EXACT CONTROLLABILITY AND BOUNDARY STABILIZATION OF A ONE-DIMENSIONAL VIBRATING STRING WITH A POINT MASS IN ITS INTERIOR

BY

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I56/31902/2015

A Research Project submitted in partial fulfillment of the requirements for the award of the Degree of Master of Science (Applied Mathematics) in the School of Pure and Applied Sciences of Kenyatta University

May, 2019

DECLARATION

I confirm that this research project is my original work and has not been presented for certification in this university or any other university. The project has been complemented by referenced works duly acknowledged. Where texts, data or graphics have been borrowed from other works, the sources are specifically accredited in accordance with the anti-plagiarism regulations.

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This research project has been submitted with my approval as the university supervisor to be presented to the Board of Postgraduate Studies

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DEDICATION

I dedicate my project report to my parents, Mr. John A. Koech and Mrs. Hellen Chepkoech, and siblings

ACKNOWLEDGEMENT

It has been a journey and a long one. I would not walk the journey alone to come this far. It is a high time to acknowledge that I am greatly indebted to my supervisor, Dr. Chepkwony Isaac. With his timely insights, inspiration and availability for consultation have I been steered to complete this research work. I too appreciate all the lecturers in the department of Mathematics, Kenyatta University, for their continued support and guidance throughout this work. I am inspired by the influence of each of my lecturers. I owe special thanks to Dr Lydia Njuguna who besides being my lecturer and a mentor, she has been a mum too. I am grateful mum.

It is with utmost appreciation and gratitude that I too acknowledge contributions of my classmate and those we studied together. Your comments, suggestions and criticism throughout this work are very much appreciated.

I owe substantially my parents for their unwavering moral and material support even when giving up was the most likely option. The family has had to endure a lot as I carried on both with this research project. May God bless you more and may you be inspired by this work.

Above all, the sound health throughout my studies is all I owe to the Almighty God

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ABSTRACT

The dynamics of hybrid systems have attracted the attention of control theorists in the recent past, the control and stabilization of these systems is of paramount importance. The complexity of the structure has increased over time. There has been need to investigate the dynamics of such complex systems as to establish their optimal operational requirements. Earlier researches on the control and stabilization of these systems have been done with homogeneous system models. These earlier works have been criticized as being unable to address the ever-growing complexities of such systems. This research project has examined the problems of boundary control and stabilization for a one- dimensional vibrating string with an interior point mass. We have investigated what happens to the singularities in waves as they cross a point mass. In the case of an interior point mass for a vibrating string with the point mass placed at the point x=0, for example, with the L^2- Dirichlet control at the left end, we have established the most reachable spaces from position of the point mass both to the left and to the right of the point mass. We have studied the problem of a one-dimensional wave equation with an interior point mass and have established a way of regulating the boundary vibrations of this system in a way that the vibration at a time t=T coincide with a function fixed in advance. This project has given an appropriate way of choosing a control *u* from the admissible set of controls U for a system with the state y so that the system is exactly controllable.

ACRONYMS AND ABBREVIATIONS

$H^m(\Omega), \mathrm{m} \in \mathbb{Z}$	The set of all m-times weakly differentiable functions, for m an integer, in the domain Ω.
$L^p(\Omega)$	The set of all functions with the domain Ω which are Lebesque integrable
$M_{(k,l)}, k, l \in \mathbb{N}$	A $k \times l$ operator (usually a matrix) for k, l natural numbers.
$E_M(t)$	The time dependent energy of the vibrating string system in the presence of a point mass of mass M.
$d^n, n \in \mathbb{N}$	The n^{th} order distributive derivative with respect to the space variable x.
δ_0	The dirac delta function with the point mass at the point $x = 0$.
T	A positive constant denoting the controllability/observability time.
C(T)	A positive constant dependent on the control time used in the controllability/observability estimates.
HUM	Hilbert's Uniqueness Method

Table 1

CHAPTER 1

INTRODUCTION

The dynamics of simple and complex hybrid systems have aroused interest in the recent past. Control theory turns out as a tool to investigate these dynamics and prescribe any future possible state of a dynamical system that is deemed controllable. Being a cross-disciplinary branch of mathematics, control theory provides a platform and a common playground for the various branches of mathematics to make contributions to real life applications of mathematics. Control theory plays a key role in control engineering, robotics, mathematical programming and the general field of cybernetics. The advancement so far made in this crucial branch of mathematics is worth a good deal of attention. Earlier works were done on the Banach spaces with a recent shift to works on the Hilbert spaces. This has been motivated by the fact that Hilbert spaces bear the property of inner product that enables control theorists to tell how good or bad a control is based on the intermediate values taken by a system being controlled. There have been a number of questions that control theorists are asking relating to the controllability of dynamical systems. A number of factors are considered in structuring responses to these questions that include, but are not limited to, the kind of modeling equations appropriate to the model, the intended end result of the control process and the cost to be incurred in running these controls.

Scott, H. and Enrique, Z. (1995) argue that with the ever growing complexities of dynamical systems and the need to have these systems controlled, it is the duty of control theorists to provide precise description of the space of exact (or any desired controllability type) controllability when the control to the system is active.

The most related concept to controllability is its dual notion, that of observability. Earlier researchers in the field of control theory have investigated and declared an observable system as controllable. The only question we seek to ask is whether all dynamical systems that are observable are exactly controllable. It is necessary for us to give an insight into the concept of observability and later link it to the concept we are investigating, that of controllability. We lay a foundation of the concept we are investigating by presenting the following background information.

1.1 Background information

Let us consider the following system

$$u_{tt} - u_{xx} = 0 \;\; in\ (0,T) \times \Omega$$

$$u = f \;\; on\ (0,T) \times \Gamma \qquad 1.1$$

$$u(0,x) = u^0, u_t(0,x) = u^1 \; for\ x \in \Omega,$$

where $\Omega \subset \mathbb{R}$ is open and Γ is its boundary. We refer to system (1.1) to make the following definitions of the three main types of controllability.

Definition 1.1.1

Let T>0. We define, for any initial data $(u^0, u^1) \in L^2(\Omega) \times H^{-1}(\Omega)$, the set of reachable states

$R(T; (u^0, u^1)) = \{u(T), u_t(T): u$ solves 1.1 $with\ f \in L^2\big((0,T) \times \Gamma_0\big)\}$, where Γ_0 is a point of Γ where the boundary control f is acting on the system.

Definition 1.1.2

The system (1.1) is approximately controllable in the time T if for every initial data $(u^0, u^1) \in L^2(\Omega) \times H^{-1}(\Omega)$, the set $R(T; (u^0, u^1))$ of reachable states is dense in $L^2(\Omega) \times H^{-1}(\Omega)$.

Definition 1.1.3

The system (1.1) is said to be exactly controllable in the time T if for every initial data $(u^0, u^1) \in L^2(\Omega) \times H^{-1}(\Omega)$, the set $R(T; (u^0, u^1))$ coincides with the set $L^2(\Omega) \times H^{-1}(\Omega)$

Definition 1.1.4

System (1.1) is null controllable in the time T if for every initial data $(u^0, u^1) \in L^2(\Omega) \times H^{-1}(\Omega)$, the set $R(T; (u^0, u^1))$ of reachable states contains the element (0,0).

At this point, let us introduce the dual notion to controllability, that of observability of a dynamical system. Let us consider the finite dimensional linear system

$\frac{du}{dt} = Au + By, \;\; 0 < t < T,$

$u(0) = u_0 \in \mathbb{R}^n,$ 1.2

where $A \in M_{(n,n)}$ and $B \in M_{(n,m)}$, m ≤ n are operators. It is convenient to study the properties of a system strongly related to the prime system (1.2), the adjoint system of (1.2). If we let A^* be adjA, then we will consider the adjoint system

$-\phi' = A^*\phi, for\ 0 < t < T,$

$\phi(T) = \phi_T$ 1.3

which runs backward in time. The system (1.3) give rise to the dual notion to controllability, the observability notion. Let us make the following definition

Definition 1.1.5

The adjoint system (1.3) is said to be *observable* in a time $T > 0$ if there exists a constant C=C(T) such that

$\int_0^T |B^*\phi|^2 dt \geq C|\phi(T)|^2$, for all $\phi(T) \in \mathbb{R}^n$, with ϕ as the corresponding solution to the system (1.3).

Our problem is to find suitable observability estimates and then apply Hilbert Uniqueness Method (HUM), introduced by Lions, (1988), to determine the controllability requirements for our vibrating system. As partake of this, we note that observability concept makes concrete the general notion that the actions of controls is sufficient to determine the state of the system for all time $t>0$ and therefore at the time T.

Having established that the concept of controllability relates closely to that of observability of dynamical systems, we are now in a position to present the following preliminary results on the existence, uniqueness and regularity of solutions and the compatibility conditions of the solutions to the initial data. We do this by acknowledging and making use of some properties of the wave equation mainly those of linearity, time- reversibility and the fact that the wave equation does not force infinite propagation speed of the data.

1.2 Statement of the Problem

Besides the good work by the control theorists as in the case of Micu and Zuazua (2003), we find that there are gaps left out in addressing some of the issues surrounding control theory problems. The controllability time T, for example, has been argued that it has to be sufficiently large for the controllability conditions of the system to be achieved. The issue of how long the controllability time has to be so as to be sufficiently large remains unaddressed by earlier works. We seek to exploit some of the attributes of our model to determine the least that the controllability time has to be. The other gap left by earlier works is in the type of controllability to be achieved. In as much as we agree that null controllability implies exact controllability, there are cases where attainment of null controllability to imply exact controllability will be of no significance. For a body whose functioning is vibrational, introducing a control that drives such a body to rest in the time T to achieve null controllability and argue for exact controllability bears no significance. We therefore seek to present equilibrium results that suit the requirements of each model.

Having identified the gap left by earlier works in control theory, we then present the following objectives that this research has achieved with the results we are presenting in chapter 5.

1.3 Objectives

1.3.1 General Objective

The general objective of this study was to examine problems of boundary control and stabilization of the one-dimensional wave equation with a point mass in its interior

1.3.2 Specific Objectives

Specifically, this study sought to:

i. Explore the preliminary results on the existence, uniqueness and regularity of solutions of both the uncontrolled and the controlled system of vibrating string
ii. Examine the controllability problem for the case where the control is active at both extremes and the case when the control is active at only one extreme
iii. Establish boundary stabilization conditions for a vibrating string system problem.

1.4 Significance of Study

Our study has been significant in that it has utilized the properties of the model such as tension, string density and the length of the string to determine the least that the controllability time can be. Earlier works estimated the controllability time as a sufficiently large time and this begged one to ask how large the control time has to be in order to be said that it is sufficiently large.

We have achieved exact controllability in this work by looking for equilibrium solutions rather than achieving null controllability to imply exact controllability. The systems that operate by way of vibration can now be designed to achieve optimal operation requirements.

CHAPTER 2

LITERATURE REVIEW

The modern control theory has its roots in the contributions of the U.S scientist R. Bellman, (1953) who keyed in the concept of *dynamic programming* and the Russian Pontragin, (1956) with the idea of *maximum principle* for nonlinear optimal control problems. Control theory is the crossroad to control engineering and the control theory as a mathematical discipline with mathematics playing an increasingly important role in modeling the ever-sophisticating systems that control theory has to deal with.

Mayr, O. (1970) gave a detailed discussion of the cause-effect-cause principle that the state of the system determines the way the control is to be exerted at any time. The second notion in control theory is that of optimization. Mayr argues that for a system under control, optimal operation is achieved by taking into account the self-regulation of the system. He adds that the state of the system at an intermediate time may be used to determine the way the control is to be applied at a later time.

E. Zuazua, (2001) introduced the idea of a cost function alongside his model and sought for a minimizer of this function as the best control of all the controls in the admissible set of controls. The basic idea in this case is that it will be of no value to run a control that will be too costly in comparison to the output of the system. Zuazua therefore sought a control that will run as a minimizer of the cost function that he designed alongside his control model.

Fattorini, H. (1975) investigated the problem of boundary control of temperature distributions in a parallelepipedon. He noted that smoothing properties of the heat equation makes it severe the conditions on u_T for solution of the controllability problem. Fatttorini found out that exact controllability of the heat equation is only possible if the controllability space is too small such as a section of the boundary of the system, the parallelepipedon. He used D.L Russel and H.O Fattorini (1971) approach of reducing the parabolic equation to moment problems and finding the solutions of these moment problems and later constructing the solution to the original problem he was considering. The heat equation has an intrinsic property of forcing an infinite propagation speed of the data (both boundary and initial) given to the system. This accounts for the finding of Fattorini (1975). He however noted that null controllability of the wave equation is always possible.

J. Lions (1988) studied the problem of exact controllability, stabilization and perturbations for distributed systems. The systematic method of HUM was introduced as a method based on uniqueness results and constructed on Hilbert spaces by using uniqueness. Lions used both boundary and local distributed controls to drive the system to rest in a given finite time,T. He noted that solutions of control problems are highly dependent on the function space from which the initial data are taken and the spaces where the controls are chosen from.

Alkahby et al. (1999) in their work on *the mathematical model for the basilar membrane as a two dimensional plate* numerically computed eigenvalues for both the annular and rectangular plate models. They showed that if the curvature of the basilar membrane is taken into account, the range of the hearing frequencies is significantly wider. They further noted that for the rectangular plate model, the hearing frequencies were only significant when the rectangular region tends to a secured region.

Fernandez, E. and Zuaua, E. (2000) considered the linear heat equation for the cost of approximate controllability. They considered the model equations in a bounded domain of $\mathbb{R}^d$ with Dirichlet boundary conditions. In their analysis of this controllability problem when the control acts on a small open subset of the domain, they noted that the heat equation is approximately controllable, null controllable and also finite-exactly controllable. By finite-approximate controllability, they meant that there exist controls by means of which one can simultaneously guarantee the approximate controllability and the exact controllability of a projection of a solution over a finite-dimensional subspace. They concluded their work by determining the speed of convergence of the limiting process in which the approximate control is obtained through a process of penalized optimal control problems.

Micu and Zuazua (2003) considered an evolution system modeled by means of an ordinary differential equation. By actions of some suitable controls on the trajectories of the system, over a time interval, they sought to find a control such that the solution of the system matches both the initial and the final states of the system. They argued that the system dynamics can be changed by acting on a subset of the domain of the modelling equation for the system. This result was due to Kalman's rank condition. They gave that the attainment of exact controllability depends on the geometric properties of the system domain. They could not verify exact controllability but established both null controllability and approximate controllability conditions for the wave equation.

Chepkwony (2006) established the controllability conditions of the cochlear model both with and without longitudinal elasticity. The cochlear model was given as approximately controllable on [0,T] when the equations modeling the original systems were reduced and solved to give the null solutions on some boundaries of the cochlear model for all times in [0,T].

Mottelet S. (2009) on the non-dissipative boundary stabilization of a canal with wave generators obtained a strong stability result using an *ad hoc* energy functionals. They noted that the approach they took can be applied to same type system as for a one dimensional wave equation in which the control and observation operator verify certain properties related to the action on the eigenfunctions of the unstabilized system.

Dustin, (2012) in studying controllability of the heat equation and making reference to the works of Zuazua (2003) sought a control he referred to as an optimal control. He found this control from the admissible set of controls as one that minimizes the cost functional J(.) of running the control. Using Kalman's rank condition, he established that a finite linear system is exactly controllable in a time T if the rank of the controllability matrix equals the dimension of the space from which the initial data is taken from.

Yong, H. (2018) investigated exact controllability for wave equations with switching controls. Yong analyzed the exact controllability problem for wave equations endowed with switching controls. By this, Yong controlled the dynamics of the system by switching among different actuators such that in each instant of time, there are few active actuators as much as possible. Yong proved that the system is exactly controllable under suitable geometric control conditions.

CHAPTER 3

GENERAL GOVERNING MODEL EQUATIONS

In this section, we present the modeling equations for the different possible systems that can be modeled using the wave equation. We first give the notation that we are using to describe the state of the system. Let us suppose the vibrating string system to be occupying $-l_1 \leq x \leq l_2$ where $l_1, l_2 > 0$ with the point mass at the point x=0, regarding the first string to be occupying $\Omega_1 = (-l_1, 0) \subset \mathbb{R}$ and the second one occupying $\Omega_2 = (0, l_2) \subset \mathbb{R}$ so that the deformations are described by

$u = u(x,t), \quad x \in \Omega_1, t > 0$

$v = v(x,t), \quad x \in \Omega_2, t > 0$,

and the function describing the position of the point mass to be $z = z(t)$. If we further suppose that the string system satisfy Dirichlet conditions at the boundary points $x = -l_1, l_2$, the equations that model the dynamics of this vibrating hybrid system in the absence of control are

$$
\begin{aligned}
&\rho_1 u_{tt} = \sigma_1 u_{xx}, && x \in \Omega_1, t > 0 \\
&\rho_2 v_{tt} = \sigma_2 v_{xx}, && x \in \Omega_2, t > 0 \\
&M z_{tt}(t) + \sigma_1 u_x(0,t) - \sigma_2 v_x(0,t) = 0, && t > 0 \\
&u(-l_1, t) = v(l_2, t) = 0, && t > 0 \\
&u(0,t) = v(0,t) = z(t), && t > 0
\end{aligned} \qquad 3.1
$$

where ρ_1 *and* ρ_2 are positive constants denoting the density of the first and the second strings respectively, and the tensions in the strings are σ_1 and σ_2 assumed to be positive constants. We uniquely determine the solution to the system (3.1) by prescribing some initial conditions, say at the time t=0, as

$u(x,0) = u^0(x) = u^0,\ \ u_t(x,0) = u^1$

$v(x,0) = v^0(x) = v^0,\ \ v_t(x,0) = v^1$ 3.2

$z(0) = z^0,\qquad\qquad z_t(0) = z^1$

The time dependent energy relation for this type of dynamical system is given to be

$$E_M(t)=\frac{1}{2}\int_{-l_1}^{0}[\rho_1|u_t(x,t)|^2+\sigma_1|u_x(x,t)|^2]dx+\frac{M}{2}|z_t(t)|^2+\frac{1}{2}\int_{0}^{l_2}[\rho_2|v_t(x,t)|^2+\sigma_2|v_x(x,t)|^2]dx$$

3.1 Existence, Uniqueness, Regularity and Compatibility conditions for solutions

In this section, we are giving some results concerning the existence, uniqueness and regularity of solutions. We first give the existence, uniqueness and regularity of solutions of the system with homogeneous boundary conditions in section (3.1.1) and then with nonhomogeneous boundary conditions in section (3.1.2). The case of nonhomogeneous boundary conditions will guide us in building a case to argue on the dynamics of the system when the control is active at both ends or at one end of our system.

Let us begin by giving the existence, uniqueness and regularity of solutions of the vibrating system with no controls, the case of homogeneous boundary conditions.

3.1.1 Homogeneous Boundary Conditions

In this section, we go back to system (3.1) with the Dirichlet boundary conditions at the end points $x = -l_1, l_2$. The solutions exhibit different degrees of regularity depending on the regularity of the initial data that the modeling system starts with. Let us first introduce the following vector spaces

$\mathcal{W}_1 := \{\phi \in H^1(\Omega_1) : \phi(-l_1) = 0\},$

$\mathcal{W}_2 := \{\psi \in H^1(\Omega_2) : \psi(l_2) = 0\}$,

$\mathcal{W} := \{(\phi, \psi) \in \mathcal{W}_1 \times \mathcal{W}_2 : \phi(0) = \psi(0)\}$,

endowed with the norms

$|| \phi||^2_{\mathcal{W}_i} = \int_{\mathcal{W}_i} |\phi_x(x)|^2 \, dx$ for i=1,2

$||(\phi, \psi)||^2_{\mathcal{W}} = || \phi||^2_{\mathcal{W}_1} + ||\psi||^2_{\mathcal{W}_2}$.

We are using the above spaces to construct appropriate Hilbert structures for the initial data in our case to find the controllability spaces for our vibrating string system. Let us note that the space $\mathcal{W}$ is algebraically and topologically equivalent to $H^1_0(-l_1, l_2)$. We however think of $\mathcal{W}$ as a subspace of $\mathcal{W}_1 \times \mathcal{W}_2$ since we assume our string system is made up of two (we treat the system as one of two different strings because of the presence of the point mass) different strings.

Let us also define the closed subspace

$W_1 := \{(\phi, \psi, z) \in \mathcal{W} \times \mathbb{R} : \phi(0) = \psi(0) = z\}$,

of $\mathcal{W} \times \mathbb{R}$ which is densely and continuously embedded in the space

$W_0 := L^2(\Omega_1) \times L^2(\Omega_2) \times \mathbb{R}$,

We then define the Hilbert space by

$\boldsymbol{\mathcal{H}} := W_1 \times W_0$,

with the product topology.

Our task is to introduce appropriate boundary controls to act on the vibrating system at the extreme points $x = -l_1, l_2$ so as to have the set of all reachable states of the system coinciding with the space $\boldsymbol{\mathcal{H}}$. In terms of the vector-valued function,

$y := (u, v, z, u', v', z')^t$,

we define the unbounded operator $\boldsymbol{\mathcal{A}}$ on $\boldsymbol{\mathcal{H}}$ by

$$\mathcal{A}\mathbf{y} = \begin{pmatrix} 0 & I \\ \mathbf{A} & 0 \end{pmatrix} \boldsymbol{y}, \text{ where } \mathbf{A} = \begin{pmatrix} \frac{\sigma 1}{\rho 1} d^2 & 0 & 0 \\ 0 & \frac{\sigma 2}{\rho 2} d^2 & 0 \\ -\frac{\sigma 1}{M} d\delta_0 & \frac{\sigma 2}{M} d\delta_0 & 0 \end{pmatrix},$$

with the domain given by

$$\mathcal{D}(\mathcal{A}) = \{y \in \mathcal{H}: u \in H^2(\Omega_1), v \in H^2(\Omega_2), (u', v', z') \in W_1\}$$

We thus present the system (3.1)-(3.2) as

$$\frac{\mathbf{dy}}{\mathbf{dt}} = \mathcal{A}\mathbf{y}, \qquad \boldsymbol{y}(\mathbf{0}) = \boldsymbol{y}^{\mathbf{0}} = (u^0, v^0, z^0, u^1, v^1, z^1)^{\boldsymbol{t}}, \qquad 3.3$$

Having defined the spaces from where our initial data will be picked from, we are in a position to give the propositions that will guide us on the kind of solutions we expect for the uncontrolled (with the homogeneous boundary conditions) system.

Proposition 3.1.1.1

a) For every $\mathbf{y^0} \in \mathcal{H}$, there exist a unique solution of (3.1)-(3.2) in the class

$$(u, v, z) \in C([0, T]; W_1) \cap C^1([0, T]; W_0) \qquad 3.4$$

For which the energy E_M remains constant along this solution trajectory.

b) If $\mathbf{y^0} \in \mathcal{D}(\mathcal{A})$**,** the corresponding solution has the following additional regularity

$$u \in C([0, T]; H^2(\Omega_1)) \cap C^1([0, T]; H^1(\Omega_1)),$$

$$v \in C([0, T]; H^2(\Omega_2)) \cap C^1([0, T]; H^1(\Omega_2)) \qquad 3.5$$

We denote solutions (3.4) by Finite-Energy solutions and further give the following regularity result for finite-energy solutions.

Proposition 3.1.1.2

For every $T > 0$, there exist some constant $C(T) > 0$ such that for every finite-energy solution

$$\int_0^T [|u_x(-l_1, \mathrm{t})|^2 + |v_x(l_2, \mathrm{t})|^2] \mathrm{dt} \leq C E_M(0) \qquad 3.6$$

Proof of proposition 3.1.1.2

This estimate is of local nature as presented in Lions, (1988). It therefore does not depend on whether there is a point mass on the string or not. We then use the conservation of the energy $E_M(t)$to obtain the upper bound in terms of the initial energy. This completes the proof giving (3.6). ■

We note that the observation inequality (3.6) guarantees that the solution of the system at the time t=0 is completely determined by the term$|u_x(-l_1,t)|^2 + |v_x(l_2,t)|^2$, a quantity observed via the controls at the end points $x = -l_1, l_2$.

In our urge to control our system, it is necessary for us to study system (3.1) in the presence of some external distributed force. We therefore are considering, for this case, the system

$$\begin{aligned}
&\rho_1 u_{tt} = \sigma_1 u_{xx} + f(x,t), && x \in \Omega_1, 0 < t < T,\\
&\rho_2 v_{tt} = \sigma_2 v_{xx} + g(x,t), && x \in \Omega_2, 0 < t < T,\\
&M z_{tt}(t) + \sigma_1 u_x(0,t) - \sigma_2 v_x(0,t) = h(t), && 0 < t < T, \qquad 3.7\\
&u(0,t) = v(0,t) = z(t), && 0 < t < T,\\
&u(-l_1,t) = v(l_2,t) = 0, && 0 < t < T,
\end{aligned}$$

The following proposition is as a result of the standard semi-group methods for establishing the existence, uniqueness and regularity of finite- energy solutions of the wave equations in the presence of an external distributed force.

Proposition 3.1.1.3

For every $\mathbf{y^0} \in \boldsymbol{\mathcal{H}}$ and $(f,g,h) \in L^1((0,T);W_0)$, there exist a unique finite-energy solution of (3.7),(3.2) in the class (3.4).

Furthermore, there exist a constant $C > 0$ for which

$$\int_0^T [|u_x(-l_1,\mathrm{t})|^2 + |v_x(l_2,\mathrm{t})|^2]\mathrm{dt} \leq C[||y^0||^2_{\mathcal{H}} + ||f||^2_{L^1\left(0,T;L^2(\Omega_1)\right)} + ||\mathrm{g}||^2_{L^1\left(0,T;L^2(\Omega_2)\right)} + ||\mathrm{h}||^2_{L^1(0,T)}] \qquad 3.8$$

We note that the propositions (3.1.1.1) through to (3.1.1.3) give the regularity of solutions for the different models where the initial data come from the space with the same regularity in both strings.

Let us further give the regularity of solutions where the initial data belong to a space with different regularities in the strings. We suppose that the initial data

$y^0 \in \mathcal{H},$ 3.9

for which

$u^0 \in H^2(\Omega_1), \quad u^1 \in \mathcal{W}_1 : u^1(0) = z^1$ 3.10

By (3.9)-(3.10), it does not imply that $y^0 \in \mathcal{D}(\mathcal{A})$ and thus the regularity of solutions that proposition 3.1.1.1(ii) provides do not apply in such a case. We however have the following regularity result

Proposition 3.1.1.4

Suppose y^0satisfies (3.9)-(3.10).Then the solution of (3.1)-(3.2) is such that, besides (3.4), we have

$u \in C\big([0,T]; H^2(\Omega_1)\big) \cap C^1([0,T]; \mathcal{W}_1),$ 3.11

Furthermore, there exists a constant $C > 0$ such that

$||u||^2_{L^\infty\left(0,T;H^2(\Omega_1)\right)} + ||u_t||^2_{L^\infty(0,T;\mathcal{W}_1)} \leq C\left[E_M(0) + ||u^0||^2_{H^2(\Omega_1)} + ||u^1||^2_{\mathcal{W}_1}\right]$ 3.12

for every solution with y^0satisfying (3.9)-(3.10).

Proof of proposition 3.1.1.4

For the proof of this proposition, it is sufficient to prove the existence of some $\lambda > 0$and $C > 0$such that (3.11) and (3.12) hold for $t \in [0, \lambda]$. By scaling $x \in \Omega_1$ and changing the time scale, we can assume that $l_1 = 1$ and we set $\rho_1 = \sigma_1 = 1$. Similarly, we may assume that $\rho_2 = \sigma_2 = 1$

by adjusting the length of the second string to say $l_2 = l > 0$. The conditions at the point mass change as a consequence and therefore we consider the system

$$
\begin{aligned}
&u_{tt} = u_{xx}, && -1 < x < 0, 0 < t < \tau,\\
&v_{tt} = v_{xx}, && 0 < x < l, 0 < t < \tau,\\
&mz_{tt}(t) + u_x(0,t) - \gamma v_x(0,t) = 0, && 0 < t < \tau, \qquad 3.13\\
&u(0,t) = v(0,t) = z(t), && 0 < t < \tau,\\
&u(-1,t) = v(l,t) = 0, && 0 < t < \tau,
\end{aligned}
$$

where $\gamma, m > 0$.

Let us define the following regions

$R_1 = \{(x,t) \in (-1,0) \times (0,1): t < -x\}$

$R_2 = \{(x,t) \in (0,l) \times (0,l): t < x\}$

where the values of $u$and v, respectively, do not depend on the presence of the point mass due to the finite speed of propagation of the wave equation. In the regions R_1 and R_2, u and v remain smooth as in the absence of the point mass. In this case therefore, we have

$$u \in C\big([0,1]; H^2(R_1)\big) \cap C^1\big([0,1]; H^1(R_1)\big).$$

Moreover,

$$||u||_{L^\infty\left(0,1;H^2(R_1)\right)} + ||u_t||_{L^\infty\left(0,1;H^1(R_1)\right)} \leq C[||u^0||_{H^2(-1,0)} + ||u^1||_{H^1(-1,0)}]$$

We further define the following regions

$$S_1 = \left\{(x,t) \in (-\tfrac{1}{2},0) \times (0,1): |2t-1| < 1 + 2x\right\},$$

$S_2 = \left\{(x,t) \in (0,\frac{1}{2}) \times (0,1): |2t-1| < 1-2x\right\}$

in which the D'Alemberts' formula give the following solutions

$$u(x,t) = \frac{1}{2}\{z(t+x) + z(t-x)\} + \frac{1}{2}\int_{t-x}^{t+x} u_x(0,s)\,ds \text{ in } S_1 \qquad 3.14$$

and,

$$v(x,t) = \frac{1}{2}\{z(t+x) + z(t-x)\} + \frac{1}{2}\int_{t-x}^{t+x} v_x(0,s)\,ds = \frac{1}{2}\{z(t+x) + z(t-x)\} + \frac{1}{2\gamma}\int_{t-x}^{t+x}[u_x(0,s) + mz''(s)]\,ds \text{ in } S_2 \qquad 3.15$$

with the second equality of (3.15) being a substitution from the third equation of system (3.13).

We deduce from (3.14) that

$$u \in C\left(\left[0,\frac{1}{2}\right]; H^2(-t,0)\right) \cap C^1\left(\left[0,\frac{1}{2}\right]; H^1(-t,0)\right). \qquad 3.16$$

We further note that the solution of the wave equation in (3.14) and the expression for u in $R_1$are such that u_x and u_t are continuous across $x = -t$. The proof is complete. ■

3.1.2 The Nonhomogeneous Case

In section (3.1.1), we have given the existence and uniqueness of solutions in the absence of controls. In this section, we give the existence and uniqueness of weak solutions when we introduce, say $L^2(0,T) - Dirichlet$, controls at the extreme points $x = -l_1, l_2$ of our system. We begin by defining what we mean by a weak solution of a system.

Let us consider the following system

$$\rho_1 u_{tt} = \sigma_1 u_{xx}, \qquad x \in \Omega_1, 0 < t < T$$

$\rho_2 v_{tt} = \sigma_2 v_{xx},$ $\quad x \in \Omega_2, 0 < t < T$

$M z_{tt}(t) + \sigma_1 u_x(0,t) - \sigma_2 v_x(0,t) = 0,$ $\quad 0 < t < T$ $\quad$ 3.17

$u(0,t) = v(0,t) = z(t),$ $\quad 0 < t < T$

$u(-l_1, t) = \alpha(t),$ $\quad 0 < t < T$

$v(l_2, t) = \beta(t),$ $\quad 0 < t < T$

$u(x,0) = u^0(x),\ u_t(x,0) = u^1(x),$ $\quad x \in \Omega_1$

$v(x,0) = v^0(x),\ v_t(x,0) = v^1(x),$ $\quad x \in \Omega_2$

$z(0) = z^0,$ $\quad z_t(0) = z^1$

with $\alpha, \beta \in L^2(0,T), u^0 \in L^2(\Omega_1), v^0 \in L^2(\Omega_2)$ and the initial velocities $u^1(x)$ and $v^1(x)$ respectively belong to the dual spaces $\mathcal{W}_1{}'$ and $\mathcal{W}_2{}'$ and $z^0, z^1 \in \mathbb{R}$. We understand the solution of (3.17) in the sense of transposition.

In order for us to give a precise definition, we take advantage of the fact that the wave equation is time- reversible. We are considering the system

$\rho_1 \phi_{tt} = \sigma_1 \phi_{xx} + f(x,t),$ $\quad x \in \Omega_1, 0 < t < T$

$\rho_2 \psi_{tt} = \sigma_2 \psi_{xx} + g(x,t),$ $\quad x \in \Omega_2, 0 < t < T$

$M \zeta_{tt}(t) + \sigma_1 \phi_x(0,t) - \sigma_2 \psi_x(0,t) = h(t),$ $\quad 0 < t < T$ $\quad$ 3.18

$\phi(0,t) = \psi(0,t) = \zeta(t),$ $\quad 0 < t < T$

$\phi(-l_1, t) = \psi(l_2, t) = 0,$ $\quad 0 < t < T$

$\phi(x,T) = \phi_t(x,T) = 0,$ $\quad x \in \Omega_1$

$\psi(x,T) = \psi_t(x,T) = 0,$ $\quad x \in \Omega_2$

$\zeta(T) = \zeta_t(T) = 0$

System (3.18) has a unique solution

$(\phi, \psi, \zeta) \in C([0,T]; W_1)$,

$(\phi_t, \psi_t, \zeta_t) \in C([0,T]; W_0)$, 3.19

for every $(f, g, h) \in L^1(0, T; W_0)$ as a consequence of proposition 3.1.1.3 satisfying

$\int_0^T [|\phi_x(-l_1, t)|^2 + |\psi_x(l_2, t)|^2] dt \leq C||(f, g, h)||^2_{L^1(0,T;W_0)}$ 3.20

To be able to define a weak solution (u, v, z) of the system (3.17) in the sense of transposition, let us multiply the equations satisfied by u and v in (3.17) by $\phi \; and \; \psi$ respectively and integrate by parts with respect to both x and t to give

$\int_0^T \int_{-l_1}^0 \phi[\rho_1 u_{tt} - \sigma_1 u_{xx}] dxdt = 0$since $\rho_1 u_{tt} - \sigma_1 u_{xx} = 0$ from the first equation of system (3.17).

Integration by parts gives

$\int_{-l_1}^0 [\rho_1 \phi u_t - \sigma_1 \phi_t u]_0^T dx + \int_0^T \int_{-l_1}^0 u[\rho_1 \phi_{tt} - \sigma_1 \phi_{xx}] dxdt = 0$

or

$\int_{-l_1}^0 [\rho_1 \phi(x, T) u_t(x, T) - \sigma_1 \phi_t(x, T) u(x, T)] dx - \int_{-l_1}^0 [\rho_1 \phi(x, 0) u_t(x, 0) -$

$\sigma_1 \phi_t(x, 0) u(x, 0)] dx + \int_0^T \int_{-l_1}^0 u(x, t) f(x, t) dxdt = 0$ 3.21

with the last double integral in the above identity having the substitution

$\rho_1 \phi_{tt} - \sigma_1 \phi_{xx} = f(x, t) \quad for \quad -l_1 < x < 0, 0 < t < T$ from (3.18)

We note that the first integral in (3.21) vanishes since

$\phi(x,T) = \phi_t(x,T) = 0, \quad x \in \Omega_1 = (-l_1, 0).$

We then rearrange (3.21) to have

$$\int_0^T \int_{-l_1}^0 u(x,t)f(x,t)dxdt = \int_{-l_1}^0 [\rho_1 \phi(x,0)u_t(x,0) - \sigma_1 \phi_t(x,0)u(x,0)]dx$$

$$\int_0^T \int_{-l_1}^0 ufdxdt = \rho_1 \int_{-l_1}^0 \phi(x,0)u^1(x)dx - \sigma_1 \int_{-l_1}^0 \phi_t(x,0)u^0(x)dx$$

Or

$$\int_0^T \int_{-l_1}^0 ufdxdt = \rho_1 \langle u^1, \phi(.,0) \rangle_{\Omega_1} - \sigma_1 \int_{-l_1}^0 u^0 \phi_t(x,0)dx \qquad 3.22$$

Similarly, multiplying through the equation that v satisfies by ψ, then integrating by parts and simplifying, we will have

$$\int_0^T \int_0^{l_2} vgdxdt = \rho_2 \langle v^1, \psi(.,0) \rangle_{\Omega_2} - \sigma_2 \int_0^{l_2} v^0 \psi_t(x,0)dx \qquad 3.23$$

and

$$\int_0^T zhdt = \sigma_1 \int_0^T u(-l_1,t)\phi_x(-l_1,t)dt - \sigma_2 \int_0^T v(l_2,t)\psi_x(l_2,t)dt + Mz_t(0)\zeta(0) - Mz(0)\zeta_t(0)$$

$$\int_0^T zhdt = \sigma_1 \int_0^T \alpha(t)\phi_x(-l_1,t)dt - \sigma_2 \int_0^T \beta(t)\psi_x(l_2,t)dt + Mz^1\zeta(0) - Mz^0\zeta_t(0). \qquad 3.24$$

Adding equations (3.22) through to (3.24), we have the identity

$$\int_0^T \int_{-l_1}^0 ufdxdt + \int_0^T \int_0^{l_2} vgdxdt + \int_0^T zhdt = -\sigma_1 \int_{-l_1}^0 u^0 \phi_t(x,0)dx - \sigma_2 \int_0^{l_2} v^0 \psi_t(x,0)dx +$$
$$\rho_1 \langle u^1, \phi(.,0) \rangle_{\Omega_1} + \rho_2 \langle v^1, \psi(.,0) \rangle_{\Omega_2} + \sigma_1 \int_0^T \alpha(t)\phi_x(-l_1,t)dt - \sigma_2 \int_0^T \beta(t)\psi_x(l_2,t)dt +$$
$$Mz^1\zeta(0) - Mz^0\zeta_t(0) \qquad 3.25$$

With this identity, we are positioned to make the following definition.

Definition 3.1.2.1

(u, v, z) is said to be a weak solution(in the sense of transposition) of (3.18) if the identity (3.25) holds for every $(f, g, h) \in L^1(0, T; W_0)$. The initial velocities (u^1, v^1, z^1) are applied to the elements $(\phi(.,0), \psi(.,0), \zeta(0))$of W_1in the sense of duality in W_1. We thus note that the two initial data that coincide in $W_0 \times W_1'$give rise to the same solution.

CHAPTER 4

RESULTS AND DISCUSSION

This section presents findings of this research on the exact controllability and stabilization of a one-dimensional wave equation with an interior point mass.

4.1 Preliminary Results on the Existence, Uniqueness and Regularity of Solutions when Control is Active.

This section presents the introductory results on the existence and uniqueness of solutions for the controlled system. We will refer to the last section of Chapter 3 that relates directly to what this research investigated. We first present some propositional results to establish the foundation to our findings.

Proposition 4.1.1

For every

$\alpha, \beta \in L^2(0,T), (u^0, v^0) \in L^2(\Omega_1) \times L^2(\Omega_2), (u^0, v^0) \in (\mathcal{W}_1' \times \mathcal{W}_2') and\ z^0, z^1 \in \mathbb{R}$,

there exists a solution of (3.17), in the sense of transposition, in the class

$$(u, v, z) \in C([0,T]; W_0), \quad 4.1$$

$$(u_t, v_t, z_t) \in C([0,T]; \mathcal{W}_1' \times \mathcal{W}_2' \times \mathbb{R}). \quad 4.2$$

Furthermore, there is a one-to-one correspondence between the initial data as elements of the quotient space $W_0 \times W_1{}'$ and the solutions of (3.17) in the class (4.1)-(4.2).

We then consider these weak solutions when the data, both initial and boundary, corresponding to the first string have one more degree of regularity. We therefore are considering a case where, say,

$$\alpha \in H^1(0,T), u^0 \in H^1(\Omega_1), u^1 \in L^2(\Omega_1), u^0(0) = z^0, \alpha(0) = u^0(-l_1). \qquad 4.3$$

For this, we have the following propositional result.

Proposition 4.1.2

Suppose that the initial and boundary data in the proposition 4.1.1 satisfy the further regularity and compatibility conditions (4.3). Then in addition to (4.1)-(4.2) we have

$$u \in C([0,T; H^1(\Omega_1)) \cap C^1([0,T]; L^2(\Omega_1)). \qquad 4.4$$

Moreover, $u_x(-l_1,t) \in L^2(\Omega_1)$ and there exist a positive constant C such that

$$\int_0^T |u_x(-l_1,t)|^2 dt \leq C\left[||\alpha||^2_{H^1(0,T)} + ||\beta||^2_{L^2(0,T)} + ||u^0||^2_{H^1(\Omega_1)} + ||u^1||^2_{L^2(\Omega_1)} + ||v^0||^2_{L^2(\Omega_2)} + ||v^1||^2_{(\mathcal{W}_2)'} + |z^0|^2 + |z^1|^1\right] \qquad 4.5$$

The regularity property (4.5) is a direct consequence of (4.4) and the local nature of the wave equation. The presence of the nonhomogeneous boundary conditions does not alter the regularity of the solution. We therefore have that the trace regularity

$$\int_0^T |u_x(-l_1,t)|^2 \leq C \int_0^T \left(||u(.,t||^2_{H^1(\Omega_1)} + ||u_t(.,t)||^2_{L^2(\Omega_1)}\right) dt \qquad 4.6$$

is local and does not depend on the whether there is a point mass. The right hand side of (4.5) defines the norm for which the solution map $(\alpha, \beta, u^0, v^0, z^0, u^1, v^1, z^1) \mapsto u$ is continuous from the space it is defined into the space (4.4). This follows from the fact that u has the regularity property (4.4).

4.2 Control at both Extremes

We now consider the problem of controlling our system from both ends $x = -l_1, l_2$ for which our system now reads

$$
\begin{aligned}
&\rho_1 u_{tt} = \sigma_1 u_{xx}, && x \in \Omega_1, 0 < t < T \\
&\rho_2 v_{tt} = \sigma_2 v_{xx}, && x \in \Omega_2, 0 < t < T \\
&M z_{tt}(t) + \sigma_1 u_x(0,t) - \sigma_2 v_x(0,t) = 0, && 0 < t < T \\
&u(0,t) = v(0,t) = z(t), && 0 < t < T \\
&u(-l_1,t) = \alpha(t), && 0 < t < T \\
&v(l_2,t) = \beta(t), && 0 < t < T \\
&u(x,0) = u^0(x),\ u_t(x,0) = u^1(x), && x \in \Omega_1 \\
&v(x,0) = v^0(x),\ v_t(x,0) = v^1(x), && x \in \Omega_2 \\
&z(0) = z^0, \qquad z_t(0) = z^1
\end{aligned}
\qquad 4.7
$$

We note that the system presents a model for our hybrid system when the control is active at both end points of the string-mass hybrid system. This presents a similar scenario as in the case of nonhomogeneous boundary conditions we presented in section 3.1.2. We now have the following results.

Theorem 4.2.1

Suppose that $T > 2\max\left(l_1\sqrt{\rho_1/\sigma_1}, l_2\sqrt{\rho_2/\sigma_1}\right)$. Then for every

$$(u^0, v^0, z^0) \in W_0, \quad (u^1, v^1, z^1) \in \mathcal{W}_1' \times \mathcal{W}_2' \times \mathbb{R} \qquad 4.8$$

There exist controls $\alpha, \beta \in L^2(0,T)$ such that the solution of (4.7) satisfies

$$\begin{aligned} &u(x,T) = u_t(x,T) = 0, \quad x \in \Omega_1 \\ &v(x,T) = v_t(x,T) = 0, \quad x \in \Omega_2 \\ &z(T) = z_t(T) = 0. \end{aligned} \qquad 4.9$$

Proof (Application of HUM)

For any $(\phi^0, \psi^0, \zeta^0, \phi^1, \psi^1, \zeta^1)^t \in \mathcal{H}$, we define

$$\mathcal{E}(\phi^0, \psi^0, \zeta^0, \phi^1, \psi^1, \zeta^1) = (u_t(0), v_t(0), z_t(0), -u(0), -v(0), -z(0)) \in \mathcal{H}' \qquad 4.10$$

where $\mathcal{H}'$ is the dual of $\mathcal{H}$ and (u, v, z) is the solution of the system

$$\begin{aligned} &\rho_1 u_{tt} = \sigma_1 u_{xx}, && x \in \Omega_1, 0 < t < T, \\ &\rho_2 v_{tt} = \sigma_2 v_{xx}, && x \in \Omega_2, 0 < t < T, \\ &M z_{tt}(t) + \sigma_1 u_x(0,t) - \sigma_2 v_x(0,t) = 0, && 0 < t < T, \\ &u(0,t) = v(0,t) = z(t), && 0 < t < T, \\ &u(-l_1,t) = -\phi_x(-l_1,t), && 0 < t < T, \\ &v(l_2,t) = \psi_x(l_2,t), && 0 < t < T, \\ &u(x,T) = u_t(x,T) = 0, && x \in \Omega_1, \\ &v(x,T) = v_t(x,T) = 0, && x \in \Omega_2, \\ &z(T) = z_t(T) = 0 \end{aligned} \qquad 4.11$$

and $(\phi, \psi, \zeta, \phi_t, \psi_t, \zeta_t)^t$ is a solution of (3.3) with $y^0 = (\phi^0, \psi^0, \zeta^0, \phi^1, \psi^1, \zeta^1)^t$. In view of proposition 3.1.1.2, we have

$\phi_x(-l_1,.), \psi_x(l_2,.) \in L^2(0,T)$.

And consequently, proposition 4.1.1 provides that (u,v,z) satisfies the regularity and compatibility conditions (4.1)-(4.2). Particularly, we have

$(u_t(0), v_t(0), z_t(0), -u(0), -v(0), -z(0))$

to be a well-defined element of $\mathcal{H}'$. This implies that

$\mathcal{E}: \mathcal{H} \longrightarrow \mathcal{H}'$

is linear and continuous.

By the transposition formula and the provision of proposition 4.2.1 (see below), we obtain

$$\langle \mathcal{E}(\phi^0,\psi^0,\zeta^0,\phi^1,\psi^1,\zeta^1), (\phi^0,\psi^0,\zeta^0,\phi^1,\psi^1,\zeta^1)\rangle = \int_0^T [|\phi_x(-l_1,t)|^2 + |\psi_x(l_2,t)|^2]\, dt$$

$$\geq CE_M(0).$$

We take into account the equivalence of $(E_M(0))^{\frac{1}{2}}$ and the norm of $(\phi^0,\psi^0,\zeta^0,\phi^1,\psi^1,\zeta^1)$ in $\mathcal{H}$ and conclude that $\mathcal{E}: \mathcal{H} \longrightarrow \mathcal{H}'$ is an isomorphism. For any$(u^0,v^0,z^0) \in W_0, (u^1,v^1,z^1) \in \mathcal{W}_1' \times \mathcal{W}_2' \times \mathbb{R}$, we have $(u^1,v^1,z^1) \in \mathcal{W}_1'$. We therefore have that the equation

$$\mathcal{E}(\phi^0,\psi^0,\zeta^0,\phi^1,\psi^1,\zeta^1) = \big(u_t(0), v_t(0), z_t(0), -u(0), -v(0), -z(0)\big) = (u^1,v^1,z^1,-u^0,-v^0,-z^0)$$

admits a unique solution $(\phi^0,\psi^0,\zeta^0,\phi^1,\psi^1,\zeta^1)^t \in \mathcal{H}$, and this is equivalent to the fact that the solution of (4.11) satisfies

$u(x,0) = u^0(x),\ u_t(x,0) = u^1(x), \qquad x \in \Omega_1$

$v(x,0) = v^0(x),\ v_t(x,0) = v^1(x), \qquad x \in \Omega_2$

$z(0) = z^0, \qquad z_t(0) = z^1$

This concludes the proof. ■

Time reversibility and linearity of system (4.7) allow us to deduce that for every initial data as in (4.8) and the final data $(\check{u}^0, \check{v}^0, \tilde{z}^0, \check{u}^1, \check{v}^1, \check{z}^1)$ satisfying the same properties, there exist controls $\alpha, \beta \in L^2(0,T)$ such that the solution of (4.7) satisfies

$u(x,T) = \check{u}^0(x),\ u_t(x,T) = \check{u}^1(x), \quad x \in \Omega_1,$

$v(x,T) = \check{v}^0(x),\ v_t(x,T) = \check{v}^1(x), \quad x \in \Omega_2,$ 4.12

$(T) = \check{z}^0, \qquad z_t(T) = \check{z}^1.$

We further note that the controllability space (4.8) is the same as that which one obtains in the absence of the point mass but the controllability time is larger in cases where $l_1\sqrt{\rho_1/\sigma_1} \neq l_2\sqrt{\rho_2/\sigma_1}$.

Theorem 4.2.2

Suppose that $T > l_1\sqrt{\rho_1/\sigma_1} + l_2\sqrt{\rho_2/\sigma_1}$ where $l_1\sqrt{\rho_1/\sigma_1} > l_2\sqrt{\rho_2/\sigma_1}$, say. Then, for every initial data as in (4.8) satisfying the additional regularity and compatibility properties

$u^0 \in \mathcal{W}_1,\ u^1 \in L^2(\Omega_1),\ u^0(0) = z^0,$ 4.13

there exist controls $\alpha \in H_0^1(0,T) and\ \beta \in L^2(0,T)$ such that the solution of (4.7) satisfies (4.9). As a consequence of Theorem 4.2.2, we deduce that we can drive our system (4.7) from any initial state in the class (4.8),(4.13) to any terminal state in the same class. This is guaranteed by the fact that our modeling equations are both linear and reversible in time. This gives us a system that is well posed in the asymmetric space (4.8),(4.13).

The control time we obtain is the same as that which we gave in the absence of the point mass in which case $M = 0$. Controllability in the space (4.8) is only possible when the controllability time, T is such that $T > 2\max\left(l_1\sqrt{\rho_1/\sigma_1}, l_2\sqrt{\rho_2/\sigma_1}\right)$.

Theorem 4.2.1 is a direct consequence of the following observability result for the solution of the uncontrolled system

$$\begin{aligned}
&\rho_1\phi_{tt} = \sigma_1\phi_{xx}, && x \in \Omega_1, 0 < t < T \\
&\rho_2\psi_{tt} = \sigma_2\psi_{xx}, && x \in \Omega_2, 0 < t < T \\
&M\zeta_{tt}(t) + \sigma_1\phi_x(0,t) - \sigma_2\psi_x(0,t) = 0, && 0 < t < T \qquad 4.14 \\
&\phi(-l_1,t) = \psi(l_2,t) = 0, && 0 < t < T \\
&\phi(0,t) = \psi(0,t) = \zeta(t), && 0 < t < T\ ,
\end{aligned}$$

when we apply Lions' HUM.

Proposition 4.2.1

Suppose $T > T_0$ where $T_0 = 2\max\left(l_1\sqrt{\rho_1/\sigma_1}, l_2\sqrt{\rho_2/\sigma_1}\right)$. Then

$$(T - T_0)E_M(0) \le \frac{1}{2}\left(\max(l_1,l_2) + \frac{M}{\rho_1+\rho_2}\right)\int_0^T [\sigma_1|\phi_x(-1,t)|^2 + \sigma_2|\psi_x(1,t)|^2]dt \qquad 4.15$$

for every finite energy solution of (4.14).

Proof

We carry on with the proof in a number of steps.

Let us first consider the space dependent energies

$$e_1(x) = \frac{1}{2}\int_{(l_1+x)\mu_1}^{T-(l_1+x)\mu_1}[\rho_1|\phi_t(x,t)|^2 + \sigma_1|\phi_x(x,t)|^2]dt\,, -l_1 \le x \le 0 \qquad 4.16$$

and

$$e_2(x) = \frac{1}{2}\int_{(l_2-x)\mu_2}^{T-(l_2-x)\mu_2}[\rho_2|\psi_t(x,t)|^2 + \sigma_2|\psi_x(x,t)|^2]dt\,, \quad 0 \le x \le l_2 \qquad 4.17$$

where $\mu_1 = \sqrt{\rho_1/\sigma_1}\,, and\ \ \mu_2 = \sqrt{\rho_2/\sigma_2}$. We take note of the physical provision that $e_1(.)$is non-increasing whereas $e_2(.)$is non-decreasing. We therefore have

$$e_1(x) \le e_1(-l_1) = \frac{\sigma_1}{2}\int_0^T[|\phi_x(x,t)|^2]dt\,, -l_1 \le x \le 0 \ \ \{\text{from (4.16)}\} \qquad 4.18$$

and

$$e_2(x) \le e_2(l_2) = \frac{\sigma_2}{2}\int_0^T[|\psi_x(x,t)|^2]dt\,, 0 \le x \le l_2.\,\{\text{from (4.17)}\} \qquad 4.19$$

Consequently (4.18)-(4.19), we deduce that

$$\int_\omega^{T-\omega}\int_{-l_1}^0[\rho_1|\phi_t(x,t)|^2 + \sigma_1|\phi_x(x,t)|^2]dxdt + \int_\omega^{T-\omega}\int_0^{l_2}[\rho_2|\psi_t(x,t)|^2 + \sigma_2|\psi_x(x,t)|^2]dxdt$$

$$\le\ \max(l_1,l_2)\int_0^T[\sigma_1|\phi_x(-l_1,t)|^2 + \sigma_2|\psi_x(l_2,t)|^2]dt, where\ \omega = \max(\mu_1 l_1, \mu_2 l_2).$$

Since $\phi(0,t) = \psi(0,t) = \zeta(t)$, we have

$$M\int_\omega^{T-\omega}|\zeta_t(t)|^2dt \le \frac{2M}{\rho_1+\rho_1}\big(e_1(0) + e_2(0)\big)$$

$$\le \frac{M}{\rho_1+\rho_1}\int_0^T (|\phi_x(x,t)|^2 + \psi_x(1,t)|^2)dt.$$

The implication is that we have the following to hold

$$(T-2\omega)E_M(0) = \int_\omega^{T-\omega} E_M(t)dt$$

$$\le \left(\frac{\max(l_1,l_2)}{2} + \frac{M}{2(\rho_1+\rho_1)}\right)\int_0^T [\sigma_1|\phi_x(-l_1,t)|^2 + \sigma_2|\psi_x(l_2,t)|^2 dt,$$

This completes the proof as the inequality (4.15) is what the result above gives. ■

We remark the following regarding the inequality (4.15):

The inequality provides explicit observability constants and in particular the dependence of these constants on the properties of the system and the mass of the point mass. We note that in the limit $M \longrightarrow 0$, we obtain the usual constant for the wave equation with piecewise constant coefficients.

We now turn our attention to a case when the control is active at only one end of the system and this is what motivates our next section

4.3 Control at one Extreme

We now consider the problem of controlling our system from only one extreme point. If we consider our system with the control active at the end point $x = l_2$, then our system now reads

$$\begin{aligned}
&\rho_1 u_{tt} = \sigma_1 u_{xx}, && x \in \Omega_1, 0 < t < T \\
&\rho_2 v_{tt} = \sigma_2 v_{xx}, && x \in \Omega_2, 0 < t < T \\
&M z_{tt}(t) + \sigma_1 u_x(0,t) - \sigma_2 v_x(0,t) = 0, && 0 < t < T \qquad 4.20 \\
&u(0,t) = v(0,t) = z(t), && 0 < t < T
\end{aligned}$$

$u(-l_1, t) = 0,$ $\quad 0 < t < T$

$v(l_2, t) = \beta(t),$ $\quad 0 < t < T$

$u(x, 0) = u^0(x),\ u_t(x, 0) = u^1(x),$ $\quad x \in \Omega_1$

$v(x, 0) = v^0(x),\ v_t(x, 0) = v^1(x),$ $\quad x \in \Omega_2$

$z(0) = z^0, \qquad z_t(0) = z^1$

The following result is of great significance in the analysis of the dynamics of our present system.

Theorem 4.3.1

Suppose that $T > 2(l_1\sqrt{\rho_1/\sigma_1} + \ l_2\sqrt{\rho_2/\sigma_1})$. Then, for every

$(u^0, u^1) \in \mathcal{W}_1 \times L^2(\Omega_1),$

$(v^0, v^1) \in L^2(\Omega_2) \times (\mathcal{W}_2)',$ 4.21

$(z^0, z^1) \in \mathbb{R}^2,$

such that

$u^0(0) = z^0,$ 4.22

there exists a control $\beta(t) \in L^2(0, T)$ such that the solution of (4.20) satisfies (4.9).

Linearity, time-reversibility and well-posedness of the system in the asymmetric space (4.21)-(4.22) and as a consequence of Theorem 4.3.1, we are guaranteed to deduce that we can drive the system (4.20) from any initial state to any final state in the class (4.21)-(4.22). The control time is the same as that we would obtain in the absence of the point mass, that is for $M = 0$. We however are observing that the controllable space is larger in the absence of the point mass and is given by (4.8). The controllable space in the case when the point mass is present is smaller and is the optimal one.

Lions' HUM give it that Theorem 4.3.1 is equivalent to the following observability results of the uncontrolled system (4.16).

Proposition 4.3.1

Suppose that $T > 2(l_1\sqrt{\rho_1/\sigma_1} + l_2\sqrt{\rho_2/\sigma_1})$. Then there exists a positive constant $C = C(T)$ for which the following holds for every finite-energy solution of (4.16)

$$||\phi(.,0)||^2_{L^2(\Omega_1)} + ||\phi_t(.,0||^2_{(\mathcal{W}_1)'} + ||\psi(.,0)||^2_{\mathcal{W}_2} + ||\psi_t(.,0)||^2_{L^2(\Omega_2)} + |\zeta(0)|^2 + |\zeta_t(0)|^2$$

$$\leq C\int_0^T |\psi_x(l_2,t)|^2 dt.$$

We note that this inequality is sharp in that the reverse holds for all $T > 0$. Consequently, this double inequality assures us that the controllable space (4.21)-(4.22) is the optimal space. This concludes the results on exact boundary controllability and now allows us to present some results on the boundary stabilization of our system.

4.4 Boundary Stabilization

In this section, we examine the problem of stabilization of the system by way of velocity feedback at the extreme ends of the system. We consider the following system (with an interest in its decay properties)

$$\rho_1 u_{tt} = \sigma_1 u_{xx}, \qquad x \in \Omega_1, 0 < t < T$$

$$\rho_2 v_{tt} = \sigma_2 v_{xx}, \qquad x \in \Omega_2, 0 < t < T$$

$Mz_{tt}(t) + \sigma_1 u_x(0,t) - \sigma_2 v_x(0,t) = 0, \qquad 0 < t < T \qquad 4.23$

$u(0,t) = v(0,t) = z(t), \qquad 0 < t < T$

$u(x,0) = u^0(x),\ u_t(x,0) = u^1(x), \qquad x \in \Omega_1$

$v(x,0) = v^0(x),\ v_t(x,0) = v^1(x), \qquad x \in \Omega_2$

$z(0) = z^0, \qquad z_t(0) = z^1$

under the following two types of boundary conditions

$u(-l_1,t) = 0, \qquad t > 0,$

$\sigma_2 v_x(l_2,t) + \tau v_t(l_2,t) = 0, \qquad t > 0, \qquad 4.24$

and

$\sigma_1 u_x(-l_1,t) - \tau u_t(-l_1,t) = 0, \qquad t > 0\ ,$

$\sigma_2 v_x(l_2,t) + \tau v_t(l_2,t) = 0, \qquad t > 0, \qquad 4.25$

with $\tau > 0$.

We introduce some damping on the system at the extreme point $x = l_2$ as is seen in (4.24) and at both extreme points for the boundary conditions (4.25). We consequently affected the energy of the solutions of (4.23)-(4.24) and (4.23),(4.25) so that

$\frac{dE_M}{dt} = -\tau|v_t(l_2,t)|^2$, and

$\frac{dE_M}{dt} = -\tau(|u_t(-l_1,t)|^2 + |v_t(l_2,t)|^2)$, respectively.

The research employed the provision of the standard semi-group theory and presented the following facts about the solutions of the system under the effect of damping at the extreme points of the system:

i. If $y^0 = (u^0, v^0, z^0, u^1, v^1, z^1)^t$ and

$y^0 \in H^1(\Omega_1) \times H^1(\Omega_2) \times \mathbb{R} \times L^2(\Omega_1) \times L^2(\Omega_2) \times \mathbb{R}$,

$u^0(0) = v^0(0) = z^0, u^0(-l_1) = 0,$ 4.26

then (4.23)-(4.24) and (4.23),(4.25) have unique finite-energy solutions in the class

$(u, v, z) \in C([0, \infty); H^1(\Omega_1) \times H^1(\Omega_2) \times \mathbb{R})$,

$(u_t, v_t, z_t) \in C([0, \infty); L^2(\Omega_1) \times L^2(\Omega_2) \times \mathbb{R})$ 4.27

And integration with respect to t, of the affected rates of energy for the system under the damping in (4.24) and (4.25) respectively, showed the following identities to hold

$E_M(t_2) - E_M(t_1) = -\tau \int_{t_1}^{t_2} |v_t(l_2, t)|^2 dt \ \ for\ (4.23) - (4.24)$, and

$E_M(t_2) - E_M(t_1) = -\tau \int_{t_1}^{t_2} [|u_t(-l_1, t)|^2 + |v_t(l_2, t)|^2] dt \ \ for\ (4.23), (4.25)$ 4.28

ii. If the initial data satisfied the following additional regularity and compatibility conditions

$(u^0, v^0) \in H^2(\Omega_1) \times H^2(\Omega_2);\ (u^1, v^1) \in H^1(\Omega_1) \times H^1(\Omega_2)$,

$u^1(0) = v^1(0) = z^1$,

$\sigma_2 v_x^0(l_2) + \tau v^1(l_2) = 0,$ 4.29

$u^1(-l_1) = 0, \quad for\ (4.23) - (4.24),$

$\sigma_1 u_x^0(-l_1) - \tau u^1(-l_1) = 0,$ for (4.23),(4.25),

Then the solutions have the following added regularity:

$(u, v) \in C\big([0, \infty); H^2(\Omega_1) \times H^2(\Omega_2)\big)$,

$(u_t, v_t) \in C\big([0, \infty); H^1(\Omega_1) \times H^1(\Omega_2)\big)$, 4.30

$z \in C^2([0, \infty); \mathbb{R})$.

The only equilibrium configuration for system (4.23) under the boundary conditions (4.24) is the zero one. The energy for the solutions of (4.23)-(4.24) is coercive and therefore $E_M(t) \longrightarrow$

$0\ as\ t \longrightarrow \infty$. On the other hand, for the system (4.23),(4.25), the energy of solutions is not coercive and thus, for every real constant $\wp$, $(u, v, z) = (\wp, \wp, \wp)$ defines solution.

CHAPTER 5

CONCLUSION AND RECOMMENDATION

5.1 Conclusion

We sought to examine a simple model for an elastic string involving an interior point mass for which we treat as a hybrid system of two strings because of the presence of the point mass. We have precisely described the space of exact controllability when control is active at both or one end of the string-mass system. We have discussed control problems chiefly by finding suitable observability estimates and applying Hilbert's Uniqueness Method (HUM) to determine the controllable spaces for the control problems when controlling from both extremes and when control is acting from one end of the system.

We saw that the presence of the point mass introduces important changes in the behavior of our system concerning the observability properties. By an explicit computation (see proposition 3.1.1.4), we established that when a wave starting from the initial data

$u^0 = \phi^0 H_0^1(\Omega_1),\;\; v^0 = 0$

$u^1 = \psi^1 \in L^2(\Omega_1),\;\; v^1 = 0$

$z^0 = z^1 = 0$

Gets to the point mass, one part is reflected and the other part is transmitted over the point mass. The part which is reflected keeps the same regularity as the initial data whereas the part that is transmitted is regularized by one degree. This means that $v(.,t) \in H^2(\Omega_2), t > 0$.

5.2 Recommendation

The results of this study indicate the important role that control theory has in analyzing the dynamics of the ever complicating dynamical systems. There has always been need for a 'vision' and the desire to actualize this vision. This is basically what exact controllability of dynamical systems is concerned with. The idea that a system can be driven from one initial state to a terminal state in a time T gives hope that one can envision a future state of a dynamical system and continually apply admissible controls to be able to realize that state. The Kenya Vision 2030 is an idea that seeks to put Kenya as per the vision statement by 2030 and the achievement of this vision is by no miracle but putting controls to ensure that we are directed towards attaining the vision.

5.3 Areas for Further Research

This research has been able to describe the space for exact controllability and boundary stabilization of a one-dimensional wave equation with an interior point mass. We recommend the following areas for further study:

i. An investigation as to whether exact controllability is possible for more than one dimensional wave equation

ii. We have used a point mass string system in our analyses of the control problems this research handled. Further study should be done to check how regularity of solutions is affected by increasing the number of point masses within the hybrid system. This is to say that some research should be done to extend the results of this research to cases of n point masses where $n \geq 2$.

iii. An investigation as to whether the results of this study remain valid for the wave equations modeling the elastic string system with spatially variable coefficients.

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