RANKS, SUBDEGREES, SUBORBITAL GRAPHS AND CYCLE INDICE SASSOCIATED WITH THE PRODUCT ACTION OF $A_n \times A_n \times A_n$ $(4 \leq n \leq 8)$ ON THE CARTESIAN PRODUCT $X \times Y \times Z$

Miano Teresa Wambui
I56/38068/2016
Department of Mathematics and Actuarial Science

A Project submitted in Partial Fullfilment of the Requirements for Award of Masters Degree (Pure Mathematics) in the School of Pure and Applied Sciences of Kenyatta University

June 2019

DECLARATION

This project is my original work and has not been presented for a degree in any other university or any other award.

Miano Teresa Wambui (B.ed. Sc.)

I56/38068/2016

Sign:________________________ Date:________________

Supervisor

I confirm that the work reported in this project was carried out by the candidate under my supervision.

Dr. Jane Rimberia

Sign:________________________ Date:________________

Department of Mathematics and Actuarial Sciences
Kenyatta University

DEDICATION

To my son Henry, my husband Patrick and my entire family.

ACKNOWLEDGMENT

I thank God almighty for the gift of life, good health and for giving me the opportunity to conduct this research.

My sincere gratitude to my supervisor Dr.Jane Rimberia for her time, support, patience and guidance throughout my research period.

My special appreciation to my entire family especially my husband Patrick for his selflessness, encouragement, patience and love which he gave me as I carried out my research work.

Finally I thank my son Henry for always giving me the zeal even without his knowledge to continue with my research when times were hard.

TABLE OF CONTENTS

REFERENCES **60**

LIST OF TABLES

LIST OF FIGURES

ABBREVIATIONS AND NOTATIONS

Symbols	-	**Representation**
$Stab_G(x)$	-	The stabilizer of point x in G
$Orb_G(y)$	-	Orbit of G containing y
$\|X\|$	-	Number of elements in the set X
$[G : H]$	-	Index of subgroup H in a group G
D_n	-	The dihedral group of order $2n$
C_n	-	The cyclic group of order n
S_n	-	Symmetric group of degree n
A_n	-	Alternating group of degree n
$PGL(2, q)$	-	Projective general linear group
$PSL(2, q)$	-	Projective special linear group
O_i	-	i^{th} G- orbit on $X \times X$
Γ_i	-	i^{th} suborbital graph corresponding to O_i
$X^{[t]}$	-	The set of all ordered t-element subsets from X
$X^{(t)}$	-	The set of all unordered t-element subsets from X
$R(G)$	-	The rank of a group G acting on the set X
$\emptyset$	-	The empty set
$X \times Y \times Z$	-	Cartesian product of the sets X, Y and Z
$G_1 \times G_2 \times G_3$	-	External direct product of groups G_1, G_2 and G_3
$Fix(g)$	-	The set of all points fixed by element g
$\left(\begin{array}{c} n \\ r \end{array} \right)$	-	n combination r
Δ	-	A suborbit of G
Δ^*	-	The suborbit of G paired with Δ
γ	-	Number of self-paired suborbits of G
$Comp_\Gamma(x)$	-	The set of vertices in the component of Γ containing x

ABSTRACT

Researchers over the years have studied group actions. The action of symmetric and alternating groups on ordered and unordered subsets of a set X have been considered leaving out the product action. In this study, we have considered the product action of $A_n \times A_n \times A_n$ on $X \times Y \times Z$, where $n \geq 4$, $X = \{1, 2, 3, \cdots, n\}$, $Y = \{n+1, \cdots, 2n\}$ and $Z = \{2n+1, \cdots, 3n\}$. This action is shown to be transitive using the Orbit-Stabilizer theorem. The subdegrees and rank have been determined using the definition of an orbit of product action. The rank is 8 and the subdegrees are; 1, $(n-1)$, $(n-1)$, $(n-1)$, $(n-1)^2$, $(n-1)^2$, $(n-1)^2$ and $(n-1)^3$. Sim's procedure has been used in the construction of the corresponding non-trivial suborbital graphs. The suborbital graphs of this action are undirected, regular and their girth is 3. All the suborbital graphs except the one corresponding to the suborbit of length $(n-1)^3$, are disconnected. The suborbital graphs corresponding to suborbits of length $(n-1)$ and $(n-1)^2$ have n^2 and n components respectively. Finally, the cycle index formulas of this action when $n = 4, 5, 6, 7, 8$, are derived using the product of monomials.

CHAPTER 1

INTRODUCTION

1.1 Background information

The collection of all permutations on a set X forms a group under the binary operation of composition of maps which is referred to as the symmetric group $S_{|X|}$ of degree $|X|$. The symmetric group has order $|X|!$. The set of all even permutations in $S_{|X|}$ forms a group referred to as the alternating group $A_{|X|}$ of degree $|X|$ and has order $\frac{|X|!}{2}$. Note that $A_{|X|}$ is a normal subgroup of $S_{|X|}$ of index 2. The direct product $G_1 \times G_2 \times \cdots \times G_n$ of the groups $G_1, G_2, \cdots, G_n$ with operations $*_1, *_2, \cdots, *_n$ respectively, is the set of $n-$tuples $(g_1, g_2, \cdots, g_n)$ where $g_i \in G_i$; with the operation defined component wise as,

$$(g_1, g_2, \cdots, g_n) * (h_1, h_2, \cdots, h_n) = (g_1 *_1 h_1 \;\;, \;\; g_2 *_2 h_2 \;\;, \cdots, g_n *_n h_n). \qquad (1.1)$$

The group $G_1 \times G_2 \times \cdots \times G_n$ has order $|G_1|\,|G_2| \cdots |G_n|$. Let G_1 act on a set X, G_2 act on a set Y and G_3 act on a set Z, then the product action of $G_1 \times G_2 \times G_3$ on $X \times Y \times Z$ is given by the rule,

$$(g_1, g_2, g_3)(x, y, z) = (g_1\,(x)\,, g_2\,(y)\,, g_3\,(z)). \qquad (1.2)$$

[See (Cameron *et al.*, 2007).]

Theorem 1.1.1. *(Armstrong, 2013) The* $G_1 \times G_2 \times G_3-$*orbit containing* $(x, y, z) \in X \times Y \times Z$ *is given by* $Orb_{G_1}(x) \times Orb_{G_2}(y) \times Orb_{G_3}(z)$ *and the stabilizer of* (x, y, z) *is given by* $Stab_{G_1}(x) \times Stab_{G_2}(y) \times Stab_{G_3}(z)$.

1.2 Definition of terms and Preliminary results

Definition 1.2.1. *A group* G *is said to act on the left of a set* X *if for each* $g \in G$ *and* $x \in X$, *there exist a unique element* $gx \in X$ *such that for all* $x \in X$ *and* $g_1,\ g_2 \in G$, $(g_1g_2)\,x = g_1\,(g_2x)$ *and* $ex = x$, *where* e *is the identity in* G.

Definition 1.2.2. *The action of a group* G *on a set* X *partitions* X *into disjoint equivalence classes referred to as orbits or transitivity classes of action. The orbit containing* $x \in X$, *is denoted by* $Orb_G(x)$.

Definition 1.2.3. *The action of a group* G *on a set* X *is said to be transitive if for all* $x, y \in X$, *there exist* $g \in G$ *such that* $gx = y$. *That is, the action is transitive if it results to only one orbit.*

Definition 1.2.4. *Let a group* G *act on a non-empty set* X *and* $g \in G$. *Then the set* $Fix(g) = \{x \in X | gx = x\}$ *is known as the fixed point set of* g.

Definition 1.2.5. *Let a group* G *act on a non-empty set* X *and* $x \in X$. *The set* $G_x = \{g \in G | gx = x\}$ *is known as the stabilizer of* x *in* G.

Theorem 1.2.6. *(Rose, 1978) Let* G *act on a set* X. *Then*

$$|Orb_G(x)| = [G : Stab_G(x)]. \tag{1.3}$$

Theorem 1.2.7. *(Rose, 1978) [Cauchy's Frobenius lemma] Let a group G act on a set X. The number of G-orbits on X is given by*

$$\frac{1}{|G|}\sum_{g\in G}|Fix(g)|. \tag{1.4}$$

Definition 1.2.8. *Let G act transitively on a set X and $x \in X$. Then G_x-orbits on X are called suborbits of G on X. The number of these suborbits is called the rank of G on X and the sizes of the suborbits are called the sub degrees of G on X.*

Definition 1.2.9. *Let Δ be a G_x-orbit on X. The set $\Delta^* = \{gx|g \in G, x \in g\Delta\}$ is also a G_x-orbit and it is known as a suborbit of G on X paired with Δ. Clearly $|\Delta^*| = |\Delta|$. Δ is said to be self-paired if $\Delta^* = \Delta$.*

Theorem 1.2.10. *(Cameron, 1999) Let the action of a group G on set X be transitive and let $g \in G$. Then the number γ of self-paired suborbits of G is given by,*

$$\gamma = \frac{1}{|G|}\sum_{g\in G}\left|Fix(g^2)\right|. \tag{1.5}$$

Definition 1.2.11. *Let the action of G on a set X be transitive. Then a block of the action is a subset $A \subseteq X$ such that either $g(A) \cap A = \emptyset$ or $g(A) = A$ for all $g \in G$. In particular $X, \emptyset$ and all singleton subsets of X are blocks known as trivial blocks. The action is said to be primitive if it does not have blocks other than the trivial ones; otherwise its imprimitive.*

Theorem 1.2.12. *(Higman, 1967) Let G act transitively on set X. Then the action is primitive if and only if all non-trivial suborbital graphs are connected.*

Definition 1.2.13. *Let $g \in S_n$ with α_i $(i = 1, 2, \cdots, n)$ cycles of length i. Then g is said to have cycle type $(\alpha_1, \alpha_2, \cdots, \alpha_n)$.*

Theorem 1.2.14. *(Krishnamurthy, 1985) Two permutations in S_n are conjugate if and only if they have the same cycle type and if $g \in S_n$ has cycle type $(\alpha_1, \alpha_2, \cdots, \alpha_n)$, then the number of permutations in S_n conjugate to g is*

$$\frac{n!}{\prod_{i=1}^{n} \alpha_i! i^{\alpha_i}}. \tag{1.6}$$

Definition 1.2.15. *A graph $G(V, E)$ is a diagram consisting of a set V, whose elements are called vertices and a set E of unordered pairs of the vertices called edges. If a graph has only one vertex and no edge it is called a trivial graph.*

Definition 1.2.16. *If a graph has no edges it is called a null graph.*

Definition 1.2.17. *A sequence of edges of the form $v_0, v_1, v_2, v_{t-1,} v_t$ in a graph G is known as a walk of length t. A walk with distinct vertices is called a path. A closed path is called a cycle or a circuit.*

Definition 1.2.18. *Let G be a graph. If for all vertices $x, y \in G$ there exist a path joining x and y, then G is said to be connected. If G is not connected then it is disconnected. A maximal connected sub-graph of G is called a component of G.*

Definition 1.2.19. *The girth of a graph G is the length of the shortest cycle in the graph.*

Definition 1.2.20. *Let G be a group and X be a finite set. The suborbitals of G on X are the G-orbits on $X \times X$.*

Theorem 1.2.21. *(Wilson and Watkins, 1990) Let Γ be a connected undirected graph. Then Γ is Eulerian if and only if every vertex is of even degree.*

Definition 1.2.22. *A suborbital graph Γ corresponding to suborbital $O \subseteq X \times X$ is a graph whose vertex set is X and xy is a directed edge if and only if $(x, y) \in O$.*

Theorem 1.2.23. *(Lauri and Mizzi, 2011) All the components of a suborbital graph are isomorphic to one another.*

Definition 1.2.24. *If a finite group G acts on a set X, $|X| = n$ and $g \in G$ has cycle type $(\alpha_1, \alpha_2, \cdots, \alpha_n)$, we define the monomial of g to be $mon(g) = t_1^{\alpha_1} t_2^{\alpha_2} \cdots t_n^{\alpha_n}$, where t_1, $t_2, \cdots, t_n$ are distinct commuting indeterminates.*

Definition 1.2.25. *The cycle index of the action of a group G on a set X is the polynomial (say over the rational field $\mathbb{Q}$) in t_1, $t_2, \ldots, t_n$ given by*

$$Z(G) = \frac{1}{|G|} \sum_{g \in G} \{mon(g)\}. \tag{1.7}$$

Note: If G has conjugacy classes $K_1, K_2, \cdots, K_m$with $g_i \in K_i$, then

$$Z(G) = \frac{1}{|G|} \sum_{i=1}^{m} |K_i| \, mon(g_i). \tag{1.8}$$

Theorem 1.2.26. *(Cameron et al., 2007) Take an $i-$cycle in a permutation $g_1 \in G_1$ and a $j-$cycle in a permutation $g_2 \in G_2$. The pair (g_1, g_2) acts on the product of the supports of these two cycle as $gcd(i, j)$ cycles each of length $lcm(i, j)$. Hence the cycle index of $G_1 \times G_2$ can be computed as follows: define $t_i^a \circ t_j^b = (t_{lcm(i,j)})^{ab(\gcd(i,j))}$,* *and extend multiplicatively to arbitrary monomials and then additively to arbitrary polynomials. Then,*

$$Z(G_1 \times G_2) = Z(G_1) \circ Z(G_2). \tag{1.9}$$

1.3 Statement of the problem

Numerous studies have been done on permutation groups acting on ordered and unordered subsets of a set X, in particular symmetric group and alternating group. Nevertheless, little has been done on the product action of permutation groups acting on the Cartesian product. Therefore in this research we will study the action of $A_n \times A_n \times A_n$ $(n \geq 4)$ on the Cartesian product $X \times Y \times Z$. In particular, we will consider ranks, subdegrees, suborbital graphs and the cycle index formula corresponding to the action.

1.4 Objectives of the study

1.4.1 General objective

To study the properties associated with the action of $A_n \times A_n \times A_n$ $(n \geq 4)$ on the Cartesian product $X \times Y \times Z$.

1.4.2 Specific objectives

i To determine whether the action of $A_n \times A_n \times A_n$ $(n \geq 4)$ on the Cartesian product $X \times Y \times Z$ is transitive.

ii To determine the ranks and subdegrees of the action of $A_n \times A_n \times A_n$ $(n \geq 4)$ on the Cartesian product $X \times Y \times Z$.

iii To construct the suborbital graphs corresponding to the action.

iv To investigate the theoretic properties arising from the constructed suborbital graphs.

v To determine the cycle index formula of the action.

1.5 Significance of the study

Graph theory has proven to be of great importance in various fields. In Biology for example, it has boosted conservation efforts whereby a vertex of a graph represents habitats of species while the edges represents migration routes of the organisms. This has been useful in combating diseases which spread and breeding patterns. Connectivity in graph theory helps to represent how one part of the brain is connected to the other in terms of their function and how they work together for various cognitive processes. In communication, the social network represents the connection between different people in the world. Transportation system use graph theory to compute the shortest and longest distance and this can be used by traders to determine the cheapest ways of distributing their products to different customers. In combinatorics, cycle index formulae are important tools in counting. For instance, they can be used in counting isomers in chemical compounds and as well as in counting graphs.

CHAPTER 2

LITERATURE REVIEW

2.1 Ranks and subdegrees

The concept of rank of a group was introduced by Higman (1964). Later Higman (1970) computed rank and the subdegrees of the symmetric group S_n acting on unordered pairs from the set $X = \{1, 2, \cdots, n\}$. It was proved that the rank is 3 while the subdegrees are $1,\ 2\,(n-2)$ and $\begin{pmatrix} n-2 \\ 2 \end{pmatrix}$.

Cameron (1973) worked on permutation groups that are transitive. It was proved that G has at most rank 4 if G is primitive on X and G_x is doubly transitive on all non-trivial suborbits except possibly one, with $|G_x| > 2$.

Kamuti (1992) studied the action of $PGL(2, q)$ and $PSL(2, q)$ on the cosets of their maximal subgroups. It was proved that when $PGL(2, q)$ acts on the cosets of its maximal dihedral subgroup of order $\frac{2(q-1)}{k}$, then its rank is $\frac{3}{4}\,(q+3)$ if $q \equiv 1 mod\ 4$, $\frac{1}{4}(3q+7)$ if $q \equiv -1 mod\ 4$ and $\frac{1}{2}(q+2)$ q is even.

Kamuti (2006) worked on the actions of $PGL(2, q)$ on the cosets of its dihedral subgroup of order $2(q-1)$. The rank was found to be $\frac{q+3}{2}$ for q odd and $\frac{q+2}{2}$ for q even. Nyaga *et al.* (2011) worked on the action of the symmetric group $\mathrm{S_n}$ on set $\mathrm{X^{(t)}}$, of unordered t-element subsets from X, where $X = \{1, 2, \cdots, n\}$. It was proved that the action is transitive, the rank is $t+1$ for $\mathrm{n \geq 2t}$ and the subdegrees are, $1, \binom{n-t}{t-1}, \binom{t}{2}\binom{n-t}{t-2}, \binom{t}{3}\binom{n-t}{t-3}, \cdots, \binom{t}{t-1}\binom{n-t}{1}, \binom{n-t}{t}$ respectively.

Rimberia *et al.* (2012) proved that for $n \geq 2t$, S_n acts transitively on the set of

t-ordered subset of $X = \{1, 2, \cdots, n\}$ with a rank of

$$R(G) = t! + t^2 (t-1)! + \frac{t^2(t-1)^2(t-2)!}{2!2!} + \frac{t^2(t-1)^2(t-2)^2(t-3)!}{3!3!} + \cdots + \frac{t^2(t-1)^2}{2!} + t^2 + 1. \tag{2.1}$$

Kimani *et al.* (2014) used table of marks to compute rank of S_n, $(n = 5, 6, 7)$ acting on ordered pairs from $X = \{1, 2, \cdots, n\}$ and proved that it is 7.

Gachago *et al.* (2015) proved that the alternating group A_n $(n = 5, 6, 7)$ acts transitively on ordered and unordered quadruples of $X = \{1, 2, \cdots, n\}$.

Magero (2015) computed the rank and subdegrees of $PSL(2, q)$ acting on the cosets of its cyclic subgroup $C_{\frac{q-1}{k}}$ where k is the greatest common divisor of $q - 1$ and 2. It was found that the rank is 1, $q + 4$ and $2(q + 3)$ for q even and odd respectively. The subdegrees were found to be, 1, $q - 1$, $q - 1, \cdots, q - 1$, $q + 2$ times for q even and 1, $\frac{q-1}{2}$, $\frac{q-1}{2}, \cdots \frac{q-1}{2}$, $2(q + 2)$ times for q odd.

Rotich (2016) computed the rank and subdegrees of $PGL(2, q)$ acting on the cosets of its cyclic subgroup C_{q-1}. It was found that the rank is $q + 4$ and subdegrees are 1 and $q - 1$, $q - 1, \cdots, q - 1$, $q + 2$ times.

2.2 Suborbital graphs

Sims (1967) introduced the concept of suborbital graphs for permutation groups that are finite acting on a set X. In the study, a suborbital graph was defined.

Kamuti (1992) constructed suborbital graphs of $PGL(2, q)$ acting on the cosets of its maximal dihedral group of order $2(q - 1)$. The method used provided another way of constructing the coxeter graph which was first constructed by Coxeter (1983).

Akbas (2001) studied the modular group acting on the rational projection line. It was proved that the suborbital graph for a modular group is a forest if and only if it has no triangles.

Kamuti *et al.* (2012) studied the action of the stabilizer of ∞ in the modular group on the set of integers. It was proved that the action is transitive and imprimitive. If $a, b \in \mathbb{Z}$ and $a \neq b$, then suborbitals $O(0, a)$ and $O(0, b)$ are disjoint. Furthermore the suborbital graph $\Gamma(0, a)$ has $|a|$ components and it is paired with $\Gamma(0, a)$.

Rimberia (2012) studied suborbits and suborbital graphs of the symmetric group S_n acting on ordered t-element subsets. It was found that all these graphs are disconnected. The girth of the graphs corresponding to self-paired orbits of $G_{[1,2,\cdots,t]}$ on $X^{[t]}$ with exactly t elements from $A = \{1, 2, \cdots, t\}$ was shown to be zero while that of paired orbits with precisely t elements from A was found to be three. For the suborbital graph corresponding to the orbit of $G_{[1,2,\cdots,t]}$ on $X^{[t]}$ with no element from A, the girth was found to be equal to three provided $n \geq 3t$.

Magero (2015) constructed suborbital graphs corresponding to the action of $PSL(2, q)$ on the cosets of its cyclic subgroup $C_{\frac{q-1}{k}}$, where k is the greatest common divisor of $q - 1$ and 2. It was found that the action is imprimitive. Also the number of self-paired suborbital is $q+2,\ q+3$ and $q+1$ for $p = 2, q \equiv 1 mod\ 4$ and $q \equiv -mod 4$ respectively.

2.3 Cycle indices

The concept of cycle index of a permutation group was introduced by Redfield (1927). The study stressed on the relationship between combinatorial analysis and permutation groups. Each permutation group was associated with a group reduced function which is nowadays known as cycle index.

Later, Pólya (1937) used the cycle index to count graphs and chemical compounds via the famous Polya's Enumeration Theorem. Through this theorem the cycle index became a very powerful tool in enumeration and hence the need arose to compute the cycle indices of permutation groups.

Kamuti (1992) computed the disjoint cycle structures of elements of $PGL\ (2, q\ )$ and PSL $(2, \mathrm{q})$ for any primitive permutation representation of these groups. He also derived general formulae for the cycle indices of these representations.

The cycle index of the reduced ordered triple group $S_n^{[3]}$ was computed by Kamuti and Obong'o (2002). Kamuti and Njuguna (2004) later extended this study to cycle index of the reduced ordered r-group $S_n^{[r]}$.

Muthoka *et al.* (2015) derived the cycle index formulas for D_n acting on unordered pairs. It was found that the cycle index formula is given by,

$$Z(D_n, X^2) = \frac{1}{2n}\left[\sum_{d|n} \phi\,(d)\, t_d^{\frac{n^2-n}{2d}} + n t_1^{\frac{n-1}{2}} t_2^{\frac{n^2-2n+1}{4}}\right], \tag{2.2}$$

when n is odd and

$$Z(D_n, X^2) = \frac{1}{2n}\left[\sum_{\substack{d|n \\ 2|d}} \phi\,(d) \left(t_{\frac{d}{2}}^{\frac{n}{d}} t_d^{\frac{n^2-2n}{2d}}\right) + \sum_{\substack{d|n \\ 2\nmid d}} \phi\,(d) \left(t_d^{\frac{n^2-n}{2d}}\right) + n\ t_1^{\frac{n}{2}} t_2^{\frac{n^2-2n}{4}}\right], \tag{2.3}$$

when n is even.

From the above, it is clear that there is a lot that has not been done on the action of $A_n \times A_n \times A_n \ \ (n \geq 4, 5, 6, 7, 8)$ on the Cartesian product $X \times Y \times Z$. Therefore, this research will determine ranks, subdegrees, suborbital graphs and the cycle index formula corresponding to the action.

CHAPTER 3

RANKS AND SUBDEGREES OF $A_n \times A_n \times A_n$ ACTING ON $X \times Y \times Z$

This chapter is divided into two sections. Section 3.1 examines the transitivity of $A_n \times A_n \times A_n$ acting on $X \times Y \times Z$ and Section 3.2 gives the rank and subdegrees of the action.

We will denote $G = A_n \times A_n \times A_n$ to be the external direct product of the alternating groups A_n on $X = \{1, 2, \cdots, n\}$, $Y = \{n+1, n+2, \cdots, 2n\}$ and $Z = \{2n+1, 2n+2, \cdots, 3n\}$ respectively. Also $X' = \{2, \cdots, n\}$, $Y' = \{n+2, \cdots, 2n\}$ and $Z' = \{2n+1, 2n+2, \cdots, 3n\}$.

3.1 Transitivity

Proposition 3.1.1. *Let K be an alternating group on set $\Omega = \{\alpha_1, \alpha_2, \cdots, \alpha_n\}$. Then the stabilizer of $Stab_K(\alpha_1)$ is the alternating group A_{n-1} on set $\Omega \setminus \{\alpha_1\}$.*

Proof: The element $g \in K$ fixes $\alpha_1 \in \Omega$ if α_1 is in its own cycle as a permutation. Therefore $Stab_K(\alpha_1)$ is isomorphic to the group of all even permutations of the set $\{\alpha_2, \alpha_3, \cdots, \alpha_n\}$, which is the alternating group A_{n-1}. □

Proposition 3.1.2. *Let G_1, G_2 and G_3 be alternating groups A_n on X, Y and Z respectively. Suppose $G = G_1 \times G_2 \times G_3$ acts on $X \times Y \times Z$. Then, the stabilizer of $(1, n+1, 2n+1)$ in G is isomorphic to $A_{n-1} \times A_{n-1} \times A_{n-1}$.*

Proof: By Proposition 3.1.1, the $Stab_{G_1}(1)$, $Stab_{G_1}(n+1)$ and $Stab_{G_1}(2n+1)$ are isomorphic to the alternating groups A_n on X', Y' and Z' respectively. By Theorem 1.1.1, we have,

$$Stab_G(1, n+1, 2n+1) = Stab_{G_1}(1) \times Stab_{G_2}(n+1) \times Stab_{G_3}(2n+1) = A_{n-1} \times A_{n-1} \times A_{n-1}. \tag{3.1}$$

□

Example 3.1.1. *The action of $A_4 \times A_4 \times A_4$ on $X \times Y \times Z$ is transitive.*

Proof: Let $G = G_1 \times G_2 \times G_3 = A_4 \times A_4 \times A_4$ act on $X \times Y \times Z$. By Proposition 3.1.2, we get,

$$\begin{aligned} |Stab_G(1,5,9)| &= |Stab_{G_1}(1) \times \ Stab_{G_2}(5) \times \ Stab_{G_3}(9)| = |A_3 \times A_3 \times A_3| \\ &= \frac{3! \times 3! \times 3!}{2 \times 2 \times 2} = 27. \end{aligned} \tag{3.2}$$

Thus, by Theorem 1.2.6, we have,

$$|Orb_G(1,5,9)| = |G : Stab_G(1,5,9)| = \frac{\frac{4!\times 4!\times 4!}{2\times 2\times 2}}{27} = \frac{1728}{27} = 64 = |X \times Y \times Z|\,. \tag{3.3}$$

Hence, the action is transitive.

□

Example 3.1.2. *The action of $A_5 \times A_5 \times A_5$ on $X \times Y \times Z$ is transitive.*

Proof: Let $G = G_1 \times G_2 \times G_3 = A_5 \times A_5 \times A_5$ act on $X \times Y \times Z$. By Proposition 3.1.2, we get,

$$\begin{aligned} |Stab_G(1,6,11)| &= |Stab_{G_1}(1) \times Stab_{G_2}(6) \times Stab_{G_3}(11)| = |A_4 \times A_4 \times A_4| \\ &= \frac{4! \times 4! \times 4!}{2 \times 2 \times 2} = 1728. \end{aligned} \tag{3.4}$$

Applying Theorem 1.2.6, we get,

$$|Orb_G(1,6,11)| = |G : Stab_G(1,6,11)| = \frac{\frac{5!\times 5!\times 5!}{2\times 2\times 2}}{1728} = \frac{216000}{1728} = 125 = |X \times Y \times Z|\,. \tag{3.5}$$

Thus the action is transitive.

□

Example 3.1.3. *The action of $A_6 \times A_6 \times A_6$ on $X \times Y \times Z$ is transitive.*

Proof: Let $G = G_1 \times G_2 \times G_3 = A_6 \times A_6 \times A_6$ act on $X \times Y \times Z$. By Proposition 3.1.2, we obtain,

$$\begin{aligned} |Stab_G(1,7,13)| &= |Stab_{G_1}(1) \times Stab_{G_2}(7) \times Stab_{G_3}(13)| = |A_5 \times A_5 \times A_5| = \frac{5! \times 5! \times 5!}{2 \times 2 \times 2} \\ &= 216000. \end{aligned} \tag{3.6}$$

Thus, by Theorem 1.2.6, we have,

$$|Orb_G(1,7,13)| = |G : Stab_G(1,7,13)| = \frac{\frac{6!\times 6!\times 6!}{2\times 2\times 2}}{216000} = \frac{46656000}{216000} = 216 = |X \times Y \times Z|\,. \tag{3.7}$$

Thus the action is transitive. □

Example 3.1.4. *The action of $A_7 \times A_7 \times A_7$ on $X \times Y \times Z$ is transitive.*

Proof: Let $G = G_1 \times G_2 \times G_3 = A_7 \times A_7 \times A_7$ act on $X \times Y \times Z$. It follows from Proposition 3.1.2, that,

$$|Stab_G(1,8,15)| = |Stab_{G_1}(1)\times\ Stab_{G_2}(8)\times\ Stab_{G_3}(15)| = |A_6 \times A_6 \times A_6| = \frac{6! \times 6! \times 6!}{2 \times 2 \times 2}$$
$$= 46656000. \tag{3.8}$$

Applying Theorem 1.2.6, we get,

$$|Orb_G(1,8,15)| = |G{:}Stab_G(1,8,15)| = \frac{\frac{7!\times 7!\times 7!}{2\times 2\times 2}}{46656000} = \frac{16003008000}{46656000} = 343 = |X \times Y \times Z|\,. \tag{3.9}$$

Hence the action is transitive. □

Example 3.1.5. *The action of $A_8 \times A_8 \times A_8$ on $X \times Y \times Z$ is transitive.*

Proof: Let $G = G_1 \times G_2 \times G_3 = A_8 \times A_8 \times A_8$ act on $X \times Y \times Z$. Then by Proposition 3.1.2,

$$\begin{aligned} |Stab_G(1,9,17)| &= |Stab_{G_1}(1)\times\ Stab_{G_2}(9)\times\ Stab_{G_3}(17)| = |A_7 \times A_7 \times A_7| \\ &= \frac{7! \times 7! \times !}{2 \times 2 \times 2} = 16003008000. \end{aligned} \tag{3.10}$$

Applying Theorem 1.2.6, we get,

$$\begin{aligned} |Orb_G(1,9,17)| &= |G{:}Stab_G(1,9,17)| = \frac{\frac{8!\times 8!\times 8!}{2\times 2\times 2}}{16003008000} = \frac{8193540096000}{16003008000} \\ &= 512 = |X \times Y \times Z|\,. \end{aligned} \tag{3.11}$$

Therefore, the action is transitive. □

Theorem 3.1.6. *The action of $A_n \times A_n \times A_n$ on $X \times Y \times Z$ is transitive.*

Proof: Let $G = G_1 \times G_2 \times G_3 = A_n \times A_n \times A_n$ act on $X \times Y \times Z$. Then by Proposition 3.1.2, we obtain,

$$\begin{aligned} |Stab_G(1, n+1, 2n+1)| &= |Stab_{G_1}(1) \times\ Stab_{G_2}(n+1) \times\ Stab_{G_3}(2n+1)| \\ &= |A_{n-1} \times A_{n-1} \times A_{n-1}| \\ &= \frac{(n-1)! \times (n-1)! \times (n-1)!}{2 \times 2 \times 2}. \end{aligned} \tag{3.12}$$

Thus, by Theorem 1.2.6, we have,

$$\begin{aligned} |Orb_G(1, n+1, 2n+1)| &= |G{:}Stab_G(1, n+1, 2n+1)| = \frac{\frac{n! \times n! \times n!}{2 \times 2 \times 2}}{\frac{(n-1)! \times (n-1)! \times (n-1)!}{2 \times 2 \times 2}} \\ &= n^3 \\ &= |X \times Y \times Z|\,. \end{aligned} \tag{3.13}$$

Hence, the action is transitive. □

3.2 Ranks and subdegrees

Example 3.2.1. *Let $G = G_1 \times G_2 \times G_3 = A_4 \times A_4 \times A_4$ act on $X \times Y \times Z$. Then the rank is 8 and the subdegrees are 1, 3, 3, 3, 9, 9, 9 and 27.*

Proof: Let $H_1 = Stab_{G_1}(1)$, $H_2 = Stab_{G_2}(5)$, $H_3 = Stab_{G_3}(9)$ and $H = Stab_G(1, 5, 9)$. Then by Proposition 3.1.2, $H = H_1 \times H_2 \times H_3$. By Proposition 3.1.1, H_1, H_2 and H_3 are isomorphic to alternating groups A_3 on the sets $X' = \{2, 3, 4\}$, $Y' = \{6, 7, 8\}$ and $Z' = \{10, 11, 12\}$ respectively. Suppose H_1 acts on $X = \{1, 2, 3, 4\}$. Then $Orb_{H_1}(1) = \{1\}$ and $Orb_{H_1}(2) = \{2, 3, 4\}$. Also let H_2 act on $Y = \{5, 6, 7, 8\}$, then $Orb_{H_2}(5) = \{5\}$ and $Orb_{H_2}(6) = \{6, 7, 8\}$. Similarly, let H_3 act on $Z = \{9, 10, 11, 12\}$, then $Orb_{H_3}(9) = \{9\}$ and $Orb_{H_3}(10) = \{10, 11, 12\}$.

By Theorem 1.1.1, we have:

$$\Delta_0 = Orb_H(1, 5, 9) = Orb_{H_1}(1) \times Orb_{H_2}(5) \times Orb_{H_3}(9) = \{1\} \times \{5\} \times \{9\} = \{(1, 5, 9)\}. \tag{3.14}$$

Therefore, $|\Delta_0| = 1$.

$$\begin{aligned} \Delta_1 = Orb_H(2, 5, 9) &= Orb_{H_1}(2, 3, 4) \times Orb_{H_2}(5) \times Orb_{H_3}(9) \\ &= \{2, 3, 4\} \times \{5\} \times \{9\} \\ &= \{(2, 5, 9), (3, 5, 9), (4, 5, 9)\}. \end{aligned} \tag{3.15}$$

Therefore, $|\Delta_1| = 3$.

$$\begin{aligned} \Delta_2 &= Orb_H(1,6,9) = Orb_{H_1}(1) \times Orb_{H_2}(6) \times Orb_{H_3}(9) \\ &= \{1\} \times \{6,7,8\} \times \{9\} \\ &= \{(1,6,9),(1,7,9),(1,8,9)\}. \end{aligned} \tag{3.16}$$

Therefore, $|\Delta_2| = 3$.

$$\begin{aligned} \Delta_3 &= Orb_H(1,5,10) = Orb_{H_1}(1) \times Orb_{H_2}(5) \times Orb_{H_3}(10) \\ &= \{1\} \times \{5\} \times \{10,11,12\} \\ &= \{(1,5,10),(1,5,11),(1,5,12)\}. \end{aligned} \tag{3.17}$$

Therefore, $|\Delta_3| = 3$.

$$\begin{aligned} \Delta_4 &= Orb_H(2,6,9) = Orb_{H_1}(2) \times Orb_{H_2}(6) \times Orb_{H_3}(9) \\ &= \{2,3,4\} \times \{6,7,8\} \times \{9\} \\ &= \{(2,6,9),(2,7,9),(2,8,9),(3,6,9),(3,7,9),(3,8,9), \\ &\quad (4,6,9),(4,7,9),(4,8,9)\}. \end{aligned} \tag{3.18}$$

Therefore, $|\Delta_4| = 9$.

$$\begin{aligned} \Delta_5 &= Orb_H(2,5,10) = Orb_{H_1}(2) \times Orb_{H_2}(5) \times Orb_{H_3}(10) \\ &= \{2,3,4\} \times \{5\} \times \{10,11,12\} \\ &= \{(2,5,10),(2,5,11),(2,5,12),(3,5,10),(3,5,11),(3,5,12), \\ &\quad (4,5,10),(4,5,11),(4,5,12)\}. \end{aligned} \tag{3.19}$$

Therefore, $|\Delta_5| = 9$.

$$\begin{aligned} \Delta_6 &= Orb_H(1,6,10) = Orb_{H_1}(1) \times Orb_{H_2}(6) \times Orb_{H_3}(10) \\ &= \{1\} \times \{6,7,8\} \times \{10,11,12\} \\ &= \{(1,6,10),(1,7,10),(1,8,10),(1,6,11),(1,7,11),(1,8,11),(1,6,12), \\ &\quad (1,7,12),(1,8,12)\}. \end{aligned} \tag{3.20}$$

Therefore, $|\Delta_6| = 9$.

$$\Delta_7 = Orb_H(2,6,10) = Orb_{H_1}(2) \times Orb_{H_2}(6) \times Orb_{H_3}(10)$$
$$= \{2,3,4\} \times \{6,7,8\} \times \{10,11,12\}$$
$$= \{(2,6,10),(2,6,11),(2,6,12),(2,7,10),(2,7,11),(2,7,12),$$
$$(2,8,10),(2,8,11),(2,8,12),(3,6,10),(3,6,11),(3,6,12), \quad (3.21)$$
$$(3,7,10),(3,7,11),(3,7,12),(3,8,10),(3,8,11),(3,8,12),$$
$$(4,6,10),(4,6,11),(4,6,12),(4,7,10),(4,7,11),(4,7,12),$$
$$(4,8,10),(4,8,11),(4,8,12)\}.$$

Therefore $|\Delta_7| = 27$.

It follows that, the subdegrees of G are 1, 3, 3, 3, 9, 9, 9 and 27. The number of elements in all the eight $H-$orbits is $1+3+3+3+9+9+9+27 = 64 = |X \times Y \times Z|$. Thus the rank is 8. □

Example 3.2.2. *Let $G = G_1 \times G_2 \times G_3 = A_5 \times A_5 \times A_5$ act on $X \times Y \times Z$. Then the rank is 8 and the subdegrees are 1, 4, 4, 4, 16, 16, 16 and 64.*

Proof: Let $H_1 = Stab_{G_1}(1)$, $H_2 = Stab_{G_2}(6)$ and $H_3 = Stab_{G_3}(11)$ acting on X, Y and Z respectively. Let $H = Stab_G(1,6,11)$. Then by Proposition 3.1.2, $H = H_1 \times H_2 \times H_3$. By Proposition 3.1.1, H_1, H_2 and H_3 are isomorphic to alternating groups A_4 on the sets X', Y' and Z' respectively. Let H_1, H_2 and H_3 act on X, Y and Z respectively. Let H_1, H_2 and H_3 act on X, Y and Z respectively. Then $Orb_{H_1}(1) = \{1\}$, $Orb_{H_1}(2) = \{2,3,4,5\}$, $Orb_{H_2}(6) = \{6\}$, $Orb_{H_2}(7) = \{7,8,9,10\}$, $Orb_{H_3}(11) = \{11\}$ and $Orb_{H_3}(12) = \{12,13,14,15\}$. By Theorem 1.1.1, we have:

$$\Delta_0 = Orb_H(1,6,11) = Orb_{H_1}(1) \times Orb_{H_2}(6) \times Orb_{H_3}(11)$$
$$= \{1\} \times \{6\} \times \{11\} = \{(1,6,11)\}. \quad (3.22)$$

$\therefore |\Delta_0| = 1$.

$$\begin{aligned}\Delta_1 &= Orb_H(2,6,11) = Orb_{H_1}(2) \times Orb_{H_2}(6) \times Orb_{H_3}(11) \\ &= \{2,3,4,5\} \times \{6\} \times \{11\} \\ &= \{(2,6,11),(3,6,11),(4,6,11),(5,6,11)\}.\end{aligned} \tag{3.23}$$

$\therefore |\Delta_1| = 4.$

$$\begin{aligned}\Delta_2 &= Orb_H(1,7,11) = Orb_{H_1}(1) \times Orb_{H_2}(7) \times Orb_{H_3}(11) \\ &= \{1\} \times \{7,8,9,10\} \times \{11\} \\ &= \{(1,7,11),(1,8,11),(1,9,11),(1,10,11)\}.\end{aligned} \tag{3.24}$$

$\therefore |\Delta_2| = 4.$

$$\begin{aligned}\Delta_3 &= Orb_H(1,6,12) = Orb_{H_1}(1) \times Orb_{H_2}(6) \times Orb_{H_3}(12) \\ &= \{1\} \times \{6\} \times \{12,13,14,15\} \\ &= \{(1,6,12),(1,6,13),(1,6,14),(1,6,15)\}.\end{aligned} \tag{3.25}$$

$\therefore |\Delta_3| = 4.$

$$\begin{aligned}\Delta_4 &= Orb_H(2,7,11) \\ &= Orb_{H_1}(2) \times Orb_{H_2}(7) \times Orb_{H_3}(11) \\ &= \{2,3,4,5\} \times \{7,8,9,10\} \times \{11\} \\ &= \{(2,7,11),(2,8,11),(2,9,11),(2,10,11),(3,7,11),(3,8,11), \\ &\quad (3,9,11),(3,10,11),(4,7,11),(4,8,11),(4,9,11),(4,10,11), \\ &\quad (5,7,11),(5,8,11),(5,9,11),(5,10,11)\}.\end{aligned} \tag{3.26}$$

$\therefore |\Delta_4| = 16.$

$$\begin{aligned}
\Delta_5 &= Orb_H(2,6,12) = Orb_{H_1}(2) \times Orb_{H_2}(6) \times Orb_{H_3}(12) \\
&= \{2,3,4,5\} \times \{6,\} \times \{12,13,14,15\} \\
&= \{(2,6,12),(2,6,13),(2,6,14),(2,6,15),(3,6,12),(3,6,13), \\
&\quad (3,6,14),(3,6,15),(4,6,12),(4,6,13),(4,6,14),(4,6,15), \\
&\quad (5,6,12),(5,6,13),(5,6,14),(5,6,15)\}.
\end{aligned} \tag{3.27}$$

$\therefore |\Delta_5| = 16.$

$$\begin{aligned}
\Delta_6 &= Orb_H(1,7,12) \\
&= Orb_{H_1}(1) \times Orb_{H_2}(7) \times Orb_{H_3}(12) \\
&= \{1\} \times \{7,8,9,10\} \times \{12,13,14,15\} \\
&= \{(1,7,12),(1,7,13),(1,7,14),(1,7,15),(1,8,12),(1,8,13), \\
&\quad (1,8,14),(1,8,15),(1,9,12),(1,9,13),(1,9,14),(1,9,15), \\
&\quad (1,10,12),(1,10,13),(1,10,14),(1,10,15)\}.
\end{aligned} \tag{3.28}$$

$\therefore |\Delta_6| = 16.$

$$\begin{aligned}
\Delta_7 &= Orb_H(2,7,12)\\
&= Orb_{H_1}(2)\times Orb_{H_2}(7)\times Orb_{H_3}(12)\\
&= \{2,3,4,5\}\times\{7,8,9,10\}\times\{12,13,14,15\}\\
&= \{(2,7,12),(2,7,13),(2,7,14),(2,7,15),(2,8,12),(2,8,13),\\
&\quad (2,8,14),(2,8,15),(2,9,12),(2,9,13),(2,9,14),(2,9,15),\\
&\quad (2,10,12),(2,10,13),(2,10,14),(2,10,15),(3,7,12),(3,7,13),\\
&\quad (3,7,14),(3,7,15),(3,8,12),(3,8,13),(3,8,14),(3,8,15),\\
&\quad (3,9,12),(3,9,13),(3,9,14),(3,9,15),(3,10,12),(3,10,13),\\
&\quad (3,10,14),(3,10,15),(4,7,12),(4,7,13),(4,7,14),(4,7,15),\\
&\quad (4,8,12),(4,8,13),(4,8,14),(4,8,15),(4,9,12),(4,9,13),\\
&\quad (4,9,14),(4,9,15),(4,10,12),(4,10,13),(4,10,14),(4,10,15),\\
&\quad (5,7,12),(5,7,13),(5,7,14),(5,7,15),(5,8,12),(5,8,13),\\
&\quad (5,8,14),(5,8,15),(5,9,12),(5,9,13),(5,9,14),(5,9,15),\\
&\quad (5,10,12),(5,10,13),(5,10,14),(5,10,15)\}.
\end{aligned} \tag{3.29}$$

$\therefore |\Delta_7| = 64.$

It follows that the subdegrees of G are 1, 4, 4, 4, 16, 16, 16 and 64. The number of elements in all the eight $H-$orbits is $1+4+4+4+16+16+16+64 = 125 = |X \times Y \times Z|$. Thus the rank is 8. □

Example 3.2.3. *Let $G = G_1 \times G_2 \times G_3 = A_6 \times A_6 \times A_6$ act on $X \times Y \times Z$. Then the rank is* 8 *and the subdegrees are* 1, 5, 5, 5, 25, 25, 25 *and* 125.

Proof: Let $H_1 = Stab_{G_1}(1)$, $H_2 = Stab_{G_2}(7)$, $H_3 = Stab_{G_3}(13)$ act on X, Y and Z respectively. Also, Let $H = Stab_G(1,7,13)$. Then by Proposition 3.1.2, $H = H_1 \times H_2 \times H_3$. It follows from Proposition 3.1.1, that, H_1, H_2 and H_3 are isomorphic to alternating groups A_5 on the sets $X' = \{2,3,4,5,6\}$, $Y' = \{8,9,10,11,12\}$ and $Z' = \{14,15,16,17,18\}$ respectively.

Let H_1,H_2 and H_3 act on $\{1,2,3,4,5,6\}$,$\{7,8,9,10,11,12\}$ and $\{13,14,15,16,17,18\}$ respectively. Then $Orb_{H_1}(1) = \{1\}$, $Orb_{H_1}(2) = \{2,3,4,5,6\}$, $Orb_{H_2}(7) = \{7\}$, $Orb_{H_2}(8) = \{8,9,10,11,12\}$, $Orb_{H_3}(13) = \{13\}$ and $Orb_{H_3}(14) = \{14,15,16,17,18\}$. Applying Theorem 1.1.1, we get:

$$\begin{aligned} \Delta_0 &= Orb_H(1,7,13) = Orb_{H_1}(1) \times Orb_{H_2}(7) \times Orb_{H_3}(13) \\ &= \{1\} \times \{7\} \times \{13\} = \{(1,7,13)\}. \end{aligned} \tag{3.30}$$

$\therefore |\Delta_0| = 1.$

$$\begin{aligned} \Delta_1 &= Orb_H(2,7,13) = Orb_{H_1}(2) \times Orb_{H_2}(7) \times Orb_{H_3}(13) \\ &= \{2,3,4,5,6\} \times \{7\} \times \{13\} \\ &= \{(x,7,13) : x \in X'\}. \end{aligned} \tag{3.31}$$

$\therefore |\Delta_1| = 5.$

$$\begin{aligned} \Delta_2 &= Orb_H(1,8,13) = Orb_{H_1}(1) \times Orb_{H_2}(8) \times Orb_{H_3}(13) \\ &= \{1\} \times \{8,9,10,11,12\} \times \{13\} \\ &= \{(1,y,13) : y \in Y'\}. \end{aligned} \tag{3.32}$$

$\therefore |\Delta_2| = 5.$

$$\begin{aligned} \Delta_3 &= Orb_H(1,7,14) = Orb_{H_1}(1) \times Orb_{H_2}(7) \times Orb_{H_3}(14) \\ &= \{1\} \times \{7\} \times \{14,15,16,17,18\} \\ &= \{(1,7,z) : z \in Z'\}. \end{aligned} \tag{3.33}$$

$\therefore |\Delta_3| = 5.$

$$\begin{aligned} \Delta_4 &= Orb_H(2,8,13) = Orb_{H_1}(2) \times Orb_{H_2}(8) \times Orb_{H_3}(13) \\ &= \{2,3,4,5,6\} \times \{8,9,10,11,12\} \times \{13\} \\ &= \{(x,y,13) : x \in X', y \in Y'\}. \end{aligned} \tag{3.34}$$

$\therefore |\Delta_4| = 25.$

$$\begin{aligned}\Delta_5 &= Orb_H(2,7,14) = Orb_{H_1}(2) \times Orb_{H_2}(7) \times Orb_{H_3}(14)\\ &= \{2,3,4,5,6\} \times \{7\} \times \{14,15,16,17,18\}\\ &= \{(x,7,z) : x \in X', z \in Z'\}.\end{aligned} \tag{3.35}$$

$\therefore |\Delta_5| = 25.$

$$\begin{aligned}\Delta_6 &= Orb_H(1,8,14) = Orb_{H_1}(1) \times Orb_{H_2}(8) \times Orb_{H_3}(14)\\ &= \{1\} \times \{8,9,10,11,12\} \times \{14,15,16,17,18\}\\ &= \{(1,y,z) : y \in Y', z \in Z'\}.\end{aligned} \tag{3.36}$$

$\therefore |\Delta_6| = 25.$

$$\begin{aligned}\Delta_7 &= Orb_H(2,8,14) = Orb_{H_1}(2) \times Orb_{H_2}(8) \times Orb_{H_3}(14)\\ &= \{2,3,4,5,6\} \times \{8,9,10,11,12\} \times \{14,15,16,17,18\}\\ &= \{(x,y,z) : x \in X', y \in Y', z \in Z'\}.\end{aligned} \tag{3.37}$$

$\therefore |\Delta_7| = 125$

Therefore the subdegrees of G are $1, 5, 5, 5, 25, 25, 25, 125$. The number of elements in all the eight $H-$orbits is $1+5+5+5+25+25+25+125 = 216 = |X \times Y \times Z|$. Thus the rank is 8. □

Example 3.2.4. *Let $G = G_1 \times G_2 \times G_3 = A_7 \times A_7 \times A_7$ act on $X \times Y \times Z$. Then the rank is* 8 *and the subdegrees are* 1, 6, 6, 6, 36, 36, 36 *and* 216.

Proof: Let $H_1 = Stab_{G_1}(1)$, $H_2 = Stab_{G_2}(8)$, $H_3 = Stab_{G_3}(15)$ and $H = Stab_G(1,8,15)$. Then by Proposition 3.1.2, $H = H_1 \times H_2 \times H_3$. By Proposition 3.1.1, H_1, H_2 and H_3 are isomorphic to alternating groups A_6 on the sets $X' = \{2,3,4,5,6,7\}$, $Y' = \{9,10,11,12,13,14\}$ and $Z' = \{16,17,18,19,20,21\}$ respectively. Suppose H_1 acts on $X = \{1,2,3,4,5,6,7\}$, then $Orb_{H_1}(1) = \{1\}$ and $Orb_{H_1}(2) = \{2,3,4,5,6,7\}$. Also let H_2 act on $Y = \{8,9,10,11,12,13,14\}$. Then $Orb_{H_2}(8) = \{8\}$ and $Orb_{H_2}(9) = \{9,10,11,12,13,14\}$. Similarly, let H_3

act on $Z = \{15, 16, 17, 18, 19, 20, 21\}$, then $Orb_{H_3}(15) = \{15\}$ and $Orb_{H_3}(16) = \{16, 17, 18, 19, 20, 21\}$. Now, by Theorem 1.1.1, we have,

$$\begin{aligned}\Delta_0 &= Orb_H(1,8,15) = Orb_{H_1}(1) \times Orb_{H_2}(8) \times Orb_{H_3}(15)\\ &= \{1\} \times \{8\} \times \{15\} \\ &= \{(1,8,15)\}.\end{aligned} \tag{3.38}$$

Therefore $|\Delta_0| = 1$.

$$\begin{aligned}\Delta_1 &= Orb_H(2,8,15) = Orb_{H_1}(2) \times Orb_{H_2}(8) \times Orb_{H_3}(15)\\ &= X' \times \{8\} \times \{15\} \\ &= \{(x,8,15) : x \in X'\}.\end{aligned} \tag{3.39}$$

Therefore $|\Delta_1| = 6$.

$$\begin{aligned}\Delta_2 &= Orb_H(1,9,15) = Orb_{H_1}(1) \times Orb_{H_2}(9) \times Orb_{H_3}(15)\\ &= \{1\} \times Y' \times \{15\} \\ &= \{(1,y,15) : y \in Y'\}\end{aligned} \tag{3.40}$$

Therefore, $|\Delta_2| = 6$.

$$\begin{aligned}\Delta_3 &= Orb_H(1,8,16) = Orb_{H_1}(1) \times Orb_{H_2}(8) \times Orb_{H_3}(16)\\ &= \{1\} \times \{8\} \times Z' \\ &= \{(1,8,z) : z \in Z'\}.\end{aligned} \tag{3.41}$$

Therefore, $|\Delta_3| = 6$.

$$\begin{aligned}\Delta_4 &= Orb_H(2,9,15) = Orb_{H_1}(2) \times Orb_{H_2}(9) \times Orb_{H_3}(15)\\ &= X' \times Y' \times \{15\} \\ &= \{(x,y,15) : x \in X', y \in Y'\}.\end{aligned} \tag{3.42}$$

Therefore, $|\Delta_4| = 36$.

$$\begin{aligned}\Delta_5 &= Orb_H(2,8,16) = Orb_{H_1}(2) \times Orb_{H_2}(8) \times Orb_{H_3}(16) \\ &= X' \times \{8\} \times Z' \\ &= \{(x,8,z) : x \in X', z \in Z'\}.\end{aligned} \tag{3.43}$$

Therefore, $|\Delta_5| = 36$.

$$\begin{aligned}\Delta_6 &= Orb_H(1,9,16) = Orb_{H_1}(1) \times Orb_{H_2}(9) \times Orb_{H_3}(16) \\ &= \{1\} \times Y' \times Z' \\ &= \{(1,y,z) : y \in Y', z \in Z'\}.\end{aligned} \tag{3.44}$$

Therefore, $|\Delta_6| = 36$.

$$\begin{aligned}\Delta_7 &= Orb_H(2,9,16) = Orb_{H_1}(2) \times Orb_{H_2}(9) \times Orb_{H_3}(16) \\ &= X' \times Y' \times Z'.\end{aligned} \tag{3.45}$$

Therefore, $|\Delta_7| = 216$.

It follows that, the subdegrees of G are 1, 6, 6, 6, 36, 36, 36 and 216. The number of elements in all the eight $H-$orbits is $1+6+6+6+36+36+36+216 = 343 = |X \times Y \times Z|$. Thus the rank is 8. □

Example 3.2.5. *Let $G = G_1 \times G_2 \times G_3 = A_8 \times A_8 \times A_8$ act on $X \times Y \times Z$. Then the rank is 8 and the subdegrees are 1, 7, 7, 7, 49, 49, 49 and 343.*

Proof: Let $H_1 = Stab_{G_1}(1)$, $H_2 = Stab_{G_2}(9)$, $H_3 = Stab_{G_3}(17)$ and $H = Stab_G(1,9,17)$. Then by Proposition 3.1.2, $H = H_1 \times H_2 \times H_3$. By Proposition 3.1.1, H_1, H_2 and H_3 are isomorphic to alternating groups A_7 on the sets $X' = \{2,3,4,5,6,7,8\}$, $Y' = \{10,11,12,13,14,15,16\}$ and $Z' = \{18,19,20,21,22,23,24\}$ respectively. Suppose H_1 acts on $X = \{1,2,3,4,5,6,7,8\}$, then $Orb_{H_1}(1) = \{1\}$ and $Orb_{H_1}(2) = \{2,3,4,5,6,7,8\}$. Also let H_2 act on $Y = \{9,10,11,12,13,14,15,16\}$, then $Orb_{H_2}(9) = \{9\}$ and $Orb_{H_2}(10) = \{10,11,12,13,14,15,16\}$. Similarly, let H_3 act on $Z = \{17,18,19,20,21,22,23,24\}$, then $Orb_{H_3}(17) = \{17\}$ and $Orb_{H_3}(18) =$

$\{18, 19, 20, 21, 22, 23, 24\}$. By Theorem 1.1.1, we have:

$$\begin{aligned}\Delta_0 &= Orb_H(1,9,17) = Orb_{H_1}(1) \times Orb_{H_2}(9) \times Orb_{H_3}(17) \\ &= \{1\} \times \{9\} \times \{17\} \\ &= \{(1,9,17)\}.\end{aligned} \tag{3.46}$$

Therefore, $|\Delta_0| = 1$.

$$\begin{aligned}\Delta_1 &= Orb_H(2,9,17) = Orb_{H_1}(2) \times Orb_{H_2}(9) \times Orb_{H_3}(17) \\ &= X' \times \{9\} \times \{17\} \\ &= \{(x,9,17) : x \in X'\}.\end{aligned} \tag{3.47}$$

Therefore, $|\Delta_1| = 7$.

$$\begin{aligned}\Delta_2 &= Orb_H(1,10,17) = Orb_{H_1}(1) \times Orb_{H_2}(10) \times Orb_{H_3}(17) \\ &= \{1\} \times Y\prime \times \{17\} \\ &= \{(1,y,17) : y \in Y'\}.\end{aligned} \tag{3.48}$$

Therefore, $|\Delta_2| = 7$.

$$\begin{aligned}\Delta_3 &= Orb_H(1,8,16) = Orb_{H_1}(1) \times Orb_{H_2}(8) \times Orb_{H_3}(16) \\ &= \{1\} \times \{9\} \times Z' \\ &= \{(1,9,z) : z \in Z'\}.\end{aligned} \tag{3.49}$$

Therefore, $|\Delta_3| = 7$.

$$\begin{aligned}\Delta_4 &= Orb_H(2,10,17) = Orb_{H_1}(2) \times Orb_{H_2}(10) \times Orb_{H_3}(17) \\ &= X' \times Y' \times \{17\} \\ &= \{(x,y,17) : x \in X', y \in Y'\}.\end{aligned} \tag{3.50}$$

Therefore, $|\Delta_4| = 49$.

$$\begin{aligned}\Delta_5 &= Orb_H(2,9,18) = Orb_{H_1}(2) \times Orb_{H_2}(9) \times Orb_{H_3}(18)\\ &= X' \times \{9\} \times Z' \\ &= \{(x,9,z) : x \in X', z \in Z'\}.\end{aligned} \tag{3.51}$$

Therefore, $|\Delta_5| = 49$.

$$\begin{aligned}\Delta_6 &= Orb_H(1,10,18) = Orb_{H_1}(1) \times Orb_{H_2}(10) \times Orb_{H_3}(18)\\ &= \{1\} \times Y' \times Z' \\ &= \{(1,y,z) : y \in Y', z \in Z'\}.\end{aligned} \tag{3.52}$$

Therefore, $|\Delta_6| = 49$.

$$\begin{aligned}\Delta_7 &= Orb_H(2,10,18) = Orb_{H_1}(2) \times Orb_{H_2}(10) \times Orb_{H_3}(18)\\ &= X' \times Y' \times Z'.\end{aligned} \tag{3.53}$$

Therefore, $|\Delta_7| = 343$.

It follows that, the subdegrees of G are 1, 7, 7, 7, 49, 49, 49 and 343. The number of elements in all the eight $H-$orbits is $1+7+7+7+49+49+49+343 = 512 = |X \times Y \times Z|$. Thus the rank is 8. □

Theorem 3.2.6. *Let $G = G_1 \times G_2 \times G_3 = A_n \times A_n \times A_n$ act on $X \times Y \times Z$. Then the rank is 8 and the subdegrees are 1, $n-1$, $n-1$, $n-1$, $(n-1)^2$, $(n-1)^2$, $(n-1)^2$ and $(n-1)^3$.*

Proof: Let $H_1 = Stab_{G_1}(1)$, $H_2 = Stab_{G_2}(n+1)$, $H_3 = Stab_{G_3}(2n+1)$ and $H = Stab_G(1, n+1, 2n+1)$. Then, by Proposition 3.1.2, $H = H_1 \times H_2 \times H_3$. By Proposition 3.1.1, H_1, H_2 and H_3 are isomorphic to alternating groups A_{n-1} on the sets $X' = \{2,3,\cdots,n\}$, $Y' = \{n+2, n+3, \cdots, 2n\}$ and $Z' = \{2n+2, 2n+3, \cdots, 3n\}$ respectively. Suppose H_1 acts on $X = \{1,2,\cdots,n\}$, then $Orb_{H_1}(1) = \{1\}$ and $Orb_{H_1}(2) = \{2,3,\cdots,n\}$. Also let H_2 act on $Y = \{n+1, n+2, \cdots, 2n\}$, then

$Orb_{H_2}(n+1) = \{n+1\}$ and $Orb_{H_2}(n+2) = \{n+2, n+3, \cdots, 2n\}$. Similarly, let H_3 act on $Z = \{2n+1, 2n+2, \cdots, 3n\}$. Then, $Orb_{H_3}(2n+1) = \{2n+1\}$ and $Orb_{H_3}(2n+2) = \{2n+2, 2n+3, \cdots, 3n\}$. It follows from Theorem 1.1.1, that:

$$\begin{aligned} \Delta_0 &= Orb_H(1, n+1, 2n+1) = Orb_{H_1}(1) \times Orb_{H_2}(n+1) \times Orb_{H_3}(2n+1) \\ &= \{(1, n+1, 2n+1)\}. \\ \Rightarrow |\Delta_0| &= 1. \end{aligned} \tag{3.54}$$

$$\begin{aligned} \Delta_1 &= Orb_H(2, n+1, 2n+1) = Orb_{H_1}(2) \times Orb_{H_2}(n+1) \times Orb_{H_3}(2n+1). \\ &= \{2, 3, \cdots, n\} \times \{n+1\} \times \{2n+1\}. \\ \Rightarrow |\Delta_1| &= |\{2, 3, \cdots, n\}|\,|\{n+1\}|\,|\{2n+1\}| = (n-1). \end{aligned} \tag{3.55}$$

$$\begin{aligned} \Delta_2 &= Orb_H(1, n+2, 2n+1) = Orb_{H_1}(1) \times Orb_{H_2}(n+2) \times Orb_{H_3}(2n+1) \\ &= \{1\} \times \{n+2, n+3, \cdots, 2n\} \times \{2n+1\}. \\ \Rightarrow |\Delta_2| &= |\{1\}|\,|\{n+2, n+3, \cdots, 2n\}|\,|\{2n+1\}| = (n-1). \end{aligned} \tag{3.56}$$

$$\begin{aligned} \Delta_3 &= Orb_H(1, n+1, 2n+2) = Orb_{H_1}(1) \times Orb_{H_2}(n+1) \times Orb_{H_3}(2n+2) \\ &= \{1\} \times \{n+1\} \times \{2n+2, 2n+3, \cdots, 3n\}. \\ \Rightarrow |\Delta_3| &= |\{1\}|\,|\{n+1\}|\,|\{2n+2, 2n+3, \cdots, 3n\}| = (n-1). \end{aligned} \tag{3.57}$$

$$\begin{aligned} \Delta_4 &= Orb_H(2, n+2, 2n+1) = Orb_{H_1}(2) \times Orb_{H_2}(n+2) \times Orb_{H_3}(2n+1) \\ &= \{2, 3, \cdots, n\} \times \{n+2, n+3, \cdots, 2n\} \times \{2n+1\}. \\ \Rightarrow |\Delta_4| &= |\{2, 3, \cdots, n\}|\,|\{n+2, n+3, \cdots, 2n\}|\,|\{2n+1\}| = (n-1)^2. \end{aligned} \tag{3.58}$$

$$\Delta_5 = Orb_H(2, n+1, 2n+2) = Orb_{H_1}(2) \times Orb_{H_2}(n+1) \times Orb_{H_3}(2n+2)$$
$$= \{2, 3, \cdots, n\} \times \{n+1\} \times \{2n+2, 2n+3, \cdots, 3n\}.$$
$$\Rightarrow |\Delta_5| = |\{2, 3, \cdots, n\}|\,|\{n+1\}|\,|\{2n+2, 2n+3, \cdots, 3n\}| = (n-1)^2. \tag{3.59}$$

$$\Delta_6 = Orb_H(1, n+2, 2n+2) = Orb_{H_1}(1) \times Orb_{H_2}(n+2) \times Orb_{H_3}(2n+2)$$
$$= \{1\} \times \{n+2, n+3, \cdots, 2n\} \times \{2n+2, 2n+3, \cdots, 3n\}.$$
$$\Rightarrow |\Delta_6| = |\{1\}|\,|\{n+2, n+3, \cdots, 2n\}|\,|\{2n+2, 2n+3, \cdots, 3n\}|$$
$$= (n-1)^2. \tag{3.60}$$

$$\Delta_7 = Orb_H(2, n+2, 2n+2) = Orb_{H_1}(2) \times Orb_{H_2}(n+2) \times Orb_{H_3}(2n+2)$$
$$= \{2, 3, \cdots, n\} \times \{n+2, n+3, \cdots, 2n\} \times \{2n+2, 2n+3, \cdots, 3n\}.$$
$$\Rightarrow |\Delta_7| = |\{2, 3, \cdots, n\}|\,|\{n+2, n+3, \cdots, 2n\}|\,|\{2n+2, 2n+3, \cdots, 3n\}|$$
$$= (n-1)^3. \tag{3.61}$$

Therefore the subdegrees of G are 1, $(n-1)$, $(n-1)$, $(n-1)$, $(n-1)^2, (n-1)^2$, $(n-1)^2$ and $(n-1)^3$. The number of elements in all the eight $H-$orbits is $1+(n-1)+(n-1)+(n-1)+(n-1)^2+(n-1)^2+(n-1)^2+(n-1)^3 = |X \times Y \times Z| = n^3$. Thus, the rank is 8. $\square$

CHAPTER 4

SUBORBITAL GRAPHS OF $A_n \times A_n \times A_n$ ACTING ON $X \times Y \times Z$

Suppose $Stab_{G_1\times G_2\times G_3}(x,y,z)$ acts on $X\times Y\times Z$ and Δ is an orbit of $Stab_{G_1\times G_2\times G_3}(x,y,z)$ on $X \times Y \times Z$. Then the suborbital O corresponding to Δ is given by,

$$\begin{aligned} O &= \{((g_1,g_2,g_3)(x,y,z),(g_1,g_2,g_3)(u,v,w) : (u,v,w) \in \Delta, \\ &\quad (g_1,g_2,g_3) \in G_1 \times G_2 \times G_3\} \\ &= \{(g_1(x),g_2(y),g_3(z)),(g_1(u),g_2(v),g_3(w)) : (u,v,w) \in \Delta, \\ &\quad (g_1,g_2,g_3) \in G_1 \times G_2 \times G_3\}. \end{aligned} \tag{4.1}$$

Suborbital graph Γ corresponding to suborbital O is formed by taking $X \times Y \times Z$ as the vertex set and a directed edge from (m,n,o) to (p,q,r) if and only if $((m,n,o),(p,q,r)) \in O$. The suborbital graph corresponding to Δ_0 is the null graph since the edge set is empty.

4.1 Suborbital graphs of $A_4 \times A_4 \times A_4$ acting on $X \times Y \times Z$

By Equation (3.15), $(2,5,9) \in \Delta_1$. Therefore suborbital O_1 corresponding to suborbit Δ_1 is given by,

$$\begin{aligned} O_1 &= \{((g_1,g_2,g_3)(1,5,9),(g_1,g_2,g_3)(2,5,9)) : (g_1,g_2,g_3) \in G_1 \times G_2 \times G_3\} \\ &= \{(g_1(1),g_2(5),g_3(9)),(g_1(2),g_2(5),g_3(9)) : (g_1,g_2,g_3) \in A_4 \times A_4 \times A_4\} \\ &= \{((m,n,o),(p,q,r)) : (m,n,o),(p,q,r) \in X \times Y \times Z,\ m \neq p,\ n = q,\ o = r\}. \end{aligned} \tag{4.2}$$

Using Equation (4.2), we obtain Figure 4.1.

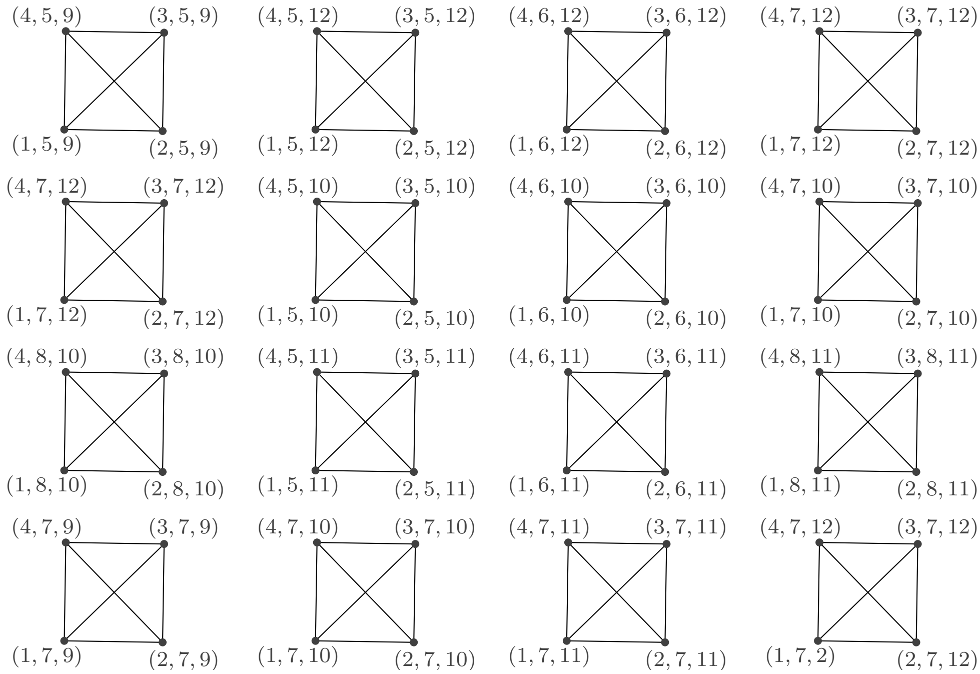

Figure 4.1: Suborbital graph Γ_1 corresponding to the action of $A_4 \times A_4 \times A_4$ on $X \times Y \times Z$

The Suborbital graph Γ_1 is undirected, regular of degree 3, has a girth of 3 and it is disconnected. It has 16 connected components and each connected component is complete.

From Equation (3.16), $(1, 6, 9) \in \Delta_2$. Then the suborbital O_2 corresponding to suborbit Δ_2 is given by,

$$\begin{aligned} O_2 &= \{((g_1, g_2, g_3)(1, 5, 9), (g_1, g_2, g_3)(1, 6, 9)) : (g_1, g_2, g_3) \in G_1 \times G_2 \times G_3\} \\ &= \{(g_1(1), g_2(5), g_3(9)), (g_1(1), g_2(6), g_3(9)) : (g_1, g_2, g_3) \in A_4 \times A_4 \times A_4\} \\ &= \{((m, n, o), (p, q, r)) : (m, n, o), (p, q, r) \in X \times Y \times Z,\ m = p,\ n \neq q,\ o = r\}. \end{aligned} \tag{4.3}$$

Applying Equation (4.3), we obtain Figure 4.2.

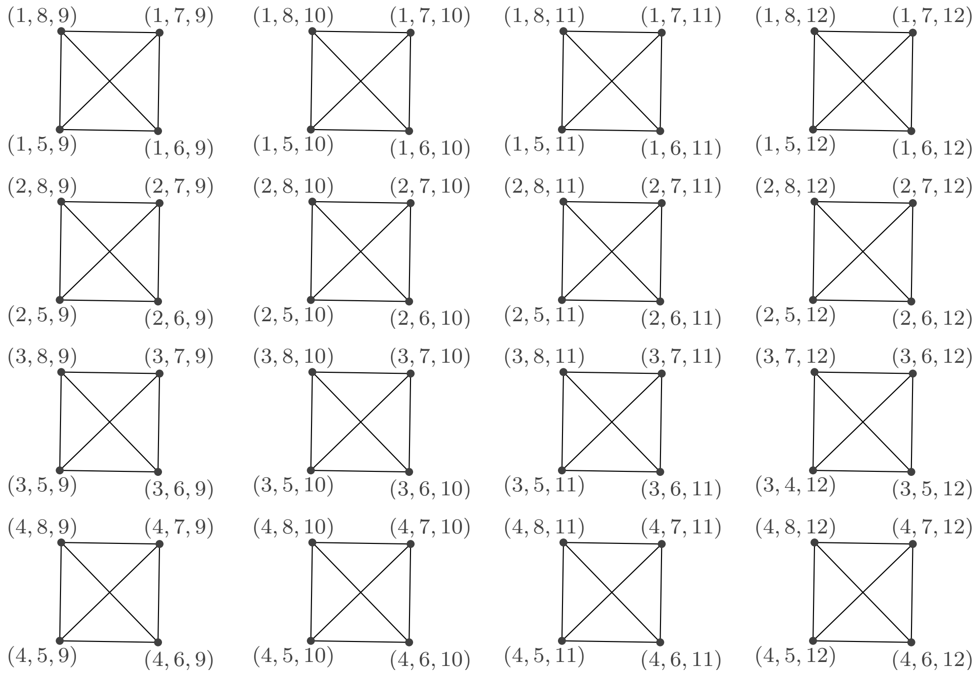

Figure 4.2: Suborbital graph $\boldsymbol{\Gamma_2}$ corresponding to the action of $\boldsymbol{A_4 \times A_4 \times A_4}$ on $\boldsymbol{X \times Y \times Z}$

The Suborbital graph Γ_2 is undirected, regular of degree 3, has a girth of 3 and it is disconnected. It has 16 connected components and each connected component is complete.

From Equation (3.17), $(1, 5, 10) \in \Delta_3$. Then the suborbital O_3 corresponding to suborbit Δ_3 is given by,

$$\begin{aligned} O_3 &= \{((g_1, g_2, g_3)(1, 5, 9), (g_1, g_2, g_3)(1, 5, 10)) : (g_1, g_2, g_3) \in G_1 \times G_2 \times G_3\} \\ &= \{(g_1(1), g_2(5), g_3(9)), (g_1(1), g_2(5), g_3(10)) : (g_1, g_2, g_3) \in A_4 \times A_4 \times A_4\} \\ &= \{((m, n, o), (p, q, r)) : (m, n, o), (p, q, r) \in X \times Y \times Z,\ m = p, n = q,\ o \neq r\}. \end{aligned} \tag{4.4}$$

Applying Equation (4.4), we obtain Figure 4.3.

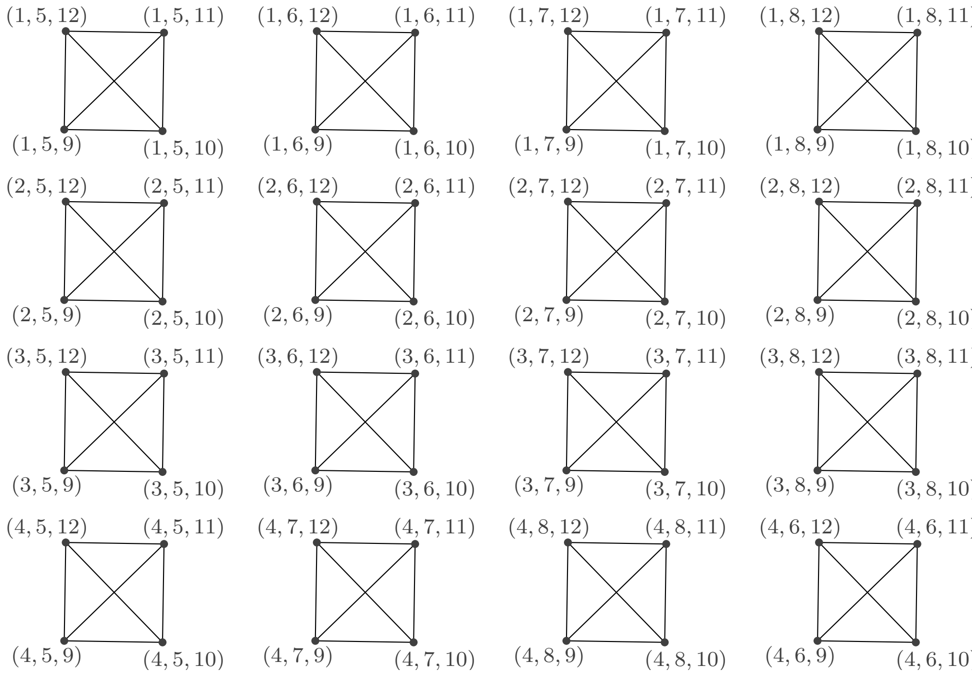

Figure 4.3: Suborbital graph Γ_3 corresponding to the action of $A_4 \times A_4 \times A_4$ on $X \times Y \times Z$

The Suborbital graph Γ_3 is undirected, regular of degree 3, has a girth of 3 and it is disconnected. It has 16 connected components and each connected component is complete.

From Equation (3.18), $(2, 7, 9) \in \Delta_4$. Therefore the suborbital O_4 corresponding to suborbit Δ_4 is given by,

$$\begin{aligned} O_4 &= \{((g_1, g_2, g_3)(1, 5, 9), (g_1, g_2, g_3)(2, 7, 9)) : (g_1, g_2, g_3) \in G_1 \times G_2 \times G_3\} \\ &= \{(g_1(1), g_2(5), g_3(9)), (g_1(2), g_2(7), g_3(9)) : (g_1, g_2, g_3) \in A_4 \times A_4 \times A_4\} \\ &= \{((m, n, o), (p, q, r)) : (m, n, o), (p, q, r) \in X \times Y \times Z,\ m \neq p,\ n \neq q,\ o = r\}. \end{aligned} \tag{4.5}$$

We obtain Figure 4.4 through application of Equation (4.5).

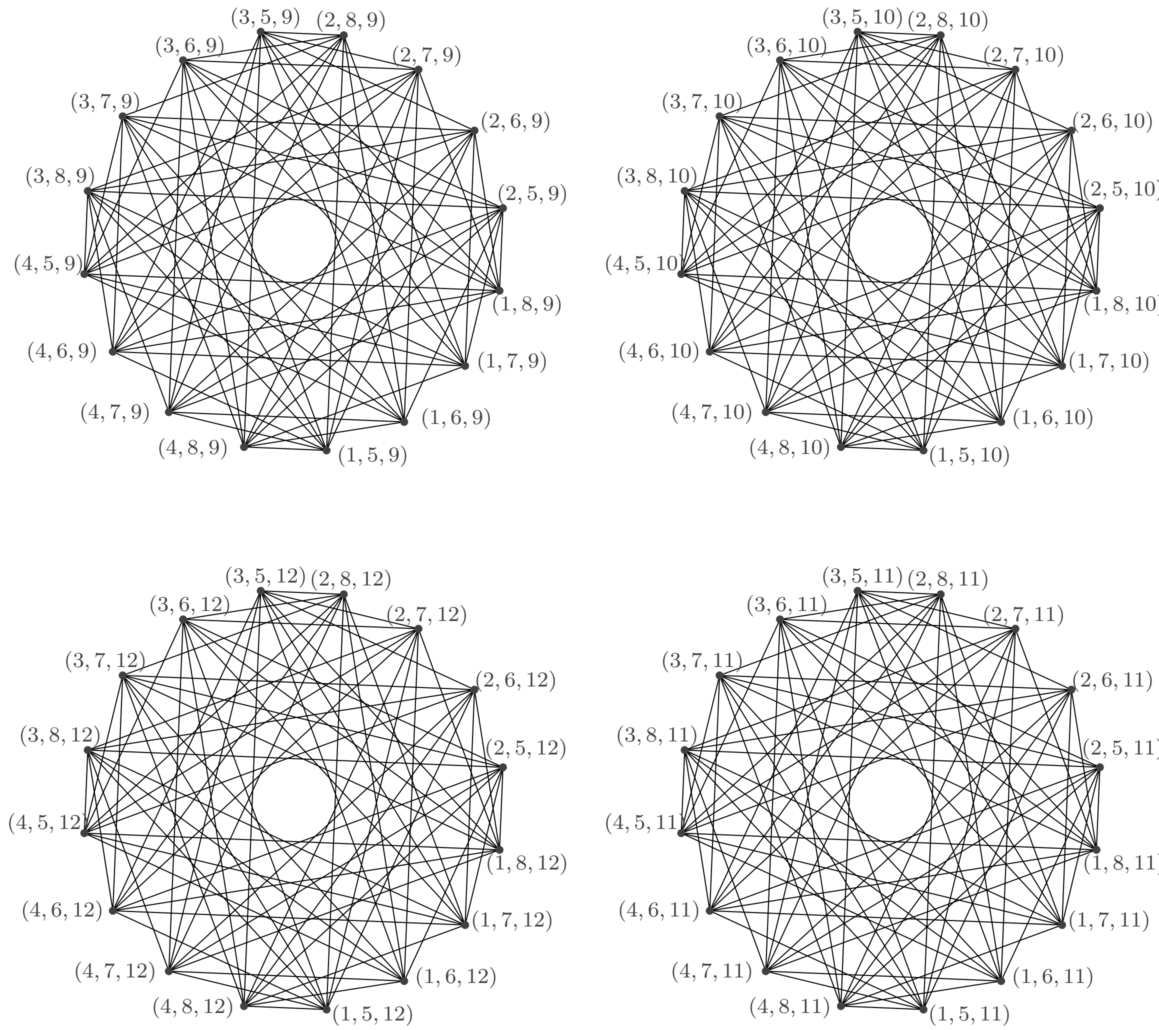

Figure 4.4: Suborbital graph Γ_4 corresponding to the action of $A_4 \times A_4 \times A_4$ on $X \times Y \times Z$

The Suborbital graph Γ_4 is undirected, regular of degree 9, has a girth of 3 and it is disconnected. It has 4 connected components.

From Equation (3.19), $(2, 5, 10) \in \Delta_5$. Therefore, suborbital O_5 corresponding to suborbit Δ_5 is given by,

$$
\begin{aligned}
O_5 &= \{((g_1, g_2, g_3)(1,5,9), (g_1, g_2, g_3)(2,5,10)) : (g_1, g_2, g_3) \in G_1 \times G_2 \times G_3\} \\
&= \{(g_1(1), g_2(5), g_3(9)), (g_1(2), g_2(5), g_3(10)) : (g_1, g_2, g_3) \in A_4 \times A_4 \times A_4\} \\
&= \{((m,n,o),(p,q,r)) : (m,n,o),(p,q,r) \in X \times Y \times Z,\ m \neq p,\ n = q,\ o \neq r\}.
\end{aligned}
\tag{4.6}
$$

Applying Equation (4.6), we obtain Figure 4.5.

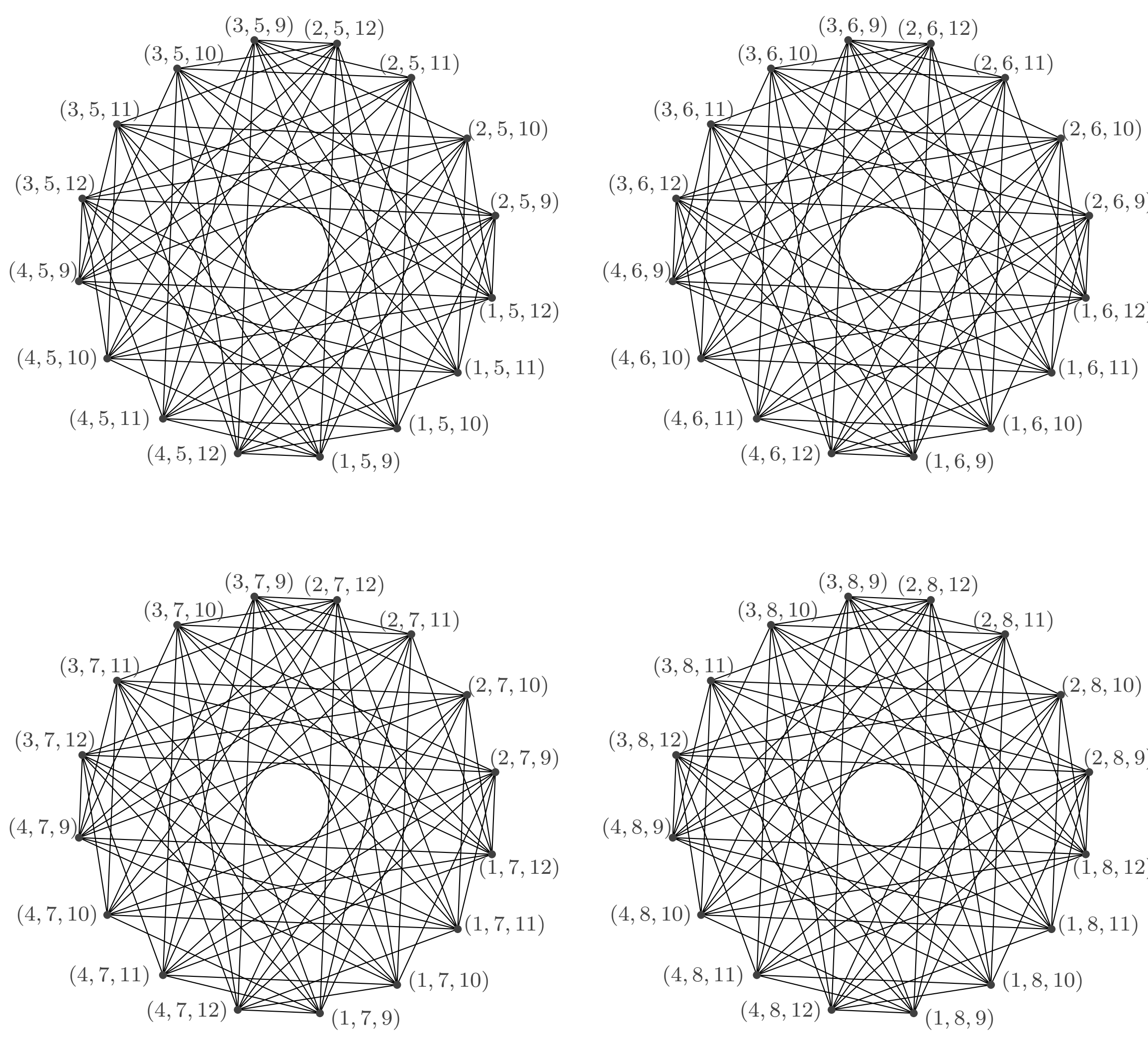

Figure 4.5: Suborbital graph Γ_5 corresponding to the action of $A_4 \times A_4 \times A_4$ on $X \times Y \times Z$

The Suborbital graph Γ_5 is undirected, regular of degree 9, has a girth of 3 and it is disconnected. It has 4 connected components.

From Equation (3.20), $(1, 6, 10) \in \Delta_6$. Therefore, the suborbital O_6 corresponding to suborbit Δ_6 is given by,

$$\begin{aligned} O_6 &= \{((g_1, g_2, g_3)(1, 5, 9), (g_1, g_2, g_3)(1, 6, 10)) : (g_1, g_2, g_3) \in G_1 \times G_2 \times G_3\} \\ &= \{(g_1(1), g_2(5), g_3(9)), (g_1(1), g_2(6), g_3(10)) : (g_1, g_2, g_3) \in A_4 \times A_4 \times A_4\} \\ &= \{((m, n, o), (p, q, r)) : (m, n, o), (p, q, r) \in X \times Y \times Z,\ m = p,\ n \neq q,\ o \neq r\}. \end{aligned} \tag{4.7}$$

Applying Equation (4.7), we obtain Figure 4.6.

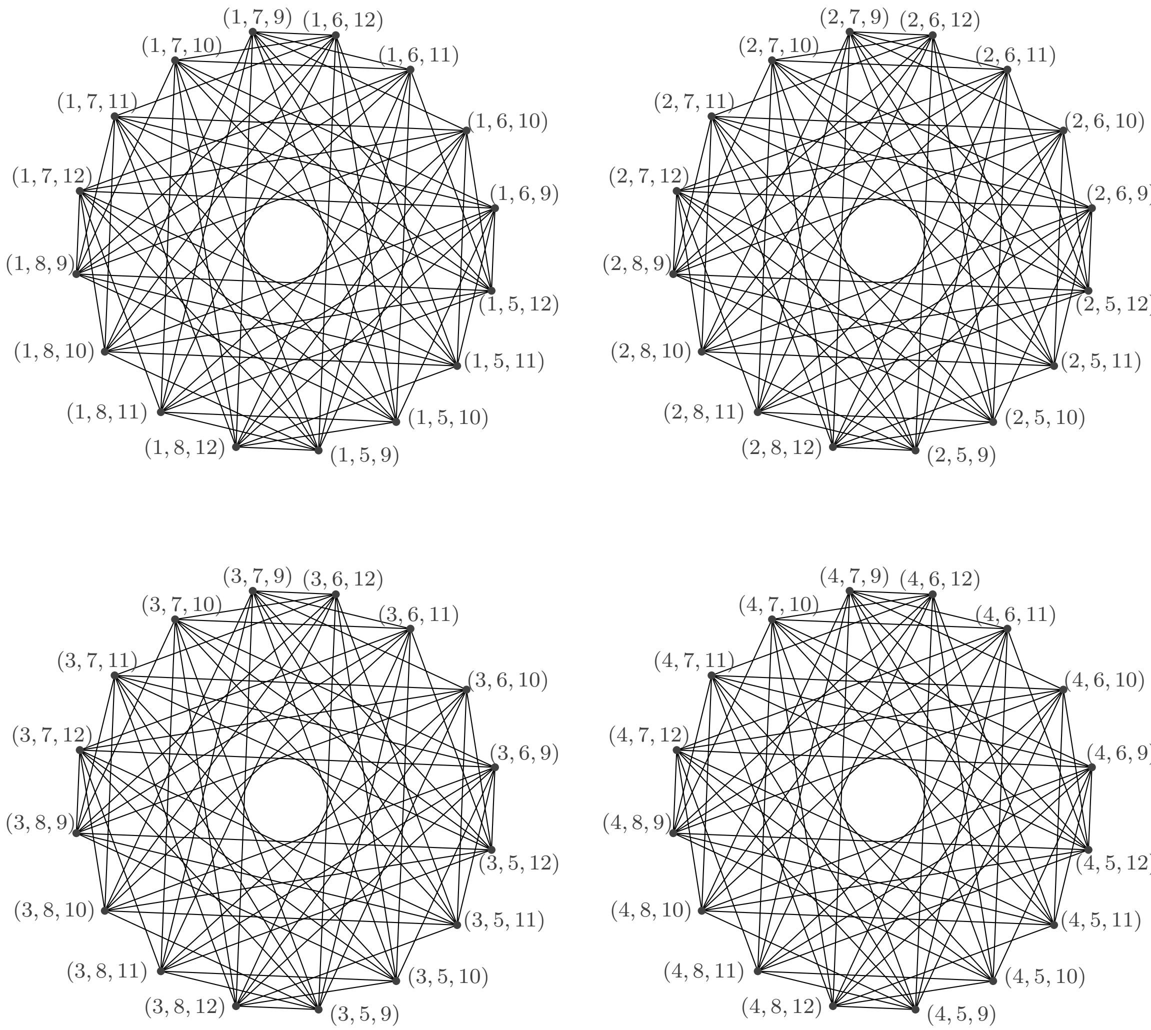

Figure 4.6: Suborbital graph Γ_6 corresponding to the action of $A_4 \times A_4 \times A_4$ on $X \times Y \times Z$

The Suborbital graph Γ_6 is undirected, regular of degree 9, has a girth of 3 and it is disconnected. It has 4 connected components.

From Equation (3.21), $(2, 6, 10) \in \Delta_7$. Therefore, the suborbital O_7 corresponding to suborbit Δ_7 is given by,

$$
\begin{aligned}
O_7 &= \{((g_1, g_2, g_3)(1,5,9), (g_1, g_2, g_3)(2,6,10)) : (g_1, g_2, g_3) \in G_1 \times G_2 \times G_3\} \\
&= \{(g_1(1), g_2(5), g_3(9)), (g_1(2), g_2(6), g_3(10)) : (g_1, g_2, g_3) \in A_4 \times A_4 \times A_4\} \\
&= \{((m, n, o), (p, q, r)) : (m, n, o), (p, q, r) \in X \times Y \times Z,\ m \neq p,\ n \neq q,\ o \neq r\}.
\end{aligned}
\tag{4.8}
$$

Using Equation (4.8), we obtain Figure 4.7.

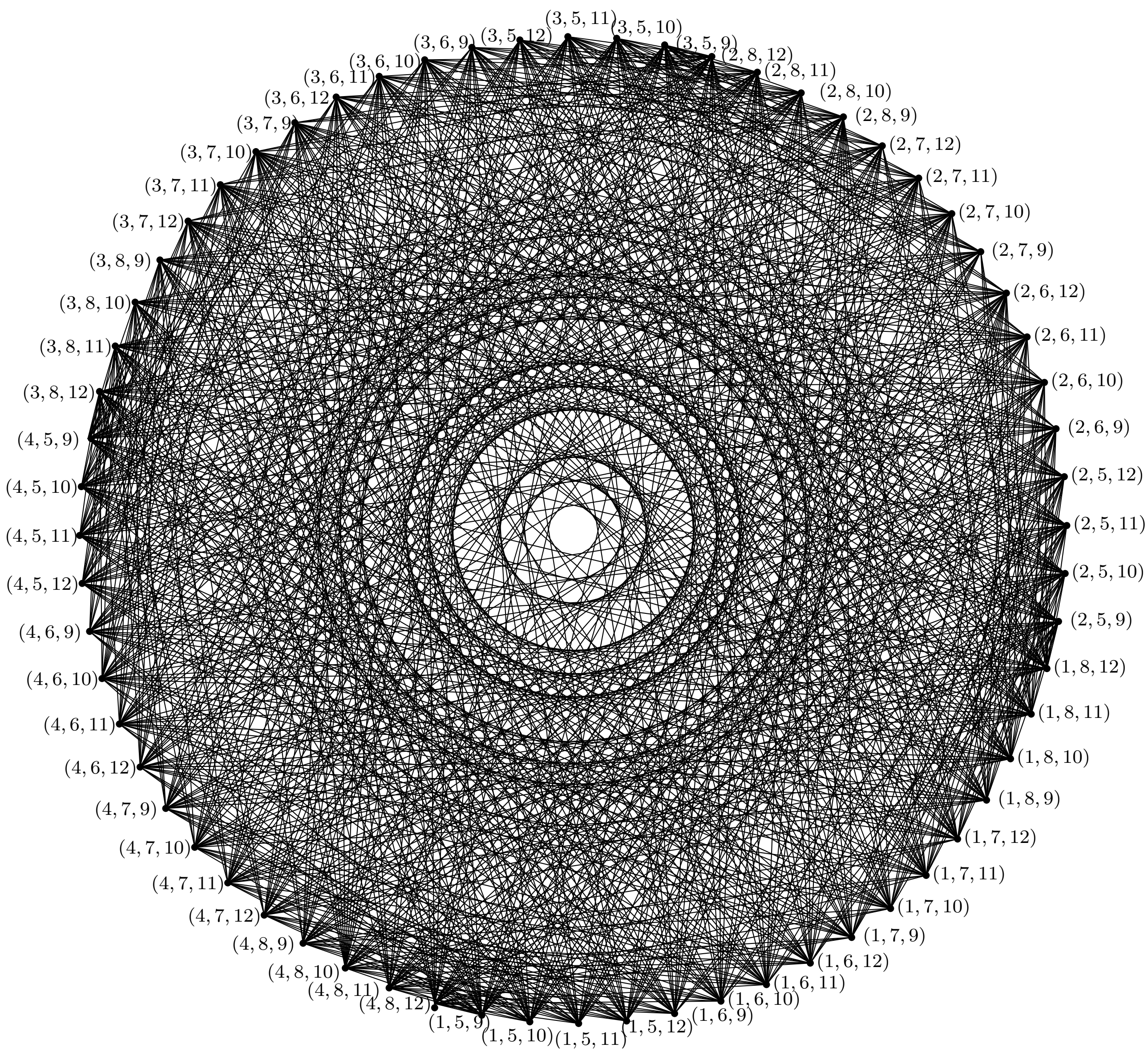

Figure 4.7: Suborbital graph Γ_7 corresponding to the action of $A_4 \times A_4 \times A_4$ on $X \times Y \times Z$

The suborbital graph Γ_7 is connected, regular of degree 27 and has girth of 3.

4.2 Suborbital graphs of the action of $A_n \times A_n \times A_n$ on $X \times Y \times Z$

From Equations (3.54) (3.55), (3.56), (3.57), (3.58), (3.59), (3.60) and (3.61), we have, $(2, n+1, 2n+1) \in \Delta_1$, $(1, n+2, 2n+1) \in \Delta_2$, $(1, n+1, 2n+2) \in \Delta_3$, $(2, n+2, 2n+1) \in \Delta_4$, $(2, n+1, 2n+2) \in \Delta_5$, $(1, n+2, 2n+2) \in \Delta_6$ and $(2, n+2, 2n+2) \in \Delta_7$. Applying Equation (4.1), we obtain,

$$\begin{aligned} O_0 &= \{((g_1, g_2, g_3)(1, n+1, 2n+1), (g_1, g_2, g_3)(1, n+1, 2n+1)) : \\ &\qquad (g_1, g_2, g_3) \in A_n \times A_n \times A_n\} \\ &= \{(g_1(1), g_2(n+1), g_3(2n+1)), (g_1(1), g_2(n+1), g_3(2n+1)) : \\ &\qquad (g_1, g_2, g_3) \in A_n \times A_n \times A_n\} \\ &= \{((m, n, o), (p, q, r)) : (m, n, o), (p, q, r) \in X \times Y \times Z,\ m = p, n = q,\ o = r\}. \end{aligned} \tag{4.9}$$

$$\begin{aligned} O_1 &= \{((g_1, g_2, g_3)(1, n+1, 2n+1), (g_1, g_2, g_3)(2, n+1, 2n+1)) : \\ &\qquad (g_1, g_2, g_3) \in A_n \times A_n \times A_n\} \\ &= \{(g_1(1), g_2(n+1), g_3(2n+1)), (g_1(2), g_2(n+1), g_3(2n+1)) : \\ &\qquad (g_1, g_2, g_3) \in A_n \times A_n \times A_n\} \\ &= \{((m, n, o), (p, q, r)) : (m, n, o), (p, q, r) \in X \times Y \times Z,\ m \neq p, n = q,\ o = r\}. \end{aligned} \tag{4.10}$$

$$\begin{aligned} O_2 &= \{((g_1, g_2, g_3)(1, n+1, 2n+1), (g_1, g_2, g_3)(1, n+2, 2n+1)) : \\ &\qquad (g_1, g_2, g_3) \in A_n \times A_n \times A_n\} \\ &= \{(g_1(1), g_2(n+1), g_3(2n+1)), (g_1(1), g_2(n+2), g_3(2n+1)) : \\ &\qquad (g_1, g_2, g_3) \in A_n \times A_n \times A_n\} \\ &= \{((m, n, o), (p, q, r)) : (m, n, o), (p, q, r) \in X \times Y \times Z,\ m = p, n \neq q,\ o = r\}. \end{aligned} \tag{4.11}$$

$$\begin{aligned}O_3 &= \{((g_1,g_2,g_3)(1,n+1,2n+1),(g_1,g_2,g_3)(1,n+1,2n+2)) : \\ &\quad (g_1,g_2,g_3) \in A_n \times A_n \times A_n\} \\ &= \{(g_1(1),g_2(n+1),g_3(2n+1)),(g_1(1),g_2(n+1),g_3(2n+2)) : \\ &\quad (g_1,g_2,g_3) \in A_4 \times A_4 \times A_4\} \\ &= \{((m,n,o),(p,q,r)) : (m,n,o),(p,q,r) \in X \times Y \times Z,\ m=p,\ n=q,\ o \neq r\}.\end{aligned} \tag{4.12}$$

$$\begin{aligned}O_4 &= \{((g_1,g_2,g_3)(1,n+1,2n+1),(g_1,g_2,g_3)(2,n+2,2n+1)) : \\ &\quad (g_1,g_2,g_3) \in A_n \times A_n \times A_n\} \\ &= \{(g_1(1),g_2(n+1),g_3(2n+1)),(g_1(2),g_2(n+2),g_3(2n+1)) : \\ &\quad (g_1,g_2,g_3) \in A_n \times A_n \times A_n\} \\ &= \{((m,n,o),(p,q,r)) : (m,n,o),(p,q,r) \in X \times Y \times Z,\ m \neq p,\ n \neq q,\ o = r\}.\end{aligned} \tag{4.13}$$

$$\begin{aligned}O_5 &= \{((g_1,g_2,g_3)(1,n+1,2n+1),(g_1,g_2,g_3)(2,n+1,2n+2)) : \\ &\quad (g_1,g_2,g_3) \in A_n \times A_n \times A_n\} \\ &= \{(g_1(1),g_2(n+1),g_3(2n+1)),(g_1(2),g_2(n+1),g_3(2n+2)) : \\ &\quad (g_1,g_2,g_3) \in A_n \times A_n \times A_n\} \\ &= \{((m,n,o),(p,q,r)) : (m,n,o),(p,q,r) \in X \times Y \times Z,\ m \neq p,\ n = q,\ o \neq r\}.\end{aligned} \tag{4.14}$$

$$\begin{aligned}O_6 &= \{((g_1,g_2,g_3)(1,n+1,2n+1),(g_1,g_2,g_3)(1,n+2,2n+2)) : \\ &\quad (g_1,g_2,g_3) \in A_n \times A_n \times A_n\} \\ &= \{(g_1(1),g_2(n+1),g_3(2n+1)),(g_1(1),g_2(n+2),g_3(2n+2)) : \\ &\quad (g_1,g_2,g_3) \in A_n \times A_n \times A_n\} \\ &= \{((m,n,o),(p,q,r)) : (m,n,o),(p,q,r) \in X \times Y \times Z,\ m = p,\ n \neq q,\ o \neq r\}.\end{aligned} \tag{4.15}$$

$$O_7 = \{((g_1, g_2, g_3)(1, n+1, 2n+1), (g_1, g_2, g_3)(2, n+2, 2n+2)) : (g_1, g_2, g_3) \in A_n \times A_n \times A_n\}$$
$$= \{(g_1(1), g_2(n+1), g_3(2n+1)), (g_1(2), g_2(n+2), g_3(2n+2)) : (g_1, g_2, g_3) \in A_n \times A_n \times A_n\}$$
$$= \{((m, n, o), (p, q, r)) : (m, n, o), (p, q, r) \in X \times Y \times Z,\ m \neq p,\ n \neq q,\ o \neq r\}. \tag{4.16}$$

Theorem 4.2.1. *All the suborbital graphs corresponding to this action are undirected.*

Proof: By Equations (4.9) to (4.16), if $((m, n, o), (p, q, r)) \in O_i$, $(i = 0, 1, \cdots, 7)$, then $((p, q, r), (m, n, o)) \in O_i$. □

Theorem 4.2.2. *All the non-trivial suborbital graphs have girth 3.*

Proof: It is enough to show that there exist a circuit of length 3 in each graph. By Equations (4.10) to (4.16), we have,

$(1, n+1, 2n+1) \to (3, n+1, 2n+1) \to (2, n+1, 2n+1) \to (1, n+1, 2n+1) \in \Gamma_1$,

$(1, n+1, 2n+1) \to (1, n+2, 2n+1) \to (1, n+3, 2n+1) \to (1, n+1, 2n+1) \in \Gamma_2$,

$(1, n+1, 2n+1) \to (1, n+1, 2n+2) \to (1, n+1, 2n+3) \to (1, n+1, 2n+1) \in \Gamma_3$,

$(1, n+1, 2n+1) \to (2, n+2, 2n+1) \to (3, n+3, 2n+1) \to (1, n+1, 2n+1) \in \Gamma_4$,

$(1, n+1, 2n+1) \to (2, n+1, 2n+2) \to (3, n+1, 2n+3) \to (1, n+1, 2n+1) \in \Gamma_5$,

$(1, n+1, 2n+1) \to (1, n+2, 2n+2) \to (1, n+3, 2n+3) \to (1, n+1, 2n+1) \in \Gamma_6$ and

$(1, n+1, 2n+1) \to (2, n+2, 2n+2) \to (3, n+3, 2n+3) \to (1, n+1, 2n+1) \in O_7$.

□

Theorem 4.2.3. *The suborbital graph Γ_7 is Eulerian if n is odd.*

Proof: If n is odd, each vertex of the graph has degree $(n-1)^3$ which is even. Therefore, by Theorem 1.2.21, Γ_7 is Eulerian. □

Lemma 4.2.4. *Let* $Comp_{\Gamma_i}(1, n+1, 2n+1)$, $(i = 1, 2, \cdots 7)$ *be the set of all vertices in the component containing* $(1, n+1, 2n+1)$ *in the suborbital graph* Γ_i. *Then,*

$$Comp_{\Gamma_1}(1, n+1, 2n+1) = \{(x, n+1, 2n+1) : x \in X\} \tag{4.17}$$

$$Comp_{\Gamma_2}(1, n+1, 2n+1) = \{(1, y, 2n+1) : y \in Y\}, \tag{4.18}$$

$$Comp_{\Gamma_3}(1, n+1, 2n+1) = \{(1, n+1, z) : z \in Z\}, \tag{4.19}$$

$$Comp_{\Gamma_4}(1, n+1, 2n+1) = \{(x, y, 2n+1) : x \in X,\ y \in Y\}, \tag{4.20}$$

$$Comp_{\Gamma_5}(1, n+1, 2n+1) = \{(x, n+1, z) : x \in X, z \in Z\}, \tag{4.21}$$

$$Comp_{\Gamma_6}(1, n+1, 2n+1) = \{(1, y, z) : y \in Y,\ z \in Z\} \tag{4.22}$$

and

$$Comp_{\Gamma_7}(1, n+1, 2n+1) = X \times Y \times Z. \tag{4.23}$$

Proof: The vertex $(1, n+1, 2n+1) \in Comp_{\Gamma_1}(1, n+1, 2n+1)$. If $x \neq 1$, then by Equation (3.55), $(x, n+1, 2n+1) \in \Delta_1$ and therefore, $(x, n+1, 2x+1) \in Comp_{\Gamma_1}(1, n+1, 2n+1)$. It follows that,

$$\{(x, n+1, 2n+1) : x \in X\} \subseteq Comp_{\Gamma_1}(1, n+1, 2n+1). \tag{4.24}$$

Suppose $(a, b, c) \in Comp_{\Gamma_1}(1, n+1, 2n+1)$. Then there exist a path $(1, n+1, 2n+1) \rightarrow (a_1, b_1, c_1) \rightarrow (a_2, b_2, c_2) \rightarrow \cdots \rightarrow (a_{l-1}, b_{l-1}, c_{l-1}) \rightarrow (a, b, c)$ of length l. It follows from Equation (4.10) that, $n+1 = b_1 = b_2 = b_3 = \cdots = b$ and $2n+1 = c_1 = c_2 = c_3 = \cdots = c$. This implies that, $(a, b, c) \in \{(x, n+1, 2n+1) : x \in X\}$. Therefore,

$$Comp_{\Gamma_1}(1, n+1, 2n+1) \subseteq \{(x, n+1, 2n+1) : x \in X\}. \tag{4.25}$$

From Expressions (4.24) and (4.25), we get Equation (4.17).

Next we prove Equation (4.18). Equations (4.18) and (4.19) are proved in a similar way as Equation (4.17).

Vertex $(1, n+1, 2n+1) \in Comp_{\Gamma_4}(1, n+1, 2n+1)$. Let $(x, y, 2n+1) \in (x, n+1, 2n+1) :$

$x \in X, y \in Y\}$. Vertex $(p, q, 2n+1)$, where $p \in X' \setminus \{x\}$ and $q \in Y' \setminus \{y\}$ is adjacent to both $(1, n+1, 2n+1)$ and $(x, y, 2n+1)$. Therefore,

$$\{(x, y, 2n+1) : x \in X, y \in Y\} \subseteq Comp_{\Gamma_4}(1, n+1, 2n+1). \tag{4.26}$$

Suppose $(a, b, c) \in Comp_{\Gamma_4}(1, n+1, 2n+1)$. Then, there exist a path $(1, n+1, 2n+1) \to (a_1, b_1, c_1) \to (a_2, b_2, c_2) \to \cdots \to (a_{l-1}, b_{l-1}, c_{l-1}) \to (a, b, c)$ of length l. By Equation (4.13), $2n+1 = c_1 = c_2 = \cdots = c$. This implies, $(a, b, c) \in \{(x, y, 2n+1) : x \in X, y \in Y\}$. Therefore,

$$Comp_{\Gamma_4}(1, n+1, 2n+1) \subseteq \{(x, y, 2n+1) : x \in X, y \in Y\}. \tag{4.27}$$

Combining Expressions (4.26) and (4.27), we obtain Equation (4.20). Equations (4.21) and (4.22) are proved in a similar way as (4.20).

Let $(x, y, z) \in X \times Y \times Z$. Consider the vertex (p, q, r), where $p \in X' \backslash \{x\}$, $q \in Y' \backslash \{y\}$ and $r \in Z' \backslash \{z\}$. By Equation (4.16), (p, q, r) is adjacent to $(1, n+1, 2n+1)$ and (x, y, z). It follows that $(x, y, z) \in Comp_{\Gamma_7}(1, n+1, 2n+1)$. Hence $X \times Y \times Z \subseteq Comp_{\Gamma_7}(1, n+1, 2n+1)$. But $X \times Y \times Z$ is the set of all vertices in Γ_7 thus $Comp_{\Gamma_7}(1, n+1, 2n+1) \subseteq X \times Y \times Z$. Thus, we get Equation (4.23). □

Theorem 4.2.5. *Each of the suborbital graphs Γ_1, Γ_2 and Γ_3 has n^2 components. Each of the suborbital graphs Γ_4, Γ_5 and Γ_6 has n components. The suborbital graph Γ_7 is connected.*

Proof: By Theorem 1.2.23, all components in a suborbital graph are isomorphic. Applying Lemma 4.2.4, we have, $|Comp_{\Gamma_1}(1, n+1, 2n+1)| = |Comp_{\Gamma_2}(1, n+1, 2n+1)| = |Comp_{\Gamma_3}(1, n+1, 2n+1)| = n$, $|Comp_{\Gamma_4}(1, n+1, 2n+1)| = |Comp_{\Gamma_5}(1, n+1, 2n+1)| = |Comp_{\Gamma_6}(1, n+1, 2n+1)| = n^2$ and $|Comp_{\Gamma_7}(1, n+1, 2n+1)| = n^3$. Since each of these graphs has n^3 vertices, the result follows. □

Corollary 4.2.6. *Each component of Γ_1, Γ_2 and Γ_3 is a complete graph on n vertices.*

Proof: By Theorem 4.2.5, each of the graphs Γ_1, Γ_2 and Γ_3 has n^2 components. By Theorem 1.2.23, it follows that each component has n vertices. By Equations (3.55), (3.56) and (3.57) the corresponding suborbits of these graphs are of length $n-1$ which is the degree of each graph. Therefore each vertex is adjacent to all the other $n-1$ vertices in its components. Hence each component is complete. □

Corollary 4.2.7. *The action of* $A_n \times A_n \times A_n$ *on* $X \times Y \times Z$ *is imprimitive.*

Proof: By Theorem 4.2.5, some non-trivial suborbital graphs of this action are disconnected. Therefore the action is imprimitive by Theorem 1.2.12. □

CHAPTER 5

CYCLE INDEX FORMULAE OF $A_n \times A_n \times A_n$ ACTING ON $X \times Y \times Z$

5.1 Cycle index formula of $A_4 \times A_4 \times A_4$ acting on $X \times Y \times Z$

We first find the cycle index of A_4. Permutations of A_4 are of the form I, (abc), $(ab)(cd)$. Applying Theorem 1.2.14, we obtain Table 5.1.

Table 5.1: Disjoint cycle structure of the elements of A_4

Permutations	Cycle type	Number	Monomial
I	$(4,0,0,0)$	1	t_1^4
(abc)	$(1,0,1,0)$	8	$t_1^1t_3^1$
$(ab)(cd)$	$(0,2,0,0)$	3	t_2^2

From Table 5.1, the cycle index formula for A_4 is given by,

$$Z\left(A_4\right) = \frac{1}{12}[t_1^4 + 8t_1^1t_3^1 + 3t_2^2]. \tag{5.1}$$

By Theorem 1.2.26, the cycle index of $A_4 \times A_4 \times A_4$ acting on $X \times Y \times Z$ is given by,

$$Z(G) = Z(G_1) \circ Z(G_2) \circ Z(G_3). \tag{5.2}$$

Therefore using Theorem 1.2.26 and Equation (5.1), we obtain,

$$\begin{aligned} Z(G_2) \circ Z(G_3) &= \frac{1}{144}\left[t_1^{16} + 8t_1^4t_3^4 + 3t_2^8 + 8t_1^4t_3^4 + 64t_1^1t_3^5 + 24t_2^2t_6^2 + 3t_2^8 + 24t_2^2t_6^2 + 9t_2^8\right] \\ &= \frac{1}{144}\left[t_1^{16} + 16t_1^4t_3^4 + 15t_2^8 + 64t_1^1t_3^5 + 48t_2^2t_6^2\right]. \end{aligned} \tag{5.3}$$

It follows that,

$$\begin{aligned} Z(G) = Z(G_1) \circ (Z(G_2) \circ Z(G_3)) = \frac{1}{1728}[&t_1^{64} + 24t_1^{16}t_3^{16} + 63t_2^{32} + 192t_1^4t_3^{20} + 512t_1^1t_3^{21} \\ &+ 360t_2^8t_6^8 + 576t_2^2t_6^{10}]. \end{aligned} \tag{5.4}$$

5.2 Cycle index formula of $A_5 \times A_5 \times A_5$ acting on $X \times Y \times Z$

Permutations of A_5 are of the form I, (abc), $(ab)(cd)$, $(abcde)$. On application of Theorem 1.2.14, we obtain Table 5.2.

Table 5.2: Disjoint cycle structures of A_5

Permutations	Cycle type	Number	Monomial
I	$(5,0,0,0,0)$	1	t_1^5
(abc)	$(2,0,1,0,0)$	20	$t_1^2t_3^1$
$(ab)(cd)$	$(1,2,0,0,0)$	15	$t_1^1t_2^2$
$(abcde)$	$(0,0,0,0,1)$	24	t_5^1

Using Table 5.2, we obtain,

$$Z(A_5) = \frac{1}{60}[t_1^5 + 20t_1^2t_3^1 + 24t_5^1 + 15t_1^1t_2^2] = Z(G_1) = Z(G_2) = Z(G_3). \tag{5.5}$$

On application of Theorem 1.2.26 and Equation (5.5), we get,

$$Z(G_2) \circ Z(G_3) = \frac{1}{3600}[t_1^{25} + 40t_1^{10}t_3^5 + 30t_1^5t_2^{10} + 225t_1^1t_2^{12} + 400t_1^4t_3^7 + 600t_1^2t_2^4t_3^1t_6^2 + 624t_5^5 + 960t_5^2t_{15}^1 + 720t_5^1t_{10}^2] \tag{5.6}$$

and

$$Z(G) = Z(G_1) \circ (Z(G_2) \circ Z(G_3)) = \frac{1}{216000}[t_1^{125} + 60t_1^{50}t_3^{25} + 450t_1^{25}t_2^{50} + 675t_1^5t_2^{60} + 3375t_1^1t_2^{62} + 1200t_1^{20}t_3^{35} + 8000t_1^8t_3^{39} + 1800t_1^{10}t_2^{20}t_3^5t_6^{10} + 13500t_1^2t_2^{24}t_3^1t_6^{12} + 18000t_1^4t_2^8t_3^7t_6^{14} + 15624t_5^{25} + 37440t_5^{10}t_{15}^5 + 28080t_5^5t_{10}^{10} + 16200t_5^1t_{10}^{12} + 28800t_5^4t_{15}^7 + 43200t_5^2t_{10}^4t_{15}^1t_{30}^2]. \tag{5.7}$$

5.3 Cycle index formula of $A_6 \times A_6 \times A_6$ acting on $X \times Y \times Z$

Permutations of A_6 are of the form I, (abc), $(ab)(cd)$, $(abcde)$, $(abcd)(ef)$ and $(abc)(def)$. Applying Theorem 1.2.14, we obtain Table 5.3.

Table 5.3: Disjoint cycle structures of A_6

Permutations	Cycle type	Number	Monomial
I	$(6,0,0,0,0,0)$	1	t_1^6
(abc)	$(3,0,1,0,0,0)$	40	$t_1^3t_3^1$
$(ab)(cd)$	$(2,2,0,0,0,0)$	45	$t_1^2t_2^2$
$(abcde)$	$(1,0,0,0,1,0)$	144	$t_1^1t_5^1$
$(abc)(def)$	$(0,0,2,0,0,0)$	40	t_3^2
$(abcd)(ef)$	$(0,1,0,1,0,0)$	90	$t_2^1t_4^1$

From Table 5.3, the cycle index of A_6 is given by,

$$Z(A_6) = \frac{1}{360}[t_1^6+40t_1^3t_3^1+45t_1^2t_2^2+144t_1^1t_5^1+40t_3^2+90t_2^1t_4^1] = Z(G_1) = Z(G_2) = Z(G_3). \tag{5.8}$$

Applying Theorem 1.2.26 and Equation (5.8), we get Equations (5.9) and (5.10).

$$\begin{aligned} Z(G_2)\circ Z(G_3) = \frac{1}{129600}[&t_1^{36} + 80t_1^{18}t_3^6 + 90t_1^{12}t_2^{12} + 2025t_1^4t_2^{16} + 16200t_1^9t_3^9 + 3600t_1^6t_2^6t_3^2t_6^2 \\ &+ 280t_1^6t_5^6 + 8280t_2^6t_4^6 + 4880t_3^{12} + 8100t_2^2t_4^8 + 11520t_1^3t_3^1t_5^3t_{15}^1 \\ &+ 12960t_1^2t_2^2t_5^2t_{10}^2 + 20736t_1^1t_5^7 + 7200t_2^3t_4^3t_6^1t_{12}^1 + 3600t_3^4t_6^4 + 25920t_2^1t_4^1t_{10}^1t_{20}^1 \\ &+ 11520t_3^2t_{15}^2 + 7200t_6^2t_{12}^2]. \end{aligned} \tag{5.9}$$

$$Z(G) = Z(G_1) \circ (Z(G_2) \circ Z(G_3)) = \frac{1}{46656000}[t_1^{216} + 120t_1^{108}t_3^{36} + 135t_1^{72}t_2^{72}$$
$$+ 6075t_1^{24}t_2^{96} + 91125t_1^{8}t_2^{104} + 4800t_1^{54}t_3^{54} + 10800t_1^{36}t_2^{36}t_3^{12}t_6^{12} + 64000t_1^{27}t_3^{63}$$
$$+ 243000t_1^{12}t_2^{48}t_3^{4}t_6^{16} + 432t_1^{36}t_5^{36} + 216000t_1^{18}t_2^{18}t_3^{18}t_6^{18} + 571320t_2^{36}t_4^{36} + 462520t_3^{72}$$
$$+ 1117800t_2^{12}t_4^{48} + 729000t_2^{4}t_4^{52} + 34560t_1^{18}t_3^{6}t_5^{18}t_{15}^{6} + 38880t_1^{12}t_2^{12}t_5^{12}t_{10}^{12}$$
$$+ 62208t_1^{6}t_5^{42} + 993600t_2^{18}t_4^{18}t_6^{6}t_{12}^{6} + 658800t_3^{24}t_6^{24} + 2985984t_1^{1}t_5^{43}$$
$$+ 874800t_1^{4}t_2^{16}t_5^{4}t_{10}^{16} + 972000t_2^{6}t_4^{24}t_6^{2}t_{12}^{8} + 243000t_3^{8}t_6^{32} + 691200t_1^{9}t_3^{9}t_5^{9}t_{15}^{9}$$
$$+ 432000t_2^{9}t_4^{9}t_6^{9}t_{12}^{9} + 1555200t_1^{6}t_2^{6}t_3^{2}t_5^{6}t_6^{2}t_{10}^{6}t_{15}^{2}t_{30}^{2} + 2488320t_1^{3}t_3^{1}t_5^{21}t_{15}^{7}$$
$$+ 2799360t_1^{2}t_2^{2}t_5^{14}t_{10}^{14} + 3576960t_2^{6}t_4^{6}t_{10}^{6}t_{20}^{6} + 2108160t_3^{12}t_{15}^{12} + 2289600t_6^{12}t_{12}^{12}$$
$$+ 3499200t_2^{2}t_4^{8}t_{10}^{2}t_{20}^{8} + 972000t_6^{4}t_{12}^{16} + 3110400t_2^{3}t_4^{3}t_6^{1}t_{10}^{3}t_{12}^{1}t_{20}^{3}t_{30}^{1}t_{60}^{1}$$
$$+ 5598720t_2^{1}t_4^{1}t_{10}^{7}t_{20}^{7} + 1555200t_3^{4}t_6^{4}t_{15}^{4}t_{30}^{4} + 2488320t_3^{2}t_{15}^{14} + 3110400t_6^{2}t_{12}^{2}t_{30}^{2}t_{60}^{2}]. \quad (5.10)$$

5.4 Cycle index formula of $A_7 \times A_7 \times A_7$ acting on $X \times Y \times Z$

Permutations of A_7 are of the form I, (abc), $(ab)(cd)$, $(abcde)$, $(abcd)(ef)$ and $(abc)(def)$. Applying Theorem 1.2.14, we obtain Table 5.4.

Table 5.4: Disjoint cycle structure of A_7

Permutations	Cycle type	Number	Monomial
I	$(7,0,0,0,0,0,0)$	1	t_1^7
(abc)	$(4,0,1,0,0,0,0)$	70	$t_1^4t_3^1$
$(ab)(cd)$	$(3,2,0,0,0,0,0)$	105	$t_1^3t_2^2$
$(abcde)$	$(2,0,0,0,1,0,0)$	504	$t_1^2t_5^1$
$(abc)(def)$	$(1,0,2,0,0,0,0)$	280	$t_1^1t_3^2$
$(abcdefg)$	$(0,0,0,0,0,0,1)$	720	t_7^1
$(abcd)(ef)$	$(1,1,0,1,0,0,0)$	630	$t_1^1t_2^1t_4^1$
$(ab)(cd)(efg)$	$(0,2,1,0,0,0,0)$	210	$t_2^2t_3^1$

$$Z\,(A_7) = \frac{1}{2520}[t_1^7 + 70t_1^4t_3^1 + 105t_1^3t_2^2 + 504t_1^2t_5^1 + 280t_1^1t_3^2 + 720t_7^1 + 630t_1^1t_2^1t_4^1 + 210t_2^2t_3^1]$$
$$= Z(G_1) = Z(G_2) = Z(G_3). \tag{5.11}$$

$$\begin{aligned} Z(G_2) \circ Z(G_3) = \frac{1}{6350400}[&t_1^{49} + 140t_1^{28}t_3^7 + 210t_1^{21}t_2^{14} + 11025t_1^9t_2^{20} \\ &+ 4900t_1^{16}t_3^{11} + 14700t_1^{12}t_2^8t_3^3t_6^2 + 1008t_1^{14}t_5^7 \\ &+ 1260t_1^7t_2^7t_4^7 + 560t_1^7t_3^{14} + 420t_2^{14}t_3^7 + 39200t_1^4t_3^{15} \\ &+ 132300t_1^3t_2^9t_4^7 + 44100t_2^{14}t_3^3t_6^2 + 78400t_1^1t_3^{16} \\ &+ 29400t_2^{16}t_3^7t_6^2 + 70560t_1^{18}t_3^2t_5^4t_{15}^1 + 105840t_1^6t_2^4t_5^3t_{10}^2 \\ &+ 88200t_1^4t_2^4t_3^1t_4^4t_6^1t_{12}^1 + 58800t_1^3t_2^2t_3^6t_6^4 + 39600t_1^1t_2^4t_4^{10} \\ &+ 44100t_2^8t_3^3t_6^4 + 254016t_1^4t_5^9 + 264600t_2^6t_3^1t_4^4t_6^1t_{12}^1 \\ &+ 117600t_2^2t_3^7t_6^4 + 635040t_1^2t_2^2t_4^2t_5^1t_{10}^1t_{20}^1 + 282240t_1^2t_3^4t_5^1t_{15}^2 \\ &+ 352800t_1^1t_2^1t_3^2t_4^1t_6^2t_{12}^2 + 211680t_2^4t_3^2t_{10}^2t_{15}^1 + 519840t_7^7 + 100800t_7^4t_{21}^1 \\ &+ 151200t_7^3t_{14}^2 + 725760t_7^2t_{35}^1 + 907200t_7^1t_{14}^1t_{28}^1 \\ &+ 403200t_7^1t_{21}^2 + 302400t_{14}^2t_{21}^1]. \end{aligned} \tag{5.12}$$

$$Z(G) = Z(G_1) \circ (Z(G_2) \circ Z(G_3)) = \frac{1}{16003008000}[t_1^{343} + 210t_1^{196}t_3^{49}$$
$$+ 315t_1^{147}t_2^{98} + 33075t_1^{63}t_2^{140} + 14700t_1^{112}t_3^{77}$$
$$+ 1157625t_1^{27}t_2^{158} + 44100t_1^{84}t_2^{56}t_3^{21}t_6^{14} + 343000t_1^{64}t_3^{93}$$
$$+ 1512t_1^{98}t_5^{49} + 1890t_{49}^{1}t_2^{49}t_4^{49} + 84t_1^{49}t_3^{98}$$
$$+ 630t_2^{98}t_3^{49} + 2315250t_1^{36}t_2^{80}t_3^{9}t_6^{20} + 1543500t_1^{48}t_2^{32}t_3^{33}t_6^{22}$$
$$+ 117600t_1^{28}t_3^{105} + 396900t_1^{21}t_2^{63}t_4^{49} + 132300t_2^{98}t_3^{21}t_6^{14}$$
$$+ 20837250t_1^{9}t_2^{69}t_4^{49} + 6945750t_2^{98}t_3^{9}t_6^{20} + 411600t_1^{16}t_3^{109}$$
$$+ 235200t_1^{7}t_3^{112} + 88200t_2^{56}t_3^{49}t_6^{14} + 16464000t_1^{4}t_3^{113}$$
$$+ 21952000t_1^{1}t_3^{114} + 211680t_1^{56}t_3^{14}t_5^{28}t_{15}^{7} + 317520t_1^{42}t_2^{28}t_5^{21}t_{10}^{14}$$
$$+ 264600t_1^{28}t_2^{28}t_3^{7}t_4^{28}t_6^{7}t_{12}^{7} + 176400t_1^{21}t_2^{14}t_3^{42}t_6^{28} + 1190700t_1^{7}t_2^{28}t_4^{70}$$
$$+ 9393300t_2^{56}t_3^{21}t_6^{28} + 125023500t_1^{3}t_2^{30}t_4^{70} + 3087000t_2^{32}t_3^{49}t_6^{22} + 13891500t_2^{56}t_3^{9}t_6^{34}$$
$$+ 27783000t_1^{12}t_2^{36}t_3^{3}t_4^{28}t_6^{9}t_{12}^{7} + 12348000t_1^{12}t_2^{8}t_3^{45}t_6^{30} + 250047000t_1^{1}t_2^{13}t_4^{79}$$
$$+ 762048t_1^{28}t_5^{63} + 793800t_2^{42}t_3^{7}t_4^{28}t_6^{7}t_{12}^{7} + 352800t_2^{14}t_3^{49}t_6^{28}$$
$$+ 83349000t_2^{42}t_3^{3}t_4^{28}t_6^{9}t_{12}^{7} + 9261000t_2^{32}t_3^{21}t_6^{36} + 16669800t_1^{18}t_2^{40}t_5^{9}t_{10}^{20}$$
$$+ 9261000t_1^{9}t_2^{20}t_3^{18}t_6^{40} + 24696000t_2^{8}t_3^{49}t_6^{30} + 24696000t_1^{3}t_2^{2}t_3^{48}t_6^{32}$$
$$+ 9261000t_2^{32}t_3^{9}t_6^{42} + 49392000t_2^{2}t_3^{49}t_6^{32} + 7408800t_1^{32}t_3^{22}t_5^{16}t_{15}^{11}$$
$$+ 9261000t_1^{16}t_2^{16}t_3^{11}t_4^{16}t_6^{11}t_{12}^{11} + 37044000t_2^{14}t_3^{21}t_6^{42} + 22226400t_1^{24}t_2^{16}t_3^{6}t_5^{12}t_6^{4}t_{10}^{8}t_{15}^{3}t_{30}^{2}$$
$$+ 128024064t_1^{8}t_5^{67} + 83349000t_1^{4}t_2^{16}t_3^{1}t_4^{40}t_6^{4}t_{12}^{10} + 250047000t_2^{18}t_3^{1}t_4^{40}t_6^{4}t_{12}^{10}$$
$$+ 37044000t_2^{8}t_3^{21}t_6^{44} + 55566000t_2^{24}t_3^{7}t_4^{16}t_6^{13}t_{12}^{11} + 83349000t_2^{24}t_3^{3}t_4^{16}t_6^{15}t_{12}^{11}$$
$$+ 53343360t_1^{16}t_3^{4}t_5^{36}\ t_{15}^{9} + 80015040t_1^{12}t_2^{8}t_5^{27}t_{10}^{18} \tag{5.13}$$

$$+ 1905120t_1^{14}t_2^{14}t_4^{14}t_5^7t_{10}^7t_{20}^7 + 846720t_1^{14}t_3^{28}t_5^7t_{15}^{14} + 1058400t_1^7t_2^7t_3^{14}t_4^7t_6^{14}t_{12}^{14}$$
$$+ 635040t_2^{28}t_3^{14}t_{10}^{14}t_{15}^7 + 59270400t_1^8t_3^{30}t_5^4t_{15}^{15} + 200037600t_1^6t_2^{18}t_4^{14}t_5^3t_{10}^9t_{20}^7$$
$$+ 74088000t_1^4t_2^4t_3^{15}t_4^4t_6^{15}t_{12}^{15} + 111132000t_1^3t_2^9t_3^6t_4^7t_6^{18}t_{12}^{14} + 66679200t_2^{28}t_3^6t_6^4t_{10}^{14}t_{15}^3t_{30}^2$$
$$+ 11854540800t_1^2t_3^{32}t_5t_{15}^{16} + 148176000t_1t_2t_3^{16}t_4t_6^{16}t_{12}^{16}$$
$$+ 44452800t_2^{16}t_3^{14}t_6^4t_{10}^8t_{15}^7t_{30}^2 + 222264000t_2^6t_3^7t_4^4t_6^{19}t_{12}^{15}$$
$$+ 374805360t_7^{49} + 1333358400t_1^8t_2^8t_3^2t_4^8t_5^4t_6^2t_{10}^4t_{12}^2t_{15}t_{20}^4t_{30}t_{60}$$
$$+ 88905600t_1^6t_2^4t_3^{12}t_5^3t_6^8t_{10}^2t_{15}^6t_{30}^4 + 6000112800t_1^2t_2^8t_4^{20}t_5t_{10}^4t_{20}^{10}$$
$$+ 333396000t_1t_2^4t_3^2t_4^{10}t_6^8t_{12}^{20} + 66679200t_2^{16}t_3^6t_6^8t_{10}^8t_{15}^3t_{30}^4$$
$$+ 480090240t_1^4t_2^4t_4^4t_5^9t_{10}^9t_{20}^9 + 213373440t_1^4t_3^8t_5^9t_{15}^{18}$$
$$+ 400075200t_2^{12}t_3^2t_4^8t_6^2t_{10}^6t_{12}^2t_{15}t_{20}^4t_{30}t_{60} + 160030080t_2^8t_3^4t_{10}^{18}t_{15}^9$$
$$+ 177811200t_2^4t_3^{14}t_6^8t_{10}^2t_{15}^7t_{30}^4 + 109166400t_7^{28}t_{21}^7 + 163749600t_7^{21}t_{14}^{14}$$
$$+ 23814000t_7^9t_{14}^{20} + 533433600t_1^2t_2^2t_3^4t_4^2t_5t_6^4t_{10}t_{12}^4t_{15}^2t_{20}t_{30}^2t_{60}^2$$
$$+ 10584000t_7^{16}t_{21}^{11} + 31752000t_7^{12}t_{14}^8t_{21}^3t_{42}^2 + 785998080t_7^{14}t_{35}^7 + 982497600t_7^7t_{14}^7t_{28}^7$$
$$+ 436665600t_7^7t_{21}^{14} + 327499200t_{14}^{14}t_{21}^7 + 84672000t_7^4t_{21}^{15} + 285768000t_7^3t_{14}^9t_{28}^7$$
$$+ 95256000t_{14}^{14}t_{21}^3t_{42}^2 + 169344000t_7t_{21}^{16} + 63504000t_{14}^8t_{21}^7t_{42}^2 + 152409600t_7^8t_{21}^2t_{35}^4t_{105}$$
$$+ 228614400t_7^6t_{14}^4t_{35}^3t_{70}^2 + 90512000t_7^4t_{14}^4t_{21}t_{28}^4t_{42}t_{84} + 127008000t_7^3t_{14}^2t_{21}^6t_{42}^4$$
$$+ 857304000t_7t_{14}^4t_{28}^{10} + 95256000t_{14}^8t_{21}^3t_{42}^4 + 548674560t_7^4t_{35}^9$$
$$+ 571536000t_{14}^6t_{21}t_{28}^4t_{42}t_{84} + 254016000t_{14}^2t_{21}^7t_{42}^4 + 1371686400t_7^2t_{14}^2t_{28}^2t_{35}t_{70}t_{140}$$
$$+ 609638400t_7^2t_{21}^4t_{35}t_{105}^2 + 762048000t_7t_{14}t_{21}^2t_{28}t_{42}^2t_{84}^2 + 457228800t_{14}^4t_{21}^2t_{70}^2t_{105}].$$

5.5 Cycle index formula of $A_8 \times A_8 \times A_8$ acting on $X \times Y \times Z$

Permutations of A_8 are of the form I, (abc),$(ab)\,(cd)$, $(abcde)$, $(abc)\,(def)$, $(abcdefg)$, $(abcd)\,(ef)$, $(abcd)\,(efgh)$, $(abcde)\,(fgh)$, $(abcdef)\,(gh)$, $(abc)\,(de)\,(fg)$, $(ab)(cd)(ef)(gh)$. Applying Theorem 1.2.14, we obtain Table 5.5.

Table 5.5: Disjoint cycle structure of elements of A_8

Permutations	Cycle type	Number	Monomial
I	$(8,0,0,0,0,0,0,0)$	1	t_1^8
(abc)	$(5,0,1,0,0,0,0,0)$	112	$t_1^5t_3^1$
$(ab)(cd)$	$(4,2,0,0,0,0,0,0)$	210	$t_1^4t_2^2$
$(abcde)$	$(3,0,0,0,1,0,0,0)$	1344	$t_1^3t_5^1$
$(abc)(def)$	$(2,0,2,0,0,0,0,0)$	1120	$t_1^2t_3^2$
$(abcdefg)$	$(1,0,0,0,0,0,1,0)$	5760	$t_1^1t_7^1$
$(abcd)(ef)$	$(2,1,0,1,0,0,0,0)$	2520	$t_1^2t_2^1t_4^1$
$(abcd)(efgh)$	$(0,0,0,2,0,0,0,0)$	1260	t_4^2
$(abcde)(fgh)$	$(0,0,1,0,1,0,0,0)$	2688	$t_3^1t_5^1$
$(abcdef)(gh)$	$(0,1,0,0,0,1,0,0)$	3360	$t_2^1t_6^1$
$(abc)(de)(fg)$	$(1,2,1,0,0,0,0,0)$	1680	$t_1^1t_2^2t_3^1$
$(ab)(cd)(ef)(gh)$	$(0,4,0,0,0,0,0,0)$	105	t_2^4

From Table 5.5, we obtain,

$$\begin{aligned} Z(A_8) = \frac{1}{20160}[&t_1^8 + 112t_1^5t_3^1 + 210t_1^4t_2^2 + 1344t_1^3t_5^1 + 1120t_1^2t_3^2 \\ &+ 5760t_1^1t_7^1 + 2520t_1^2t_2^1t_4^1 + 1260t_4^2 + 2688t_3^1t_5^1 + 3360t_2^1t_6^1 \\ &+ 1680t_1^1t_2^2t_3^1 + 105t_2^4] = Z(G_1) = Z(G_2) = Z(G_3). \end{aligned} \tag{5.14}$$

$$
\begin{aligned}
Z(G_2)\circ Z(G_3) = \frac{1}{406425600}[&t_1^{64} + 224t_1^{40}t_3^{8} + 420t_1^{32}t_2^{16} \\
&+ 44100t_1^{16}t_2^{24} + 12544t_1^{25}t_3^{13} + 47040t_1^{20}t_2^{10}t_3^{4}t_6^{2} \\
&+ 2688t_1^{24}t_5^{8} + 5040t_1^{16}t_2^{8}t_4^{8} + 2240t_1^{16}t_3^{16} \\
&+ 3360t_1^{8}t_2^{16}t_3^{8} + 55335t_2^{32} + 250880t_1^{10}t_3^{18} \\
&+ 1058400t_1^{8}t_2^{12}t_4^{8} + 705600t_1^{4}t_2^{18}t_3^{4}t_6^{2} + 376320t_1^{5}t_2^{10}t_3^{9}t_6^{2} \\
&+ 301056t_1^{15}t_3^{3}t_5^{5}t_{15} + 564480t_1^{12}t_2^{6}t_5^{4}t_{10}^{2} + 564480t_1^{10}t_2^{5}t_3^{2}t_4^{5}t_6t_{12} \\
&+ 470400t_1^{8}t_2^{4}t_3^{8}t_6^{4} + 1254400t_1^{4}t_3^{20} + 376320t_2^{20}t_6^{4} \\
&+ 529200t_2^{16}t_4^{8} + 6350400t_1^{4}t_2^{6}t_4^{12} + 2822400t_1t_2^{12}t_3^{5}t_6^{4} \\
&+ 1806336t_1^{9}t_5^{11} + 8467200t_1^{2}t_2^{9}t_3^{2}t_4^{5}t_6t_{12} + 37632t_1^{2}t_2^{4}t_3^{10}t_6^{4} \\
&+ 11520t_1^{8}t_7^{8} + 6773760t_1^{6}t_2^{3}t_4^{3}t_5^{2}t_{10}t_{20} + 3010560t_1^{6}t_3^{6}t_5^{2}t_{15}^{2} \\
&+ 56448t_1^{4}t_2^{2}t_3^{4}t_4^{2}t_6^{2}t_{12}^{2} + 4515840t_1^{3}t_2^{6}t_3^{3}t_5t_{10}^{2}t_{15} + 282240t_2^{12}t_{10}^{4} \\
&+ 2358720t_2^{8}t_6^{8} + 5376t_3^{8}t_5^{8} + 8734320t_4^{16} \\
&+ 12042240t_2^{5}t_6^{9} + 602112t_3^{8}t_5^{5}t_{15} + 1290240t_1^{5}t_3t_7^{5}t_{21} \\
&+ 2419200t_1^{4}t_2^{2}t_7^{4}t_{14}^{2} + 16934400t_2^{4}t_4^{2}t_6^{4}t_{12}^{2} + 18816000t_2^{2}t_6^{10} \\
&+ 6021120t_3^{8}t_5^{2}t_{15}^{2} + 1128960t_3^{4}t_5^{4}t_6^{2}t_{10}^{2} + 7225344t_3^{3}t_5^{8}t_{15} \\
&+ 4515840t_4^{10}t_{12}^{2} + 33177600t_1t_7^{9} + 9031680t_3^{4}t_5t_6^{2}t_{10}^{2}t_{15} \\
&+ 7225344t_3^{3}t_5^{5}t_{15}^{2} + 15482880t_1^{3}t_5t_7^{3}t_{35} + 29030400t_1^{2}t_2t_4t_7^{2}t_{14}t_{28} \\
&+ 12902400t_1^{2}t_3^{2}t_7^{2}t_{21}^{2} + 19353600t_1t_2^{2}t_3t_7t_{14}^{2}t_{21} + 1209600t_2^{4}t_{14}^{4} \\
&+ 9031680t_2^{3}t_6^{3}t_{10}t_{30} + 13547520t_3^{2}t_5^{2}t_6t_{10}t_{12}t_{20} + 3386880t_4^{6}t_{20}^{2} \\
&+ 11289600t_4^{4}t_{12}^{4} + 564480t_6^{4}t_{10}^{4} + 18063360t_6^{4}t_{10}t_{30} \\
&+ 38707200t_2t_6t_{14}t_{42} + 30965760t_3t_5t_{21}t_{35} + 14515200t_4^{2}t_{28}^{2} \\
&+ 6773760t_{12}^{2}t_{20}^{2}]
\end{aligned}
\tag{5.15}
$$

$$
\begin{aligned}
Z(G) = Z(G_1) \circ (Z(G_2) \circ Z(G_3)) \frac{1}{8193540096000}[&t_1^{512} + 336t_1^{320}t_3^{64} \\
&+ 630t_1^{256}t_2^{128} + 132300t_1^{128}t_2^{192} \\
&+ 37632t_1^{200}t_3^{104} + 141120t_1^{160}t_2^{80}t_3^{32}t_6^{16} \\
&+ 9261000t_1^{64}t_2^{224} + 4032t_1^{192}t_5^{64} + 7560t_1^{128}t_2^{64}t_4^{64} \\
&+ 3360t_1^{128}t_3^{128} + 5040t_1^{64}t_2^{128}t_3^{64} + 22160565t_2^{256} \\
&+ 1404928t_1^{125}t_3^{129} + 14817600t_1^{80}t_2^{120}t_3^{16}t_6^{24} + 7902720t_1^{100}t_2^{50}t_3^{52}t_6^{26} \\
&+ 752640t_1^{80}t_3^{144} + 3175200t_1^{64}t_2^{96}t_4^{64} + 2116800t_1^{32}t_2^{144}t_3^{32}t_6^{16} \\
&+ 1128960t_1^{40}t_2^{80}t_3^{72}t_6^{16} + 333396000t_1^{32}t_2^{112}t_4^{64} + 222264000t_1^{16}t_2^{152}t_3^{16}t_6^{24} \\
&+ 42147840t_1^{50}t_3^{154} + 903168t_1^{120}t_3^{24}t_5^{40}t_{15}^{8} + 1693440t_1^{96}t_2^{48}t_5^{32}t_{10}^{16} \\
&+ 1693440t_1^{80}t_2^{40}t_3^{16}t_4^{40}t_6^{8}t_{12}^{8} + 1411200t_1^{64}t_2^{32}t_3^{64}t_6^{32} + 3763200t_1^{32}t_3^{160} \\
&+ 297480960t_2^{160}t_6^{32} + 418332600_2t_4^{64} + 421478400t_1^{20}t_3^{164} \\
&+ 237081600t_1^{20}t_2^{90}t_3^{36}t_6^{34} + 63221760t_1^{25}t_2^{50}t_3^{77}t_6^{26} + 19051200t_1^{32}t_2^{48}t_4^{96} \\
&+ 8467200t_1^{8}t_2^{96}t_3^{40}t_6^{32} + 1404928000t_1^{8}t_3^{168} + 355622400t_1^{40}t_2^{60}t_3^{8}t_4^{40}t_6^{12}t_{12}^{8} \\
&+ 158054400t_1^{40}t_2^{20}t_3^{72}t_6^{36} + 4000752000t_1^{16}t_2^{56}t_4^{96} + 1778112000t_1^{4}t_2^{98}t_3^{20}t_6^{42} \\
&+ 5419008t_1^{72}t_5^{88} + 177811200t_1^{48}t_2^{72}t_5^{16}t_{10}^{24} + 148176000t_1^{32}t_2^{48}t_3^{32}t_6^{48} \\
&+ 25401600t_1^{16}t_2^{72}t_3^{16}t_4^{40}t_6^{8}t_{12}^{8} + 11289600t_1^{16}t_2^{32}t_3^{80}t_6^{32} + 2000376000t_2^{64}t_4^{96} \\
&+ 50577408t_1^{75}t_3^{39}t_5^{25}t_{15}^{13} + 94822640t_1^{50}t_2^{25}t_3^{26}t_4^{25}t_6^{13}t_{12}^{13} + 5334336000t_1^{8}t_2^{76}t_3^{8}t_4^{40}t_6^{12}t_{12}^{8} \\
&+ 1011548160t_2^{100}t_6^{52} + 948326400t_1^{5}t_2^{60}t_3^{41}t_6^{44} + 1264435200t_1^{10}t_2^{20}t_3^{82}t_6^{36} \\
&+ 16003008000t_1^{8}t_2^{28}t_4^{112} + 189665280t_1^{60}t_2^{30}t_3^{12}t_5^{20}t_6^{6}t_{10}^{10}t_{15}^{4}t_{30}^{2} + 790272000t_1^{16}t_2^{8}t_3^{80}t_6^{40} \\
&+ 2844979200t_2^{80}t_4^{40}t_6^{16}t_{12}^{8} + 741632000t_1t_2^{62}t_3^{21}t_6^{54} + 2370816000t_1^{8}t_2^{36}t_3^{40}t_6^{52} \\
&+ 6322176000t_1^{4}t_2^{8}t_3^{84}t_6^{40} + 2133734400t_1^{20}t_2^{30}t_3^{4}t_4^{60}t_6^{6}t_{12}^{12} + 17280t_1^{64}t_7^{64}
\end{aligned}
\tag{5.16}
$$

$$+ 20321280t_1^{48}t_2^{24}t_4^{24}t_5^{16}t_{10}^{8}t_{20}^{8} + 9031680t_1^{48}t_3^{48}t_5^{16}t_{15}^{16} + 16934400t_1^{32}t_2^{16}t_3^{32}t_4^{16}t_6^{16}t_{12}^{16}$$

$$+ 13547520t_1^{24}t_2^{48}t_3^{24}t_5^{8}t_{10}^{16}t_{15}^{8} + 2844979200t_1^{10}t_2^{45}t_3^{18}t_4^{25}t_6^{17}t_{12}^{13} + 223110720t_2^{96}t_{10}^{32}$$

$$+ 1192474080t_2^{64}t_6^{64} + 8064t_3^{64}t_5^{64} + 45909823680t_4^{128}$$

$$+ 2427715584t_1^{27}t_5^{97} + 320060t_1^{4}t_2^{38}t_3^{4}t_4^{60}t_6^{6}t_{12}^{12} + 9483264000t_1^{2}t_2^{24}t_3^{42}t_6^{56}$$

$$+ 606928896t_1^{45}t_3^{9}t_5^{55}t_{15}^{11} + 1137991680t_1^{36}t_2^{18}t_5^{44}t_{10}^{22} + 21337344000t_1^{2}t_2^{49}t_3^{10}t_4^{25}t_6^{21}t_{12}^{13}$$

$$+ 1011548160t_1^{30}t_3^{54}t_5^{10}t_{15}^{18} + 4267468800t_1^{24}t_2^{36}t_4^{24}t_5^{8}t_{10}^{12}t_{20}^{8} + 1896652800t_1^{20}t_2^{10}t_3^{36}t_4^{10}t_6^{18}t_{12}^{18}$$

$$+ 3556224000t_1^{16}t_2^{24}t_3^{16}t_4^{16}t_6^{24}t_{12}^{16} + 2844979200t_1^{12}t_2^{54}t_3^{12}t_5^{4}t_6^{6}t_{10}^{18}t_{15}^{4}t_{30}^{2} + 12680478720t_2^{40}t_6^{72}$$

$$+ 1806336t_3^{64}t_5^{40}t_{15}^{8} + 1517322240t_1^{15}t_2^{30}t_3^{27}t_5^{5}t_6^{6}t_{10}^{10}t_{15}^{9}t_{30}^{2} + 32369541120t_2^{25}t_6^{77}$$

$$+ 101154816t_3^{64}t_5^{25}t_{15}^{13} + 3870720t_1^{40}t_3^{8}t_7^{40}t_{21}^{8} + t_1^{32}t_2^{16}t_7^{32}t_{14}^{16}$$

$$+ 2275983360t_1^{30}t_2^{15}t_3^{6}t_4^{15}t_5^{10}t_6^{3}t_{10}^{5}t_{12}^{3}t_{15}^{2}t_{20}^{5}t_{30}t_{60} + 18966522800t_1^{24}t_2^{12}t_3^{24}t_5^{8}t_6^{12}t_{10}^{4}t_{15}^{8}t_{30}^{4}$$

$$+ 5057740800t_1^{12}t_3^{60}t_5^{4}t_{15}^{20} + 9483264000t_1^{8}t_2^{4}t_3^{40}t_4^{4}t_6^{20}t_{12}^{20} + 28449792000t_1^{4}t_2^{18}t_3^{20}t_4^{10}t_6^{26}t_{12}^{18}$$

$$+ 1517322240t_2^{60}t_6^{12}t_{10}^{20}t_{30}^{4} + 2133734400t_2^{48}t_4^{24}t_{10}^{16}t_{20}^{8} + 17831923200t_2^{32}t_4^{16}t_6^{32}t_{12}^{16}$$

$$+ 18232704000t_2^{16}t_6^{80} + 18063360t_3^{64}t_5^{16}t_{15}^{16} + 3386880t_3^{32}t_5^{32}t_6^{16}t_{10}^{16} + 21676032t_3^{24}t_5^{64}\ t_{15}^{8}$$

$$+ 46955704320t_4^{80}t_{12}^{16} + 101154816000t_2^{10}t_6^{82} + 2023096320t_3^{64}t_5^{10}t_{15}^{18}$$

$$+ 25804812800t_1^{12}t_2^{18}t_4^{36}t_5^{4}t_{10}^{6}t_{20}^{12} + 21337344000t_1^{8}t_2^{12}t_3^{8}t_4^{24}t_6^{12}t_{12}^{24}$$

$$+ 11379916800t_1^{3}t_2^{36}t_3^{15}t_5t_6^{12}t_{10}^{12}t_{15}^{5}t_{30}^{4} + 88510464000t_2^{4}t_6^{84} + 10115481600t_3^{64}t_5^{4}t_{15}^{20}$$

$$+ 91039334400t_2^{20}t_4^{10}t_6^{36}t_{12}^{18} + 379330560t_3^{32}t_5^{20}t_6^{16}t_{10}^{10}t_{15}^{4}t_{30}^{2} + 14566293500t_3^{9}t_5^{64}t_{15}^{11}$$

$$+ 13655900160t_1^{18}t_2^{9}t_4^{9}t_5^{22}t_{10}^{11}t_{20}^{11} + 6069288960t_1^{18}t_3^{18}t_5^{22}t_{15}^{22} + 762048000t_1^{16}t_2^{24}t_7^{16}t_{14}^{24}$$

$$+ 9103933440t_1^{9}t_2^{18}t_3^{9}t_5^{11}t_{10}^{22}t_{15}^{11} + 99532800t_1^{8}t_7^{72}$$

$$+ 34139750400t_1^{6}t_2^{27}t_3^{6}t_4^{15}t_5^{2}t_6^{3}t_{10}^{9}t_{12}^{3}t_{15}^{2}t_{20}^{5}t_{30}t_{60} + 15173222400t_1^{6}t_2^{12}t_3^{30}t_5^{2}t_6^{12}t_{10}^{4}t_{15}^{10}t_{30}^{4}$$

$$+ 568995840t_2^{36}t_{10}^{44} + 64012032000t_2^{16}t_4^{24}t_6^{16}t_{12}^{24} + 27095040t_3^{32}t_5^{8}t_6^{16}t_{10}^{16}t_{15}^{8}$$

$$+ 449391616t_3^{24}t_5^{40}t_{15}^{16} + 355622400t_3^{16}t_5^{16}t_6^{24}t_{10}^{24} + 216760320t_1^{25}t_3^{13}t_7^{25}t_{21}^{13}$$

$$+ 12138577920t_4^{50}t_{12}^{26} + 191102976000t_1t_7^{73} + 3034644480t_3^{32}t_5^{5}t_6^{16}t_{10}^{10}t_{15}^{9}t_{30}^{2}$$

$$+ 812851200t_1^{20}t_2^{10}t_3^{4}t_6^{2}t_7^{20}t_{14}^{10}t_{21}^{4}t_{42}^{2} + 142248960000t_2^{8}t_4^{4}t_6^{40}t_{12}^{20} + 3793305600t_3^{32}t_5^{8}t_6^{16}t_{10}^{4}t_{15}^{8}t_{30}^{4}$$

$$+ 4551966720t_3^{12}t_5^{32}t_6^{6}t_{10}^{16}t_{15}^{4}t_{30}^{2} + 2427715584t_3^{24}t_5^{25}t_{15}^{21} + 30346444800t_3^{32}t_5^{2}t_6^{16}t_{10}^{4}t_{15}^{10}t_{30}^{4}$$

$$+ 5689958400t_3^{16}t_5^{4}t_6^{24}t_{10}^{18}t_{15}^{4}t_{30}^{2} + 29132587010t_3^{9}t_5^{40}t_{15}^{19} + 46448640t_1^{24}t_5^{8}t_7^{24}t_{35}^{8}$$

$$+ 87091200t_1^{16}t_2^{8}t_4^{8}t_7^{16}t_{14}^{8}t_{28}^{8} + 38707200t_1^{16}t_3^{16}t_7^{16}t_{21}^{16}$$

$$+ 22759833600t_1^{12}t_2^{6}t_3^{12}t_4^{6}t_5^{4}t_6^{6}t_{10}^{2}t_{12}^{6}t_{15}^{4}t_{20}^{2}t_{30}^{2}t_{60}^{2} + 58060800t_1^{8}t_2^{16}t_3^{8}t_7^{8}t_{14}^{16}t_{21}^{8}$$

$$+ 956188800t_2^{32}t_{14}^{32} + 9510359040t_2^{24}t_6^{24}t_{10}^{8}t_{30}^{8} + 24277155840t_3^{24}t_5^{16}t_{15}^{24}$$

$$+ 40642560t_3^{16}t_5^{16}t_6^{8}t_{10}^{8}t_{12}^{8}t_{20}^{8} + 35216778240t_4^{48}t_{20}^{16} + 11738926800t_4^{32}t_{12}^{32}$$

$$+ 446221440t_6^{32}t_{10}^{32} + 22759833600t_3^{16}t_5t_6^{24}t_{10}^{12}t_{15}^{5}t_{30}^{4} + 1147643600t_1^{5}t_3t_7^{45}t_{21}^{9}$$

$$+ 20901888000t_1^{4}t_2^{2}t_7^{36}t_{14}^{18} + 24277155840t_3^{24}t_5^{10}t_{15}^{26} + 4551966720t_3^{12}t_5^{20}t_6^{6}t_{10}^{10}t_{15}^{8}t_{30}^{4}$$

$$+ 19421724670t_3^{9}t_5^{25}t_{15}^{24} + 4335206400t_1^{10}t_3^{18}t_7^{10}t_{21}^{18} + 18289152000t_1^{8}t_2^{12}t_4^{8}t_7^{8}t_{14}^{12}t_{28}^{8}$$

$$+ 12192768000t_1^{4}t_2^{18}t_3^{4}t_6^{2}t_7^{4}t_{14}^{18}t_{21}^{4}t_{42}^{2} + 48554311680t_2^{15}t_6^{27}t_{10}^{5}t_{30}^{9}$$

$$+ 4551966720t_3^{16}t_5^{10}t_6^{8}t_{10}^{5}t_{12}^{8}t_{15}^{2}t_{20}^{5}t_{30}t_{60} + 36415733760t_3^{12}t_5^{8}t_6^{6}t_{10}^{16}t_{15}^{12}t_{30}^{2}$$

$$+ 5534937600t_3^{8}t_5^{8}t_6^{12}t_{10}^{12}t_{12}^{8}t_{20}^{8} + 60692889600t_4^{20}t_{12}^{36} + 3034644480t_6^{32}t_{10}^{20}t_{30}^{4}$$

$$+ 6502809600t_1^{5}t_2^{10}t_3^{9}t_6^{2}t_7^{5}t_{14}^{10}t_{21}^{9}t_{42}^{2} + 36415733760t_3^{12}t_5^{5}t_6^{6}t_{10}^{10}t_{15}^{13}t_{30}^{4}$$

$$+ 5202247680t_1^{15}t_3^{3}t_5^{5}t_7^{15}t_{15}t_{21}^{3}t_{35}^{5}t_{105} + 9754214400t_1^{12}t_2^{6}t_5^{4}t_7^{12}t_{10}^{2}t_{14}^{6}t_{35}^{4}t_{70}^{2}$$

$$+ 9754214400t_1^{10}t_2^{5}t_3^{2}t_4^{5}t_6t_7^{10}t_{12}t_{14}^{5}t_{21}^{2}t_{28}^{5}t_{42}t_{84} + 8128512000t_1^{8}t_2^{4}t_3^{8}t_6^{4}t_7^{8}t_{14}^{4}t_{21}^{8}t_{42}^{4}$$

$$+ 2167603200t_1^{4}t_3^{20}t_7^{4}t_{21}^{20} + 6502809600t_2^{20}t_6^{4}t_{14}^{20}t_{42}^{4} + 9144576000t_2^{16}t_4^{8}t_{14}^{16}t_{28}^{8}$$

$$+ 68279500800t_2^{12}t_4^{6}t_6^{12}t_{10}^{4}t_{12}^{6}t_{20}^{2}t_{30}^{4}t_{60}^{2} + 75866112000t_2^{6}t_6^{30}t_{10}^{2}t_{30}^{10}$$

$$+ 45519667200t_3^{16}t_5^{4}t_6^{8}t_{10}^{2}t_{12}^{8}t_{15}^{4}t_{20}^{2}t_{30}^{2}t_{60}^{2} + 68279500800t_3^{8}t_5^{2}t_6^{12}t_{10}^{9}t_{12}^{8}t_{15}^{2}t_{20}^{5}t_{30}t_{60}$$

$$+ 54623600640t_3^{6}t_5^{16}t_6^{3}t_{10}^{8}t_{12}^{3}t_{15}^{2}t_{20}^{8}t_{30}t_{60} + 18207866880t_4^{30}t_{12}^{6}t_{20}^{10}t_{60}^{2}$$

$$+ 75866112000t_4^{8}t_{12}^{40} + 1902071808t_6^{32}t_{10}^{8}t_{30}^{8} + 4267468800t_6^{16}t_{10}^{16}t_{12}^{8}t_{20}^{8}$$

$$+ 2275983360t_6^{12}t_{10}^{32}t_{30}^4 + 97108623360t_6^{32}t_{10}^5t_{30}^9 + 109734912000t_1^4t_2^6t_4^{12}t_7^4t_{14}^6t_{28}^{12}$$
$$+ 48771072000t_1t_2^{12}t_3^5t_6^4t_7t_{14}^{12}t_{21}^5t_{42}^4 + 51209625600t_3^4t_5^4t_6^6t_{10}^6t_{12}^{12}t_{20}^{12}$$
$$+ 151732224000t_6^{32}t_{10}^2t_{30}^{10} + 31213486080t_1^9t_5^{11}t_7^9t_{35}^{11} + 133772083200t_1^3t_5t_7^{27}t_{35}^9$$
$$+ 146313216000t_1^2t_2^9t_3^2t_4^5t_6t_7^2t_{12}t_{14}^9t_{21}^2t_{28}^5t_{42}t_{84} + 6502896000t_1^2t_2^4t_3^{10}t_6^4t_7^2t_{14}^4t_{21}^{10}t_{42}^4$$
$$+ 250822656000t_1^2t_2t_4t_7^{18}t_{14}^9t_{28}^9 + 111476736000t_1^2t_3^2t_7^{18}t_{21}^{18}$$
$$+ 167215104000t_1t_2^2t_3t_7^9t_{14}^{18}t_{21}^9 + 18207866880t_2^9t_6^9t_{10}^{11}t_{30}^{11}$$
$$+ 10450944000t_2^4t_{14}^{36} + 54623600640t_3^6t_5^{10}t_6^3t_{10}^5t_{12}^3t_{15}^4t_{20}^5t_{30}^2t_{60}^2 + 6827950080t_4^{18}t_{20}^{22}$$
$$+ 2275983360t_6^{12}t_{10}^{20}t_{30}^8 + 136559001600t_6^{16}t_{10}^4t_{12}^8t_{20}^2t_{30}^4t_{60}^2$$
$$+ 117050572800t_1^6t_2^3t_4^3t_5^2t_7^6t_{10}t_{14}^3t_{20}t_{28}^3t_{35}^2t_{70}t_{140} + 52022476800t_1^6t_3^6t_5^2t_7^6t_{15}^2t_{21}^6t_{35}^2t_{105}^2$$
$$+ 97542144000t_1^4t_2^2t_3^4t_4^2t_6^2t_7^4t_{12}^2t_{14}^2t_{21}^4t_{28}^2t_{42}^2t_{84}^2 + 78033715200t_1^3t_2^6t_3^3t_5t_7^3t_{10}^2t_{14}^6t_{15}t_{21}^3t_{35}t_{70}^2t_{105}$$
$$+ 4877107200t_2^{12}t_{10}^4t_{14}^{12}t_{70}^4 + 40758681600t_2^8t_6^8t_{14}^8t_{42}^8 + 92897280t_3^8t_5^8t_{21}^8t_{35}^8$$
$$+ 150929049600t_4^{16}t_{28}^{16} + 45519667200t_4^{12}t_{12}^{12}t_{20}^4t_{60}^4 + 72831467520t_6^{12}t_{10}^8t_{30}^{12}$$
$$+ 70433556480t_{12}^{16}t_{20}^{16} + 72831467520t_6^{12}t_{10}^5t_{30}^{13} + 208089907200t_2^5t_6^9t_{14}^5t_{42}^9$$
$$+ 10404495360t_3^8t_5^5t_{15}t_{21}^8t_{35}^5t_{105} + 36415733760t_{12}^{16}t_{20}^{10}t_{60}^2$$
$$+ 292626432000t_2^4t_4^2t_6^4t_{12}^2t_{14}^4t_{28}^2t_{42}^4t_{84}^2 + 325140480000t_2^2t_6^{10}t_{14}^2t_{42}^{10}$$
$$+ 104044953600t_3^8t_5^2t_{15}^2t_{21}^8t_{35}^2t_{105}^2 + 19508428800t_3^4t_5^4t_6^2t_{10}^2t_{21}^4t_{35}^4t_{42}^2t_{70}^2$$
$$+ 124853944300t_3^3t_5^8t_{15}t_{21}^3t_{35}^8t_{105} + 78033715200t_4^{10}t_{12}^2t_{28}^{10}t_{84}^2 + 910393334400t_{12}^{16}t_{20}^4t_{60}^4$$
$$+ 27311800320t_{12}^6t_{20}^{16}t_{60}^2 + 334430208000t_2t_6t_{14}^9t_{42}^9$$
$$+ 156067430460t_3^4t_5t_6^2t_{10}^2t_{15}t_{21}^4t_{35}t_{42}^2t_{70}^2t_{105} + 124853944300t_3^3t_5^5t_{15}^2t_{21}^3t_{35}^5t_{105}^2$$
$$+ 267544166400t_3t_5t_{21}^9t_{35}^9 + 125411328000t_4^2t_{28}^{18} + 27311800320t_{12}^6t_{20}^{10}t_{60}^4$$
$$+ 156067430400t_2^3t_6^3t_{10}t_{14}^3t_{30}t_{42}^3t_{70}t_{210} + 234101145600t_3^2t_5^2t_6t_{10}t_{12}t_{20}t_{21}^2t_{35}^2t_{42}t_{70}t_{84}t_{140}$$
$$+ 58525286400t_4^6t_{20}^2t_{28}^6t_{140}^2 + 195084288000t_4^4t_{12}^4t_{28}^4t_{84}^4 + 9754214400t_6^4t_{10}^4t_{42}^4t_{70}^4$$
$$+ 312134860800t_6^4t_{10}t_{30}t_{42}^4t_{70}t_{210} + 117050572800t_{12}^2t_{20}^2t_{84}^2t_{140}^2].$$

The cycle indices obtained in this chapter can also be obtained by use of the GAP system as given in the appendix.

CHAPTER 6

CONCLUSION AND RECOMMENDATIONS

This chapter consists of the conclusion of this research and recommendations for further research.

6.1 Conclusion

This research was set to investigate the product action of $A_n \times A_n \times A_n$ on $X \times Y \times Z$ for particular cases when $n = 4, 5, 6, 7$ and 8 and its properties. This was done in Chapters 3,4 and 5.

Transitivity of the product action of $A_n \times A_n \times A_n$ on $X \times Y \times Z$ was determined in chapter 3 and the action was found to be transitive. In the same chapter the rank was found to be 8 and the subdegrees were 1,$(n-1)$,$(n-1)$,$(n-1)$,$(n-1)^2$,$(n-1)^2$,$(n-1)^2$,$(n-1)^3$.

In chapter four, non-trivial suborbital graphs were constructed. All the non-trivial suborbital graphs have a girth of 3 and are undirected. Some of the non-trivial suborbital graphs were found to be disconnected and in particular the suborbital graph Γ_7 corresponding to Δ_7 was found to be connected.

In chapter five, the cycle index formulas for the direct product of $A_n \times A_n \times A_n$ were derived as indicated in Equations 5.4, 5.7, 5.10, 5.13 and 5.16.

6.2 Recommendations

This research can be extended by considering the product action of $A_n \times A_n \times ... \times A_n$ $(n \geq 4)$ on $Y_1 \times Y_2 \times ... \times Y_n$.

REFERENCES

Akbas (2001). Suborbital graphs for the modular group. *Bulletin of the London Mathematical Society* **33**:647-652.

Armstrong (2013). Groups and symmetry. Springer-Verlag New York Inc.

Cameron, P. J. (1973). Primitive groups with most suborbits doubly transitive. *Geometriae Dedicata*, **1**: 434 – 446.

Cameron, P. J. (1999). *Permutation Groups.* London Mathematical Society Student Texts. Cambridge University Press.

Cameron, P. J., Gewurz, D. A. and Merola, F. (2007). Product action. *Discrete Mathematics*, **308**(2): 386 – 394.

Cameron, P. J., Maimani, H., Omidi, G. and Tayfeh-Rezaie, B. (2006). 3-designs from $PSL(2, q)$. *Discrete Mathematics*, **306**(23): 3063 – 3073.

Coxeter, H. S. (1983). My graph. *Proceedings of London Mathematical Society*, **46**: 117–136.

Gachago, J. K., Kinyanjui, J., Rimberia, J., Kimani, P. and Muchemi, Jacob, K. (2015). Transitivity action of A_n $(n = 4, 5, 6, 7)$ on unordered and ordered quadrupples. *Mathematial Theory and Modeling*, **5**(6), 111 – 123.

Higman, D. G. (1964). Finite permutation groups of rank 3. *Mathematische Zeitschrift*, **86**: 145–156.

Higman, D. (1967). Intersection matrices for finite permutation groups. *Journal of Algebra*, **6**(1): 22 – 42.

Higman, D. (1970). Characterization of families of rank 3 permutation groups by subdegrees. *Arch. Math*, **21**: 151 – 156.

Kamuti, I. N. (1992). *Combinatorial formulas, invariants and structures associated with primitive permutation representations of $PGL(2, q)$ and $PSL(2, q)$.* PhD thesis, University of Southampton.

Kamuti, I. N. (2006). Subdegrees of primitive permutation representations of $PGL(2, q)$. *East African Journal of Physical Sciences*, **7**: 25–41.

Kamuti, I. N. and Njuguna, L. (2004). On the cycle index of the reduced ordered $r-$group. *East African Journal of Physical Sciences*, **5(2)**: 99–108.

Kamuti, I. N. and Obong'o, J. (2002). The derivation of cycle index of $S_n^{[3]}$. *Questionnes Mathematicae*, **25**: 437–444.

Kamuti, I. N., Inyangala, E. B. and Rimberia, J. K. (2012). Action of Γ_∞ on $\mathbb{Z}$ and the corresponding suborbital graphs. *International Mathematical Forum*, **7**: 1483–1490.

Kimani, P., Rimberia, J., Lao, H. and Muthoka, G. (2014). Application of marks to computation of ranks and subdegrees of the symmetric group acting on ordered pairs and on ordered triples. *Mathematial Theory and Modeling*, **4**(13): 95 – 102.

Krishnamurthy (1985). *Combinatorics, theory and applications.* Affiliated East-West press Private Limited, New Delhi.

Lauri, J. and Mizzi, R. (2011). Two-fold automorphisms of graphs. *Australasian Journal of Combinatorics*, **49**: 165–176.

Magero, F. B. (2015). *Cycle indices, subdegrees and suborbital graphs of $PSL(2, q)$ acting on the cosets of some of its subgroups.* PhD thesis, Kenyatta University.

Muthoka, G., Kamuti, I., Lao, H. and Kimani, P. (2015). Cycle index formulas for D_n acting on unordered pairs. *Mathematial Theory and Modeling*, **5**(1): 16 – 22.

Nyaga, L., Kamuti, I., Mwathi, C. and Akanga, J. (2011). Ranks and subdegrees of the symmetric group acting on unordered element subsets. *International Electronic Journal of Pure and Applied Mathematics*, **3**: 147–163.

Pólya, G. (1937). Kombinatorishe anzahlbestimmungen fur gruppen, graphen and chemische verbindungen. *Acta Mathematica*, **68**: 145–254.

Redfield, J. H. (1927). The theory of group-reduced distributions. *American Journal of Mathematics*, **49**: 433–455.

Rimberia, J. K. (2012). *Ranks and subdegrees of the symmetric groups S_n acting on ordered r-element subsets.* Ph.d thesis, Kenyatta University.

Rose, J. S. (1978). *A Course on Groups Theory.* Cambridge University press, Cambridge.

Rotich, S. K. (2016). *Cycle indices, subdegrees and suborbital graphs of $PGL(2, q)$ acting on the cosets of its subgroups.* PhD thesis, Kenyatta University.

Sims, C. C. (1967). Graphs and finite permutation groups. *Mathematica Zeitchrift*, **95**: 76–86.

Wilson, J. and Watkins, J. (1990). *Graphs: an introductory approach : a first course in discrete mathematics.* Wiley.

APPENDIX

Cycle index of $A_4 \times A_4 \times A_4$ acting on $X \times Y \times Z$

```
gap> g:=AlternatingGroup(4);
Alt( [ 1 .. 4 ] )
gap> G:=DirectProduct(g,g,g);
Group([ (1,2,3), (2,3,4), (5,6,7), (6,7,8), (9,10,11), (10,11,12) ])
gap> O:=Orbit(G,(1,5,9));
[ (1,5,9), (2,5,9), (1,6,9), (1,5,10), (3,5,9), (2,6,9), (2,5,10), (1,7,9), (1,6,10),
(1,5,11), (4,5,9), (3,6,9),
  (3,5,10), (2,7,9), (2,6,10), (2,5,11), (1,8,9), (1,7,10), (1,6,11), (1,5,12),
(4,6,9), (4,5,10), (3,7,9), (3,6,10),
  (3,5,11), (2,8,9), (2,7,10), (2,6,11), (2,5,12), (1,8,10), (1,7,11), (1,6,12),
(4,7,9), (4,6,10), (4,5,11),
  (3,8,9), (3,7,10), (3,6,11), (3,5,12), (2,8,10), (2,7,11), (2,6,12), (1,8,11),
(1,7,12), (4,8,9), (4,7,10),
  (4,6,11), (4,5,12), (3,8,10), (3,7,11), (3,6,12), (2,8,11), (2,7,12), (1,8,12),
(4,8,10), (4,7,11), (4,6,12),
  (3,8,11), (3,7,12), (2,8,12), (4,8,11), (4,7,12), (3,8,12), (4,8,12) ]
gap> C:=CycleIndex(G,O);
1/1728*x_1^64+1/72*x_1^16*x_3^16+7/192*x_2^32+1/9*x_1^4*x_3^20+8/27*x_1*x_3^21+5/24*x_
2^8*x_6^8+1/3*x_2^2*x_6^10
```

Cycle index of $A_5 \times A_5 \times A_5$ acting on $X \times Y \times Z$

```
gap> h:=AlternatingGroup(5);
Alt( [ 1 .. 5 ] )
gap> H:=DirectProduct(h,h,h);
Group([ (1,2,3,4,5), (3,4,5), (6,7,8,9,10), (8,9,10), (11,12,13,14,15), (13,14,15) ])
gap> O1:=Orbit(H,(1,6,11));
[ (1,6,11), (2,6,11), (1,7,11), (1,6,12), (3,6,11), (2,7,11), (2,6,12), (1,8,11),
(1,7,12), (1,6,13), (4,6,11),
  (3,7,11), (3,6,12), (2,8,11), (2,7,12), (2,6,13), (1,9,11), (1,8,12), (1,7,13),
(1,6,14), (5,6,11), (4,7,11),
  (4,6,12), (3,8,11), (3,7,12), (3,6,13), (2,9,11), (2,8,12), (2,7,13), (2,6,14),
(1,10,11), (1,9,12), (1,8,13),
  (1,7,14), (1,6,15), (5,7,11), (5,6,12), (4,8,11), (4,7,12), (4,6,13), (3,9,11),
(3,8,12), (3,7,13), (3,6,14),
  (2,10,11), (2,9,12), (2,8,13), (2,7,14), (2,6,15), (1,10,12), (1,9,13), (1,8,14),
(1,7,15), (5,8,11), (5,7,12),
  (5,6,13), (4,9,11), (4,8,12), (4,7,13), (4,6,14), (3,10,11), (3,9,12), (3,8,13),
(3,7,14), (3,6,15), (2,10,12),
  (2,9,13), (2,8,14), (2,7,15), (1,10,13), (1,9,14), (1,8,15), (5,9,11), (5,8,12),
(5,7,13), (5,6,14), (4,10,11),
  (4,9,12), (4,8,13), (4,7,14), (4,6,15), (3,10,12), (3,9,13), (3,8,14), (3,7,15),
(2,10,13), (2,9,14), (2,8,15),
  (1,10,14), (1,9,15), (5,10,11), (5,9,12), (5,8,13), (5,7,14), (5,6,15), (4,10,12),
(4,9,13), (4,8,14), (4,7,15),
  (3,10,13), (3,9,14), (3,8,15), (2,10,14), (2,9,15), (1,10,15), (5,10,12), (5,9,13),
(5,8,14), (5,7,15), (4,10,13),
  (4,9,14), (4,8,15), (3,10,14), (3,9,15), (2,10,15), (5,10,13), (5,9,14), (5,8,15),
(4,10,14), (4,9,15), (3,10,15),
  (5,10,14), (5,9,15), (4,10,15), (5,10,15) ]
gap> C1:=CycleIndex(H,O1);
1/216000*x_1^125+1/3600*x_1^50*x_3^25+1/4800*x_1^25*x_2^50+1/320*x_1^5*x_2^60+1/64*x_1
*x_2^62+1/180*x_1^20*x_3^35+1/27\
*x_1^8*x_3^39+1/120*x_1^10*x_2^20*x_3^5*x_6^10+1/16*x_1^2*x_2^24*x_3*x_6^12+1/12*x_1^4
*x_2^8*x_3^7*x_6^14+217/3000*x_5\
^25+13/75*x_5^10*x_15^5+13/100*x_5^5*x_10^10+3/40*x_5*x_10^12+2/15*x_5^4*x_15^7+1/5*x_
5^2*x_10^4*x_15*x_30^2
```

Cycle index of $A_6 \times A_6 \times A_6$ acting on $X \times Y \times Z$

```
gap> g2:=AlternatingGroup(6);
Alt( [ 1 .. 6 ] )
gap> G1:=DirectProduct(g2,g2,g2);
<permutation group of size 46656000 with 6 generators>
gap> O2:=Orbit(G1,(1,7,13));
[ (1,7,13), (2,7,13), (1,8,13), (1,7,14), (3,7,13), (2,8,13), (2,7,14), (1,9,13),
(1,8,14), (1,7,15), (4,7,13),
  (3,8,13), (3,7,14), (2,9,13), (2,8,14), (2,7,15), (1,10,13), (1,9,14), (1,8,15),
(1,7,16), (5,7,13), (4,8,13),
  (4,7,14), (3,9,13), (3,8,14), (3,7,15), (2,10,13), (2,9,14), (2,8,15), (2,7,16),
(1,11,13), (1,10,14), (1,9,15),
  (1,8,16), (1,7,17), (6,7,13), (5,8,13), (5,7,14), (4,9,13), (4,8,14), (4,7,15),
(3,10,13), (3,9,14), (3,8,15),
  (3,7,16), (2,11,13), (2,10,14), (2,9,15), (2,8,16), (2,7,17), (1,12,13), (1,11,14),
(1,10,15), (1,9,16), (1,8,17),
  (1,7,18), (6,8,13), (6,7,14), (5,9,13), (5,8,14), (5,7,15), (4,10,13), (4,9,14),
(4,8,15), (4,7,16), (3,11,13),
  (3,10,14), (3,9,15), (3,8,16), (3,7,17), (2,12,13), (2,11,14), (2,10,15), (2,9,16),
(2,8,17), (2,7,18), (1,12,14),
  (1,11,15), (1,10,16), (1,9,17), (1,8,18), (6,9,13), (6,8,14), (6,7,15), (5,10,13),
(5,9,14), (5,8,15), (5,7,16),
  (4,11,13), (4,10,14), (4,9,15), (4,8,16), (4,7,17), (3,12,13), (3,11,14), (3,10,15),
(3,9,16), (3,8,17), (3,7,18),
  (2,12,14), (2,11,15), (2,10,16), (2,9,17), (2,8,18), (1,12,15), (1,11,16),
(1,10,17), (1,9,18), (6,10,13),
  (6,9,14), (6,8,15), (6,7,16), (5,11,13), (5,10,14), (5,9,15), (5,8,16), (5,7,17),
(4,12,13), (4,11,14), (4,10,15),
  (4,9,16), (4,8,17), (4,7,18), (3,12,14), (3,11,15), (3,10,16), (3,9,17), (3,8,18),
(2,12,15), (2,11,16), (2,10,17),
  (2,9,18), (1,12,16), (1,11,17), (1,10,18), (6,11,13), (6,10,14), (6,9,15), (6,8,16),
(6,7,17), (5,12,13),
  (5,11,14), (5,10,15), (5,9,16), (5,8,17), (5,7,18), (4,12,14), (4,11,15), (4,10,16),
(4,9,17), (4,8,18), (3,12,15),
  (3,11,16), (3,10,17), (3,9,18), (2,12,16), (2,11,17), (2,10,18), (1,12,17),
(1,11,18), (6,12,13), (6,11,14),
  (6,10,15), (6,9,16), (6,8,17), (6,7,18), (5,12,14), (5,11,15), (5,10,16), (5,9,17),
(5,8,18), (4,12,15), (4,11,16),
  (4,10,17), (4,9,18), (3,12,16), (3,11,17), (3,10,18), (2,12,17), (2,11,18),
(1,12,18), (6,12,14), (6,11,15),
  (6,10,16), (6,9,17), (6,8,18), (5,12,15), (5,11,16), (5,10,17), (5,9,18), (4,12,16),
(4,11,17), (4,10,18),
  (3,12,17), (3,11,18), (2,12,18), (6,12,15), (6,11,16), (6,10,17), (6,9,18),
(5,12,16), (5,11,17), (5,10,18),
  (4,12,17), (4,11,18), (3,12,18), (6,12,16), (6,11,17), (6,10,18), (5,12,17),
(5,11,18), (4,12,18), (6,12,17),
  (6,11,18), (5,12,18), (6,12,18) ]
gap> C2:=CycleIndex(G1,O2);
1/46656000*x_1^216+1/388800*x_1^108*x_3^36+1/345600*x_1^72*x_2^72+1/7680*x_1^24*x_2^96
+1/512*x_1^8*x_2^104+1/9720*x_1^\
54*x_3^54+1/4320*x_1^36*x_2^36*x_3^12*x_6^12+1/729*x_1^27*x_3^63+1/192*x_1^12*x_2^48*x
_3^4*x_6^16+1/108000*x_1^36*x_5^\
36+1/216*x_1^18*x_2^18*x_3^18*x_6^18+529/43200*x_2^36*x_4^36+11563/1166400*x_3^72+23/9
60*x_2^12*x_4^48+1/64*x_2^4*x_4^\
52+1/1350*x_1^18*x_3^6*x_5^18*x_15^6+1/1200*x_1^12*x_2^12*x_5^12*x_10^12+1/750*x_1^6*x
_5^42+23/1080*x_2^18*x_4^18*x_6^\
6*x_12^6+61/4320*x_3^24*x_6^24+8/125*x_1*x_5^43+3/160*x_1^4*x_2^16*x_5^4*x_10^16+1/48*
x_2^6*x_4^24*x_6^2*x_12^8+1/192*\
x_3^8*x_6^32+2/135*x_1^9*x_3^9*x_5^9*x_15^9+1/108*x_2^9*x_4^9*x_6^9*x_12^9+1/30*x_1^6*
x_2^6*x_3^2*x_5^6*x_6^2*x_10^6*x\
_15^2*x_30^2+4/75*x_1^3*x_3*x_5^21*x_15^7+3/50*x_1^2*x_2^2*x_5^14*x_10^14+23/300*x_2^6
*x_4^6*x_10^6*x_20^6+61/1350*x_3\
^12*x_15^12+53/1080*x_6^12*x_12^12+3/40*x_2^2*x_4^8*x_10^2*x_20^8+1/48*x_6^4*x_12^16+1
/15*x_2^3*x_4^3*x_6*x_10^3*x_12*\
x_20^3*x_30*x_60+3/25*x_2*x_4*x_10^7*x_20^7+1/30*x_3^4*x_6^4*x_15^4*x_30^4+4/75*x_3^2*
x_15^14+1/15*x_6^2*x_12^2*x_30^2\
*x_60^2
```

Cycle index of $A_7 \times A_7 \times A_7$ acting on $X \times Y \times Z$

```
gap> g3:=AlternatingGroup(7);
Alt( [ 1 .. 7 ] )
gap> G2:=DirectProduct(g3,g3,g3);
<permutation group of size 16003008000 with 6 generators>
gap> O3:=Orbit(G2,(1,8,15));
[ (1,8,15), (2,8,15), (1,9,15), (1,8,16), (3,8,15), (2,9,15), (2,8,16), (1,10,15),
(1,9,16), (1,8,17), (4,8,15),   (3,9,15), (3,8,16), (2,10,15), (2,9,16), (2,8,17), (1,11,15), (1,10,16), (1,9,17),
(1,8,18), (5,8,15), (4,9,15),
  (4,8,16), (3,10,15), (3,9,16), (3,8,17), (2,11,15), (2,10,16), (2,9,17), (2,8,18),
(1,12,15), (1,11,16), (1,10,17),
  (1,9,18), (1,8,19), (6,8,15), (5,9,15), (5,8,16), (4,10,15), (4,9,16), (4,8,17),
(3,11,15), (3,10,16), (3,9,17),
  (3,8,18), (2,12,15), (2,11,16), (2,10,17), (2,9,18), (2,8,19), (1,13,15), (1,12,16),
(1,11,17), (1,10,18),
  (1,9,19), (1,8,20), (7,8,15), (6,9,15), (6,8,16), (5,10,15), (5,9,16), (5,8,17),
(4,11,15), (4,10,16), (4,9,17),
  (4,8,18), (3,12,15), (3,11,16), (3,10,17), (3,9,18), (3,8,19), (2,13,15), (2,12,16),
(2,11,17), (2,10,18),
  (2,9,19), (2,8,20), (1,14,15), (1,13,16), (1,12,17), (1,11,18), (1,10,19), (1,9,20),
(1,8,21), (7,9,15), (7,8,16),
  (6,10,15), (6,9,16), (6,8,17), (5,11,15), (5,10,16), (5,9,17), (5,8,18), (4,12,15),
(4,11,16), (4,10,17), (4,9,18),
  (4,8,19), (3,13,15), (3,12,16), (3,11,17), (3,10,18), (3,9,19), (3,8,20), (2,14,15),
(2,13,16), (2,12,17),
  (2,11,18), (2,10,19), (2,9,20), (2,8,21), (1,14,16), (1,13,17), (1,12,18),
(1,11,19), (1,10,20), (1,9,21),
  (7,10,15), (7,9,16), (7,8,17), (6,11,15), (6,10,16), (6,9,17), (6,8,18), (5,12,15),
(5,11,16), (5,10,17), (5,9,18),
  (5,8,19), (4,13,15), (4,12,16), (4,11,17), (4,10,18), (4,9,19), (4,8,20), (3,14,15),
(3,13,16), (3,12,17),
  (3,11,18), (3,10,19), (3,9,20), (3,8,21), (2,14,16), (2,13,17), (2,12,18),
(2,11,19), (2,10,20), (2,9,21),
  (1,14,17), (1,13,18), (1,12,19), (1,11,20), (1,10,21), (7,11,15), (7,10,16),
(7,9,17), (7,8,18), (6,12,15),
  (6,11,16), (6,10,17), (6,9,18), (6,8,19), (5,13,15), (5,12,16), (5,11,17),
(5,10,18), (5,9,19), (5,8,20),
  (4,14,15), (4,13,16), (4,12,17), (4,11,18), (4,10,19), (4,9,20), (4,8,21),
(3,14,16), (3,13,17), (3,12,18),
  (3,11,19), (3,10,20), (3,9,21), (2,14,17), (2,13,18), (2,12,19), (2,11,20),
(2,10,21), (1,14,18), (1,13,19),
  (1,12,20), (1,11,21), (7,12,15), (7,11,16), (7,10,17), (7,9,18), (7,8,19),
(6,13,15), (6,12,16), (6,11,17),
  (6,10,18), (6,9,19), (6,8,20), (5,14,15), (5,13,16), (5,12,17), (5,11,18),
(5,10,19), (5,9,20), (5,8,21),
  (4,14,16), (4,13,17), (4,12,18), (4,11,19), (4,10,20), (4,9,21), (3,14,17),
(3,13,18), (3,12,19), (3,11,20),
  (3,10,21), (2,14,18), (2,13,19), (2,12,20), (2,11,21), (1,14,19), (1,13,20),
(1,12,21), (7,13,15), (7,12,16),
  (7,11,17), (7,10,18), (7,9,19), (7,8,20), (6,14,15), (6,13,16), (6,12,17),
(6,11,18), (6,10,19), (6,9,20),
  (6,8,21), (5,14,16), (5,13,17), (5,12,18), (5,11,19), (5,10,20), (5,9,21),
(4,14,17), (4,13,18), (4,12,19),
  (4,11,20), (4,10,21), (3,14,18), (3,13,19), (3,12,20), (3,11,21), (2,14,19),
(2,13,20), (2,12,21), (1,14,20),
  (1,13,21), (7,14,15), (7,13,16), (7,12,17), (7,11,18), (7,10,19), (7,9,20),
(7,8,21), (6,14,16), (6,13,17),
  (6,12,18), (6,11,19), (6,10,20), (6,9,21), (5,14,17), (5,13,18), (5,12,19),
(5,11,20), (5,10,21), (4,14,18),
  (4,13,19), (4,12,20), (4,11,21), (3,14,19), (3,13,20), (3,12,21), (2,14,20),
(2,13,21), (1,14,21), (7,14,16),
  (7,13,17), (7,12,18), (7,11,19), (7,10,20), (7,9,21), (6,14,17), (6,13,18),
(6,12,19), (6,11,20), (6,10,21),
  (5,14,18), (5,13,19), (5,12,20), (5,11,21), (4,14,19), (4,13,20), (4,12,21),
(3,14,20), (3,13,21), (2,14,21),
  (7,14,17), (7,13,18), (7,12,19), (7,11,20), (7,10,21), (6,14,18), (6,13,19),
(6,12,20), (6,11,21), (5,14,19),
```

```
  (5,13,20), (5,12,21), (4,14,20), (4,13,21), (3,14,21), (7,14,18), (7,13,19),
(7,12,20), (7,11,21), (6,14,19),
  (6,13,20), (6,12,21), (5,14,20), (5,13,21), (4,14,21), (7,14,19), (7,13,20),
(7,12,21), (6,14,20), (6,13,21),
  (5,14,21), (7,14,20), (7,13,21), (6,14,21), (7,14,21) ]
gap> C3:=CycleIndex(G2,O3);
1/16003008000*x_1^343+1/76204800*x_1^196*x_3^49+1/50803200*x_1^147*x_2^98+1/483840*x_1
^63*x_2^140+1/1088640*x_1^112*x_\
3^77+1/13824*x_1^27*x_2^158+1/362880*x_1^84*x_2^56*x_3^21*x_6^14+1/46656*x_1^64*x_3^93
+1/10584000*x_1^98*x_5^49+1/8467\
200*x_1^49*x_2^49*x_4^49+1/19051200*x_1^49*x_3^98+1/25401600*x_2^98*x_3^49+1/6912*x_1^
36*x_2^80*x_3^9*x_6^20+1/10368*x\
_1^48*x_2^32*x_3^33*x_6^22+1/136080*x_1^28*x_3^105+1/40320*x_1^21*x_2^63*x_4^49+1/1209
60*x_2^98*x_3^21*x_6^14+1/768*x_\
1^9*x_2^69*x_4^49+1/2304*x_2^98*x_3^9*x_6^20+1/3888*x_1^16*x_3^109+1/68040*x_1^7*x_3^1
12+1/181440*x_2^56*x_3^49*x_6^14\
+1/972*x_1^4*x_3^113+1/729*x_1*x_3^114+1/75600*x_1^56*x_3^14*x_5^28*x_15^7+1/50400*x_1
^42*x_2^28*x_5^21*x_10^14+1/6048\
0*x_1^28*x_2^28*x_3^7*x_4^28*x_6^7*x_12^7+1/90720*x_1^21*x_2^14*x_3^42*x_6^28+1/13440*
x_1^7*x_2^28*x_4^70+71/120960*x_\
2^56*x_3^21*x_6^28+1/128*x_1^3*x_2^30*x_4^70+1/5184*x_2^32*x_3^49*x_6^22+1/1152*x_2^56
*x_3^9*x_6^34+1/576*x_1^12*x_2^3\
6*x_3^3*x_4^28*x_6^9*x_12^7+1/1296*x_1^12*x_2^8*x_3^45*x_6^30+1/64*x_1*x_2^13*x_4^79+1
/21000*x_1^28*x_5^63+1/20160*x_2\
^42*x_3^7*x_4^28*x_6^7*x_12^7+1/45360*x_2^14*x_3^49*x_6^28+1/192*x_2^42*x_3^3*x_4^28*x
_6^9*x_12^7+1/1728*x_2^32*x_3^21\
*x_6^36+1/960*x_1^18*x_2^40*x_5^9*x_10^20+1/1728*x_1^9*x_2^20*x_3^18*x_6^40+1/648*x_2^
8*x_3^49*x_6^30+1/648*x_1^3*x_2^\
2*x_3^48*x_6^32+1/1728*x_2^32*x_3^9*x_6^42+1/324*x_2^2*x_3^49*x_6^32+1/2160*x_1^32*x_3
^22*x_5^16*x_15^11+1/1728*x_1^16\
*x_2^16*x_3^11*x_4^16*x_6^11*x_12^11+1/432*x_2^14*x_3^21*x_6^42+1/720*x_1^24*x_2^16*x_
3^6*x_5^12*x_6^4*x_10^8*x_15^3*x\
_30^2+1/125*x_1^8*x_5^67+1/192*x_1^4*x_2^16*x_3*x_4^40*x_6^4*x_12^10+1/64*x_2^18*x_3*x
_4^40*x_6^4*x_12^10+1/432*x_2^8*\
x_3^21*x_6^44+1/288*x_2^24*x_3^7*x_4^16*x_6^13*x_12^11+1/192*x_2^24*x_3^3*x_4^16*x_6^1
5*x_12^11+1/300*x_1^16*x_3^4*x_5\
^36*x_15^9+1/200*x_1^12*x_2^8*x_5^27*x_10^18+1/8400*x_1^14*x_2^14*x_4^14*x_5^7*x_10^7*
x_20^7+1/18900*x_1^14*x_3^28*x_5\
^7*x_15^14+1/15120*x_1^7*x_2^7*x_3^14*x_4^7*x_6^14*x_12^14+1/25200*x_2^28*x_3^14*x_10^
14*x_15^7+1/270*x_1^8*x_3^30*x_5\
^4*x_15^15+1/80*x_1^6*x_2^18*x_4^14*x_5^3*x_10^9*x_20^7+1/216*x_1^4*x_2^4*x_3^15*x_4^4
*x_6^15*x_12^15+1/144*x_1^3*x_2^\
9*x_3^6*x_4^7*x_6^18*x_12^14+1/240*x_2^28*x_3^6*x_6^4*x_10^14*x_15^3*x_30^2+1/135*x_1^
2*x_3^32*x_5*x_15^16+1/108*x_1*x\
_2*x_3^16*x_4*x_6^16*x_12^16+1/360*x_2^16*x_3^14*x_6^4*x_10^8*x_15^7*x_30^2+1/72*x_2^6
*x_3^7*x_4^4*x_6^19*x_12^15+1735\
21/7408800*x_7^49+1/120*x_1^8*x_2^8*x_3^2*x_4^8*x_5^4*x_6^2*x_10^4*x_12^2*x_15*x_20^4*
x_30*x_60+1/180*x_1^6*x_2^4*x_3^\
12*x_5^3*x_6^8*x_10^2*x_15^6*x_30^4+3/80*x_1^2*x_2^8*x_4^20*x_5*x_10^4*x_20^10+1/48*x_
1*x_2^4*x_3^2*x_4^10*x_6^8*x_12^\
20+1/240*x_2^16*x_3^6*x_6^8*x_10^8*x_15^3*x_30^4+3/100*x_1^4*x_2^4*x_4^4*x_5^9*x_10^9*
x_20^9+1/75*x_1^4*x_3^8*x_5^9*x_\
15^18+1/40*x_2^12*x_3^2*x_4^8*x_6^2*x_10^6*x_12^2*x_15*x_20^4*x_30*x_60+1/100*x_2^8*x_
3^4*x_10^18*x_15^9+1/90*x_2^4*x_\
3^14*x_6^8*x_10^2*x_15^7*x_30^4+361/52920*x_7^28*x_21^7+361/35280*x_7^21*x_14^14+1/672
*x_7^9*x_14^20+1/30*x_1^2*x_2^2*\
x_3^4*x_4^2*x_5*x_6^4*x_10*x_12^4*x_15^2*x_20*x_30^2*x_60^2+1/1512*x_7^16*x_21^11+1/50
4*x_7^12*x_14^8*x_21^3*x_42^2+36\
1/7350*x_7^14*x_35^7+361/5880*x_7^7*x_14^7*x_28^7+361/13230*x_7^7*x_21^14+361/17640*x_
14^14*x_21^7+1/189*x_7^4*x_21^15\
+1/56*x_7^3*x_14^9*x_28^7+1/168*x_14^14*x_21^3*x_42^2+2/189*x_7*x_21^16+1/252*x_14^8*x
_21^7*x_42^2+1/105*x_7^8*x_21^2*\
x_35^4*x_105+1/70*x_7^6*x_14^4*x_35^3*x_70^2+1/84*x_7^4*x_14^4*x_21*x_28^4*x_42*x_84+1
/126*x_7^3*x_14^2*x_21^6*x_42^4+\
3/56*x_7*x_14^4*x_28^10+1/168*x_14^8*x_21^3*x_42^4+6/175*x_7^4*x_35^9+1/28*x_14^6*x_21
*x_28^4*x_42*x_84+1/63*x_14^2*x_\
21^7*x_42^4+3/35*x_7^2*x_14^2*x_28^2*x_35*x_70*x_140+4/105*x_7^2*x_21^4*x_35*x_105^2+1
/21*x_7*x_14*x_21^2*x_28*x_42^2*\
x_84^2+1/35*x_14^4*x_21^2*x_70^2*x_105
```

Cycle index of $A_8 \times A_8 \times A_8$ acting on $X \times Y \times Z$

```
gap> g4:=AlternatingGroup(8);
Alt( [ 1 .. 8 ] )
gap> G3:=DirectProduct(g4,g4,g4);
<permutation group of size 8193540096000 with 6 generators>
gap> O4:=Orbit(G3,(1,9,17));
[ (1,9,17), (2,9,17), (1,10,17), (1,9,18), (3,9,17), (2,10,17), (2,9,18), (1,11,17),
(1,10,18), (1,9,19), (4,9,17),
  (3,10,17), (3,9,18), (2,11,17), (2,10,18), (2,9,19), (1,12,17), (1,11,18),
(1,10,19), (1,9,20), (5,9,17),
  (4,10,17), (4,9,18), (3,11,17), (3,10,18), (3,9,19), (2,12,17), (2,11,18),
(2,10,19), (2,9,20), (1,13,17),
  (1,12,18), (1,11,19), (1,10,20), (1,9,21), (6,9,17), (5,10,17), (5,9,18), (4,11,17),
(4,10,18), (4,9,19),
  (3,12,17), (3,11,18), (3,10,19), (3,9,20), (2,13,17), (2,12,18), (2,11,19),
(2,10,20), (2,9,21), (1,14,17),
  (1,13,18), (1,12,19), (1,11,20), (1,10,21), (1,9,22), (7,9,17), (6,10,17), (6,9,18),
(5,11,17), (5,10,18),
  (5,9,19), (4,12,17), (4,11,18), (4,10,19), (4,9,20), (3,13,17), (3,12,18),
(3,11,19), (3,10,20), (3,9,21),
  (2,14,17), (2,13,18), (2,12,19), (2,11,20), (2,10,21), (2,9,22), (1,15,17),
(1,14,18), (1,13,19), (1,12,20),
  (1,11,21), (1,10,22), (1,9,23), (8,9,17), (7,10,17), (7,9,18), (6,11,17), (6,10,18),
(6,9,19), (5,12,17),
  (5,11,18), (5,10,19), (5,9,20), (4,13,17), (4,12,18), (4,11,19), (4,10,20),
(4,9,21), (3,14,17), (3,13,18),
  (3,12,19), (3,11,20), (3,10,21), (3,9,22), (2,15,17), (2,14,18), (2,13,19),
(2,12,20), (2,11,21), (2,10,22),
  (2,9,23), (1,16,17), (1,15,18), (1,14,19), (1,13,20), (1,12,21), (1,11,22),
(1,10,23), (1,9,24), (8,10,17),
  (8,9,18), (7,11,17), (7,10,18), (7,9,19), (6,12,17), (6,11,18), (6,10,19), (6,9,20),
(5,13,17), (5,12,18),
  (5,11,19), (5,10,20), (5,9,21), (4,14,17), (4,13,18), (4,12,19), (4,11,20),
(4,10,21), (4,9,22), (3,15,17),
  (3,14,18), (3,13,19), (3,12,20), (3,11,21), (3,10,22), (3,9,23), (2,16,17),
(2,15,18), (2,14,19), (2,13,20),
  (2,12,21), (2,11,22), (2,10,23), (2,9,24), (1,16,18), (1,15,19), (1,14,20),
(1,13,21), (1,12,22), (1,11,23),
  (1,10,24), (8,11,17), (8,10,18), (8,9,19), (7,12,17), (7,11,18), (7,10,19),
(7,9,20), (6,13,17), (6,12,18),
  (6,11,19), (6,10,20), (6,9,21), (5,14,17), (5,13,18), (5,12,19), (5,11,20),
(5,10,21), (5,9,22), (4,15,17),
  (4,14,18), (4,13,19), (4,12,20), (4,11,21), (4,10,22), (4,9,23), (3,16,17),
(3,15,18), (3,14,19), (3,13,20),
  (3,12,21), (3,11,22), (3,10,23), (3,9,24), (2,16,18), (2,15,19), (2,14,20),
(2,13,21), (2,12,22), (2,11,23),
  (2,10,24), (1,16,19), (1,15,20), (1,14,21), (1,13,22), (1,12,23), (1,11,24),
(8,12,17), (8,11,18), (8,10,19),
  (8,9,20), (7,13,17), (7,12,18), (7,11,19), (7,10,20), (7,9,21), (6,14,17),
(6,13,18), (6,12,19), (6,11,20),
  (6,10,21), (6,9,22), (5,15,17), (5,14,18), (5,13,19), (5,12,20), (5,11,21),
(5,10,22), (5,9,23), (4,16,17),
  (4,15,18), (4,14,19), (4,13,20), (4,12,21), (4,11,22), (4,10,23), (4,9,24),
(3,16,18), (3,15,19), (3,14,20),
  (3,13,21), (3,12,22), (3,11,23), (3,10,24), (2,16,19), (2,15,20), (2,14,21),
(2,13,22), (2,12,23), (2,11,24),
  (1,16,20), (1,15,21), (1,14,22), (1,13,23), (1,12,24), (8,13,17), (8,12,18),
(8,11,19), (8,10,20), (8,9,21),
  (7,14,17), (7,13,18), (7,12,19), (7,11,20), (7,10,21), (7,9,22), (6,15,17),
(6,14,18), (6,13,19), (6,12,20),
  (6,11,21), (6,10,22), (6,9,23), (5,16,17), (5,15,18), (5,14,19), (5,13,20),
(5,12,21), (5,11,22), (5,10,23),
  (5,9,24), (4,16,18), (4,15,19), (4,14,20), (4,13,21), (4,12,22), (4,11,23),
(4,10,24), (3,16,19), (3,15,20),
  (3,14,21), (3,13,22), (3,12,23), (3,11,24), (2,16,20), (2,15,21), (2,14,22),
(2,13,23), (2,12,24), (1,16,21),
  (1,15,22), (1,14,23), (1,13,24), (8,14,17), (8,13,18), (8,12,19), (8,11,20),
(8,10,21), (8,9,22), (7,15,17),
  (7,14,18), (7,13,19), (7,12,20), (7,11,21), (7,10,22), (7,9,23), (6,16,17),
```

```
(6,15,18), (6,14,19), (6,13,20),
  (6,12,21), (6,11,22), (6,10,23), (6,9,24), (5,16,18), (5,15,19), (5,14,20),
(5,13,21), (5,12,22), (5,11,23),
  (5,10,24), (4,16,19), (4,15,20), (4,14,21), (4,13,22), (4,12,23), (4,11,24),
(3,16,20), (3,15,21), (3,14,22),
  (3,13,23), (3,12,24), (2,16,21), (2,15,22), (2,14,23), (2,13,24), (1,16,22),
(1,15,23), (1,14,24), (8,15,17),
  (8,14,18), (8,13,19), (8,12,20), (8,11,21), (8,10,22), (8,9,23), (7,16,17),
(7,15,18), (7,14,19), (7,13,20),
  (7,12,21), (7,11,22), (7,10,23), (7,9,24), (6,16,18), (6,15,19), (6,14,20),
(6,13,21), (6,12,22), (6,11,23),
  (6,10,24), (5,16,19), (5,15,20), (5,14,21), (5,13,22), (5,12,23), (5,11,24),
(4,16,20), (4,15,21), (4,14,22),
  (4,13,23), (4,12,24), (3,16,21), (3,15,22), (3,14,23), (3,13,24), (2,16,22),
(2,15,23), (2,14,24), (1,16,23),   (1,15,24), (8,16,17), (8,15,18), (8,14,19), (8,13,20), (8,12,21), (8,11,22),
(8,10,23), (8,9,24), (7,16,18),
  (7,15,19), (7,14,20), (7,13,21), (7,12,22), (7,11,23), (7,10,24), (6,16,19),
(6,15,20), (6,14,21), (6,13,22),
  (6,12,23), (6,11,24), (5,16,20), (5,15,21), (5,14,22), (5,13,23), (5,12,24),
(4,16,21), (4,15,22), (4,14,23),
  (4,13,24), (3,16,22), (3,15,23), (3,14,24), (2,16,23), (2,15,24), (1,16,24),
(8,16,18), (8,15,19), (8,14,20),
  (8,13,21), (8,12,22), (8,11,23), (8,10,24), (7,16,19), (7,15,20), (7,14,21),
(7,13,22), (7,12,23), (7,11,24),
  (6,16,20), (6,15,21), (6,14,22), (6,13,23), (6,12,24), (5,16,21), (5,15,22),
(5,14,23), (5,13,24), (4,16,22),
  (4,15,23), (4,14,24), (3,16,23), (3,15,24), (2,16,24), (8,16,19), (8,15,20),
(8,14,21), (8,13,22), (8,12,23),
  (8,11,24), (7,16,20), (7,15,21), (7,14,22), (7,13,23), (7,12,24), (6,16,21),
(6,15,22), (6,14,23), (6,13,24),
  (5,16,22), (5,15,23), (5,14,24), (4,16,23), (4,15,24), (3,16,24), (8,16,20),
(8,15,21), (8,14,22), (8,13,23),
  (8,12,24), (7,16,21), (7,15,22), (7,14,23), (7,13,24), (6,16,22), (6,15,23),
(6,14,24), (5,16,23), (5,15,24),
  (4,16,24), (8,16,21), (8,15,22), (8,14,23), (8,13,24), (7,16,22), (7,15,23),
(7,14,24), (6,16,23), (6,15,24),
  (5,16,24), (8,16,22), (8,15,23), (8,14,24), (7,16,23), (7,15,24), (6,16,24),
(8,16,23), (8,15,24), (7,16,24),
  (8,16,24) ]
gap> C4:=CycleIndex(G3,O4);
1/8193540096000*x_1^512+1/24385536000*x_1^320*x_3^64+1/13005619200*x_1^256*x_2^128+1/6
1931520*x_1^128*x_2^192+1/217728\
000*x_1^200*x_3^104+1/58060800*x_1^160*x_2^80*x_3^32*x_6^16+1/884736*x_1^64*x_2^224+1/
2032128000*x_1^192*x_5^64+1/1083\
801600*x_1^128*x_2^64*x_4^64+1/2438553600*x_1^128*x_3^128+1/1625702400*x_1^64*x_2^128*
x_3^64+70351/26011238400*x_2^256\
+1/5832000*x_1^125*x_3^129+1/552960*x_1^80*x_2^120*x_3^16*x_6^24+1/1036800*x_1^100*x_2
^50*x_3^52*x_6^26+1/10886400*x_1\
^80*x_3^144+1/2580480*x_1^64*x_2^96*x_4^64+1/3870720*x_1^32*x_2^144*x_3^32*x_6^16+1/72
57600*x_1^40*x_2^80*x_3^72*x_6^1\
6+1/24576*x_1^32*x_2^112*x_4^64+1/36864*x_1^16*x_2^152*x_3^16*x_6^24+1/194400*x_1^50*x
_3^154+1/9072000*x_1^120*x_3^24*\
x_5^40*x_15^8+1/4838400*x_1^96*x_2^48*x_5^32*x_10^16+1/4838400*x_1^80*x_2^40*x_3^16*x_
4^40*x_6^8*x_12^8+1/5806080*x_1^\
64*x_2^32*x_3^64*x_6^32+1/2177280*x_1^32*x_3^160+527/14515200*x_2^160*x_6^32+527/10321
920*x_2^128*x_4^64+1/19440*x_1^2\
0*x_3^164+1/34560*x_1^20*x_2^90*x_3^36*x_6^34+1/129600*x_1^25*x_2^50*x_3^77*x_6^26+1/4
30080*x_1^32*x_2^48*x_4^96+1/967\
680*x_1^8*x_2^96*x_3^40*x_6^32+1/5832*x_1^8*x_3^168+1/23040*x_1^40*x_2^60*x_3^8*x_4^40
*x_6^12*x_12^8+1/51840*x_1^40*x_\
2^20*x_3^72*x_6^36+1/2048*x_1^16*x_2^56*x_4^96+1/4608*x_1^4*x_2^98*x_3^20*x_6^42+1/151
2000*x_1^72*x_5^88+1/46080*x_1^4\
8*x_2^72*x_5^16*x_10^24+1/55296*x_1^32*x_2^48*x_3^32*x_6^48+1/322560*x_1^16*x_2^72*x_3
^16*x_4^40*x_6^8*x_12^8+1/725760\
*x_1^16*x_2^32*x_3^80*x_6^32+1/4096*x_2^64*x_4^96+1/162000*x_1^75*x_3^39*x_5^25*x_15^1
3+1/86400*x_1^50*x_2^25*x_3^26*x\
_4^25*x_6^13*x_12^13+1/1536*x_1^8*x_2^76*x_3^8*x_4^40*x_6^12*x_12^8+1/8100*x_2^100*x_6
^52+1/8640*x_1^5*x_2^60*x_3^41*x\
_6^44+1/6480*x_1^10*x_2^20*x_3^82*x_6^36+1/512*x_1^8*x_2^28*x_4^112+1/43200*x_1^60*x_2
^30*x_3^12*x_5^20*x_6^6*x_10^10*\
```

```
x_15^4*x_30^2+1/10368*x_1^16*x_2^8*x_3^80*x_6^40+1/2880*x_2^80*x_4^40*x_6^16*x_12^8+1/
1728*x_1*x_2^62*x_3^21*x_6^54+1/\
3456*x_1^8*x_2^36*x_3^40*x_6^52+1/1296*x_1^4*x_2^8*x_3^84*x_6^40+1/3840*x_1^20*x_2^30*
x_3^4*x_4^60*x_6^6*x_12^12+1/474\
163200*x_1^64*x_7^64+1/403200*x_1^48*x_2^24*x_4^24*x_5^16*x_10^8*x_20^8+1/907200*x_1^4
8*x_3^48*x_5^16*x_15^16+1/483840\
*x_1^32*x_2^16*x_3^32*x_4^16*x_6^16*x_12^16+1/604800*x_1^24*x_2^48*x_3^24*x_5^8*x_10^1
6*x_15^8+1/2880*x_1^10*x_2^45*x_\
3^18*x_4^25*x_6^17*x_12^13+527/19353600*x_2^96*x_10^32+118301/812851200*x_2^64*x_6^64+
1/1016064000*x_3^64*x_5^64+75909\
1/135475200*x_4^128+1/3375*x_1^27*x_5^97+1/256*x_1^4*x_2^38*x_3^4*x_4^60*x_6^6*x_12^12
+1/864*x_1^2*x_2^24*x_3^42*x_6^5\
6+1/13500*x_1^45*x_3^9*x_5^55*x_15^11+1/7200*x_1^36*x_2^18*x_5^44*x_10^22+1/384*x_1^2*
x_2^49*x_3^10*x_4^25*x_6^21*x_12\
^13+1/8100*x_1^30*x_3^54*x_5^10*x_15^18+1/1920*x_1^24*x_2^36*x_4^24*x_5^8*x_10^12*x_20
^8+1/4320*x_1^20*x_2^10*x_3^36*x\_4^10*x_6^18*x_12^18+1/2304*x_1^16*x_2^24*x_3^16*x_4^1
6*x_6^24*x_12^16+1/2880*x_1^12*x
_2^54*x_3^12*x_5^4*x_6^6*x_10^18\
*x_15^4*x_30^2+13/8400*x_2^40*x_6^72+1/4536000*x_3^64*x_5^40*x_15^8+1/5400*x_1^15*x_2^
30*x_3^27*x_5^5*x_6^6*x_10^10*x_\
15^9*x_30^2+8/2025*x_2^25*x_6^77+1/81000*x_3^64*x_5^25*x_15^13+1/2116800*x_1^40*x_3^8*
x_7^40*x_21^8+1/1128960*x_1^32*x\
_2^16*x_7^32*x_14^16+1/3600*x_1^30*x_2^15*x_3^6*x_4^15*x_5^10*x_6^3*x_10^5*x_12^3*x_15
^2*x_20^5*x_30*x_60+1/4320*x_1^2\
4*x_2^12*x_3^24*x_5^8*x_6^12*x_10^4*x_15^8*x_30^4+1/1620*x_1^12*x_3^60*x_5^4*x_15^20+1
/864*x_1^8*x_2^4*x_3^40*x_4^4*x_\
6^20*x_12^20+1/288*x_1^4*x_2^18*x_3^20*x_4^10*x_6^26*x_12^18+1/5400*x_2^60*x_6^12*x_10
^20*x_30^4+1/3840*x_2^48*x_4^24*\
x_10^16*x_20^8+39/17920*x_2^32*x_4^16*x_6^32*x_12^16+323/145152*x_2^16*x_6^80+1/453600
*x_3^64*x_5^16*x_15^16+1/2419200\
*x_3^32*x_5^32*x_6^16*x_10^16+1/378000*x_3^24*x_5^64*x_15^8+1733/302400*x_4^80*x_12^16
+1/81*x_2^10*x_6^82+1/4050*x_3^6\
4*x_5^10*x_15^18+1/320*x_1^12*x_2^18*x_4^36*x_5^4*x_10^6*x_20^12+1/384*x_1^8*x_2^12*x_
3^8*x_4^24*x_6^12*x_12^24+1/720*\
x_1^3*x_2^36*x_3^15*x_5*x_6^12*x_10^12*x_15^5*x_30^4+7/648*x_2^4*x_6^84+1/810*x_3^64*x
_5^4*x_15^20+1/90*x_2^20*x_4^10*\
x_6^36*x_12^18+1/21600*x_3^32*x_5^20*x_6^16*x_10^10*x_15^4*x_30^2+2/1125*x_3^9*x_5^64*
x_15^11+1/600*x_1^18*x_2^9*x_4^9\
*x_5^22*x_10^11*x_20^11+1/1350*x_1^18*x_3^18*x_5^22*x_15^22+1/10752*x_1^16*x_2^24*x_7^
16*x_14^24+1/900*x_1^9*x_2^18*x_\
3^9*x_5^11*x_10^22*x_15^11+1/82320*x_1^8*x_7^72+1/240*x_1^6*x_2^27*x_3^6*x_4^15*x_5^2*
x_6^3*x_10^9*x_12^3*x_15^2*x_20^\
5*x_30*x_60+1/540*x_1^6*x_2^12*x_3^30*x_5^2*x_6^12*x_10^4*x_15^10*x_30^4+1/14400*x_2^3
6*x_10^44+1/128*x_2^16*x_4^24*x_\
6^16*x_12^24+1/302400*x_3^32*x_5^8*x_6^16*x_10^16*x_15^8+113/378000*x_3^24*x_5^40*x_15
^16+1/23040*x_3^16*x_5^16*x_6^24\
*x_10^24+1/37800*x_1^25*x_3^13*x_7^25*x_21^13+1/675*x_4^50*x_12^26+8/343*x_1*x_7^73+1/
2700*x_3^32*x_5^5*x_6^16*x_10^10\
*x_15^9*x_30^2+1/10080*x_1^20*x_2^10*x_3^4*x_6^2*x_7^20*x_14^10*x_21^4*x_42^2+5/288*x_
2^8*x_4^4*x_6^40*x_12^20+1/2160*\
x_3^32*x_5^8*x_6^16*x_10^4*x_15^8*x_30^4+1/1800*x_3^12*x_5^32*x_6^6*x_10^16*x_15^4*x_3
0^2+1/3375*x_3^24*x_5^25*x_15^21\
+1/270*x_3^32*x_5^2*x_6^16*x_10^4*x_15^10*x_30^4+1/1440*x_3^16*x_5^4*x_6^24*x_10^18*x_
15^4*x_30^2+4/1125*x_3^9*x_5^40*\
x_15^19+1/176400*x_1^24*x_5^8*x_7^24*x_35^8+1/94080*x_1^16*x_2^8*x_4^8*x_7^16*x_14^8*x
_28^8+1/211680*x_1^16*x_3^16*x_7\
^16*x_21^16+1/360*x_1^12*x_2^6*x_3^12*x_4^6*x_5^4*x_6^6*x_10^2*x_12^6*x_15^4*x_20^2*x_
30^2*x_60^2+1/141120*x_1^8*x_2^1\
6*x_3^8*x_7^8*x_14^16*x_21^8+527/4515840*x_2^32*x_14^32+13/11200*x_2^24*x_6^24*x_10^8*
x_30^8+2/675*x_3^24*x_5^16*x_15^\
24+1/201600*x_3^16*x_5^16*x_6^8*x_10^8*x_12^8*x_20^8+1733/403200*x_4^48*x_20^16+1733/1
20960*x_4^32*x_12^32+527/9676800\
*x_6^32*x_10^32+1/360*x_3^16*x_5*x_6^24*x_10^12*x_15^5*x_30^4+1/735*x_1^5*x_3*x_7^45*x
_21^9+1/392*x_1^4*x_2^2*x_7^36*x\
_14^18+2/675*x_3^24*x_5^10*x_15^26+1/1800*x_3^12*x_5^20*x_6^6*x_10^10*x_15^8*x_30^4+8/
3375*x_3^9*x_5^25*x_15^24+1/1890\
*x_1^10*x_3^18*x_7^10*x_21^18+1/448*x_1^8*x_2^12*x_4^8*x_7^8*x_14^12*x_28^8+1/672*x_1^
4*x_2^18*x_3^4*x_6^2*x_7^4*x_14^\
18*x_21^4*x_42^2+4/675*x_2^15*x_6^27*x_10^5*x_30^9+1/1800*x_3^16*x_5^10*x_6^8*x_10^5*x
_12^8*x_15^2*x_20^5*x_30*x_60+1/\
```

```
225*x_3^12*x_5^8*x_6^6*x_10^16*x_15^12*x_30^2+1/960*x_3^8*x_5^8*x_6^12*x_10^12*x_12^8*
x_20^8+1/135*x_4^20*x_12^36+1/27\
00*x_6^32*x_10^20*x_30^4+1/1260*x_1^5*x_2^10*x_3^9*x_6^2*x_7^5*x_14^10*x_21^9*x_42^2+1
/225*x_3^12*x_5^5*x_6^6*x_10^10*\
x_15^13*x_30^4+1/1575*x_1^15*x_3^3*x_5^5*x_7^15*x_15*x_21^3*x_35^5*x_105+1/840*x_1^12*
x_2^6*x_5^4*x_7^12*x_10^2*x_14^6\
*x_35^4*x_70^2+1/840*x_1^10*x_2^5*x_3^2*x_4^5*x_6*x_7^10*x_12*x_14^5*x_21^2*x_28^5*x_4
2*x_84+1/1008*x_1^8*x_2^4*x_3^8*\
x_6^4*x_7^8*x_14^4*x_21^8*x_42^4+1/378*x_1^4*x_3^20*x_7^4*x_21^20+1/1260*x_2^20*x_6^4*
x_14^20*x_42^4+1/896*x_2^16*x_4^\
8*x_14^16*x_28^8+1/120*x_2^12*x_4^6*x_6^12*x_10^4*x_12^6*x_20^2*x_30^4*x_60^2+1/108*x_
2^6*x_6^30*x_10^2*x_30^10+1/180*\
x_3^16*x_5^4*x_6^8*x_10^2*x_12^8*x_15^4*x_20^2*x_30^2*x_60^2+1/120*x_3^8*x_5^2*x_6^12*
x_10^9*x_12^8*x_15^2*x_20^5*x_30\
*x_60+1/150*x_3^6*x_5^16*x_6^3*x_10^8*x_12^3*x_15^2*x_20^8*x_30*x_60+1/450*x_4^30*x_12
^6*x_20^10*x_60^2+1/108*x_4^8*x_\
12^40+13/5600*x_6^32*x_10^8*x_30^8+1/1920*x_6^16*x_10^16*x_12^8*x_20^8+1/3600*x_6^12*x
_10^32*x_30^4+8/675*x_6^32*x_10^\
5*x_30^9+3/224*x_1^4*x_2^6*x_4^12*x_7^4*x_14^6*x_28^12+1/168*x_1*x_2^12*x_3^5*x_6^4*x_
7*x_14^12*x_21^5*x_42^4+1/160*x_\
3^4*x_5^4*x_6^6*x_10^6*x_12^12*x_20^12+1/54*x_6^32*x_10^2*x_30^10+2/525*x_1^9*x_5^11*x
_7^9*x_35^11+4/245*x_1^3*x_5*x_7\
^27*x_35^9+1/56*x_1^2*x_2^9*x_3^2*x_4^5*x_6*x_7^2*x_12*x_14^9*x_21^2*x_28^5*x_42*x_84+
1/126*x_1^2*x_2^4*x_3^10*x_6^4*x\
_7^2*x_14^4*x_21^10*x_42^4+3/98*x_1^2*x_2*x_4*x_7^18*x_14^9*x_28^9+2/147*x_1^2*x_3^2*x
_7^18*x_21^18+1/49*x_1*x_2^2*x_3\
*x_7^9*x_14^18*x_21^9+1/450*x_2^9*x_6^9*x_10^11*x_30^11+1/784*x_2^4*x_14^36+1/150*x_3^
6*x_5^10*x_6^3*x_10^5*x_12^3*x_1\
5^4*x_20^5*x_30^2*x_60^2+1/1200*x_4^18*x_20^22+1/3600*x_6^12*x_10^20*x_30^8+1/60*x_6^1
6*x_10^4*x_12^8*x_20^2*x_30^4*x_\
60^2+1/70*x_1^6*x_2^3*x_4^3*x_5^2*x_7^6*x_10*x_14^3*x_20*x_28^3*x_35^2*x_70*x_140+2/31
5*x_1^6*x_3^6*x_5^2*x_7^6*x_15^2\
*x_21^6*x_35^2*x_105^2+1/84*x_1^4*x_2^2*x_3^4*x_4^2*x_6^2*x_7^4*x_12^2*x_14^2*x_21^4*x
_28^2*x_42^2*x_84^2+1/105*x_1^3*\
x_2^6*x_3^3*x_5*x_7^3*x_10^2*x_14^6*x_15*x_21^3*x_35*x_70^2*x_105+1/1680*x_2^12*x_10^4
*x_14^12*x_70^4+39/7840*x_2^8*x_\
6^8*x_14^8*x_42^8+1/88200*x_3^8*x_5^8*x_21^8*x_35^8+1733/94080*x_4^16*x_28^16+1/180*x_
4^12*x_12^12*x_20^4*x_60^4+2/225\
*x_6^12*x_10^8*x_30^12+1733/201600*x_12^16*x_20^16+2/225*x_6^12*x_10^5*x_30^13+8/315*x
_2^5*x_6^9*x_14^5*x_42^9+2/1575*\
x_3^8*x_5^5*x_15*x_21^8*x_35^5*x_105+1/225*x_12^16*x_20^10*x_60^2+1/28*x_2^4*x_4^2*x_6
^4*x_12^2*x_14^4*x_28^2*x_42^4*x\
_84^2+5/126*x_2^2*x_6^10*x_14^2*x_42^10+4/315*x_3^8*x_5^2*x_15^2*x_21^8*x_35^2*x_105^2
+1/420*x_3^4*x_5^4*x_6^2*x_10^2*\
x_21^4*x_35^4*x_42^2*x_70^2+8/525*x_3^3*x_5^8*x_15*x_21^3*x_35^8*x_105+1/105*x_4^10*x_
12^2*x_28^10*x_84^2+1/90*x_12^16\
*x_20^4*x_60^4+1/300*x_12^6*x_20^16*x_60^2+2/49*x_2*x_6*x_14^9*x_42^9+2/105*x_3^4*x_5*
x_6^2*x_10^2*x_15*x_21^4*x_35*x_\
42^2*x_70^2*x_105+8/525*x_3^3*x_5^5*x_15^2*x_21^3*x_35^5*x_105^2+8/245*x_3*x_5*x_21^9*
x_35^9+3/196*x_4^2*x_28^18+1/300\
*x_12^6*x_20^10*x_60^4+2/105*x_2^3*x_6^3*x_10*x_14^3*x_30*x_42^3*x_70*x_210+1/35*x_3^2
*x_5^2*x_6*x_10*x_12*x_20*x_21^2\
*x_35^2*x_42*x_70*x_84*x_140+1/140*x_4^6*x_20^2*x_28^6*x_140^2+1/42*x_4^4*x_12^4*x_28^
4*x_84^4+1/840*x_6^4*x_10^4*x_42\
^4*x_70^4+4/105*x_6^4*x_10*x_30*x_42^4*x_70*x_210+1/70*x_12^2*x_20^2*x_84^2*x_140^2
gap>
```