

**MATHEMATICAL MODELLING OF VARIABLE VISCOSITY
HYDROMAGNETIC BOUNDARY LAYER FLOW WITH
THERMAL RADIATION AND NEWTONIAN HEATING**

BY

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DECEMBER, 2018

DECLARATION

I hereby declare that this is my own work and has not been submitted to any University, in part or whole, for a degree award.

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This project has been submitted for examination to the School of Pure and Applied Sciences with my approval as the university supervisor.

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DEDICATION

I would like to dedicate my project work to my parents; Rosemary Achola and Alexander Achola and my uncle Alfred Musisi Okello for their financial support towards my education.

ACKNOWLEDGEMENTS

I would like to thank the almighty God for the gift of life and good health throughout my entire study period. I am also grateful to my supervisor, Dr. Winifred N. Mutuku for taking her time to guide me and to go through my work; without her support i could not have completed this work. I also wish to appreciate my family members and my course mates for their moral support and encouragement. Their encouragement ignited hope in me and made me remain energized throughout the study period.

ABSTRACT

Magnetohydrodynamic boundary layer flow involving a fluid of varying viscosity subject to thermal radiation and Newtonian heating has various applications in industry and engineering some of the applications include designing of cooling systems used in electronic devices, cooling of nuclear reactors, harvesting of solar energy, thermal insulation, heat exchangers and in geothermal reservoirs. Heat transfer by thermal radiation is of significance to engineering processes that occurs at high temperature and plays a major role in designing of equipment used in Nuclear reactors, gas turbines and equipment for propelling air crafts, missiles, satellites and rockets. In this study we use the fourth order Runge-Kutta method and the shooting technique to find the numerical solution to the equations of fluid flow governing the boundary layer flow of a varying viscosity electrically conducting fluid that is subjected to a constant magnetic field in the presence of thermal radiation and Newtonian heating. The graphical results depicting the effects of various thermophysical parameters on the velocity and temperature profiles of the fluid are presented and then discussed quantitatively. From the study we note that the velocity of the fluid increases with the increase in the values of magnetic parameter and variable viscosity parameter. Furthermore temperature of the fluid increases with increase in the values of magnetic field parameter, Brinkmann number and local Biot number and decreases with the increase in thermal radiation parameter and variable viscosity parameter.

TABLE OF CONTENTS

DECLARATION	i
DEDICATION	ii
ACKNOWLEDGEMENTS	iii
ABSTRACT	iv
LIST OF FIGURES	vi
NOMECLATURE	vii
CHAPTER ONE: INTRODUCTION.....	1
1.1 Mathematical Modelling	1
1.2 Hydromagnetic	4
1.3 Boundary layer flow	8
1.4 Variable viscosity	10
1.5 Thermal Radiation	12
1.6 Newtonian Heating	13
1.7 Statement of the problem	14
1.8 Objectives of the Study	15
1.8.1 General Objectives	15
1.8.2 Specific Objectives	15
1.9 Significance of the Study	15
CHAPTER TWO: LITERATURE REVIEW	16
CHAPTER THREE: EQUATIONS GOVERNING FLUID DYNAMICS	19
3.1 Introduction	19
3.2 The Continuity Equation	19
3.3 Momentum (Navier-Stokes) Equation	20
3.4 Energy Equation	21
CHAPTER FOUR.....	23
4.1 Mathematical Formulation	23
4.2 Numerical Solution	28
CHAPTER FIVE	29
5.0 Results and Discussion	29
5.1 Effect of Parameter Variation on Velocity Profiles	29
5.2 Effect of Parameter Variation on Temperature Profiles	31
CHAPTER SIX	35
6.0 Conclusion and Recommendations	35
6.1 Suggestion for Further work	35
REFERENCES.....	36

LIST OF FIGURES

Figure 1.1: MHD micropump and MHD generator	7
Figure 1.2: MHD Propulsion in Jamato (Mitsubishi 1991) and MHD Blood Flow Meters	8
Figure 1.3: The boundary layer flow	10
Figure 1.4: Variation of Viscosity with Shear Rate for Newtonian and non-Newtonian fluids	12
Figure 1.5: Modes of Heat transfer	13
Figure 4.1: Flow configuration and coordinate system	23
Figure 5.1: Velocity profile for varying Ha	30
Figure 5.2: Velocity profile for varying a	30
Figure 5.3: Temperature profile for varying Ha	32
Figure 5.4: Temperature profile for varying Br	32
Figure 5.5: Temperature profile for varying Bi	33
Figure 5.6: Temperature profile for varying Ra	33
Figure 5.7: Temperature profile for varying a	34

NOMECLATURE

(u, v)	Velocity components	ρ	Density
(x, y)	Coordinates	ψ	Stream function
T	Temperature	σ	Electrical conductivity
T_∞	Free stream temperature	μ	Dynamic viscosity
B_0	Constant applied magnetic field	U_∞	Free stream velocity
β	Thermal expansion coefficient	ϕ	Nanoparticle concentration
Bi	Local Biot number	Pr	Prandtl number
Ra	Thermal radiation parameter	a	Variable viscosity parameter
Ha	Magnetic field intensity parameter	Br	Brinkmann number
θ	Dimensionless temperature	η	Transverse distance

Subscripts

f	Fluid
s	Solid
nf	Nanofluid

Abbreviations

MHD	Magnetohydrodynamics
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CHAPTER ONE

INTRODUCTION

The chapter one of this project focusses on the main terminologies used in the dissertation. Mathematical modelling and its application in different fields is introduced and discussed in details. Hydromagnetic (MHD) and its applications [i.e. MHD pump, MHD propulsion, Metallurgy, MHD generator and MHD Flowmeters] are also explored. Boundary layer in fluid dynamics, variable viscosity in fluids, heat transfer as a result of thermal radiation and the concept of Newtonian heating are explained in detail. The statement of the problem, objectives governing the study and the importance of the study are also highlighted.

1.1 Mathematical Modelling

Mathematical modelling refers to the art of translating problems from an application area into tractable mathematical formulations whose theoretical and numerical analysis provides insight, answers, and guidance useful for the originating application. Several mathematical models have been developed such as statistical models, Deterministic ODE models, stochastic models and discrete or continuous flow models. Mathematical modelling has many applications in different fields.

In Electrical engineering; mathematical modelling and simulations is important during the designing stage and the operational stage of electrical power systems. These electrical power systems are usually large, complex and are spread over a wider geographical area and are usually made of many different electrical devices (Kundur, 1994). Mathematical models are also used when establishing the stability of electrical circuits, when analyzing microchips and during the optimization of power supply networks. The Circuit

Differential equations (Kirchhoff laws) are normally used in describing electrical circuits. In mechanical engineering; mathematical modelling is of significance in crash simulations, (Omar *et al.* 1998; Pawlus *et al.* 2009) and in structural optimization of motor vehicle chassis under different amount of load/ stress, (Kurdi & Rahman, 2010; Lu *et al.* 2010). In civil and structural engineering mathematical models are of importance in designing buildings capable of resisting earthquakes to guarantee safety and security of building occupants and assets, (Lu *et al.* 2007; Rahgozar *et al.* 2010). Mathematical modelling also helps in ascertaining the structural integrity and stability of skyscrapers, bridges and dams when subjected to natural forces such as typhoons and earthquakes, (Li *et al.* 2005; Taranath, 2005).

In the field of Agriculture, mathematical models are of significance in the following areas; in modelling of agricultural production systems, Norton & Schiefer (1980; Audsley (1981; Lambert & McCarl (1985), determining the amount of feed and energy supply required by ruminant animals; this minimizes wastage and enhances efficiency when it comes to production, (Baldwin, 1995; Kebreab *et al.*, 2004). Mathematical models are also important in addressing areas such as Body fat development in beef cattle, Allen *et al.* (1976) animal growth, absorption of water and mineral salts by plant roots, Roose *et al.* (2001); Roose & Fowler (2004), infestation of plants by weeds (Kiniry *et al.* 1992) and induced resistance to diseases by plants.

In combating crime; mathematical models are important in the identification of fingerprints and recognizing faces from crime scenes, (Samaria & Harter, 1994; Li *et al.* 2006; Ram *et al.* 2010). The fingerprints and image of faces captured by CCTV cameras in crime scenes are cross checked with those ones stored in the database to nab the

suspects. The automated fingerprint identification system (AFIS), Moses *et al.* (2010) can be used to check the identity of applicants in recruitment process, to restrict access to important military and telecommunication installations, in border control and in restricting access of those people logging into electronic devices such as laptops and mobile phones by using fingerprint as the password. Mathematical models have been successfully used to reconstruct deformed fingerprints of criminals who want to evade arrest after committing crime.

In meteorological science; mathematical modelling is important in areas such as: modelling of occurrence and evolution of tornadoes, Dotzek *et al.* (2003); Gubar *et al.* (2008), modelling evolution and occurrence of hurricanes, Bender *et al.* (2007), and models on occurrence of tsunamis, (Titov *et al.*, 2005; Wang & Power, 2011). These models can be of use in averting major disasters from happening as a result of timely prediction of occurrence of these natural disasters. Rescue measures such as evacuation of people to safer places, fixing of power outages due to destruction of electrical transmission lines and purchasing of medical supplies can be planned for in advance.

In the field of medicine; mathematical modelling in areas such as evolution of tumour cells, Kim *et al.* (2007); Cristini *et al.* (2009); Deisboeck & Stamatakos (2010), Stochastic and continuum models on tumour response to radiation therapy have been developed, (Borkenstein *et al.* 2004; Harting *et al.* 2007). The continuum model uses diffusion in 3-D to illustrate the growing tumour. The model has been widely used in the study of brain tumour (glioblastoma multiforme and low-grade glioma), (Rockne *et al.* 2009; Nawrocki & Zubik-Kowal, 2015; Pérez-García *et al.*, 2015). The other mathematical models used in the study of cancer cells include; non-stochastic model that

studies tumour kinetics after irradiation, Huang et al. (2010); Chvetsov (2013), models on chronic myelogenous leukemia (CML) and its interaction with T cells, Fokas *et al.* 1991), models on cervical cancer resulting from Human Papillomavirus infection, Lee & Tameru (2012) and models on throat cancer. Other mathematical models in medicine are models on osteoarthritis that are used to assess the articular cartilage function and possible failure sites in joints. Mathematical models and equations governing mechanics and fluid flow have also been used to explain the flow of synovial fluid during the displacement of bone element and cartilage when subjected to various stress configurations.

1.2 Hydromagnetic

Hydromagnetic (MHD) was first founded by Hannes Alfvén (1908-1995). MHD as a discipline deals with dynamics of conductive fluids in magnetic fields. These conductive fluids comprise of Liquid metals (gallium, mercury and molten iron), plasmas (such as solar atmosphere) and strong electrolytes. In MHD; as the magnetic field and the conducting fluid comes into contact, the electric current of density \mathbf{j} is induced into the conducting fluid resulting into induced magnetic field. The total field \mathbf{B} interacts with induced current resulting into Lorentz force $\mathbf{F} = \mathbf{j} \times \mathbf{B}$. The applications of MHD are broad and they include; MHD pump, MHD propulsion, metallurgy, MHD generators and MHD flow meters.

In MHD pump, the magnetic field and electric field are perpendicular to each other and to the axis of the duct carrying the conductive liquid. The conductive liquid permits the flow of electric current in the duct. The Lorentz force arising from the interaction of magnetic field and electric field provides the pumping action. MHD pumps have no movable parts

reducing chances of mechanical failure. They are of importance in the following areas; MHD micropumps are used as microsyringes for diabetics, Jang & Lee, (2000), MHD pumps are used in fusion research to create high impact velocities and in cooling of nuclear reactors by pumping sodium coolant in the reactor core.

The MHD propulsion; an alternative to use of mechanical propellers in propelling marine vessels such as military submarines overcomes the problem of cavitation noise associated with the movement of propellers which is an advantage in military where stealth is necessary, (Bednarczyk, 1989; Lin *et al.* 1991). MHD propulsion is achieved when the seawater is drawn into MHD pumps within the submarine where the magnetic field and electric current channeled through the seawater interacts giving rise to Lorentz force. The Lorentz force, forces water out of the vessel making the vessel to accelerate in the opposite direction. With MHD propulsion; greater speeds of the seawater vehicles can be attained provided large magnets of high magnetic field strength are built for the marine vessel thruster duct, (Mitchell & Gubser, 1988; Doss & Geyer, 1990). The first marine vessel to be propelled using MHD was 'Yamato' which was built by Mitsubishi in Japan in 1991.

In the metallurgical industries; electromagnetic stirring of liquid metals during the formation of the alloys is important in ensuring proper mixing of the metal elements in the alloy as this leads to a fine-structured homogeneous ingot, (Khristinich *et al.* 2003; Dolezel *et al.* 2010; Hertwich & Foliforov, 2011). Magnetic damping another application of magnetic field in the metallurgical industry slows down the motion of rotating liquid metals which results into a more quiescent process that reduces the amount of contamination arising from violent fluid motion. In magnetic damping the static magnetic

field converts the kinetic energy of the rotating liquid metal into heat energy via joule dissipation. Levitation of liquid metals is another application of magnetic fields. In this process an induction coil of high frequency induces opposing currents in the neighbouring conductor thus repelling the conducting material. The 'basket' formed as a result of the process helps in levitating and melting very reactive metals. Other uses of MHD in metallurgy include; electrolysis of aluminium oxide to aluminium, electromagnetic (non-contact) casting of aluminium and vacuum-arc remelting of titanium and nickel-based super alloys.

The MHD generator unlike the traditional electric generators has the ability to work under very high temperatures. It works by converting thermal or kinetic energy directly into electrical energy. MHD generators have no movable parts reducing chances of mechanical failure. The simplest MHD generator has a gas nozzle that serves as a combustion chamber injecting pulses of the gas into the duct. The walls of the duct act as electrodes. The first MHD generator was developed using copper disks and horse shoe magnet in 1831 by Michael Faraday. The powerful electromagnet in Faraday's generator serves as a source of magnetic field through which plasma flows. The current flowing across the plasma between the two installed electrodes which are perpendicular to magnetic field serves as the main electrical output of the MHD generator.

The MHD flowmeters that works based on Faraday's law of electromagnetic induction are used to determine liquid flow through a pipe. In MHD flowmeters, a magnetic field generated is passed through the conductive fluid flowing inside the pipe and this leads to generation of voltage in the fluid as per the Faraday's law. The voltage generated in the fluid is proportional to the speed of the flowing fluid. MHD flowmeters can also be used

to determine the rate of blood flow through blood vessels. The first use of MHD blood flowmeters was by Kolin (1936). MHD blood flowmeters are used in surgery to determine the amount of blood flowing through a vessel before, during and after the surgery, (Wyatt, 1968; Bevir, 1969).

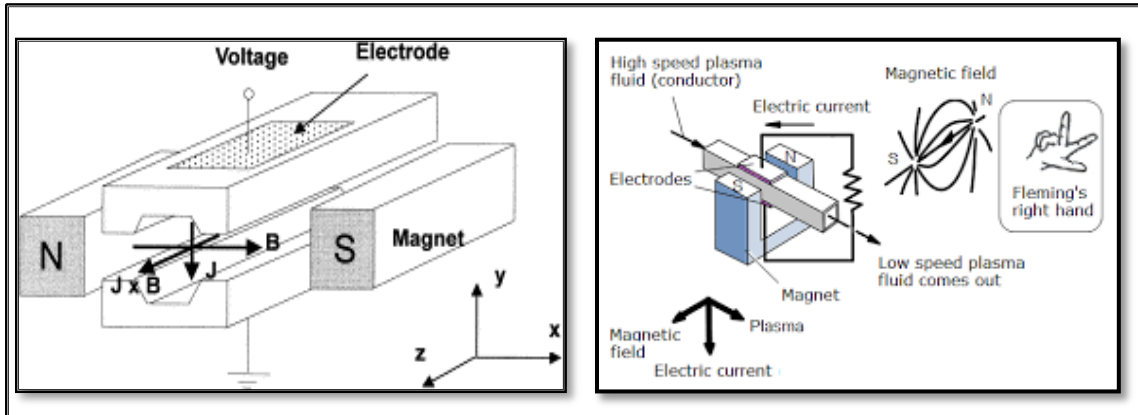


Figure 1.1: MHD micropump and MHD generator (Image Source: Sciencedirect website)

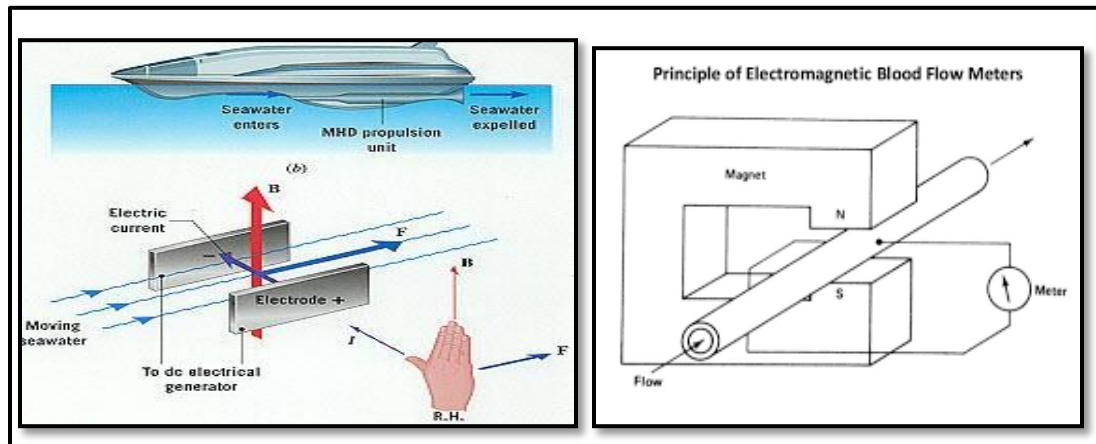


Figure 1.2: MHD Propulsion in Jamato (Mitsubishi 1991) and MHD Blood Flow Meters

1.3 Boundary layer flow

The idea of boundary layer in fluid flow was first introduced by Ludwig Prandtl in 1904. Boundary layer flow has vital importance in fluid dynamics. It is of importance in determining friction drag of bodies moving in fluids; such as viscous drag on aerodynamics (airplanes, rockets, and projectiles such as missiles), hydrodynamics (ships, submarines and torpedoes), automobiles (motor vehicles) and engineering structures such as buildings and bridges.

A boundary layer in fluid flow can be described as a thin layer of viscous fluid just neighbouring the surface or the wall in which the fluid has a zero velocity at the wall/plate and a free stream velocity u_0 far away from plate. This zero velocity at the walls is due to the wetting or sticking of the fluid on the surface of the wall as a result of adhesive forces between the wall and the fluid. This condition is known as the ‘no slip

condition'. The fluid above the surface of the plate is moving with shearing happening between its layers. The shear stress happening between the surface of the plate and the first moving layer of the fluid adjacent to the plate is known as the wall shear stress (T_W).

The boundary layer thickness (δ) which is a function of the Reynolds number refers to the distance from the solid wall to the height above the surface of the wall where the velocity of the fluid is 99% of the free stream velocity u_0 .

In boundary layer flow, the hydrodynamic and thermal boundary layers are of great significance. In hydrodynamic boundary layer, the fluid velocity is zero at the plate and its value increases to a free stream value u_0 far away from the plate. In thermal boundary layer, the temperature of the fluid varies from the wall temperature T_0 to the free stream value T_∞ far away from the wall. The fluid particles in contact with the solid wall acquire temperature equal to that of the wall. If the wall temperature is higher compared to the rest of the fluid, the fluid particles in contact with the wall exchanges heat with those in the neighbouring layers leading to the development of a thermal gradient in the fluid. The understanding of hydrodynamic and thermal boundary layer is of significance in fluid mechanics since velocity is an important component in mass, momentum and energy equations while temperature gradient in the thermal boundary layer influences heat transfer in the fluid.

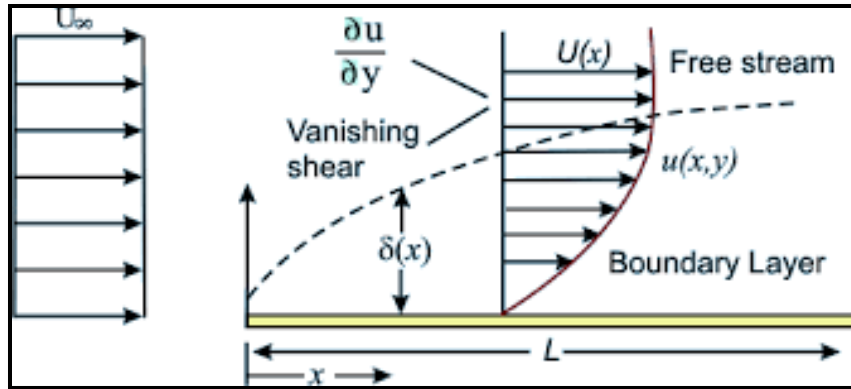


Figure 1.3: The boundary layer flow

1.4 Variable viscosity

Viscosity of a fluid can be termed as friction involving a fluid. It is a measure of internal resistance to the flow of a fluid when subjected to shear stress or tensile stress. Fluids that show resistance to stress are termed as viscous fluids while those that do not exhibit resistance to shear stress are referred to as ideal fluids or inviscid fluids.

Viscosity of a fluid depends on various factors such as temperature, pressure and shear rate. The temperature affects the viscosity of both liquids and gases. In the case of liquids increasing temperature of the liquid leads to a reduction in its viscosity as this can be explained using the particle theory. Liquids generally have particles which are closely packed compared with the arrangement of particles in gases and therefore raising the temperature of a liquid increases the energy level of its molecules which in turn leads to an increase in the intermolecular distances and this weakens intermolecular forces of attraction increasing the ability of the liquid to flow over a surface (fluidity). In gases, increasing the temperature of a gas increases the rate of movement of the gas molecules

which results in the increased rate of collisions among the gas molecules leading to an increase in the viscosity of a gas due to the transfer of momentum between the stationary and moving gas molecules.

The effect of applied pressure on the viscosity of liquids is low; this is attributed to the fact that most liquids are incompressible at medium or low pressure. When the applied pressure is increased to higher values, there is decrease in the intermolecular distances between the liquid molecules leading to increased internal friction among the molecules of the liquid which results in the increased flow resistance. In gases applied pressure increases viscosity of gases since gases are compressible.

In Newtonian fluids (fluids obeying Newton's law of viscosity) viscosity does not depend on shear rate; such fluids include water, organic solvent, honey etc. In non-Newtonian fluids, viscosity of the fluid depends on shear rate (shear thinning or thickening). In shear thickening fluids, the viscosity of the fluid increases with the shear rate; example of this kind of fluids is a mixture of water and cornstarch. In shear thinning fluids, viscosity decreases with the increasing shear rate e.g. blood.

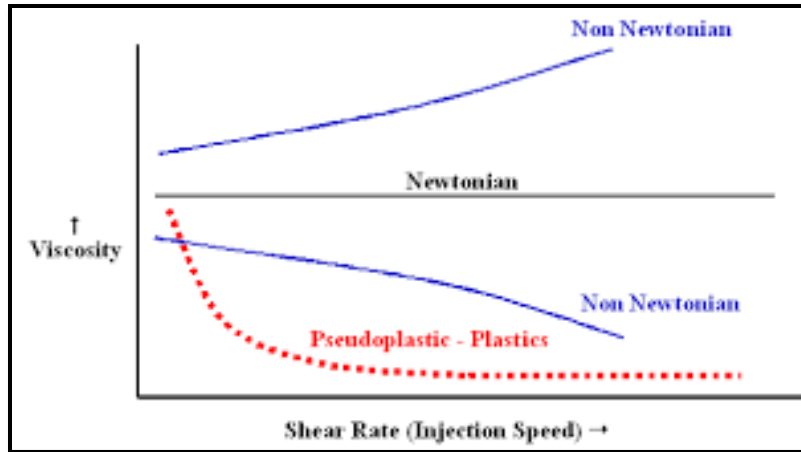


Figure 1.4: Variation of Viscosity with Shear Rate for Newtonian and non-Newtonian fluids

1.5 Thermal Radiation

Mechanisms by which heat energy can be transferred from one point to another are conduction, convection and radiation. In thermal radiation heat transfer takes place in the form of electromagnetic waves and at the speed of light ($c = 3.0 \times 10^8 \text{ms}^{-1}$). In thermal radiation heat energy can be propagated through space or vacuum (i.e. does not require material medium for transmission). It is the fastest means of heat transfer.

The total radiant energy from a heated surface can be arrived at by applying the Stefan-Boltzmann law which states that; the rate of outward radiative energy per unit area emitted by an object with absolute temperature T is proportional to the fourth power of T . Mathematically Stefan-Boltzmann law is expressed as:

$$E = \epsilon\sigma T^4 \quad (1.1)$$

If the heated surface happens to be that of a blackbody (i.e. bodies considered to be perfect absorbers and perfect emitters), then $\epsilon = 1$ and the rate of outward radiative energy per unit area for a blackbody will be given by:

$$E = \sigma^* T^4 \quad (1.2)$$

Where in the equations (1.1) and (1.2) above, ϵ – refers to the emissivity of the surface, σ^* which is a constant (Stefan-Boltzmann constant) has a value of $\sigma^* = 5.670367 \times 10^{-8} Wm^{-2}K^{-4}$ and T stands for absolute temperature of the surface expressed in Kelvin (K). Heat transfer mechanism from the sun to the earth is by thermal radiation.

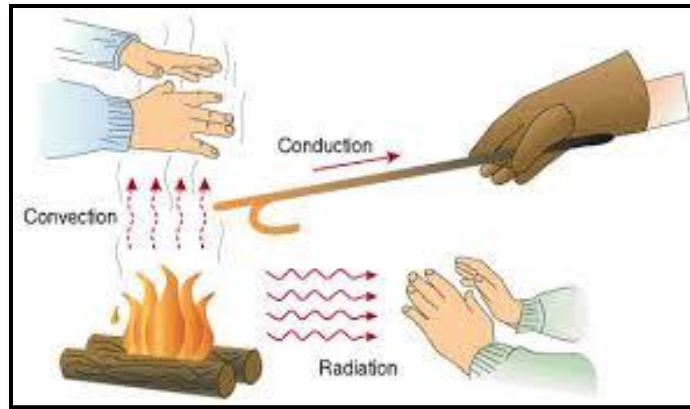


Figure 1.5: Modes of Heat transfer (Image Source: Green-Mechanic website)

1.6 Newtonian Heating

Newtonian heating makes use of the Newton's law of heating which is used to describe temperature variation of a body placed in a medium or surroundings of higher temperature. The law states that the rate at which the temperature of a body changes is proportional to the difference between the temperature of the body and the ambient temperature (temperature of the surrounding). Let $T(t)$ denote the temperature of the

body at time t , M^* denote the ambient temperature, and $\frac{dT}{dt}$ be the rate of change of the body temperature, then Newton's law of heating can be expressed as:

$$\frac{dT}{dt} \propto |M^* - T| \quad (1.3)$$

But since the surrounding is hotter than the object, $M^* > T$, this implies therefore that $M^* - T$ is positive and therefore $|M^* - T| = (M^* - T)$. Equation 1.3 can therefore be written as;

$$\frac{dT}{dt} = K(M^* - T) \quad (1.4)$$

Where K denotes the constant of proportionality. Equation (1.4) can be solved using the separation of variables to give:

$$T = M^* - Ae^{-Kt} \quad \text{Where } A = e^{-C} \quad (1.5)$$

The equation(1.5) is the solution to the Newton's law of heating and it gives the temperature of the body at any given time, t .

1.7 Statement of the problem

This study investigates effects of thermophysical parameters (Magnetic field intensity, variable viscosity, thermal radiation, Brinkmann number and local Biot number) on velocity and temperature profile of varying viscosity MHD boundary layer flow of fluid subject to thermal radiation and Newtonian heating putting into consideration the significance of the study in the field of engineering and its numerous industrial applications such as in designing of cooling systems for electronic devices, solar energy harvesting, cooling of geothermal reservoirs, enhanced oil recovery, thermal insulation and in heat exchangers.

1.8 Objectives of the Study

1.8.1 General Objectives

To investigate theoretically magnetohydrodynamic (MHD) Boundary layer flow with Variable viscosity, Thermal radiation and Newtonian heating.

1.8.2 Specific Objectives

- i. To determine effects of variable viscosity, thermal radiation and Newtonian heating on velocity of fluid.
- ii. To examine the effects of variable viscosity, thermal radiation and Newtonian heating on the temperature profile.
- iii. To analyze the effects of pertinent parameters such as Hartman number, Brinkmann number, Local Biot number, variable viscosity parameter and thermal radiation parameter.

1.9 Significance of the Study

Problems involving MHD boundary layer flow of a fluid of varying viscosity subject to thermal radiation and Newtonian heating are of great importance to engineering and industrial applications due to their vast applications in thermal insulation, heat exchangers, geothermal reservoir, cooling of nuclear reactors, enhanced oil recovery, solar energy collection, designing of cooling systems for electronic devices, packed-bed catalytic reactors etc. Heat transfer by thermal radiation is also of great significance to engineering processes occurring at high temperatures and it is important in the designing of equipment in Nuclear power plants, gas turbines, and propulsion devices for air craft, missiles, satellites and space vehicles.

CHAPTER TWO

LITERATURE REVIEW

Magnetohydrodynamic boundary layer flow involving viscous fluid with thermal radiation is of significance to most engineering and industrial applications, (Woods, 1975; Aïboud & Saouli, 2010). The applications include designing of electronic cooling systems for electronic devices, solar energy harvesting, thermal insulation, geothermal reservoirs, cooling of nuclear reactors during the production of nuclear energy, in the enhanced oil recovery, etc.

Heat transfer through thermal radiation is of significance to most engineering and industrial processes occurring at higher temperatures and its knowledge is important in the design of most engineering equipment, (Sparrow & Cess, 1978; Raptis *et al.* 2004; Mbeledogu *et al.* 2007) such equipment are used as devices for propelling air crafts, missiles, satellites and hypersonic flights, in rocket combustion chambers, gas turbines and in gas cooled nuclear reactors.

Studies involving hydromagnetic/ magnetohydrodynamic boundary layer flow has been researched on extensively. (Hartmann & Lazarus, 1937) did pioneering work involving MHD flow of an electrically conducting viscous liquid. Makinde (2010); Rajput & Prasad (2013; Jat & Neemawat (2014) investigated the problem under different flow conditions. Suneetha *et al.* (2011) looked at the concept of radiation and mass transfer and their effects on MHD free convection dissipating fluid when there is a heat source/sink. Suneetha *et al.* (2009) came up with analysis of the effects of thermal radiation on MHD free convective flow past an impulsively started vertical plate with varying temperature and concentration. Abd El-Naby *et al.* (2004) presented a finite

difference solution of radiation effects on MHD unsteady free convection flow over a vertical porous plate. Samad & Rahman (2008) investigated thermal radiation interaction with unsteady MHD flow past a vertical porous plate immersed in a porous medium. Makinde & Sibanda (2008) studied magnetohydrodynamic mixed convective flow and heat transfer past a vertical plate in a porous medium with constant wall suction. Hossain & Munir (2000) did the analysis of a 2-D mixed convection flow involving a viscous incompressible fluid whose viscosity depended on temperature past a vertical plate. Fang (2004) analyzed influence of fluid property variation on boundary layers of a stretching surface. Mahmoud (2007) studied variable viscosity effects on hydromagnetic boundary layer flow along a continuously moving vertical plate in the presence of radiation. Hossain *et al.* (2001) discussed the effects of radiation on free convection flow of a fluid with variable viscosity from a porous vertical plate.

Poornima & Reddy (2013) studied steady free convective boundary layer flow involving a radiating nanofluid along a non-linear stretching sheet subject to transverse magnetic field. Kandasamy *et al.* (2013) studied MHD boundary layer flow of a nanofluid in the presence of thermal stratification due to solar radiation taking into consideration the effects of Brownian motion and thermophoresis. Motsumi & Makinde (2012) numerically investigated the effects of thermal radiation and viscous dissipation on the boundary layer flow of a nanofluid over a permeable moving flat plate. Bataller (2008); Makinde & Aziz (2010); Makinde & Bég (2010) conducted studies involving hydromagnetic boundary layer flow with heat transfer based on thermodynamic point of view restricting their studies on first law analysis. Narusawa (2001) looked at entropy generation minimization based on second law of thermodynamics. Bejan (1982, 1998)

introduced the theoretical concept of entropy generation minimization based on second-law analysis into heat transfer and thermal design problems. Makinde & Aziz (2010) in their study gave analytical and numerical analysis of the second law for variable viscosity plane Poiseuille flow with asymmetric convective heat transfer. Makinde (2010) investigated generation of entropy in a liquid film with falling varying viscosity on an inclined heated plate subject to convective cooling. Mahmud *et al.* (2003) investigated magnetic field effect on generation of entropy in a mixed convection channel flow. Makinde (2010) applied second law analysis to the problem of inherent irreversibility in a reactive hydromagnetic channel flow. Chen *et al.* (2010) investigated double-diffusive (natural) convection in vertical annuluses with opposing temperature and concentration gradients. In their investigation they found out that lattice Boltzmann-based numerical method was effective compared to traditional CFD in determining irreversibility associated with viscosity. In this study we extend the recent work by examining the mathematical modelling of varying viscosity MHD boundary layer flow subject to thermal radiation and Newtonian heating.

CHAPTER THREE

EQUATIONS GOVERNING FLUID DYNAMICS

3.1 Introduction

The equations of fluid flow (fluid dynamics) consist of three important equations namely:

- i. The Equation of continuity which is derived from conservation of mass for a system (i.e. mass can neither be created nor destroyed).
- ii. Navier-Stokes (Momentum) Equation which is derived from Newton's second law of motion $(\vec{F} = m\vec{a} = m \frac{d\vec{u}}{dt})$
- iii. The energy equation which is derived from the first law of thermodynamics (i.e. the amount of heat added to the system dQ is equal to change in internal energy dE plus the amount of energy lost due to work done on the system dW that is $dQ = dE + dW$)

3.2 The Continuity Equation

To arrive at the continuity equation we apply the mass conservation principle on an infinitesimal volume of fluid element within a moving fluid. The equation of continuity for differential element in the Cartesian coordinate system is of the form:

$$\frac{\partial \rho}{\partial t} + \frac{\partial(\rho u)}{\partial x} + \frac{\partial(\rho v)}{\partial y} + \frac{\partial(\rho w)}{\partial z} = 0 \quad \text{Or} \quad \frac{\partial \rho}{\partial t} + \nabla \cdot (\rho \vec{V}) = 0 \quad (3.1)$$

Equation(3.1) is the Continuity equation for a compressible fluid in a rectangular Cartesian coordinate system. In equation (3.1) above ρ is the density of the fluid, (u, v, w) is velocity component of the fluid in (x, y, z) directions respectively and $\nabla \cdot (\vec{V})$

refers to divergence of the velocity vector (i.e. the rate at which volume of a moving fluid element changes per unit volume.)

$$\text{We define } \nabla \equiv \vec{i} \frac{\partial}{\partial x} + \vec{j} \frac{\partial}{\partial y} + \vec{k} \frac{\partial}{\partial z} \quad (3.2)$$

If the flow of the fluid is steady (i.e. density is not a function of time), $\frac{\partial \rho}{\partial t} = 0$ and equation (3.1) simplifies to:

$$\frac{\partial(\rho u)}{\partial x} + \frac{\partial(\rho v)}{\partial y} + \frac{\partial(\rho w)}{\partial z} = 0 \quad \text{Or } \nabla \cdot (\rho \vec{V}) = 0 \quad (3.3)$$

For incompressible flow (density is constant), the material derivative of density is zero (that is $\frac{\partial \rho}{\partial t} = 0$) and the continuity equation for incompressible flow becomes

$$\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} + \frac{\partial w}{\partial z} = 0 \quad \text{Or } \nabla \cdot (\vec{V}) = 0 \quad (3.4)$$

3.3 Momentum (Navier-Stokes) Equation

The momentum equation for a flowing fluid is of the form

$$\frac{\partial V}{\partial t} + (V \cdot \nabla)V = \frac{1}{\rho} [-\nabla P + \mu \nabla^2 V] + F \quad (3.5)$$

Where F in the equation (3.5) above represents forces acting on flowing fluid.

If forces acting on flowing fluid are as a result of gravity, thermal expansion and the Lorentz force created by magnetic field, then the Navier-Stokes equation (3.5) takes the form:

$$\frac{\partial V}{\partial t} + (V \cdot \nabla)V = \frac{1}{\rho} [-\nabla P + \mu \nabla^2 V] + \rho \beta g \Delta T + \frac{1}{\rho} j \times B \quad (3.6)$$

In equation(3.6) above; V denotes fluid's velocity, ρ represents density of fluid, P denotes pressure, μ represents dynamic viscosity, g stands for gravitational force, β

denotes thermal expansion coefficient, j represents electric current and B stands for magnetic field. If the flowing fluid is a nanofluid (fluid containing nanoparticles), the quantities ρ, μ and β becomes ρ_{nf}, μ_{nf} and β_{nf} respectively which are defined as:

$$\begin{aligned}\rho_{nf} &= (1 - \phi)\rho_f + \phi\rho_s, \\ \mu_{nf} &= \frac{\mu_f}{(1-\phi)^{2.5}}, \text{ and} \\ \beta_{nf} &= (1 - \phi)\beta_f + \phi\beta_s\end{aligned}\quad (3.7)$$

For a nanofluid equation (3.6) will take the form:

$$\frac{\partial V}{\partial t} + (V \cdot \nabla)V = \frac{1}{\rho_{nf}} [-\nabla P + \mu_{nf} \nabla^2 V] + (\rho\beta)_{nf} g \Delta T + \frac{1}{\rho_{nf}} j \times B \quad (3.8)$$

3.4 Energy Equation.

The energy equation which is arrived at by applying the first law of thermodynamics takes the form:

$$\rho C_p \left(\frac{\partial T}{\partial t} + u \frac{\partial T}{\partial x} + v \frac{\partial T}{\partial y} + w \frac{\partial T}{\partial z} \right) = k \left(\frac{\partial^2 T}{\partial x^2} + \frac{\partial^2 T}{\partial y^2} + \frac{\partial^2 T}{\partial z^2} \right) + q^* + \mu \Phi \quad (3.9)$$

Where $\Phi = 2 \left[\left(\frac{\partial u}{\partial x} \right)^2 + \left(\frac{\partial v}{\partial x} \right)^2 + \left(\frac{\partial w}{\partial x} \right)^2 \right] + \left(\frac{\partial u}{\partial y} + \frac{\partial v}{\partial x} \right)^2 + \left(\frac{\partial v}{\partial z} + \frac{\partial w}{\partial y} \right)^2 + \left(\frac{\partial w}{\partial x} + \frac{\partial u}{\partial z} \right)^2$

In the equation (3.9) above, (ρC_p) refers to heat capacitance of the fluid, (u, v, w) is the velocity component of the fluid in (x, y, z) directions respectively, T refers to local temperature of the fluid, k is the thermal conductivity of the fluid and q^* is the heat flux.

3.5 Magnetohydrodynamic (MHD) Flow

If the flowing fluid happens to be in a magnetic field, then the equations governing such a flow are Navier-Stokes (momentum) equation and Maxwell's equations of electromagnetism. The Maxwell's equations of electromagnetism are:

$$\nabla \times \mathbf{B} = \mu_0 \mathbf{j} \quad (3.10)$$

$$\frac{\partial \mathbf{B}}{\partial t} = -\nabla \times \mathbf{E} \quad (3.11)$$

$$\mathbf{j} = \sigma(\mathbf{E} + \mathbf{V} \times \mathbf{B}) \quad (3.12)$$

Equation 3.10 is the Ampere's law; Equation 3.11 is the Faraday's law and equation 3.12 is the Ohm's law. In the above equations, μ_0 is magnetic permeability, σ denotes the electrical conductivity of the fluid, \mathbf{j} represents electric current density, \mathbf{E} stands for electric field and \mathbf{B} is the magnetic field.

CHAPTER FOUR

In the chapter, we formulate a mathematical model on MHD boundary layer flow involving a fluid of varying viscosity subject to thermal radiation and Newtonian heating. The equations representing the flow and the associated boundary conditions are given and solved numerically.

4.1 Mathematical Formulation

We consider steady 2-D MHD boundary layer flow of a varying viscosity electrically conducting fluid with heat transfer over a horizontal plate placed in a stream of this fluid. The fluid is at the temperature T_∞ and is subjected to magnetic field and thermal radiation. The lower surface of the plate is exposed to a heated fluid of temperature T_f that provides a heat transfer coefficient h_f . The fluid located on the upper side of the plate is subjected to Newtonian heating and a variation in fluid property as a result of temperature is limited to viscosity. A constant magnetic field B_0 is imposed perpendicular to the flow. The induced magnetic field arising from flowing conductive fluid and the electric field due to polarization of charges are considered negligible.

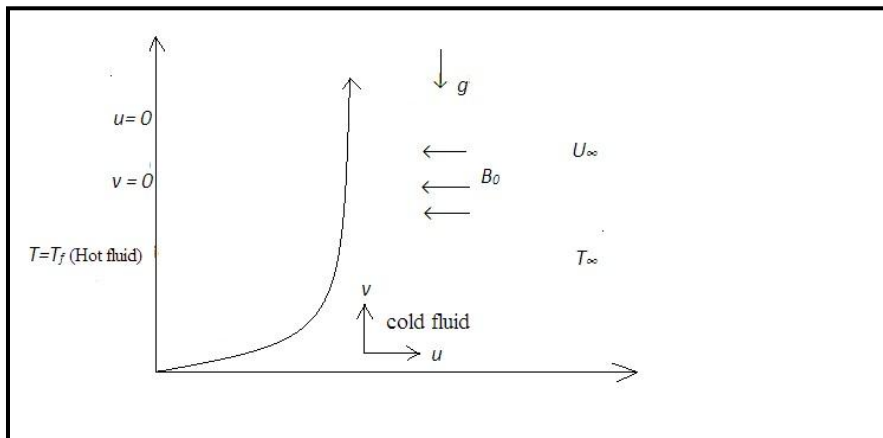


Figure 4.1: Flow configuration and coordinate system

The equations of fluid flow for the above flow configuration are;

$$\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} = 0 \quad (4.1)$$

$$u \frac{\partial u}{\partial x} + v \frac{\partial u}{\partial y} = \frac{1}{\rho} \frac{\partial}{\partial y} \left(\mu \frac{\partial u}{\partial y} \right) + g\beta(T - T_\infty) - \frac{\sigma\beta_0^2}{\rho} (u - U_\infty) \quad (4.2)$$

$$u \frac{\partial T}{\partial x} + v \frac{\partial T}{\partial y} = \frac{1}{\rho c_p} \frac{\partial q_r}{\partial y} + \frac{\mu}{\rho c_p} \left(\frac{\partial u}{\partial y} \right)^2 + \frac{\sigma\beta_0^2}{\rho c_p} (u - U_\infty)^2 \quad (4.3)$$

With the following boundary conditions

$$\begin{aligned} u(x, 0) = 0, \quad v(x, 0) = 0, \quad -k \frac{\partial T}{\partial y}(x, 0) = h_f (T_f - T(x, 0)) \\ u(x, \infty) = U_\infty, \quad T(x, \infty) = T_\infty \end{aligned} \quad (4.4)$$

Where (u, v) denotes the fluid's velocity in (x, y) directions respectively, U_∞ refers to the free stream velocity of the fluid, c_p is the specific heat at constant pressure, T represents temperature, T_∞ is the free stream temperature of the fluid, ρ denotes density of the fluid, σ denotes fluid's electrical conductivity, k stands for fluid's thermal conductivity, g represents gravitational acceleration and μ stands for fluid's dynamical viscosity.

The dynamical viscosity μ given by equation (4.5) is an inverse linear function of temperature, Lai *et al.* (1991).

$$\mu(T) = \frac{\mu_\infty}{1 + \gamma(T - T_\infty)} \quad (4.5)$$

In equation(4.5) μ_∞ refers to the viscosity of the cold fluid and γ represents a constant.

The radiative heat flux simplifies to equation (4.6) upon application of Rosseland approximation for radiation, Sparrow *et al.* (1978).

$$q_r = -\frac{4\sigma^*}{3k^*} \frac{\partial T^4}{\partial y} \quad (4.6)$$

In equation (4.6), σ^* refers to the Stephan-Boltzmann constant and k^* refers to the mass absorption coefficient. Writing T^4 using truncated Taylor series about T_∞ to be a linear function of T by letting temperature difference to be sufficiently small within the flow:

$$T^4 \approx 4T_\infty^3 T - 3T_\infty^4. \quad (4.7)$$

Introducing the following dimensionless quantities and the stream function Ψ

$$\eta = y \sqrt{\frac{U_\infty}{\nu x}}, \quad \Psi = \sqrt{\nu x U_\infty} f(\eta), \quad v = \frac{\mu_\infty}{\rho}, \quad \theta(\eta) = \frac{T - T_\infty}{T_f - T_\infty} \quad (4.8)$$

Thermal conductivity of the fluid is taken to be linear function of temperature and is written as:

$$k(T) = k_\infty (1 + \gamma(T - T_\infty)) \quad (4.9)$$

The equation of continuity (4.1) is satisfied by the stream function by defining $u = \frac{\partial \Psi}{\partial y}$

and $v = -\frac{\partial \Psi}{\partial x}$ as follows:

$$\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} = \frac{\partial^2 \Psi}{\partial x \partial y} - \frac{\partial^2 \Psi}{\partial x \partial y} = 0 \quad (4.10)$$

Equations (4.1) to (4.4) are solved together with their boundary conditions as follows:

$$u = \frac{\partial \Psi}{\partial y} = \frac{\partial \Psi}{\partial \eta} \cdot \frac{\partial \eta}{\partial y} = U_\infty f'(\eta) \quad (4.11a)$$

$$v = \frac{\mu_\infty}{\rho} \quad (4.11b)$$

$$\frac{\partial u}{\partial y} = \frac{\partial u}{\partial \eta} \cdot \frac{\partial \eta}{\partial y} = \left(\frac{U_\infty^3}{\nu x} \right)^{1/2} f''(\eta) \quad (4.11c)$$

$$v \frac{\partial u}{\partial y} = \frac{\mu_\infty}{\rho} \left(\frac{U_\infty^3}{vx} \right)^{1/2} f''(\eta) \quad (4.11d)$$

$$\frac{\partial}{\partial y} \left(\mu \frac{\partial u}{\partial y} \right) = \mu \left(\frac{U_\infty^3}{vx} \right)^{1/2} f'''(\eta) \quad (4.11e)$$

Substituting equations 4.11 (a-e) into equation (4.2) and introducing parameters Ha and Gr as defined then simplifying gives

$$\frac{d^3 f}{d\eta^3} + \frac{1}{2}(1 + a\theta)f \frac{d^2 f}{d\eta^2} - \frac{a}{(1+a\theta)} \frac{d\theta}{d\eta} \frac{d^2 f}{d\eta^2} + Gr(1 + a\theta)\theta - Ha(1 + a\theta) \left(\frac{df}{d\eta} - 1 \right) = 0 \quad (4.12)$$

For equation (4.3) we proceed as follows:

$$\theta(\eta) = \frac{T - T_\infty}{T_f - T_\infty} \text{ Making } T \text{ the subject gives}$$

$$T = \theta(\eta)(T_f - T_\infty) + T_\infty \quad (4.13a)$$

$$\frac{\partial T}{\partial x} = \frac{\partial T}{\partial \eta} \cdot \frac{\partial \eta}{\partial x} = -\frac{1}{2} y \left(\frac{U_\infty}{vx^3} \right)^{1/2} \theta'(\eta)(T_f - T_\infty) \quad (4.13b)$$

$$\frac{\partial T}{\partial y} = \frac{\partial T}{\partial \eta} \cdot \frac{\partial \eta}{\partial y} = \left(\frac{U_\infty}{vx} \right)^{1/2} \left(\theta'(\eta)(T_f - T_\infty) \right) \quad (4.13c)$$

$$u \frac{\partial T}{\partial x} = -\frac{1}{2} y \left(\frac{U_\infty^3}{vx^3} \right)^{1/2} f'(\eta) \theta'(\eta)(T_f - T_\infty) \quad (4.13d)$$

$$v \frac{\partial T}{\partial y} = \frac{\mu_\infty}{\rho} \left(\frac{U_\infty}{vx} \right)^{1/2} \theta'(\eta)(T_f - T_\infty) \quad (4.13e)$$

Substituting equations (4.13) (a-e) into (4.3) and introducing parameters Pr , Br and Ha and then simplifying equation (4.3) becomes:

$$\frac{d^2 \theta}{d\eta^2} + \frac{1}{2} Pr \beta f \frac{d\theta}{d\eta} + \frac{Br \beta}{(1+a\theta)} \left(\frac{d^2 f}{d\eta^2} \right)^2 + \beta Br Ha \left(\frac{df}{d\eta} - 1 \right)^2 = 0 \quad (4.14)$$

The boundary conditions (4.4) are transformed as follows:

$u = U_\infty f'(\eta)$ transforms $u(x, 0) = 0$ to $\frac{df}{d\eta}(0) = 0$ and $u(x, \infty) = U_\infty$ is transformed to

$$\frac{df}{d\eta}(\infty) = 1.$$

$v = -\frac{\partial\Psi}{\partial x} = \frac{1}{4}\chi^{-1}U_\infty(f(\eta))^2$ transforms $v(x, 0) = 0$ to $f(0) = 0$.

$T = T_\infty + \theta(\eta)(T_f - T_\infty)$ transforms $T(x, \infty) = T_\infty$ to $\theta(\infty) = 0$ and

$$-k\frac{\partial T}{\partial y}(x, 0) = h_f(T_f - T(x, 0)) \text{ is transformed to } \frac{d\theta}{d\eta}(0) = Bi[\theta(0) - 1]$$

Therefore for equations (4.12) and (4.14) the boundary conditions are:

$$f(0) = 0, \quad \frac{df}{d\eta}(0) = 0, \quad \frac{d\theta}{d\eta}(0) = Bi[\theta(0) - 1], \quad \frac{df}{d\eta}(\infty) = 1, \quad \theta(\infty) = 0 \quad (4.15)$$

For the absence of thermal radiation $\beta = 3Ra/(3Ra + 4)$ or $\beta = 1$. In equations (4.12)

and (4.14) the prime symbol denotes derivative with respect to η and $Ha = \frac{\sigma B_0^2 x}{\rho U_\infty}$ refers

to local magnetic field parameter, $Bi = \frac{h_f}{k} \sqrt{\frac{vx}{U_\infty}}$ is the local convective heat exchange

parameter, $Br = \frac{\mu_\infty U_\infty^2}{k(T_f - T_\infty)}$ denotes the Brinkmann number, $Pr = \frac{\nu}{\alpha}$ stands for the Prandtl

number, $a = \gamma(T_f - T_\infty)$ refers to the viscosity variation parameter, and $Ra = \frac{kk^*}{4\sigma^* T_\infty^3}$ is

the thermal radiation parameter.

4.2 Numerical Solution

Numerical solutions are obtained by solving equations (4.12) and (4.14) subject to the boundary conditions (4.15) using fourth order Runge-Kutta integration scheme together with a shooting technique. The computations are performed using MAPLE computer programme that uses symbolic and computational computer language MAPLE. The method entails transforming equations (4.12) and (4.14) that are of third order in f and second order in θ into system of first order differential equations. The system of first order ordinary differential equations are obtained by letting

$$f_1 = f, \quad f_2 = f', \quad f_3 = f'', \quad f_4 = \theta, \quad f_5 = \theta' \quad (4.16)$$

Where prime denotes derivative with respect to η .

The set of higher order non-linear boundary value problem with their respective boundary conditions are reduced to first order differential equations with appropriate initial conditions as shown below:

$$f_1' = f_2 \quad (4.17a)$$

$$f_2' = f_3 \quad (4.17b)$$

$$f_3' = -\frac{1}{2}(1 + af_4)f_1f_3 + \frac{a}{(1+af_4)}f_5f_3 - Gr(1 + af_4)f_4 + Ha(1 + af_4)(f_2 - 1) \quad (4.17c)$$

$$f_4' = f_5 \quad (4.17d)$$

$$f_5' = -\frac{1}{2}Pr\beta f_1f_5 - \frac{Br\beta}{(1+af_4)}f_3^2 - \beta BrHa(f_2 - 1)^2 \quad (4.17e)$$

Subject to the initial conditions

$$f_1(0) = 0, \quad f_2(0) = 0, \quad f_3(0) = B_i[f_4(0) - 1], \quad f_2(\infty) = 1, \quad f_4(\infty) = 0 \quad (4.18)$$

CHAPTER FIVE

5.0 Results and Discussion

Numerical computations were performed for values of the physical parameters involved namely; Thermal radiation parameter (Ra), Prandtl number (Pr), Magnetic field intensity parameter (Ha), Variable viscosity parameter (a), local Biot number (Bi), and Brinkmann number (Br). For illustration of the results, numerical values were plotted in figure 5.1 to 5.7 and a detailed description of the effects of the above parameters on velocity and temperature profile was done.

5.1 Effect of Parameter Variation on Velocity Profiles

In figure 5.1 velocity as a function of η is plotted for different values of magnetic field parameter (Ha). At the surface of the plate the fluid has got zero velocity since the fluid ‘sticks’ to the wall as a result of adhesive forces i.e. the ‘no slip condition’. Far away from the surface of the plate the velocity increases to U_∞ (free stream velocity) with increasing values of (Ha). This is true since when magnetic field is applied normal to the flow a resistive force develops in the fluid that opposes the flow of the fluid making the fluid’s velocity to overshoot towards the surface of the plate.

In figure 5.2 velocity as a function of η is plotted for different values of variable viscosity parameter (a). From the figure we observe that as (a) increases in value (i.e. as viscosity of fluid decreases) there is increase in fluid’s velocity. This is true since a decline in fluid’s viscosity results in a reduction in momentum boundary layer making the velocity gradient of the fluid to increase.

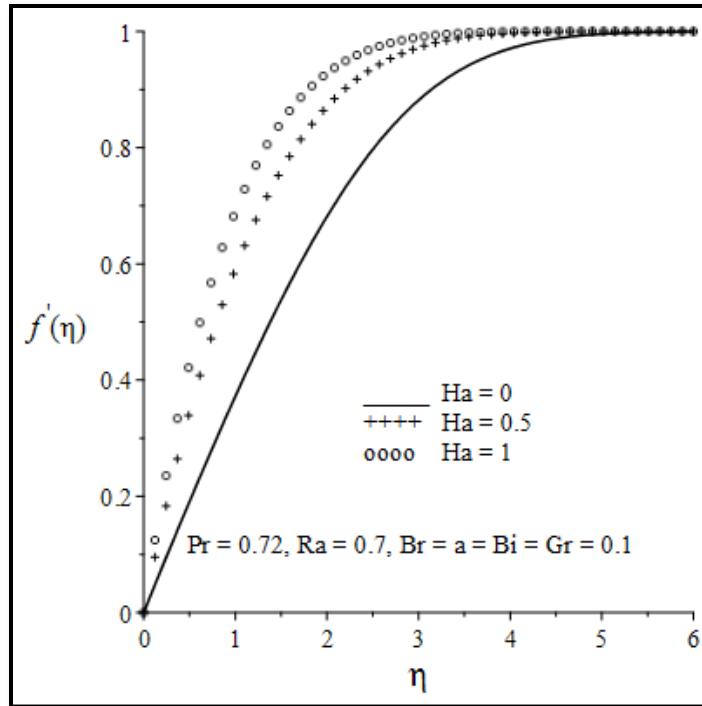


Figure 5.1: Velocity profile for varying Ha

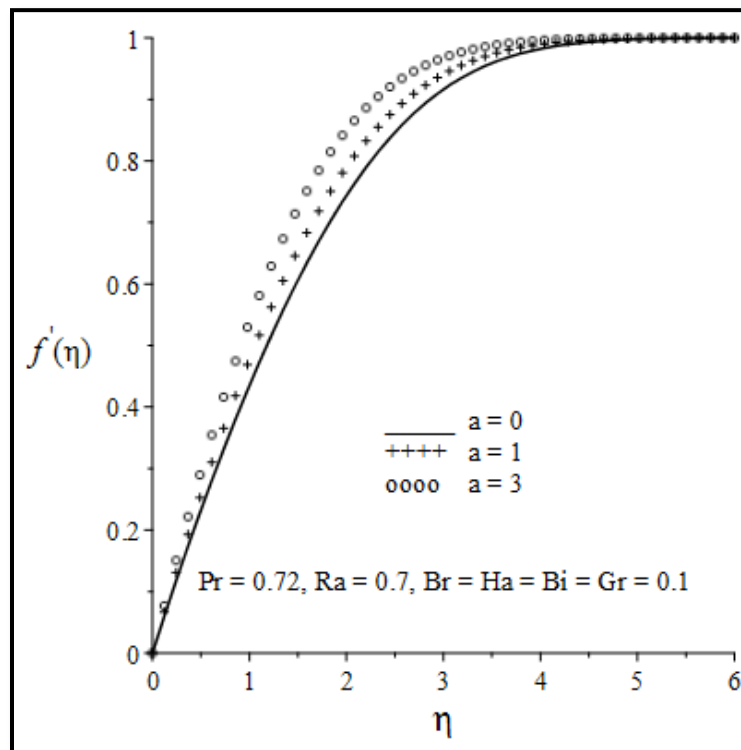


Figure 5.2: Velocity profile for varying a

5.2 Effect of Parameter Variation on Temperature Profiles

The figures 5.3-5.7 are graphs depicting variation of θ as a function of η for different values of Ha, Br, Bi, Ra and a . From the graphs we note that the highest fluid temperature occurs on the plate's surface. Far away from the surface, the fluid's temperature decreases exponentially to free stream zero value as indicated in the boundary conditions. In figure 5.3 we observe increase in fluid's temperature for increasing values of Ha this is because imposing magnetic field normal to the flow generates a resistive force (Lorentz force) in the fluid that opposes the flow of the fluid resulting in the increased friction between the layers of fluid which increases the fluid temperature and the thermal boundary layer. Similar trend repeats itself in figure 5.4 and 5.5 as the values of Br and Bi increases the temperature of the fluid increases. This is based on the fact that there is generation of energy due to viscous heating and Newtonian heating in the thermal boundary layer. In figure 5.6 and 5.7 the increase in values of Ra and a (i.e. decrease in fluid's viscosity) makes the temperature of the fluid to decline which results in the decline in the thickness of thermal boundary layer.

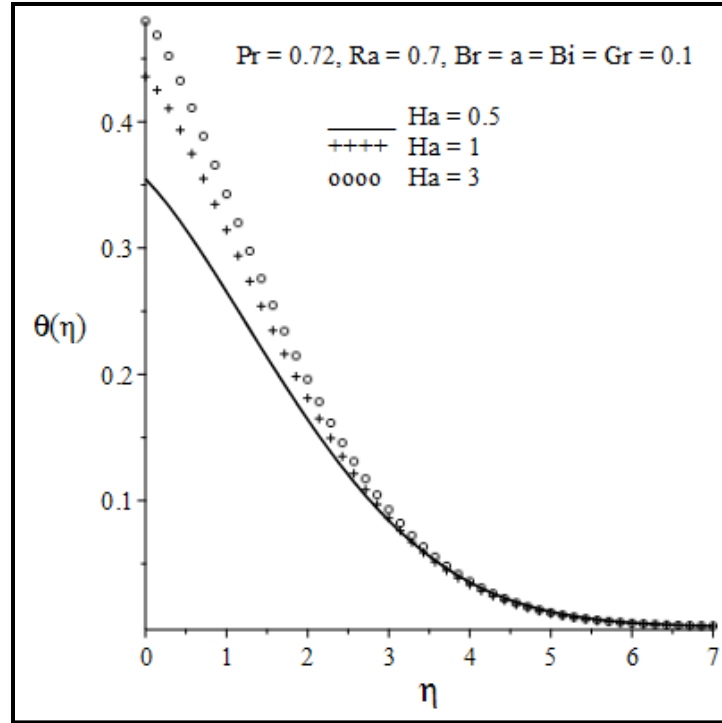


Figure 5.3: Temperature profile for varying Ha

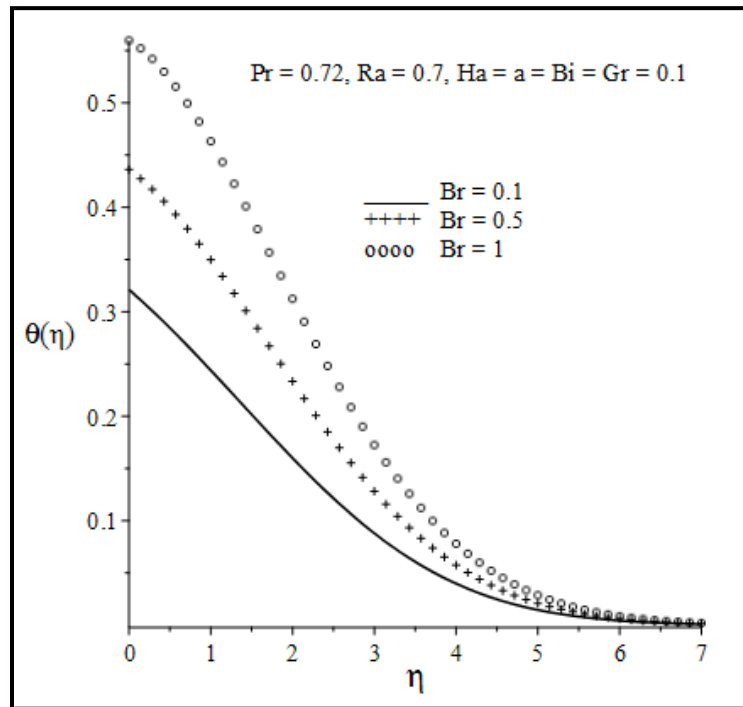


Figure 5.4: Temperature profile for varying Br

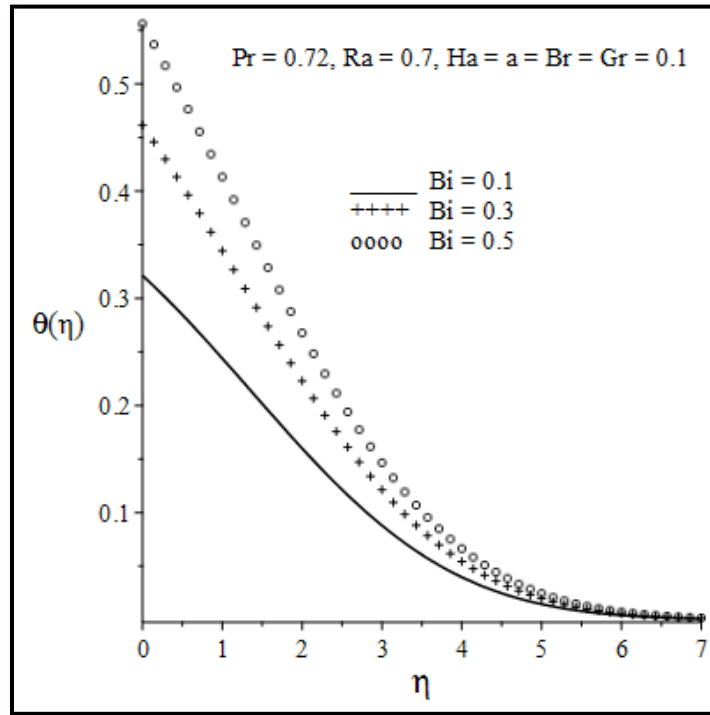


Figure 5.5: Temperature profile for varying Bi

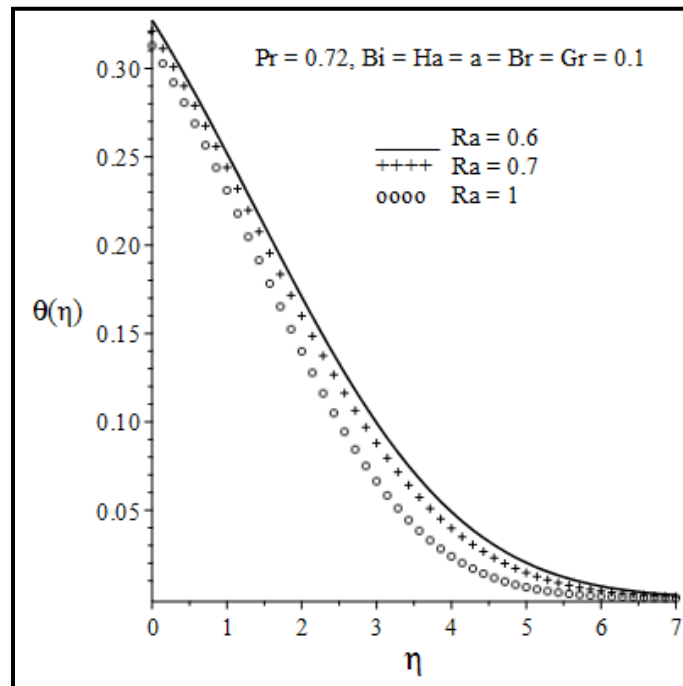


Figure 5.6: Temperature profile for varying Ra

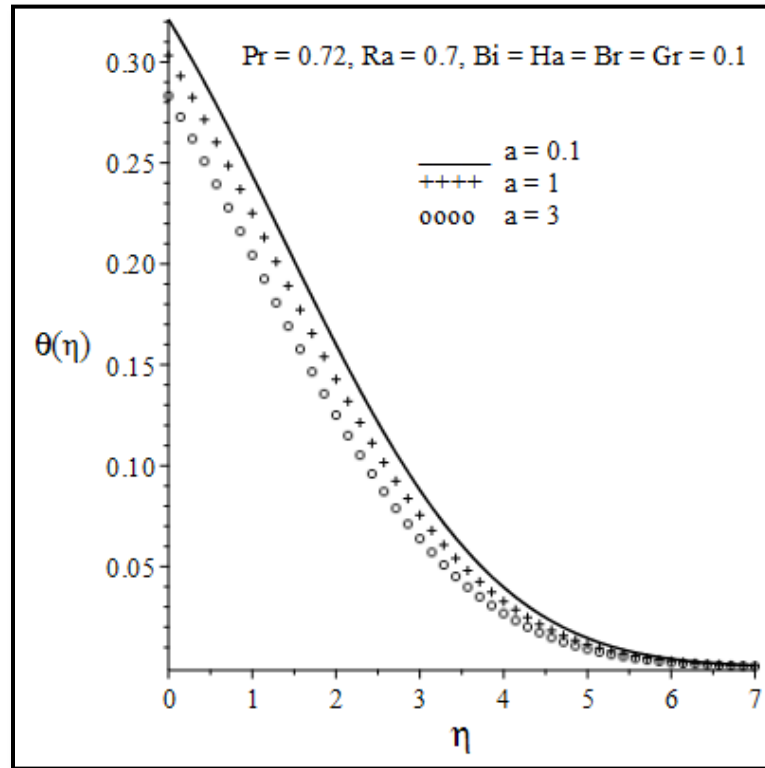


Figure 5.7: Temperature profile for varying a

CHAPTER SIX

6.0 Conclusion and Recommendations.

In this study we investigated steady 2-D MHD boundary Layer Flow involving heat transfer over a horizontal plate in a stream of conducting fluid with varying viscosity at temperature T_∞ subject to thermal radiation and Newtonian heating. The model equations governing the flow were formulated and numerically solved using shooting technique with a fourth order Runge-Kutta integration scheme. The effects of thermal radiation parameter, magnetic field intensity parameter, variable viscosity parameter, local Biot number and Brinkman number on velocity profile and temperature profile of the fluid were represented graphically. From the result we make the following conclusions:

- i. There is an increase in the velocity of the fluid as the values of magnetic parameter (Ha) and variable viscosity parameter (a) increases.
- ii. There is a reduction in thickness of the velocity boundary layer with Ha and a .
- iii. There is an increase in fluid temperature with increase in values of Ha , Br , Bi .
- iv. There is a decrease in the temperature of the fluid as values of Ra and a increases
- v. There is an increase in the thickness of the thermal boundary layer with Ha , Br , Bi and a decrease in the thickness with increasing values of Ra and a .

6.1 Suggestion for Further work

Further studies can be conducted on mathematical modelling of variable viscosity hydromagnetic boundary layer flow with thermal radiation and Newtonian heating over an inclined plate with an inclined magnetic field to the plate.

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