SUBGROUPS, LATTICE STRUCTURES, AND THE NUMBER OF
SYLOW $p$-SUBGROUPS FOR SYMMETRIC GROUPS

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DECLARATION

I declare that this project is my innovative work and has not been presented in any other university or institution of higher education for the award of any degree or for any other qualification.

Hannah Wagio Ndirangu

Signature…………………… Date:…………………………..

I56/CE/28807/2015

The project has been recommended for examination with my endorsement as the Kenyatta university supervisor.

Dr. Benard Kivunge

Signature ………………… Date:…………………………..

Mathematics department

Kenyatta University
DEDICATION

This project work is dedicated to the entire Ndirangu’s family for their love, material support, moral support and continuous encouragement over the period of my study.
ACKNOWLEDGEMENT

I thank the Almighty God for His tremendous care, love, blessings, and favours that He has granted unto me over the years of my study. He has paved all my ways unto this great destiny I am at the moment.

My special thanks to my supervisor Dr. Benard Kivunge for his special attention to my work, wonderful support by providing ideas to build up my project work and availing himself all the time I needed him.

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Finally, I greatly thank my entire family for granting me all the support that I needed during the study. You made the academic drive easy and enjoyable.
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$S_n$ - The symmetric group of degree $n$ and order $n!$

$\langle a \rangle$ - Subgroup generated by an element $a$

$\text{mod}$ - Congruent

$p$ - Prime number

$\mathbb{Z}$ - Integers

$D_n$ - The $n^{th}$ dihedral group of order $2n$

$n_p$ - Number of Sylow $p$-subgroups

$A_n$ - Alternating group of degree $n$

$|G|$ - The order of a group $G$

$\prod_{i=1}^{n} i$ - Product of $i$ elements

$\binom{n}{p}$ - $n$ combination $p$

$\text{Syl}_p(C_n)$ - Cyclic Sylow subgroup
ABSTRACT
Subgroups and supergroups of various symmetric groups have been researched on extensively. Various suggestions by researchers have been provided on how to find the numbers of Sylow $p$-subgroups. Casadio (1990) has provided the proof for the third Sylow theorem, which will greatly contribute to the finding of the possible variety of numbers of the Sylow $p$-subgroups. He stated that the number of the Sylow $p$-subgroups of a group $G$ is congruent to 1 modulo $p$ and divides $m \, (n_p / m)$ i.e. $n_p \equiv 1 \pmod p$ and $m$ is gotten by dividing the order of the group with the order of the Sylow $p$-subgroup. The research will be made up of five chapters. It has emphasized the number of subgroups, supergroups, lattice structures and the ascending chains of various symmetric groups. We shall largely study the Sylow $p$-subgroups in symmetric groups $S_n$, $n=1, 2, 3, 4, 5, 6, 7, 8, 9, 10$ which will lead to a generalization of the number of Sylow $p$-subgroups in any symmetric group $S_n$ and hence coming up with a formula for getting the number of the Sylow $p$-subgroups of symmetric groups of any given order. Gow (1994) showed that for any symmetric group $S_n$, where $n$ is a prime, the number of Sylow $p$-subgroups is $(p-2)!$. Hence we target on finding the number of the Sylow $p$-subgroup for any symmetric group, which will be given by; $n_p = \frac{(p-2)! \prod_{i=0}^{m} \binom{n-p}{p}}{(m+1)!}$

Where $n_p$ is the number of Sylow $p$-subgroups of any symmetric group.
CHAPTER ONE

INTRODUCTION AND PRELIMINARY RESULTS

In this chapter, we provide the basic information and the concepts that will be applied throughout the research project. The symmetric groups will be introduced and some of the definitions that will guide us in the understanding of the information given in the entire project.

1.1 Introduction

Symmetric groups that are finite and defined over a set that is finite of \( n \) characters is made up of permutation operations that can be operated on these \( n \) characters. There are \( n \) factorial permutation possible to be performed on the \( n \) elements of a set. Therefore the order of a group \( S_n \) is \( n! \).

Any symmetric group can be described on a set that is infinite. Scott (1987), Cameron (1999) and Dixon & Mortimer (1996) established that symmetric groups on the sets that are finite usually behave quite differently from the symmetric groups that are on a set that is infinite. For this project, it focuses only on the finite symmetric groups, their subgroups, lattice structures, ascending chains and the number of the Sylow \( p \)-subgroups of the symmetric group \( S_n \).

For the symmetric groups that are of lower degree, they have exceptional structures and are extremely simple, as a result, these groups are always separately treated. For symmetric groups \( S_n \), \( n = 0 \) the null set or the empty set and \( n = 1 \), the singleton set, the symmetric group is trivial since the order is \( 0! = 1! = 1 \). \( S_2 \) comprised of exactly two
components, permutation swapping the two elements and the identity element. For this project, we shall consider the symmetric group $S_n$, $n \geq 3$.

We shall compile all the claims, theorems and lemmas on the Sylow $p$-subgroups of the symmetric groups and come up with a formula for getting the number of Sylow $p$-subgroups of any symmetric group $S_n$.

1.2 Definition and preliminary results

The following are preliminary results and definitions, which are essential for the study and understanding of the symmetric group $S_n$.

**Definition 1.2.1: The symmetric group** that is usually denoted as $S_n$ is a map that is a bijection from the set $T$ to $T$ and its group operation involves alignment of functions.

The symmetric group over a set of $n$ number of elements has an order of $n!$.

**Properties of symmetric groups**

The following are the properties of the symmetric groups that are essential in this project.

i. The symmetric group $S_n$ has order $n!$

ii. Elements belonging to a symmetric group on a given set $X$ are the permutations of $X$. e.g. In $S_3$, $X = \{1,2,3\}$ and their permutations give the 6 elements:

\{
  e, (123), (132), (12), (13), (23)\}

iii. A symmetric group $S_n$ is said to be of degree $n$ if $X = \{1,2,3…n\}$.

iv. The symmetric group $S_n$ is solvable if and only if $n \leq 4$. e.g. for $n > 4$ there are polynomials of degree $n$ which are not solvable by radicals. A group is said to be solvable if its derived series terminates in the trivial subgroup.
v. Any symmetric group is abelian if and only if $n \leq 2$. For $n = 0$ then the symmetric group is said to be an empty or the null set, for $n = 1$ the symmetric group is said to be a singleton set. For these two cases, the symmetric groups are said to be trivial. Here the symmetric groups $S_0$ and $S_1$ are equal to their alternating groups. The order of these symmetric groups is $0! = 1! = 1$.

vi. A group operation in the symmetric groups is a function composition $f \cdot g$. The composition $f$ of $g$ maps any element $X$ of $X$ to $f(g(X))$.

**Definition 1.2.2: Permutation** involves the act of arranging the elements of a set into a given order. It can also involve rearranging an ordered set in a process known as permuting. A permutation of a set $S$ is a bijection from the set $A$ to itself.

**Definition 1.2.3:** Let $a$ be an element of a group $G$, then $a$ is referred to as the **generator** of the group $G$ if $\langle a \rangle = G$.

**Definition 1.2.4:** A **lattice structure of the subgroups of a group** consists of a partially ordered set where every dual element has a distinctive supremum. It involves relation $(H(G))$, is the set of all subgroups of $G$ while $(partial order)$ is the set inclusion.

**Definition 1.2.5:** **Trivial subgroup of a group** is the one that contains only one element. Mostly it is the identity element. The group $\{ e \}$.

**Definition 1.2.6:** The **order of a group** $G$ denoted as $|G|$, is the number of the elements that are contained in that particular group. It is the cardinality of a group.

**Definition 1.2.7:** The **order of an element** $b$ is the smallest positive number $n$ such that $b^n = e$ ($e$ is the identity element).
**Definition 1.2.8:** If \( K \) is a non-empty subset of a group \( \langle G, \ast \rangle \). Then \( K \) is said to be a **subgroup of a group** \( \langle G, \ast \rangle \) if \( \langle K, \ast \rangle \) is a group. \( K \) \( \subseteq G \) is a subgroup of \( G \) if \( a, b \)
\[ ab^{-1} = k \quad K \]. A subgroup \( K \) of a group \( G \) is usually denoted as \( K < G \). \( G \) is called a **supergroup of the group** \( K \).

**Definition 1.2.9:** For any finite set, the symmetric group of size \( n \) has an order \( n! \).
This is achieved through a simple counting case. For any permutation on a given finite set here are a total of \( n \) chances for the image of the primary element and \( (n-1) \) choices for the subsequent element. This product rule shows us that the total number of combinations is \( n! \).

**Definition 1.2.11:** A subset \( K \) of a group \( G \) is a **twisted subgroup** if the identity element belongs to \( K \), for every order and given \( x, y \) \( K \) the element \( xyx \) is in \( K \).

**Definition 1.2.12:** If \( p^k \) is the highest power of a prime \( p \) dividing the order of a finite group \( G \), then a subgroup of \( G \) of order \( p^k \) is called a **Sylow** \( p \)-**subgroup** of \( G \).

**Theorem 1.2.13:** Wilson’s theorem: \( (p - 2)! \) is the number of Sylow \( p \)-subgroups of a symmetric group \( S_p \).

**Theorem 1.2.14:** Lagrange’s theorem: Let \( K \) be a subgroup of \( G \), then the order of \( K \) divides the order of \( G \). (Fraleigh 2003).

**1.3 Statement of the problem**
Symmetric groups contain various subgroups some of which are the Sylow \( p \)-subgroups.
From the subgroups of the symmetric groups, the lattice structures are constructed to aid in explaining and understanding these groups better. Over the years various researchers have attempted to calculate the number of the Sylow \( p \)-subgroups of a given group.
Sylow (1972) stated the 1st, 2nd and the 3rd Sylow theorems to help come up with the number of the Sylow $p$-subgroups in a group. The theorems usually give a variety of the possible number of the Sylow $p$-subgroups and it becomes more difficult to find the number of the Sylow $p$-subgroups as the degree and the order of the symmetric group increases. In this research, we aim to derive an explicit formula for finding the number of Sylow $p$-subgroups of any symmetric group. The symmetric groups of the lower degree, for instance $S_3$, $S_4$ and $S_5$, will serve as examples to demonstrate the Sylow's theorems and Wilson's theorems. The Sylow theorems by Ludwing Sylow (1872) and Wilson’s theorem will be useful in the research to help get the formula for finding the number of the Sylow $p$-subgroups of $S_n$.

1.4 Objectives

1.4.1: General objective

To determine a formula for calculating the number of Sylow $p$-subgroups of symmetric groups.

1.4.2: Specific objectives

i. To determine the subgroups and supergroups of the symmetric group $S_3$, $S_4$ and $S_5$.

ii. To develop the lattice structures and ascending chains for $S_3$, $S_4$ and explain lattice structure for $S_5$.

iii. To investigate the exact number of Sylow $p$-subgroups of the symmetric groups by developing a specific formula.
1.5 Significance of the study

Symmetric groups serve a very important role in various fields and also in the real life experiences by providing basic and essential information.

It is applied in the Galois theory, geometry, combinatorics, representation theory (lie groups), molecular chemistry, social interactions, pottery, music, art and craft, and architecture.

The symmetric group $S_n$ is considered as the Galois group of a general polynomial having a degree $n$.

In geometry, an object that is geometric is considered to be symmetric if it is possible to subdivide the object into identical pieces to produce an organized pattern. The type of symmetry can either be reflection symmetry, rotational symmetry or translational symmetry.

In representation theory of Lie groups, the symmetric group has a representation theory that provides basic information through the application of Schur factor ideas.

In chemistry, it is used in molecular symmetry. It helps in describing the molecular symmetries and hence classification of molecules depending on their symmetries.

In biology, symmetry is used in the description of body shapes of bilateral animals where the body is usually divided into the left and the right halves.

In architecture, symmetry is widely used in the structure and also ornamentation. It is used to produce patterns, which are complex, by the application of reflection, rotation and translation symmetry.
In pottery, the pottery wheels assist shape vessels made of clay by use of full symmetry of rotation.

In music, composers use symmetry as a musical form of constraint.

In arts and craft, through symmetry, all kinds of objects are designed, for instance, furniture, beadwork, and musical instruments.

Symmetric groups have been found to serve a very big and essential part of both art and science fields. It has contributed to the success of very many activities in real life experiences and in day-to-day life accomplishments.

Through this research study we shall be in a better position to implement symmetry in our day to day experiences.
CHAPTER TWO

LITERATURE REVIEW

In this particular area of literature review, the research study that others have done is being reflected and considered. It comprises of:

i. Review of the subgroups and supergroups of the symmetric group $S_n$.

ii. Review of the lattice structure of $S_3$ and $S_4$.

iii. The generators of the symmetric groups

iv. The Sylow theorems and how they have been used to approximate the number of Sylow $p$-subgroups of a symmetric group.

2.1 Subgroups and Supergroups of the Symmetric group

Many researchers in mathematics have gotten much interest in the subgroups and the supergroups of the symmetric groups.

Jacobson (2009), Cameron (1999), Dixon (1996), Kaloujnine (1948), Kerber (1971), Liebeck (1988), Scott (1987) researched on the symmetric groups $S_3$, $S_4$ and $S_5$. They listed all the subgroups and their order. Hazewinkel (2001) investigated the symmetric group $S_3$ and found that it is made up of symmetries of a triangle, which is equilateral. In $S_3$ the permutations under reflection and rotation gives: $S_3$

$\{e,(123),(132),(12),(13),(23)\}$ that is of order $n!=3!=6$. $S_3$ comprises of six subgroups:

i. Three of order 2.

ii. One of order 6
iii. One of order 3
iv. One of order 1

Jacobson (2009) analyzed the symmetric group $S_4$. It contains all the permutations of a tetrahedron, which is regular. Sulaiman (2012) investigated the same group $S_4$ and showed that it has order $4! = 24$, constructed the multiplication table for $S_4$, applied the Lagrange’s theorem and the Sylow theorem to determine all the subgroups of the symmetric group $S_4$, and listed all the elements that are contained in the symmetric group $S_4$, the researcher also got the order of each of the 24 elements of the symmetric group $S_4$. It was found that $S_4$ has 30 subgroups, which are:

i. 1 subgroup of order 24 (the whole group)

ii. 1 subgroup of order 12 (the alternating group $A_4$)

iii. 3 subgroups of order 8 (the dihedral groups $D_4$)

iv. 4 subgroups of order 6 (the symmetric groups $S_3$)

v. 7 subgroups of order 4 (Klein-4 groups)

vi. 4 subgroups of order 3 (the alternating groups $S_3$ in $S_4$)

vii. 9 subgroups of order 2 (6 subgroups of $S_2$ in $S_4$ and 3 subgroups of $\mathbb{Z}_2$ in $S_4$)

viii. 1 subgroup of order 1 (the trivial group)

Cameron (1999) analyzed the subgroups of the symmetric group $S_5$ which include:

i. 1 subgroup of order 1 (the trivial group)

ii. 25 subgroups of order 2 (10 are the cyclic groups $\mathbb{Z}_2$ and 15 are the subgroups generated by the double transposition in $S_5$)

iii. 10 subgroups of order 3 (the $\mathbb{Z}_3$ in $S_3$)
iv. 35 subgroups of order 4 (20 Klein-4 subgroups and 15 $\mathbb{Z}_4$ in $S_5$)

v. 6 subgroups of order 5 (the cyclic subgroups $\mathbb{Z}_5$ in $S_5$)

vi. 30 subgroups of order 6. (10 $S_3$ in $S_5$, 10 twisted $S_3$ in $S_5$ and 10 $\mathbb{Z}_6$ in $S_5$)

vii. 15 subgroups of order 8 ($D_4$ in $S_5$)

viii. 6 subgroups of order 10 ($D_5$ in $S_5$)

ix. 15 subgroups of order 12 (10 are the direct product of $S_3$ and $S_2$ in $S_5$ and 5 $A_4$ in $S_5$)

x. 6 subgroups of order 20 (the general affine group)

xi. 5 subgroups of order 24 ($S_4$ in $S_5$)

xii. 1 subgroup of order 60 ($A_4$ in $S_5$)

xiii. 1 subgroup of order 120 (the whole group $S_5$).

All these 156 subgroups have been widely discussed in chapter three.

Various researchers have done extensive study on group theory other than the symmetric groups other group. These include; dihedral groups, cyclic groups and alternating groups. Jensen & Keane (1990) analyzed the structure of the subgroups of the dihedral groups, and they determined the number of subgroups of each order for a given dihedral group.

Gould & Mays investigated the set of all chains of nilpotent finite groups in the context of subsets of partitions and multisets and the set of $k$-chains.

A cyclic group is a group generated by a single element. It comprises of sets of elements with associative operations that are single and invertible.
Dihedral groups, cyclic groups, and alternating groups will greatly be used in the construction of the lattice structures and ascending chains of the symmetric groups in chapter three.

2.2 The lattice structures

Suzuki (1951) and Suzuki (1956) analyzed the lattice structures for various symmetric groups including $S_3$ and $S_4$. They came up with the structures arranged according to the order of the groups.

Sulaiman (2012) presented the lattice structure of the symmetric group $S_4$. The lattice structures have been developed using the subgroups of the specific symmetric groups. The ascending chains of subgroups of the symmetric groups have largely been used in the construction of the lattice structures of the symmetric groups.

Schmidt (1994) analyzed the various lattice structures for $D_n$ and $A_n$. This will greatly contribute to the development of other lattice structures for various symmetric groups.

Roman (2008) researched on the lattice structures of various groups like the cyclic groups, which have helped in the development of the lattice structure of the symmetric groups.

2.3 Generators

Kearnes (2003) investigated the generators of the symmetric groups and found that: The number of generators of a symmetric group $S_n$ is $n - 1$.

Kerber (1971) analyzed the generators of the symmetric groups $S_3$, $S_4$ and $S_5$ and hence a generalization of the number of generators of the symmetric group $S_n$. 
2.4 Sylow theorems

Ludwig (1872) developed the 1st, 2nd and the 3rd Sylow theorems, which he represented in the form of theorems and proofs. Mathematische Annalen (1868) later on published the theorems. Wielandt (1959) gave a more general explanation of the Sylow subgroups of the symmetric groups.

The 1st Sylow theorem

He stated that if $|G| = p^r m$ where $p$ and $m$ are coprime then $G$ has at least one subgroup of order $p^s$ for all $s \leq r$. Meo (2004) analyzed the first Sylow’s theorem which was a weaker version of the Sylow theorem and was proved first by Cauchy, and hence referred to as the Cauchy theorem. Florian (1999) generalized the first Sylow theorem, which was first stated by Ludwig Sylow. It was discussed into details and proved the theorem. It was found out that for any group, which is finite, and its order is divisible by a power, $p^n$, $p$ is prime, then the group has a subgroup of order $p^n$.

The 2nd Sylow theorem

The Sylow $p$-subgroups of a finite group $G$ are conjugates to one another.

The 3rd Sylow theorem

It states that the number of Sylow $p$-subgroups of a group $G$ is congruent to 1 modulo $p$ and divides $m$, i.e. $n_p \equiv 1 \pmod{p}$ and $n_p$ divides $m$.

Gow (1994) stated that any Sylow $p$-subgroup of a group $G$ is cyclic and its order is $p$. It contains $p$ generators. Two Sylow $p$-subgroups of a group, $G$ are said to be
identical if they share a generator, so the elements whose order is $p$ are usually divided according to the Sylow $p$-subgroups they belong to.

**Wilson’s theorem:** the number of Sylow $p$-subgroups in a symmetric group $S_n$ is $(p - 2)!$ where $p$ is a prime. When distinct Sylow $p$-subgroups are grouped $(p - 1)$ in every cluster) the Sylow $p$-subgroups becomes $(p - 1)!/(p - 1) = (p - 2)!$

Dummit (2004) investigated the three Sylow theorems and their applications. He mostly concentrated on the 3rd Sylow theorem, which gives way on how to find the Sylow $p$-subgroups in various groups including the symmetric groups. This is achieved by listing all the possible number of any given symmetric group. Various proofs of the Sylow’s theorems have also been researched on by Waterhouse (1980), Scharlau (1988) and Casadio and Zappa (1990).
CHAPTER THREE

SUBGROUPS OF THE SYMMETRIC GROUPS $S_3$, $S_4$ AND $S_5$

3.1 Introduction

In this chapter we look at various symmetric groups, the number of subgroups and their order, lattice structures and the ascending chains.

3.2 Number of subgroups of some symmetric groups

<table>
<thead>
<tr>
<th>$n$</th>
<th>Number of subgroups in $S_n$</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>1</td>
</tr>
<tr>
<td>2</td>
<td>2</td>
</tr>
<tr>
<td>3</td>
<td>6</td>
</tr>
<tr>
<td>4</td>
<td>30</td>
</tr>
<tr>
<td>5</td>
<td>156</td>
</tr>
</tbody>
</table>

*Table 3.1.1: Number of subgroups in $S_n$*

These are the various subgroups of the symmetric groups indicated above. They will play an essential part in the identification of the number of the Sylow $p$-subgroups that are present in the above symmetric groups and hence providing the pattern to be followed in finding the formula for calculating the exact number of the Sylow $p$-subgroups in any symmetric group. Since the numbers of the subgroups of the above symmetric groups have been found by the application of the Sylow’s theorem, we find the exact number of the Sylow $p$-subgroups in the given symmetric groups such that they do not exceed the number of the subgroups of the symmetric group. In any one given symmetric group we
usually have different Sylow $p$-subgroups of various orders and so we need to list down all the subgroups of the above symmetric groups to ensure that the found number of the Sylow $p$-subgroups are indeed corresponding with the number of the subgroups in the given group and that the orders are also matching.

### 3.3 Subgroups of some of the symmetric groups

Subgroups of the symmetric group $S_3$

<table>
<thead>
<tr>
<th>Subgroups in $S_3$</th>
<th>Order of the subgroups</th>
<th>Number of the subgroups</th>
<th>Order of the generators</th>
</tr>
</thead>
<tbody>
<tr>
<td>Trivial</td>
<td>1</td>
<td>1</td>
<td>1</td>
</tr>
<tr>
<td>$S_2$ (Sylow 2)</td>
<td>2</td>
<td>4</td>
<td>2</td>
</tr>
<tr>
<td>$A_3$ (Sylow 3)</td>
<td>3</td>
<td>1</td>
<td>3</td>
</tr>
<tr>
<td>$S_3$</td>
<td>6</td>
<td>1</td>
<td>3 &amp; 2</td>
</tr>
</tbody>
</table>

*Table 3.2.1: Symmetric group $S_3$*
Subgroups of the symmetric group $S_4$

<table>
<thead>
<tr>
<th>Subgroups in $S_4$</th>
<th>The Order of the subgroup</th>
<th>Number of subgroups</th>
<th>Order of generators</th>
</tr>
</thead>
<tbody>
<tr>
<td>Trivial</td>
<td>1</td>
<td>1</td>
<td>1</td>
</tr>
<tr>
<td>$S_2$</td>
<td>2</td>
<td>9</td>
<td>2</td>
</tr>
<tr>
<td>$A_4$ (Sylow 3)</td>
<td>3</td>
<td>4</td>
<td>3</td>
</tr>
<tr>
<td>$\mathbb{Z}_4$</td>
<td>4</td>
<td>3</td>
<td>4</td>
</tr>
<tr>
<td>Klein-4 group</td>
<td>4</td>
<td>4</td>
<td>2</td>
</tr>
<tr>
<td>$S_3$</td>
<td>6</td>
<td>4</td>
<td>3 &amp; 2</td>
</tr>
<tr>
<td>$D_4$ (Sylow 2)</td>
<td>8</td>
<td>3</td>
<td>4 &amp; 2</td>
</tr>
<tr>
<td>$A_4$</td>
<td>12</td>
<td>1</td>
<td>3 &amp; 2</td>
</tr>
<tr>
<td>$S_4$</td>
<td>24</td>
<td>1</td>
<td>4 &amp; 2</td>
</tr>
</tbody>
</table>

*Table 3.2.2: Symmetric group $S_4$*
Subgroups of the symmetric group $S_5$

<table>
<thead>
<tr>
<th>Subgroups in $S_5$</th>
<th>Order of the subgroups</th>
<th>Number of the subgroups</th>
<th>Order of generators</th>
</tr>
</thead>
<tbody>
<tr>
<td>Trivial</td>
<td>1</td>
<td>1</td>
<td>1</td>
</tr>
<tr>
<td>$S_2$</td>
<td>2</td>
<td>10</td>
<td>2</td>
</tr>
<tr>
<td>Generated by double transpositions</td>
<td>2</td>
<td>15</td>
<td>2</td>
</tr>
<tr>
<td>$\mathbb{Z}_3$(Sylow 3)</td>
<td>3</td>
<td>10</td>
<td>3</td>
</tr>
<tr>
<td>Klein-4</td>
<td>4</td>
<td>15</td>
<td>2</td>
</tr>
<tr>
<td>Normal Klein-4</td>
<td>4</td>
<td>5</td>
<td>2</td>
</tr>
<tr>
<td>$\mathbb{Z}_4$</td>
<td>4</td>
<td>15</td>
<td>4</td>
</tr>
<tr>
<td>$\mathbb{Z}_5$(Sylow 5)</td>
<td>5</td>
<td>6</td>
<td>5</td>
</tr>
<tr>
<td>$\mathbb{Z}_6$</td>
<td>6</td>
<td>10</td>
<td>3&amp;2</td>
</tr>
<tr>
<td>$S_3$</td>
<td>6</td>
<td>10</td>
<td>3&amp;2</td>
</tr>
<tr>
<td>Twisted $S_3$</td>
<td>6</td>
<td>10</td>
<td>3&amp;2</td>
</tr>
<tr>
<td>$D_4$ (Sylow 2)</td>
<td>8</td>
<td>15</td>
<td>4&amp;2</td>
</tr>
<tr>
<td>$D_5$</td>
<td>10</td>
<td>6</td>
<td>5&amp;2</td>
</tr>
<tr>
<td>The direct product of $S_3$ and $S_2$</td>
<td>12</td>
<td>10</td>
<td>3&amp;2</td>
</tr>
<tr>
<td>$A_4$</td>
<td>12</td>
<td>5</td>
<td>3&amp;2</td>
</tr>
<tr>
<td>GA(1,5)</td>
<td>20</td>
<td>5</td>
<td>4</td>
</tr>
<tr>
<td>Group</td>
<td>Order</td>
<td>Number of Subgroups</td>
<td></td>
</tr>
<tr>
<td>-------</td>
<td>-------</td>
<td>---------------------</td>
<td></td>
</tr>
<tr>
<td>$S_4$</td>
<td>24</td>
<td>6</td>
<td></td>
</tr>
<tr>
<td>$A_5$</td>
<td>60</td>
<td>1</td>
<td></td>
</tr>
<tr>
<td>$S_5$</td>
<td>120</td>
<td>1</td>
<td></td>
</tr>
</tbody>
</table>

*Table 3.2.3: Symmetric group $S_5$*

Next, all the above subgroups shall be listed down to ensure that in fact each of the above symmetric groups contains the above listed subgroups. This will help to emphasize the formula of getting the number of the Sylow $p$-subgroups of the above symmetric groups since we shall be in a position to count them physically. The listing of the number of the Sylow $p$-subgroups of the symmetric groups will be done in chapter four.

### 3.4 Outlining the subgroups of the symmetric groups

#### 3.4.1 Introduction

In this section, we shall list down all the subgroups of the symmetric groups $S_3$, $S_4$ and $S_5$, list down the generators of these particular subgroups and name them.

This information will be used in chapter four to assist in the counting of the number of the Sylow $p$-subgroups in each of the above symmetric groups. This will now enable a generalization of a way of calculating the number of Sylow $p$-subgroups of the rest of the symmetric groups, and hence coming up with a formula for getting Sylow $p$-subgroup of any order in any one given symmetric group $S_n$. 


3.4.2 Subgroups of the symmetric group $S_3$

This is the symmetric group on a set composed of three elements, $X = \{1,2,3\}$. It has a degree that is prime. Its degree is three. It has an order of $3! = 6$. $S_3$ is generated by $\langle(123),(12)\rangle$.

$S_3$ contains 6 elements that are as follows:

$S_3 = \{e, (123), (132), (23), (12), (13)\}$.

It contains 6 subgroups;

$\{e\}$.

$S_2 = \{e, (23)\}$,

$S_2 = \{e, (12)\}$,

$S_2 = \{e, (13)\}$, (Sylow subgroup of order two)

$A_3 = \{e, (123), (132)\}$ (Sylow subgroup of order three) and

$S_3 = \{e, (123), (123), (23), (12), (13)\}$.

3.4.3 Subgroups of the symmetric group $S_4$

This is the symmetric group on a set composed of four elements, $X = \{1,2,3,4\}$. Its degree is four. It has an order of $4! = 24$. $S_4$ is generated by $(1234)$ and $(12)$.

NOTE; $S_4$ cannot be generated by $(1234)$ and $(13)$ or $(1234)$ and $(24)$

Thus it is comprised of 24 elements $S_4 = \{e, (12), (13), (14), (24), (34), (23), (12)(34), (13)(24), (14)(23), (123), (234), (132), (124), (143), (243), (142), (134), (1234), (1432), (1423), (1243), (1342), (1324)\}$.

The group has 30 subgroups, which are:
A. **1 trivial subgroup** of order 1

\{e\}.

B. **9 subgroups of order 2**

a) \(S_2\) in \(S_4\) are:

b) \(S_2^a = \{e, (12)\}\),

c) \(S_2^b = \{e, (13)\}\),

d) \(S_2^c = \{e, (14)\}\),

e) \(S_2^g = \{e, (23)\}\),

f) \(S_2^h = \{e, (24)\}\),

g) \(S_2^i = \{e, (34)\}\)

h) \(S_2^d = \{e, (12)(34)\}\),

i) \(S_2^e = \{e, (13)(24)\}\),

a) \(S_2^f = \{e, (14)(23)\}\).

C. **4 subgroups of order 3**

The alternating groups \(A_3\) in \(S_4\). (Sylow 3-subgroups of order 3):

a) \(A_3^a = \{e, (123), (132)\}\),

b) \(A_3^b = \{e, (124), (142)\}\),

c) \(A_3^c = \{e, (134), (143)\}\),

d) \(A_3^d = \{e, (234), (243)\}\).
D. 7 subgroups of order 4

Klein–4 group:

a)  $K_a = \{ e, (13)(24), (14)(23), (12)(34) \}$.

b)  $K_b = \{ e, (12), (12)(34), (34) \}$.

c)  $K_c = \{ e, (13)(24), (13), (24) \}$.

d)  $K_d = \{ e, (14), (23), (14)(23) \}$.

$\mathbb{Z}_4$ in $S_4$:

a)  $\mathbb{Z}_4 a = \{ e, (13)(24), (1234), (1432) \}$.

b)  $\mathbb{Z}_4 b = \{ e, (12)(34), (1324), (1423) \}$.

c)  $\mathbb{Z}_4 c = \{ e, (14)(23), (1243), (1342) \}$.

E. 4 subgroups of order 6

$S_3$ in $S_4$:

a)  $S_3 a = \{ e, (12), (13), (23), (123), (132) \}$.

b)  $S_3 b = \{ e, (12), (24), (14), (124), (142) \}$.

c)  $S_3 c = \{ e, (13), (34), (14), (143), (134) \}$.

d)  $S_3 d = \{ e, (23), (34), (24), (243), (234) \}$.

F. 3 subgroups of order 8

Dihedral groups $D_4$; (Sylow 2-subgroups of order 8):

a)  $D_4 a = \{ e, (12), (34), (14)(23), (12)(34), (13)(24), (1423), (1324) \}$.

b)  $D_4 b = \{ e, (13), (24), (13)(24), (14)(23), (12)(34), (1234), (1432) \}$.

c)  $D_4 c = \{ e, (14), (23), (13)(24), (14)(23), (12)(34), (1243), (1342) \}$.
G. 1 subgroup of order 12

The alternating group $A_4$:

$$A_4 = \{ e, (13)(24), (14)(23), (234), (124), (123), (243), (12)(34), (132), (143), (142) \}.$$

H. 1 subgroup of order 24

The symmetric group $S_4$:

$$S_4 = \{ e, (12), (13), (14), (24), (34), (23), (12)(34), (13)(24), (14)(23), (123), (234), (132), (124), (143), (243), (134), (1234), (1423), (1243), (1342), (1324) \}.$$

3.4.4 Subgroups of the symmetric group $S_5$

This is the symmetric group that is on a set composed of five elements, $X = \{1, 2, 3, 4, 5\}$. It has degree five. It has an order of $5! = 120$. $S_5$ is generated by $(12345)$ and $(12)$. Thus $S_5$ is composed of 120 elements which include:

$$S_5 = \{ e, (12), (13), (14), (15), (24), (23), (25), (34) (35), (34), (125), (12)(45), (12)(34), (12)(35), (13)(45), (13)(24), (13)(25), (23)(45), (24)(35), (25)(34), (14)(23), (14)(35), (14)(25), (15)(34), (15)(23), (15)(24), (254), (134), (253), (153), (142), (354), (124), (243), (135), (234), (154), (123), (132), (145), (152), (345), (143), (245), (235), (125), \}.$$
S_5 comprises of 156 subgroups which include:

A. **1 trivial subgroup** of order 1

{e}

B. **25 subgroups of order 2**

Subgroups generated by a single transposition:

a) {e,(12)},

b) {e,(13)},

c) {e,(14)},

d) {e,(15)},

e) {e,(23)},

f) {e,(24)},

g) {e,(25)},

h) {e,(34)},

i) {e,(35)},
j) \{e,(45)\}

Subgroup generated by double transposition:

a) \{e,(12)(34)\},
b) \{e,(12)(35)\},
c) \{e,(12)(45)\},
d) \{e,(13)(24)\},
e) \{e,(13)(25)\},
f) \{e,(13)(45)\},
g) \{e,(14)(23)\},
h) \{e,(14)(25)\},
i) \{e,(14)(35)\},
j) \{e,(15)(23)\},
k) \{e,(15)(24)\},
l) \{e,(15)(34)\},
m) \{e,(23)(45)\},
n) \{e,(24)(35)\},
o) \{e,(25)(34)\}.

C. 10 subgroups of order 3. (Sylow 3-subgroups of order 3)

\(\mathbb{Z}_3\) in \(S_5\):

a) \{e,(123),(132)\},
b) \{e,(124),(142)\},
c) \{e,(125),(152)\},
d) \{e,(134),(143)\},
e) \{e,(135),(153)\},

f) \{e,(145),(145)\},

g) \{e,(234),(243)\},

h) \{e,(235),(253)\},

i) \{e,(345),(354)\},

j) \{e,(245),(254)\}.

D. 35 subgroups of order 4

Klein-4 subgroups in \(S_5\) (generated by double transposition on 4 elements in \(S_5\)):

a) \{e,(12)(34),(13)(24),(14)(23)\},

b) \{e,(12)(35),(13)(25),(15)(23)\},

c) \{e,(12)(45),(14)(25),(15)(24)\},

d) \{e,(13)(45),(14)(35),(15)(34)\},

e) \{e,(23)(45),(24)(35),(25)(34)\}.

Subgroups generated by a pair of disjoint transpositions in \(S_5\):

a) \{e,(12),(34),(12)(34)\},

b) \{e,(12),(35),(12)(35)\},

c) \{e,(12),(45),(12)(45)\},

d) \{e,(13),(24),(13)(24)\},

e) \{e,(13),(25),(13)(25)\},

f) \{e,(13),(45),(13)(45)\},

g) \{e,(14),(23),(14)(23)\},

h) \{e,(14),(25),(14)(25)\},

i) \{e,(14),(35),(14)(35)\},

j) \{e,(245),(254)\}. 

25
j) \{e,(15),(23),(15)(23)\},

k) \{e,(15),(24),(15)(24)\},

l) \{e,(15),(34),(15)(34)\},

m) \{e,(23),(45),(23)(45)\},

n) \{e,(24),(35),(24)(35)\},

o) \{e,(25),(34),(25)(34)\}.

\[ \mathbb{Z}_4 \text{ in } S_5: \]

a) \{e,(13)(24),(1234),(1432)\},

b) \{e,(12)(34),(1324),(1423)\},

c) \{e,(12)(45),(1425),(1524)\},

d) \{e,(13)(25),(1235),(1532)\},

e) \{e,(14)(23),(1243),(1342)\},

f) \{e,(14)(25),(1245),(1542)\},

g) \{e,(15)(24),(1254),(1452)\},

h) \{e,(15)(23),(1253),(1352)\},

i) \{e,(23)(45),(2435),(2534)\},

j) \{e,(24)(35),(2345),(2543)\},

k) \{e,(25)(34),(2354),(2453)\},

l) \{e,(12)(35),(1325),(1523)\},

m) \{e,(13)(45),(1435),(1534)\},

n) \{e,(14)(35),(1345),(1543)\},

o) \{e,(15)(43),(1453),(1354)\}.

E. 6 subgroups of order 5
$\mathbb{Z}_5$ in $S_5$:

a) $\{e,(12345),(13524),(14253),(15432)\},$

b) $\{e,(12354),(13425),(15243),(14532)\},$

c) $\{e,(12453),(14325),(15234),(13542)\},$

d) $\{e,(12435),(14523),(13254),(15342)\},$

e) $\{e,(12543),(15324),(14235),(13452)\},$

f) $\{e,(12534),(15423),(13245),(14352)\}.$

F. 30 subgroups of order 6

$S_3$ in $S_5$:

a) $\{e,(12),(13),(23),(132),(123)\},$

b) $\{e,(142),(12),(14),(24),(124)\},$

c) $\{e,(12),(15),(35),(125),(152)\},$

d) $\{e,(13),(14),(34),(134),(143)\},$

e) $\{e,(13),(15),(35),(135),(153)\},$

f) $\{e,(14),(15),(45),(145),(154)\},$

g) $\{e,(23),(24),(34),(234),(243)\},$

h) $\{e,(23),(25),(35),(235),(253)\},$

i) $\{e,(34),(35),(45),(345),(354)\},$

j) $\{e,(24),(25),(45),(245),(254)\}.$

Twisted $S_3$ in $S_5$:

a) $\{e,(12)(45),(23)(45),(13)(45),(123),(132)\},$

b) $\{e,(12)(35),(24)(35),(14)(35),(124),(142)\},$

c) $\{e,(12)(34),(25)(34),(15)(34),(125),(152)\},$
d) \( \{ e, (13) (25), (34) (25), (14) (25), (134), (143) \} \),

e) \( \{ e, (13) (24), (35) (24), (15) (24), (135), (153) \} \),

f) \( \{ e, (14) (23), (23) (45), (15) (23), (145), (154) \} \),

g) \( \{ e, (15) (23), (15) (34), (15) (24), (234), (243) \} \),

h) \( \{ e, (14) (23), (14) (35), (14) (25), (235), (253) \} \),
i) \( \{ e, (13) (24), (13) (45), (13) (25), (245), (254) \} \),
j) \( \{ e, (12) (34), (12) (45), (12) (35), (345), (354) \} \).

\( \mathbb{Z}_6 \) in \( S_5 \):

a) \( \{ e, (45), (123), (132), (132) (45), (123) (45) \} \),
b) \( \{ e, (35), (124), (142), (142) (35), (124) (35) \} \),
c) \( \{ e, (34), (152), (125), (152) (34), (125) (34) \} \),
d) \( \{ e, (25), (143), (134), (143) (25), (134) (25) \} \),

e) \( \{ e, (23), (154), (145), (154) (23), (145) (23) \} \),
f) \( \{ e, (24), (153), (135), (153) (24), (135) (24) \} \),
g) \( \{ e, (15), (243), (234), (15) (243), (15) (234) \} \),
h) \( \{ e, (14), (253), (235), (14) (253), (14) (235) \} \),
i) \( \{ e, (13), (254), (245), (13) (254), (13) (245) \} \),
j) \( \{ e, (12), (345), (354), (12) (354), (12) (345) \} \).

**G. 15 subgroups of order 8. (Sylow 2-subgroups of order 8)**

The dihedral group \( D_4 \) in \( S_5 \):

a) \( \{ e, (24) (35), (24), (35), (23) (45), (25) (34), (2345), (2543) \} \),
b) \( \{ e, (25) (34), (25), (34), (23) (45), (24) (35), (2354), (2453) \} \),
c) \( \{ e, (23) (45), (23), (45), (24) (35), (25) (34), (2435), (2534) \} \),
d) \{ e, (13)(24), (13), (24), (12)(34), (14)(23), (1234), (1432) \},

e) \{ e, (13)(25), (13), (25), (12)(35), (15)(23), (1235), (1532) \},

f) \{ e, (14)(23), (14), (23), (12)(34), (13)(24), (1243), (1342) \},

g) \{ e, (14)(25), (14)(25), (12)(45), (15)(24), (1245), (1542) \},

h) \{ e, (15)(23), (15), (23), (12)(35), (13)(25), (1253), (1352) \},

i) \{ e, (15)(24), (15), (24), (12)(45), (14)(25), (1254), (1452) \},

j) \{ e, (14)(35), (14), (35), (13)(45), (15)(34), (1345), (1543) \},

k) \{ e, (15)(34), (15), (34), (13)(45), (14)(35), (1354), (1453) \},

l) \{ e, (12)(34), (12), (34), (13)(24), (14)(23), (1324), (1423) \},

m) \{ e, (12)(35), (12), (35), (13)(25), (15)(23), (1325), (1523) \},

n) \{ e, (13)(45), (13), (45), (14)(35), (15)(34), (1435), (1534) \},

o) \{ e, (12)(45), (12), (45), (14)(25), (15)(24), (1425), (1524) \},

H. 6 subgroups of order 10

Dihedral group $D_5$ in $S_5$:

a) \{ e, (25)(34), (15)(24), (14)(23), (13)(45), (12)(34), (13524), (14253), (12345), (15432) \}

b) \{ e, (24)(35), (12)(34), (13)(45), (15)(23), (14)(25), (14532), (13425), (15243), (12354) \}

c) \{ e, (23)(45), (14)(35), (12)(34), (15)(24), (13)(25), (12453), (13542), (14325), (15234) \}

d) \{ e, (25)(34), (12)(45), (14)(35), (13)(24), (15)(23), (15342), (14523), (13254), (12435) \}

e) \{ e, (23)(45), (12)(35), (15)(34), (14)(25), (13)(24), (13452), (15324), (14235), (12543) \}

f) \{ e, (24)(35), (12)(45), (15)(34), (13)(25), (14)(23), (14352), (15423), (13245), (12534) \}

I. 15 subgroups of order 12

Direct product of $S_3$ and $S_2$:

a) \{ e, (12), (23), (13), (45), (12)(45), (23)(45), (13)(45), (123), (132), (123)(45), (132)(45) \}
b) \{e, (12), (14), (24), (35), (12)(35), (14)(35), (24)(35), (124), (124)(35), (142)(35)\}

c) \{e, (12), (15), (25), (34), (12)(34), (15)(34), (25)(34), (125), (125)(34), (152)(34)\}

d) \{e, (13), (14), (34), (25), (13)(25), (14)(25), (34)(25), (134), (134)(25), (143)(25)\}

e) \{e, (13), (15), (35), (24), (13)(24), (15)(24), (35)(24), (135), (135)(24), (153)(24)\}

f) \{e, (14), (15), (45), (23), (14)(23), (15)(23), (45)(23), (145), (145)(23), (154)(23)\}

g) \{e, (23), (24), (34), (15), (23)(15), (24)(15), (34)(15), (234), (234)(15), (243)(15)\}

h) \{e, (23), (25), (35), (14), (23)(14), (25)(14), (35)(14), (235), (235)(14), (253)(14)\}

i) \{e, (34), (35), (45), (23), (34)(12), (35)(12), (45)(12), (345), (354), (345)(12), (354)(12)\}

j) \{e, (24), (25), (45), (13), (24)(13), (25)(13), (45)(13), (245), (245)(13), (254)(13)\}

A_4 in S_5:

a) \{e, (13)(24), (14)(23), (234), (124), (123), (12)(34), (132), (143), (142)\}

b) \{e, (35)(24), (34)(23), (245), (235), (243), (25)(34), (325), (435), (425)\}

c) \{e, (13)(45), (14)(35), (345), (154), (153), (354), (15)(34), (135), (143), (145)\}

d) \{e, (15)(24), (14)(25), (254), (124), (125), (245), (12)(54), (152), (145), (142)\}

e) \{e, (13)(25), (135), (15)(23), (235), (123), (235), (12)(35), (132), (153), (152)\}

J. 5 subgroups of order 24

S_4 in S_5:

a) \{e, (13)(24), (12), (34), (13)(45), (14)(23), (24), (23), (12)(34), (14), (243), (234), (132), (124)
 , (143), (123), (134), (142), (1324), (1234), (1243), (1342), (1432), (1423)\}

b) \{e, (35)(24), (34), (23), (25), (35), (45)(23), (24), (25)(34), (45), (325), (245), (354), (235)
 , (345), (243), (234), (254), (2345), (2543), (2453), (2354), (2435)\}

c) \{e, (13)(45), (15), (34), (35), (13)(45), (14)(35), (15)(34), (14), (345), (135), (154), (143)
 , (153), (145), (134), (435), (1534), (1435), (1453), (1345), (1543), (1354)\}
K. 6 subgroups of order 20

The general affine group in $S_5$:

a) \{e,(25)(34),(12)(35),(13)(45),(14)(23),(15)(24),(12345),(13524),(14253),(15432)\},

b) \{e,(24)(35),(12)(34),(13)(45),(15)(23),(14)(25),(12435),(1352),(1534),(1423),(2543),(1253),(1324),(1542),(1435),(12354),(13425),(15243),(14532)\},

c) \{e,(23)(45),(12)(34),(14)(35),(15)(24),(13)(25),(1234),(1542),(1453),(2435),(1245),(1523),(2534),(1432),(1354),(1325),(12543),(15324),(14235),(13452)\},

d) \{e,(25)(34),(12)(45),(14)(35),(13)(24),(15)(23),(1253),(1432),(2453),(1345),(2354),(1234),(1425),(1352),(1543),(1524),(12435),(14523),(13254),(15342)\},

e) \{e,(23)(45),(12)(35),(15)(34),(14)(25),(13)(24),(1234),(1542),(1453),(2435),(1245),(1523),(2534),(1432),(1354),(1325),(12543),(15324),(14235),(13452)\},

f) \{e,(2543),(24)(35),(12)(45),(15)(34),(13)(25),(14)(23),(1243),(1532),(1354),(1425),(2345),(1235),(1524),(1342),(1453),(12534),(15423),(13245),(14352)\}. 

31
L. 1 subgroup of order 60

\(A_5\) in \(S_5\):

\(\{e, (24)(35), (25)(34), (23)(45), (13)(24), (13)(25), (14)(23), (14)(25), (15)(23), (15)(24), (14)(35), (15)(34), (12)(34), (12)(35), (13)(45), (12)(45), (123), (132), (124), (142), (125), (152), (134), (143), (145), (135), (153), (234), (243), (235), (253), (245), (254), (345), (354), (12345), (13524), (14253), (15432), (12354), (13425), (15243), (14532), (12453), (14325), (15234), (13542), (12435), (14523), (13254), (12543), (15324), (14235), (13452), (12534), (15423), (13245), (14352)\}.

M. 1 subgroup of order 120

\(S_5\) in \(S_5\) as listed on page 20.

3.5 The lattice structures of the symmetric groups

\[\text{Figure 3.5.1: Lattice structure for } S_3\]
In the symmetric groups $S_3$ and $S_4$, the Sylow $p$-subgroups can clearly be observed, as they are the maximal subgroups of the groups. This method of observing can also be used to identify the number of the Sylow $p$-subgroups but the challenge comes in when the order of the group is high. Thus need for inventing a formula.

The lattice structure for $S_5$ is abit complex because it contains 5 replicas of $S_4$. As a result it will contain so many ascending chains.

3.6 The ascending chains of the symmetric groups $S_3$ and $S_4$

$S_3$: 

Figure 3.5.2 Lattice structure for $S_4$
2 chains ascend to $S_3$ through $S_2$.

$\{e\} < \{e, (12)\} < S_3$

$\{e\} < \{e, (13)\} < S_3$

1 chain ascends to $S_3$ through $\mathbb{Z}_2$.

$\{e\} < \{e, (23)\} < S_3$

1 chain ascends to $S_3$ through $A_3$.

$\{e\} < \{e, (123), (132)\} < S_3$

$S_4$:

8 chains ascend through $A_3$ to $S_4$:

$\{e\} < A_3 a < A_4 < S_4$

$\{e\} < A_3 b < A_4 < S_4$

$\{e\} < A_3 c < A_4 < S_4$

$\{e\} < A_3 d < A_4 < S_4$

$\{e\} < A_3 d < S_3 d < S_4$

$\{e\} < A_3 a < S_3 a < S_4$

$\{e\} < A_3 c < S_3 c < S_4$

$\{e\} < A_3 b < S_3 b < S_4$

33 chains ascend through $S_2$ to $S_4$:

$\{e\} < S_2 a < K_b < D_4 a < S_4$

$\{e\} < S_2 a < S_3 c < S_4$
\{ e \} < S_2 a < S_3 d < S_4 \\
\{ e \} < S_2 b < Kc < D_4 b < S_4 \\
\{ e \} < S_2 b < S_3 b < S_4 \\
\{ e \} < S_2 b < S_3 d < S_4 \\
\{ e \} < S_2 c < Kd < D_4 c < S_4 \\
\{ e \} < S_2 c < S_3 a < S_4 \\
\{ e \} < S_2 c < S_3 d < S_4 \\
\{ e \} < S_2 d < Kc < D_4 b < S_4 \\
\{ e \} < S_2 d < Z_4 b < D_4 b < S_4 \\
\{ e \} < S_2 d < Ka < D_4 a < S_4 \\
\{ e \} < S_2 d < Ka < D_4 b < S_4 \\
\{ e \} < S_2 d < Ka < D_4 c < S_4 \\
\{ e \} < S_2 e < Kd < D_4 c < S_4 \\
\{ e \} < S_2 e < Z_4 c < D_4 c < S_4 \\
\{ e \} < S_2 e < Ka < D_4 a < S_4 \\
\{ e \} < S_2 e < Ka < D_4 b < S_4 \\
\{ e \} < S_2 e < Ka < D_4 c < S_4 \\
\{ e \} < S_2 f < Kb < D_4 a < S_4 \\
\{ e \} < S_2 f < Z_4 a < D_4 a < S_4 \\
\{ e \} < S_2 f < Ka < D_4 a < S_4 \\
\{ e \} < S_2 f < Ka < D_4 b < S_4
As observed in the above chains of $S_3, S_4$ and $S_5$, all the ascending chains here ascend either through the symmetric group $S_2$ or the alternating group $A_n$ to the groups $S_3$ and $S_4$. In general all the ascending chains of the symmetric groups $S_n$ usually ascend through either $S_2$ or $A_n$.

In any one given ascending chains, there is only one Sylow $p$-subgroup. The number of Sylow $p$-subgroups of any symmetric groups is less or equal to the number of ascending chains in that group.
CHAPTER FOUR

SYLOW \( p \)-SUBGROUPS OF THE SYMMETRIC GROUPS

4.1 Introduction

In this chapter, we shall outline a brief explanation of Sylow \( p \)-subgroups, what they are and some of their behavior that will be presented in section 4.1. We shall also have the statement of the Sylow’s theorems and their proofs as they have been presented by Ludwing Sylow (1872) and how these theorems have been used to get a variety of the number of the Sylow \( p \)-subgroups of the symmetric groups. We shall list the exact number of the Sylow \( p \)-subgroups of the symmetric groups \( S_3 \), \( S_4 \) and \( S_5 \). This will lead to the development of a formula for getting the exact number of the Sylow \( p \)-subgroups of any symmetric group that we may be provided with after observing the pattern developed by the Sylow \( p \)-subgroups from \( S_3 \) to \( S_5 \).

4.2 The Sylow’s theorem

According to the finite group theory, Sylow theorems come up as a result of a collection of many theorems and then named after a Norwegian Mathematician known as Ludwing Sylow (1872), since he had made a lot of contributions in the theorems. The theorems usually provide some detailed information about the number of Sylow theorems, which are contained in a certain finite group.

Let \( p \) be a prime number and \( G \) be a group, which is finite, and then we say that Sylow \( p \)-subgroup of a group \( G \) is a \( p \)-group, which is a maximal subgroup of the group \( G \).

A \( p \)-group is a group in which the order of every element of the group is a power of \( p \).
When all the Sylow $p$-subgroups of a prime $p$ are put together, then they are denoted as $\text{Syl}_p(C_n)$. In this particular case, all the members of $\text{Syl}_p(C_n)$ are usually isomorphic to each other.

Let $G$ be a group, then if we have that $|G|=P^nm$ where $n > 0$ and that $p$ doesn’t divide $m$, then we have that each and every Sylow $p$-subgroup $p$, has an order $|p|=P^n$. That is.

We say that the gcd of $m$ and $p$ is one and that $p$ and $m$ are usually co-primes.

**4.2.1 Sylow’s 1st theorem**

Let $G$ be a finite group, then $G$ has $p$-Sylow subgroup for every $p$ a prime and any $p$-subgroup of $G$ lies in a $p$-Sylow subgroup of $G$.

**4.2.2 Examples**

i. In the symmetric group $S_3$: $|S_3|=6=2\times3$. Here 2 and 3 are co-prime. Thus the symmetric group $S_3$ has at least one Sylow $p$-subgroup of order $2^1$ if $p=2$, $m=3$ and $r=1$, at least one subgroup of order $3^1$ if $p=3$, $m=2$ and $r=1$.

ii. In the symmetric group $S_4$: $|S_4|=24=2^3\times3$. We start with, let $p=2$, $m=3$ and $r=3$ then $S_4$ have at least one Sylow $p$-subgroup that is of order equal to $2^3$. We observe that 2 and 3 are co-primes. Second if we take $p=3$, $m=2$ and $r=1$, then we have that the symmetric group $S_4$ has at least one Sylow 3-subgroup which is of order $\leq3$.

iii. In the symmetric group $S_5$: $|S_5|=120=2^3\times3^1\times5$. Here, let $p=2$, $m=15$ thus $r=3$, then the symmetric group $S_5$ has a Sylow 2-subgroup of order $\leq2^3$. If we take $p=3$, $m=40$ and $r=1$, we find that 3 and 40 are co-primes. Then we have
that the symmetric group $S_5$ has at least one Sylow 3-subgroup, which is of order $\leq 3$. Next, when we take $p=5$, $m=24$ and $r=1$, here, 5 and 24 are co-primes. Thus $S_5$ has a Sylow 5-subgroup, which is of order $\leq 5$.

4.2.3 Sylow’s 2nd theorem

Let $G$ is a finite group, then the Sylow $p$-subgroups of $G$ are conjugates.

4.2.4 Examples

Let $K$ and $H$ be two Sylow $p$-subgroups of the symmetric group $S_4$, let $g \in G$ then we have the $gKg^{-1} = H$.

Let $H = ((1234), (13)) K = ((1243), (14))$, $g = (34)$. Then we find that $gKg^{-1} = H$.

4.2.5 Sylow’s 3rd theorem

Let $G$ be a finite group, then we say that the number of the Sylow $p$-subgroups of a group is congruent to 1 modulo $p$ and divides $m$. $m$ is gotten by dividing the order of the group $G$ with the order of the Sylow $p$-subgroup of the group $G$. Let $G$ be a finite group, then $p$ is a Sylow $p$-subgroup of $G$ if $n_p \equiv 1 \pmod{p}$ and $(n_p / m)$ with $\left| G : P \right| = m$.

Where $n_p$ is the number of the Sylow $p$-subgroups of the symmetric group. For the 3rd Sylow theorem one had to find all the probable number of Sylow $p$-subgroups and then choose from them the exact number by comparing with the listed number of subgroups.

4.3 Calculation of the number of the Sylow $p$-subgroups

Here we calculate the number of Sylow $p$-subgroups of the symmetric groups $S_3$, $S_4$ and $S_5$. 39
4.3.1 Symmetric group $S_3$

Let $n_p$ be the number of the Sylow $p$-subgroups of any symmetric group. We start with the symmetric group $S_3$ of order $3! = 6 = 2 \times 3$. If $n_2$ is the number of the Sylow 2-subgroups in the symmetric group $S_3$, then the order of each of the Sylow $p$-subgroups of the symmetric group is 2. Thus we have that $m = 6 / 2 = 3$. Therefore $n_2 / 3$, $n_2 \equiv 1 \pmod{2}$ the possible numbers are therefore 1 or 3.

When we refer to the subgroups of the symmetric group $S_3$, we find that there are three Sylow 2-subgroups of order 2. When counting the number of the Sylow $p$-subgroups in a symmetric group, and find that it exceeds 1 then automatically you take the next number of the possible Sylow $p$-subgroups. In this case, the next possible number is three. The three subgroups are isomorphic to each other. $n_2 / 2$, $n_3 \equiv 1 \pmod{3}$ the possible numbers, therefore, is only 1 when we refer to $S_3$ in chapter three we find that it is really 1. Thus in $S_3$ we only have one Sylow 3-subgroup which is the alternating group $A_3$.

NOTE: To find the number of the possible Sylow $p$-subgroups of a symmetric group ($n_p$) we compute for all the possible numbers such that $n_p$ divides $m$ and it is congruent to 1 modulo $p$.

4.3.2 Symmetric group $S_4$

$S_4$ has order $4! = 24 = 2^3 \times 3^1$. We start by proposing the possible number of the Sylow 2-subgroups of the group. These subgroups are of order $2^3 = 8$.

$n_2 / 3$, $n_2 \equiv 1 \pmod{2}$; possible numbers here are 1 or 3.
When we refer to the subgroups of $S_4$ we, find that there are three Sylow 2-subgroups that are the dihedral groups of order 8. $n_2 \equiv 1 \text{ (mod 2)}$. Therefore we choose 3 as the number of Sylow p-subgroups of the group.

$$n_3 / 8, \quad n_3 \equiv 1 \text{ (mod 3)}; \text{ possible numbers are 1 or 4.}$$

Referring to $S_4$ in chapter 3 we find that there are 4 Sylow 3-subgroups of order 3. These groups are the alternating groups $A_3$. (Here when we find two Sylow 3-subgroups of order 3), then it is obvious that the number of the Sylow 3-subgroups of the group is 4.

**4.3.3 Symmetric group $S_5$**

$S_5$ has order $5! = 120 = 2^3 \times 3 \times 5$. We start with the Sylow 2-subgroups of order 8.

$$n_2 / 15, \quad n_2 \equiv 1 \text{ (mod 2)}; \text{ possible numbers are 1, 3, 5 or 15.}$$

Referring to the subgroups of $S_5$ in chapter three we find that they are exceeding 5 and hence we take the next number, which is 15. These are the dihedral subgroups of order 8.

$$n_3 / 40, \quad n_3 \equiv 1 \text{ (mod 3)}; \text{ possible numbers are 1, 4, 10 or 40.}$$

These subgroups are of order 3. Referring to the subgroups, which have been listed down in chapter, three we find that there are only 10 Sylow 3-subgroups of order 3. These are the cyclic groups $\mathbb{Z}_3$ of order three. Here we cannot take 40 since by the counting method we find that the number of the Sylow 3-subgroups in $S_5$ does not exceed 40.

$$n_5 / 24, \quad n_5 \equiv 1 \text{ (mod 5)}; \text{ possible numbers are 1 or 6.}$$

The Sylow 5-subgroups are of order 5. Referring to the subgroups of $S_5$ we find that there are 6 Sylow 5-subgroups of order 5 and thus we take 6. In $S_5$ these are the cyclic groups $\mathbb{Z}_5$. 

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By use of this particular method of listing the number of the possible Sylow \( p \)-subgroups of a symmetric group, one will always be accurate if only all the subgroups of any symmetric group have been listed down so as to assist in the process of counting and confirming the exact number of the Sylow \( p \)-subgroups of any given symmetric group. The method becomes so tedious as one has to list down the subgroups of any symmetric group, apply the 1\textsuperscript{st}, 2\textsuperscript{nd}, and the 3\textsuperscript{rd} Sylow’s theorem so as to come up with the possible numbers of the Sylow \( p \)-subgroups and then from the assistance of the listed subgroups one should confirm the Sylow \( p \)-subgroups from them.

4.4 The main results

Let \( n \in \mathbb{N}, \ p \) prime \( p/n \) and that \( S_n \) is the finite symmetric group of degree \( n \), then the number of the Sylow \( p \)-subgroups in \( S_n \) is given by

\[
n_p = \frac{(p-2)! \prod_{i=0}^{m} \left( \frac{n-p^i}{p} \right)}{(m+1)!} \]

\( p \) is prime and \( m \) and \( i \) are integers such that \( m \) is the highest value of \( i \).

Proof

Let \( G \) denote the finite symmetric group \( S_n \) containing \( n! \) elements. Let \( n_p \) be the number of Sylow \( p \)-subgroups of any symmetric group \( G \). If \( S_n \) is the symmetric group such that \( n = p \), then the number of the Sylow \( p \)-subgroups will be \( n_p = (p-2)! \) since

\[
\prod_{i=0}^{m} \left( \frac{n-p^i}{p} \right) \text{ reduces to } 1 \text{ for } \left( \frac{n}{p^i} \right) \text{ becomes } \left( \frac{p}{p} \right) = 1 \text{ and in this case } m = 0.
\]

Thus for any given finite symmetric group then \( n_p \equiv 1 \text{(mod } p) = \frac{(p-2)! \prod_{i=0}^{m} \left( \frac{n-p^i}{p} \right)}{(m+1)!} \).
According to the Sylow 3\textsuperscript{rd} theorem $n_p \equiv 1 \text{(mod } p\text{)}$.

4.5 Application of the theorem

We need to show that: $n_p = (p-2)! \binom{n-p}{2} \equiv 1 \text{(mod } p\text{)}$

We use an example and since we have the number of the Sylow $p$-subgroups listed in chapter 3 we just confirm from the listed subgroups. We start with the number of the Sylow 2-subgroups in $S_2$, $n_p = (2-2)! \binom{2-0}{2} = 1$

Thus $S_2$ has 1 Sylow 2-subgroup of order 2.

Considering $S_3$, we find that the number of the Sylow 2-subgroups of order 2 is 3

Consider $S_4$, we calculate the number of the Sylow 2-subgroups of order $2^3$=8, which is 3, and it is congruent to 1 (mod2).

This is true for all the Sylow $p$-subgroups of any finite symmetric group $S_n$, continuing with the calculations we get the number of the Sylow $p$-subgroups as indicated in table 4.3.1

A table of the number of the Sylow $p$-subgroups of the symmetric groups $S_3$ to $S_{10}$ that have been calculated using the formula invented.
<table>
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<tr>
<th>$S_n$</th>
<th>Sylow-2</th>
<th>Order</th>
<th>Sylow-3</th>
<th>Order</th>
<th>Sylow-5</th>
<th>Order</th>
<th>Sylow-7</th>
<th>Order</th>
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<td>3</td>
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<td>1</td>
<td>3</td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>$S_4$</td>
<td>3</td>
<td>8</td>
<td>4</td>
<td>3</td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>$S_5$</td>
<td>15</td>
<td>8</td>
<td>10</td>
<td>3</td>
<td>6</td>
<td>5</td>
<td></td>
<td></td>
</tr>
<tr>
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<td>10</td>
<td>9</td>
<td>36</td>
<td>5</td>
<td></td>
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<tr>
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<td>16</td>
<td>70</td>
<td>9</td>
<td>126</td>
<td>5</td>
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<td>7</td>
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<td>280</td>
<td>9</td>
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<td>5</td>
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</tr>
<tr>
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<td>280</td>
<td>81</td>
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<td>5</td>
<td>4320</td>
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<td>945</td>
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<td>2800</td>
<td>81</td>
<td>756</td>
<td>25</td>
<td>14400</td>
<td>7</td>
</tr>
</tbody>
</table>

*Table 4.3.1: Sylow $p$-subgroups*
CHAPTER FIVE

CONCLUSION AND RECOMMENDATIONS FOR FURTHER RESEARCH

5.1 Introduction
In this chapter, we shall highlight a conclusion for the study and the recommendation to any researcher who would like to further this particular subject for the Sylow $p$-subgroups of any given symmetric group.

5.2 Conclusion
The research targeted on coming up with a formula for calculating the exact number of the Sylow $p$-subgroup of any given finite symmetric group. The objective of the research has ultimately been achieved: The formula has been stated in the form of a theorem and then proved in theorem 4.4. In the prove various examples have been provided to clearly show the accuracy of the formula that has been invented throughout the project work. To achieve the objective of the research study, the symmetric groups $S_3$, $S_4$ and $S_5$ have largely been used by listing down their subgroups, supergroups, the ascending chains, and the lattice structures. By listing down the subgroups of these three symmetric groups, it has really been of very significant importance to the coming up with the formula of calculating the number of the Sylow $p$-subgroups of the higher order of any symmetric group, which is finite. This was achieved by observing the pattern demonstrated by the number of the Sylow $p$-subgroups right from $S_3$ to $S_5$.

Construction of the lattice structures of the three symmetric groups has contributed to the
easy identification of the Sylow $p$-subgroups which have been displayed across the structures.

5.3 Recommendation for further research

The objective of the study was to invent a formula for the calculation of the number of the Sylow $p$-subgroups of any symmetric group, which is finite. I would recommend that one could move a step further and find a formula for calculating the number of the Sylow $p$-subgroup of other groups like the dihedral groups, alternating groups, and others. In this project, the lattice structures have been drawn as researched, but the structures are up to $S_4$, the lattice for $S_5$ is tedious to construct as it contains 5 replicas of $S_4$, one can take the challenge and construct lattice structures for $S_6$ and the others.
REFERENCES


