$$\Lambda - (1-\tau)\beta P^* S^* - \mu P^* + \gamma R^* = 0$$

$$(1-\tau)\beta P^* S^* - (\tau + \sigma + \mu) S^* = 0$$

$$\tau S^* - (\gamma + \sigma + \mu) R^* = 0 \tag{2.15}$$

From the third equation of system (2.15), we have

$$R^* = \frac{(\tau S^*)}{\gamma + \sigma + \mu} \tag{2.16}$$

In view of the second equation of system (2.15), we solve for P^* to obtain

$$P^* = \frac{\gamma + \sigma + \mu}{(1-\tau)\beta} \tag{2.17}$$

Substituting equation (2.16) and equation (2.17) in the first equation of system (2.15) and solving for S^* , we get

$$S^* = \frac{\big((1-\tau)\beta\Lambda - \mu(\tau+\sigma+\mu)\big)(\gamma+\sigma+\mu)}{\big((\tau+\sigma+\mu)(\gamma+\sigma+\mu) - \gamma\tau\big)(1-\tau)\beta} \tag{2.18}$$

Finally, we substitute equation (2.18) in equation (2.16) for S^* to get

$$R^* = \frac{\big((1-\tau)\beta\Lambda - \mu(\tau+\sigma+\mu)\big)\tau}{\big((\tau+\sigma+\mu)(\gamma+\sigma+\mu) - \gamma\tau\big)(1-\tau)\beta} \tag{2.19}$$

Therefore, the EE is

$$E^*\begin{bmatrix} P^* \\ S^* \\ R^* \end{bmatrix} = E^*\begin{bmatrix} \dfrac{\gamma+\sigma+\mu}{(1-\tau)\beta} \\ \dfrac{\big((1-\tau)\beta\Lambda - \mu(\tau+\sigma+\mu)\big)(\gamma+\sigma+\mu)}{\big((\tau+\sigma+\mu)(\gamma+\sigma+\mu) - \gamma\tau\big)(1-\tau)\beta} \\ \dfrac{(1-\tau)\beta\Lambda - \mu(\gamma+\sigma+\mu)\tau}{\big((\tau+\sigma+\mu)(\gamma+\sigma+\mu) - \gamma\tau\big)(1-\tau)\beta} \end{bmatrix}$$

2.2.3 The Basic Reproduction Number (R_S)

This refers to expected number of secondary smoking cases arising from a single individual during his or her entire smoking period, in a purely potential smokers' population. Applying next generation matrix method by Van den Driessche and Watmough [5] to establish R_S, the infectious compartment is S only. Thus, in view of system (2.3), the equation of compartment S is

$$\frac{dS}{dt} = (1-\tau)\beta PS - (\tau + \sigma + \mu)S \tag{2.20}$$

Equation (2.20) can be expressed as

$$\frac{dS}{dt} = f_j - v_j = (1-\tau)\beta PS + (\tau + \sigma + \mu)S$$

Implying that

$$f_j = (1 - \tau)\beta PS \; and \; v_j = (\tau + \sigma + \mu)S$$

Thus the jacobian of f_j $at\ E^0\ is\ F = \frac{(1-\tau)\beta\Lambda}{\mu}$ and that of v_j at E^0 is $V = (\tau + \sigma + \mu)$ and $V^{-1} = \frac{1}{\tau+\sigma+\mu}$
Therefore

$$R_S = \rho(FV^{-1}) = \frac{(1-\tau)\beta\Lambda}{\mu(\tau+\sigma+\mu)}$$

2.3 Stability analysis

In this section we analyze local and global stability of the Smoking-free equilibrium and endemic equilibrium.

2.3.1 Local stability of smoking-free equilibrium

Here we linearize system of equations $(2.2 - 2.4)$ in order to study local stability. By doing so we obtain Jacobian Matrix

$$J = \begin{bmatrix} -(1-\tau)\beta S - \mu & -(1-\tau)\beta P & \gamma \\ (1-\tau)\beta S & (1-\tau)\beta P - (\tau+\sigma+\mu) & 0 \\ 0 & \tau & -(\gamma+\sigma+\mu) \end{bmatrix} \tag{2.21}$$

Theorem 3. Smoking-free equilibrium (E^0) of the system of equations (2.2 – 2.4) is locally asymptotically stable when $R_S < 1$.

Proof. Setting the Jacobian Matrix (2.21) at E^0, we get

$$J(E^0) = \begin{bmatrix} -\mu & -(1-\tau)\beta\Lambda & \gamma \\ 0 & \frac{(1-\tau)\beta\Lambda}{\mu} - (\tau+\sigma+\mu) & 0 \\ 0 & \tau & -(\gamma+\sigma+\mu) \end{bmatrix}$$

The eigenvalues of $J(E^0)$ are given by $|J(E^0) - I\lambda| = 0$, that is

$$\begin{vmatrix} -\mu-\lambda & -\frac{(1-\tau)\beta\Lambda}{\mu} & \gamma \\ 0 & \frac{(1-\tau)\beta\Lambda}{\mu} - (\tau+\sigma+\mu) - \lambda & 0 \\ 0 & \tau & -(\gamma+\sigma+\mu) - \lambda \end{vmatrix} = 0$$

Thus, the characteristic equation is given by

$$(\mu+\lambda)\left[\left(\frac{(1-\tau)\beta\Lambda}{\mu} - (\tau+\sigma+\mu)\right) - \lambda\right]\{(\tau+\sigma+\mu)+\lambda\} = 0 \tag{2.22}$$

From equation (2.22), it can be seen that the eigenvalues are;

$$\lambda 1 = -\mu < 0$$

$$\lambda_2 = -(\tau+\sigma+\mu) < 0$$
$$\lambda_3 = \frac{(1-\tau)\beta\Lambda}{\mu} - (\tau+\sigma+\mu) < 0 \;\; \text{, for } R_S = \frac{(1-\tau)\beta\Lambda}{\mu(\tau+\sigma+\mu)} < 1$$

Since all eigenvalues are negative when $R_S < 1, E^0$ is locally asymptotically stable.

2.3.2 Global stability of smoking free equilibrium

Using the approach by Castillo-Chavez et al. [6] in Castillo-Chavez theorem, the system (2.2 – 2.4) can be written as

$$\frac{dX}{dt} = \mathbf{G}(X, Y)$$

$$\frac{dY}{dt} = \mathbf{H}(X, Y), \mathbf{H}(X, 0) = 0$$

Where $X \in R = (P)$,potential smokers/non-smoking individuals and $Y \in R^2(S, R)$, smoking compartments we consider the following conditions for global stability of smoking-free equilibrium point

$$E^0 = (P^0, 0{,}0) = \left(\frac{\Lambda}{\mu}, 0{,}0\right) = (X^0, 0)\ for\ X^0 = \frac{\Lambda}{\mu}$$

i. $\frac{dX}{dt} = \mathbf{G}(X, O), X^0$ is globally asymptotically stable.

ii. $\mathbf{H}(X, Y) = WY - \widehat{\mathbf{H}}(X, Y),\ \widehat{\mathbf{H}}(X, Y) \geq 0\ for\ (X, Y) \in \mathbf{\Omega}$

where $W = D_Y\mathbf{H}(X^{\mathbf{0}}, 0)$ is an M-matrix (in that the off diagonal elements of W are positive) and Ω is the region where the equations of the model make epidemiological sense. The following theorem holds if conditions 1 and 2 are satisfied by system $(2.2 - 2.4)$.

Theorem 4. *The smoking free equilibrium point* $E^0 = (X^0, 0)$ *of the system of equations* $(2.2 - 2.4)$ *is globally asymptotically stable given that* $R_S < 1$ *and the conditions (i) and (ii)are satisfied.*

Proof. Because $X = (P)$ and $Y = (S, R)$, condition (i) above $(\frac{dX}{dt} = \mathbf{G}(X, 0))$ can be written

$$as\ \frac{dP}{dt} = \Lambda - \mu P$$

which upon solving we obtain

$$\Lambda - \mu P(t) = (\Lambda \mu P(0) e^{-\mu t}$$

or

$$P(t) = \frac{\Lambda - (\Lambda - \mu P(0))e^{-\mu t}}{\mu}$$

$$\Rightarrow P(t) \longrightarrow \frac{\Lambda}{\mu} as\ t \to \infty$$

hence E^0 is globally asymptotically stable.

Now, we rearrange condition (ii) to obtain

$$\widehat{\mathbf{H}}(X, Y) = WY - \mathbf{H}(X, Y) \tag{2.23}$$

By definition

$$\mathbf{H}(X,Y) = \begin{bmatrix} (1-\tau)\beta PS - (\tau+\sigma+\mu)S \\ \tau S - (\gamma+\sigma+\mu)R \end{bmatrix}$$

$$W = D_Y\mathbf{H}(X^0,0) = \begin{bmatrix} (1-\tau)\beta P^0 - (\tau+\sigma+\mu) & 0 \\ \tau & -(\gamma+\sigma+\mu) \end{bmatrix}$$

$$WY = \begin{bmatrix} (1-\tau)\beta P^0 S - (\tau+\sigma+\mu)S \\ \tau S - (\gamma+\sigma+\mu)R \end{bmatrix}$$

And it can be seen that matrix W is an M-matrix since all off diagonal entries of matrix W are positive.

Therefore

$$\widehat{\mathbf{H}}(X,Y) = \begin{bmatrix} (1-\tau)\beta P^0 S - (1-\tau)\beta PS \\ 0 \end{bmatrix}$$

Since $0 < \tau < 1$ and $P^0 \geq P \ \forall \ (X,Y) \in \Omega, \mathbf{H}(X,Y)\ 0$

Thus condition (ii) can be written as

$$\frac{dY}{dt} \leq WY$$

Finally, in view of matrix W, its characteristic equation is given by

$$\big((1-\tau)\beta P^0 - (\tau+\sigma+\mu) - \lambda\big)(-(\gamma+\sigma+\mu) - \lambda) = 0$$

which gives

$$\lambda_1 = (1-\tau)\beta P^0 - (\tau+\sigma+\mu)$$

$$\lambda_2 = -(\gamma+\sigma+\mu)$$

clearly λ_1 and λ_2 are negative when $R_S < 1$ This completes the proof.

2.3.3 Local stability endemic equilibrium

Now we use the following theorem to investigate the stability of Endemic equilibrium.

Theorem 5. *The endemic equilibrium point* (E^*) *of system of equations* $(2.2 - 2.4)$ *is locally asymptotically stable when* $R_S > 1$

Proof. Expressing the jacobian matrix (2.21) at the endemic equilibrium point, we have

$$J(E^*) = \begin{bmatrix} -\dfrac{\big((1-\tau)\beta\Lambda - \mu(\tau+\sigma+\mu)\big)(\gamma+\sigma+\mu)}{\big((\tau+\sigma+\mu)(\gamma+\sigma+\mu) - \gamma\tau\big)} - \mu & -(\tau+\sigma+\mu) & \gamma \\ \dfrac{\big((1-\tau)\beta\Lambda - \mu(\tau+\sigma+\mu)\big)(\gamma+\sigma+\mu)}{((\tau+\sigma+\mu)(\gamma+\sigma+\mu) - \gamma\tau} & 0 & 0 \\ 0 & \tau & -(\gamma+\sigma+\mu) \end{bmatrix}$$

The eigenvalues of $J(E^*)$ are given by $|J(E^*) - I\lambda| = 0$, that is

$$\begin{vmatrix} -\frac{\left((1-\tau)\beta\Lambda-\mu(\tau+\sigma+\mu)\right)(\gamma+\sigma+\mu)}{(\tau+\sigma+\mu)(\gamma+\sigma+\mu)-\gamma\tau}+\mu+\lambda & -(\tau+\sigma+\mu) & \gamma \\ \frac{(1-\tau)\beta\Lambda-\mu(\tau+\sigma+\mu)(\gamma+\sigma+\mu)}{((\tau+\sigma+\mu)(\gamma+\sigma+\mu)-\gamma\tau} & -\lambda & 0 \\ 0 & \tau & -((\gamma+\sigma+\mu)+\lambda \end{vmatrix} = 0$$

This gives the following characteristic equation

$$\begin{aligned} &-\left(\frac{\left((1-\tau)\beta\Lambda-\mu(\tau+\sigma+\mu)\right)(\gamma+\sigma+\mu)}{\left((\tau+\sigma+\mu)(\gamma+\sigma+\mu)-\gamma\tau\right)}+\mu+\lambda\right)\left[\left((\gamma+\sigma+\mu)\lambda+\lambda^2\right)\right] \\ &-\frac{\left((1-\tau)\beta\Lambda-\mu(\tau+\sigma+\mu)\right)(\gamma+\sigma+\mu)(\tau+\sigma+\mu)}{\left((\tau+\sigma+\mu)(\gamma+\sigma+\mu)-\gamma\tau\right)}\left((\gamma+\sigma+\mu)+\lambda\right) \\ &+\frac{\left((1-\tau)\beta\Lambda-\mu(\tau+\sigma+\mu)\right)(\gamma+\sigma+\mu)\gamma\tau}{\left((\tau+\sigma+\mu)(\gamma+\sigma+\mu)-\gamma\tau\right)}=0 \end{aligned} \tag{2.24}$$

Which upon expansion, we obtain

$$\begin{aligned} &\lambda^3+\left(\frac{\left((1-\tau)\beta\Lambda-\mu(\tau+\sigma+\mu)\right)(\gamma+\sigma+\mu)}{\left((\tau+\sigma+\mu)(\gamma+\sigma+\mu)-\gamma\tau\right)}+\mu+(\gamma+\sigma+\mu)\right)\lambda^2 \\ &+\Big(\frac{\left((1-\tau)\beta\Lambda-\mu(\tau+\sigma+\mu)\right)(\gamma+\sigma+\mu)^2}{\left((\tau+\sigma+\mu)(\gamma+\sigma+\mu)-\gamma\tau\right)}+\mu(\gamma+\sigma+\mu) \\ &+\frac{\left((1-\tau)\beta\Lambda-\mu(\tau+\sigma+\mu)\right)(\gamma+\sigma+\mu)(\tau+\sigma+\mu)}{\left((\tau+\sigma+\mu)(\gamma+\sigma+\mu)-\gamma\tau\right)}\Big)\lambda \\ &+\frac{\left((1-\tau)\beta\Lambda-\mu(\tau+\sigma+\mu)\right)(\gamma+\sigma+\mu)(\sigma+\mu)[(\tau+\sigma+\mu)+\gamma]}{\left((\tau+\sigma+\mu)(\gamma+\sigma+\mu)-\gamma\tau\right)}=0 \end{aligned} \tag{2.25}$$

Equation (2.25) can be written as

$$\lambda^3 + a\lambda^2 + b\lambda + c = 0 \tag{2.26}$$

Where

$$a=\frac{\left((1-\tau)\beta\Lambda-\mu(\tau+\sigma+\mu)\right)(\gamma+\sigma+\mu)}{\left((\tau+\sigma+\mu)(\gamma+\sigma+\mu)-\gamma\tau\right)}+\mu+(\gamma+\sigma+\mu)$$

$$b=\frac{\left((1-\tau)\beta\Lambda-\mu(\tau+\sigma+\mu)\right)(\gamma+\sigma+\mu)^2}{\left((\tau+\sigma+\mu)(\gamma+\sigma+\mu)-\gamma\tau\right)}+\mu(\gamma+\sigma+\mu)+\frac{\left((1-\tau)\beta\Lambda-\mu(\tau+\sigma+\mu)\right)(\gamma+\sigma+\mu)(\tau+\sigma+\mu)}{\left((\tau+\sigma+\mu)(\gamma+\sigma+\mu)-\gamma\tau\right)}$$

$$c=\frac{\left((1-\tau)\beta\Lambda-\mu(\tau+\sigma+\mu)\right)(\gamma+\sigma+\mu)(\sigma+\mu)[(\tau+\sigma+\mu)+\gamma]}{((\tau+\sigma+\mu)(\gamma+\sigma+\mu)-\gamma\tau)}$$

Clearly, when $R_S=\frac{(1-\tau)\beta\Lambda}{\mu(\tau+\sigma+\mu)}>1$, $a,b,c>0$ and

$$\begin{aligned} ab-c=&\Bigg[\frac{\left((1-\tau)\beta\Lambda-\mu(\tau+\sigma+\mu)\right)(\gamma+\sigma+\mu)}{((\tau+\sigma+\mu)(\gamma+\sigma+\mu)-\gamma\tau}+\mu \\ &+(\gamma+\sigma+\mu)\Bigg]\Bigg[\frac{\left((1-\tau)\beta\Lambda-\mu(\tau+\sigma+\mu)\right)(\gamma+\sigma+\mu)^2}{\left((\tau+\sigma+\mu)(\gamma+\sigma+\mu)-\gamma\tau\right)}+\mu(\gamma+\sigma+\mu) \\ &+\frac{\left((1-\tau)\beta\Lambda-\mu(\tau+\sigma+\mu)\right)(\gamma+\sigma+\mu)(\tau+\sigma+\mu)}{\left((\tau+\sigma+\mu)(\gamma+\sigma+\mu)-\gamma\tau\right)}\Bigg] \\ &-\Bigg[\frac{\left((1-\tau)\beta\Lambda-\mu(\tau+\sigma+\mu)\right)(\gamma+\sigma+\mu)(\sigma+\mu)[\tau+\sigma+\mu)+\gamma]}{((\tau+\sigma+\mu)(\gamma+\sigma+\mu)-\gamma\tau)}\Bigg] \end{aligned}$$

which upon expansion, gives

$$ab - c =$$
$$\left[\frac{((1-\tau)\beta\Lambda-\mu(\tau+\sigma+\mu))(\gamma+\sigma+\mu)}{((\tau+\sigma+\mu)(\gamma+\sigma+\mu)-\gamma\tau} + \mu\right]\left[\frac{((1-\tau)\beta\Lambda-\mu(\tau+\sigma+\mu))(\gamma+\sigma+\mu)^2}{((\tau+\sigma+\mu)(\gamma+\sigma+\mu)-\gamma\tau)} + \mu(\gamma+\sigma+\mu) +\right.$$
1− τβΛ− μτ+σ+μγ+σ+μτ+σ+μ(τ+σ+μγ+σ+μ− γτ+γ+σ+μ1− τβΛ− μτ+σ+μγ+σ+μ2τ+σ+μγ+ σ+μ− γτ+μγ+σ+μ+1− τβΛ− μτ+σ+μγ+σ+μγτ)(τ+σ+μγ+σ+μ− γτ>0
(2.27)

Thus, applying Routh-Hurwitz criterion of a polynomial of degree three, since $a, b, c > 0$ and $ab - c > 0$ when $RS > 1$, the system (2.1) is locally asymptotically stable.

2.3.4 Global stability of endemic equilibrium

In this section we use LaSalle [7] approach to prove global stability of E^*.

Theorem 6. *The Endemic Equilibrium Point E^* of the system of equations* $(2.2 - 2.4)$ *is globally asymptotically stable if $R_S > 1$.*

Proof. Consider the following Lyapunov function

$$V(P,S,R) = (P - P^* \ln\frac{P}{P^*} + L(S - S^* \ln\frac{S}{S^*}) + M(R - R^* \ln\frac{R}{R^*})$$

Differentiating V with respect to t, we obtain

$$\frac{dV}{dt} = \left(1 - \frac{P^*}{P}\right)\frac{dP}{dt} + L\left(1 - \frac{S^*}{S}\right)\frac{dS}{dt} + M\left(1 - \frac{R^*}{R}\right)\frac{dR}{dt}$$

Substituting for $\frac{dP}{dt}, \frac{dS}{dt}$ and $\frac{dR}{dt}$ from system $(2.2 - 2.4)$ in equation (2.28) gives

$$\frac{dV}{dt} = \left(1 - \frac{P^*}{P}\right)(\Lambda - (1-\tau)\beta PS - \mu P + \gamma R) + L\left(1 - \frac{S^*}{S}\right)\left((1-\tau)\beta PS - (\tau+\sigma+\mu)S\right) + M\left(1 - \frac{R^*}{R}\right)(\tau S - (\gamma+\sigma+\mu)R) \quad (2.29)$$

Upon rearrangement of system (2.15), we get

$$\begin{aligned} \Lambda &= (1-\tau)\beta P^*S^* + \mu P^* - \gamma R^* \\ (\tau+\sigma+\mu) &= (1-\tau)\beta P^* \\ (\gamma+\sigma+\mu) &= \frac{\tau S^*}{R^*} \end{aligned} \quad (2.30)$$

Substituting system (2.30) in equation (2.29), we obtain

$$\frac{dV}{dt} = \left(1 - \frac{P^*}{P}\right)\left((1-\tau)\beta P^*S^* + \mu P^* - \gamma R^* - (1-\tau)\beta PS - \mu P + \gamma R\right) + L\left(1 - \frac{S^*}{S}\right)((1-\tau)\beta PS - (1-\tau)\beta PS^* + M\left(1 - \frac{R^*}{R}\right)\left(\tau S - \frac{\tau S^* R}{R^*}\right)$$

Which can be expressed as

$$\frac{dV}{dt} = -\frac{\mu(P-P^*)^2}{P} + \left(1 - \frac{1}{x}\right)\left((1-\tau)\beta P^*S^* - \gamma R^* - (1-\tau)\beta P^*S^*xy + \gamma R^*z\right) + L\left(1 - \frac{1}{y}\right)((1-\tau)\beta P^*S^*xy - (1-\tau)\beta P^*S^*y + M\left(1 - \frac{1}{z}\right)(\tau S^*Y - \tau S^*z) \quad (2.31)$$

Where$x = \frac{P}{P^*}$, $y = \frac{S}{S^*}$ and $z = \frac{R}{R^*}$
Factorizing equation (2.31) gives

$$\frac{dV}{dt} = -\mu\frac{(P-P^*)^2}{P} + (1-\tau)\beta P^*S^*\left(1+y-xy-\frac{1}{x}\right) + \gamma R\left(z+\frac{1}{x}-1-\frac{z}{x}\right) + L(1-\tau)\beta P^*S^*(1+xy-$$
$$y-x+M\tau S*1+y-z-yz \qquad (2.32)$$

To determine L and M, we set the coefficients of x, y and z of equation (2.32) equal to zero.
Thus, we obtain,

$$L(1-\tau)\beta P^*S^* = 0$$
$$M\tau S^* = 0$$

For L=1,

$$(1-\tau)\beta P^*S^* = 0$$
$$M\tau S^* = (1-\tau)\beta P^*S^*$$
$$M = \frac{(1-\tau)\beta P^*}{\tau}$$

Upon substitution for L=1 and $M = \frac{(1-\tau)\beta P^*}{\tau}$ in (2.32), we obtain

$$\frac{dV}{dt} == -\mu\frac{(P-P^*)^2}{P} + (1-\tau)\beta P^*S^*\left(1+y-xy-\frac{1}{x}\right) + \gamma R\left(z+\frac{1}{x}-1-\frac{z}{x}\right) + (1-\tau)\beta P^*S^*(1+$$
$$xy-y-x+1-\tau\beta P*S*1+y-z-yz \qquad (2.33)$$

which can be simplified to get

$$\frac{dV}{dt} = -\mu\frac{(P-P^*)^2}{P} + (1-\tau)\beta P^*S^*\left(3+y-x-z-\frac{y}{z}-\frac{1}{x}\right) + \gamma R\left(z+\frac{1}{x}-1-\frac{z}{x}\right) \qquad (2.34)$$

By geometric mean, the inequality $\frac{dV}{dt} \leq 0$ in Ω. The equality $\frac{dV}{dt} = 0$ if $x = y = z = 1$ and $P = P^*, S = S^*, R = R^*$. Therefore, using LaSalle's invariance principle, the endemic equilibrium point of the system of equations $(2.2 - 2.4)$ is globally asymptotically stable.

3. Numerical Simulation

Using MATLAB ode45 solver and parameter values in Table (3.1), we carry out numerical simulation of the model.

Table 3.1. Parameter values of the model

Parameter symbol	Value	Source
Λ	9.1×10^{-4}/day	Gul (2011)
μ	3.1×10^{-4}/day	Gul (2011)
γ	0.0011/day	Estimate
β	10^6 3.8×10^{-2}/day	Gul (2011)
τ	$0 < \tau < 1$	Assumed
σ	5.0×10^{-1}/day	Estimate

Fig. (3.2) demonstrates the impact of rehabilitation on smokers. From the figure it can be observed that in the absence of the rehabilitation (τ = 0.0), there is a very high increase of smokers followed by a decrease and finally converges to zero. And it can be seen that as the rate of rehabilitation increases, the peaks' level of the trajectories decreases and eventually converge to zero. Also from the figure, it can be observed that at τ =

0.99999, the trajectory of the smokers converges to zero without any increase and it takes the shortest time to converge.

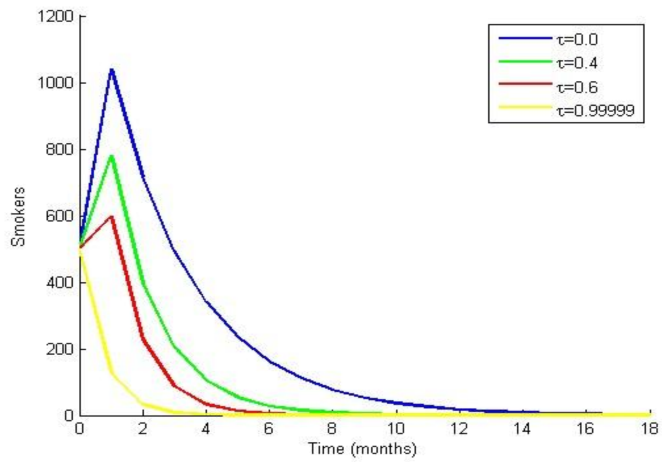

Fig. 3.2. The impact of rehabilitation on smokers

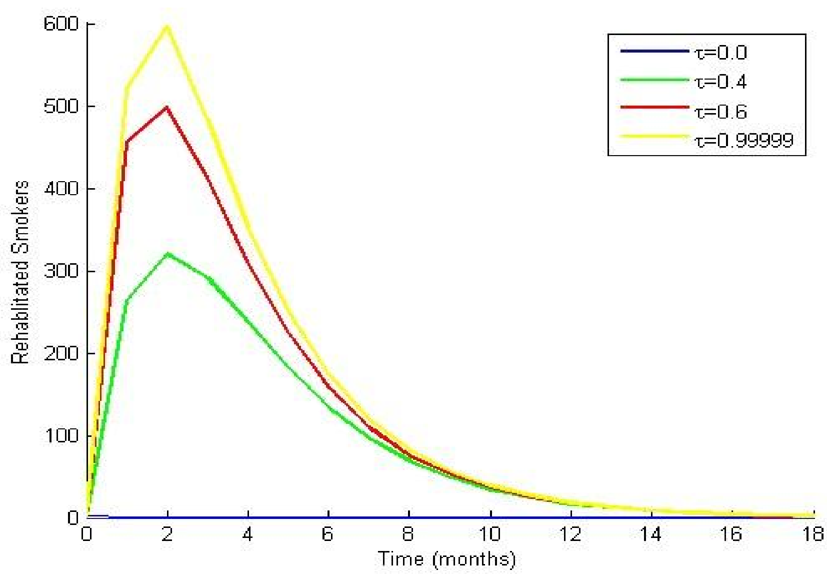

Fig. 3.3. The trend of rehabilitated individuals at different rates of rehabilitation

This implies that when rehabilitation is implemented effectively, it can help in eradication of smoking in the community. Fig. (4.3) shows that the number of rehabilitated individuals increase with increase of rehabilitation rate and the trajectories converge to zero with time.

4. Conclusion

In this study, we formulated mathematical model of the effect of inpatient rehabilitation of tobacco smokers on smoking dynamics. We studied the stability of the smoking free and endemic equilibrium. The results of the disease-free equilibrium showed that the model is both locally and globally asymptotically stable when $R_S < 1$. This implies that when R_S is below unity, the spread of smoking reduces. Next, we studied the endemic equilibrium which we found to be both locally and globally asymptotically stable when $R_S > 1$. Numerical

Simulation shows that when rehabilitation is implemented effectively it can help in minimizing smoking in the community.

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Competing Interests

Authors have declared that no competing interests exist.

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