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ON THE SPECTRUM OF C_1 AS AN OPERATOR ON bv_0

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Abstract

In 1985 John Reade determined the spectrum of C_1 regarded as an operator on the space c_0 of all null sequences normed by $\|x\| = \sup_{n\geq 0}|x_n|$. It is the purpose of this paper to determine the spectrum of C_1 regarded as an operator on the space bv_0 of all sequences x such that $x_k \to 0$ as $k \to \infty$ and $\|x\| = \sum_{k=0}^{\infty}|x_{k+1} - x_k| < \infty$.

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NOTATION. s; c_0; l_1; bv_0; bs; T^*; $X^*B(X)$; A^t; $\sigma(T)$; $O(1)$; $o(1)$; $\asymp$; $\mathrm{Re}(z)$; will denote the set of all sequences; convergent to zero sequences, that is, null sequences; sequences such that $\sum_{k=0}^{\infty}|x_k| < \infty$; sequences such that $\sum_{k=0}^{\infty}|x_{k+1} - x_k| < \infty$; bounded series, that is, sequences x such that $\sup_{n\geq 0}|\sum_{k=0}^{n} x_k| < \infty$; the adjoint operator of T; the space of all continuous linear functionals on X, that is, the continuous dual of X; the linear space of all bounded linear operators, say, T on X into itself; the transposed matrix of A; the spectrum of T; capital order, that is, $x_n = O(1)$ if there exists $M \in \mathbf{R}^+$ such that $|x_n| \leq M$ for all n; small order, that is, $x_n = o(1)$ as $n \to \infty$, that is, $\lim_{n\to\infty} x_n = 0$; lies between two positive constant multiples, for example $a_n \asymp b_n$ means that there exist m, $M \in \mathbf{R}^+$ such that $mb_n \leq a_n \leq Mb_n$; the real part of the complex number z, respectively.

1. Introduction

In his 1985 paper Reade considers the operator which converts a sequence $(x_n)_0^\infty$ into its sequence of averages

$$\left(\frac{x_0 + x_1 + \cdots + x_n}{n+1}\right)_{n=0}^{\infty}.$$

He shows it is a bounded operator on c_0. We shall denote this operator by $C_1 = (C, 1)$ and call it the Cesàro operator. It can be represented by the matrix

$$\begin{pmatrix} 1 & 0 & 0 & 0 & \cdots \\ \frac{1}{2} & \frac{1}{2} & 0 & 0 & \cdots \\ \frac{1}{3} & \frac{1}{3} & \frac{1}{3} & 0 & \cdots \\ \cdots & \cdots & \cdots & & \end{pmatrix}.$$

The key to determining the spectrum $\sigma(C_1)$ of a bounded linear operator $C_1 \colon bv_0 \to bv_0$ on a Banach space bv_0 is the determination of all eigenvalues of $C_1^* \in B(bv_0^*)$, that is, the determination of all $\lambda \in \mathbb{C}$ such that $(C_1 - \lambda I)^{-1} \in B(bv_0)$.

1.1 THEOREM. *Let $T \in B(X)$, where X is any Banach space. Then the spectrum of T^* is identical with the spectrum of T. Furthermore, $R_\lambda(T^*) = (R_\lambda(T))^*$ for $\lambda \in \rho(T) = \rho(T^*)$, where $R_\lambda(T) = (T - \lambda I)^{-1}$ and $\rho(T) = \{\lambda \in \mathbb{C} \colon (T - \lambda I)^{-1}$ exists$\}$.*

PROOF. See [2, page 568] and [3, page 71].

1.2. LEMMA. *$C_1 \colon bv_0 \to bv_0$ and $C_1 \in B(bv_0)$ with $\|C_1\|_{bv_0} = 1$.*

PROOF. Since $C_1 \colon bv_0 \to bv_0$, write $y_n = C_1 x$ and define $x_n = a_0 + a_1 + \cdots + a_n$. Then $y_n \to 0$ as $n \to \infty$ and

$$\begin{aligned} \sum_{n=0}^{\infty} |y_n - y_{n+1}| &= \sum_{n=0}^{\infty} \frac{1}{(n+1)(n+2)} \left| \sum_{\nu=1}^{n+1} \nu a_\nu \right| \\ &\le \sum_{n=0}^{\infty} \frac{1}{(n+1)(n+2)} \sum_{\nu=1}^{n} \nu |a_\nu| \\ &= \sum_{\nu=1}^{\infty} \nu |a_\nu| \sum_{n=\nu-1}^{\infty} \frac{1}{(n+1)(n+2)} \\ &\le \sum_{\nu=1}^{\infty} |a_\nu| = \sum_{\nu=0}^{\infty} |x_\nu - x_{\nu-1}|. \end{aligned}$$

Direct manipulation gives

$$\|C_1\|_{(bv_0,bv_0)} = \sup\{1,0,0,\dots\} = 1.$$

Clearly $\lim_{n\to\infty} \frac{1}{1+n} = 0$ and hence $C_1 \in B(bv_0)$.

1.3. LEMMA. *Each bounded linear operator $T\colon X \to Y$, where $X = c_0, l_p$ and $Y = c_0, l_p$ $(1 \le p < \infty)$, l_∞ (where l_p denotes sequences x such that $\sum_{k=0}^{\infty} |x_k|^p < \infty$ and l_∞ denotes bounded sequences) determines and is determined by an infinite matrix of complex numbers.*

PROOF. See Taylor [13, pages 221–223].

1.4. LEMMA. *Let $C_1\colon bv_0 \to bv_0$. Then $C_1^*\colon bv_0^* \to bv_0^*$ is given by C_1^t and $C_1^t \in B(bs)$.*

PROOF. Since bv_0 has AK and bv_0^* is isomorphic to bs under the map $h\colon bv_0^* \to bs$, $h(f) = (t_0, t_1, t_2, \dots)$, where $t_n = f(\delta^n)$, $n \ge 0$, $f \in bv_0^*$, we have (see Lemma 1.3)

$$C_1^* = C_1^t = \begin{pmatrix} 1 & \frac{1}{2} & \frac{1}{3} & \frac{1}{4} & \cdots \\ 0 & \frac{1}{2} & \frac{1}{3} & \frac{1}{4} & \cdots \\ 0 & 0 & \frac{1}{3} & \frac{1}{4} & \cdots \\ \cdots & \cdots & \cdots & & \end{pmatrix}.$$

But for any operator T on a normed linear space X, $\|T\|_X = \|T^*\|_{X^*}$ (see [2, page 478], [3, page 54] and [7, page 232]), so

$$\|C_1\|_{bv_0} = \|C_1^*\|_{bv_0^*} = \|C_1^t\|_{bs} = 1.$$

Thus $C_1^* \in B(bv_0^*)$, that is, $C_1^t \in B(bs)$ since it is also clear that each column of C_1^t is null (C_1 being a normal matrix).

1.5. COROLLARY. *$C_1 \in B(bv_0)$ has not eigenvalues.*

PROOF. The proof follows from the fact that $C_1 \in B(c_0)$ has no eigenvalues (see [10]) since $bv_0 \subset c_0$.

1.6. LEMMA. *Let*

$$Z_n = \prod_{\nu=0}^{n} \left(1 - \frac{1}{\lambda(\nu+1)}\right), \qquad \lambda \ne 0, \lambda \in \mathbb{C}.$$

Then the partial sums of $\sum_{n=1}^{\infty} Z_n$ are bounded if and only if $\operatorname{Re}(1/\lambda) \ge 1$.

PROOF. When $\lambda = 1$, $Z_n = 0$ for all n and so the partial sums of $\sum_{n=0}^{\infty} Z_n$ are certainly bounded.

Let C be a constant depending only on λ which may be different at each occurrence and A a non-zero constant. We have that

$$\log_e(1-u) = -\sum_{n=1}^{\infty} \frac{1}{n} u^n = -u + O(u^2) \tag{1.1}$$

uniformly in $|u| \le \frac{1}{2}$, $u \in \mathbb{C}$. Now given $\lambda \ne 0$ there is ν_0 such that $|\lambda|(\nu+1) > 2$ for $\nu \ge \nu_0$,

$$\begin{aligned} \log_e Z_n &= \sum_{\nu=0}^{n} \log\left(1 - \frac{1}{\lambda(\nu+1)}\right) \\ &= C - \frac{1}{\lambda} \sum_{\nu=\nu_0} \frac{1}{1+\nu} + \sum_{\nu=\nu_0}^{n} t_\nu \end{aligned} \tag{1.2}$$

where $t_\nu = O(1/\nu^2)$, and

$$\sum_{\nu=\nu_0}^{n} t_\nu = \sum_{\nu=\nu_0}^{\infty} t_\nu - \sum_{\nu=n+1}^{\infty} t_\nu = C + O\left(\frac{1}{n}\right) \tag{1.3}$$

Also

$$\sum_{\nu=\nu_0}^{n} \frac{1}{\nu+1} = C + \log n + O\left(\frac{1}{n}\right) \tag{1.4}$$

since if $C = \sum_{\nu=0}^{n} \frac{1}{\nu+1} - \log n$, then

$$C_{n+1} - C_n = \frac{1}{2+n} - \log\left(\frac{n+1}{n}\right) = O\left(\frac{1}{n^2}\right)$$

Therefore

$$\begin{aligned} C_{n+1} &= C + \sum_{\nu=0}^{n} (C_{\nu+1} - C_\nu) \\ &= C + \sum_{\nu=0}^{\infty} (C_{\nu+1} - C_\nu) - \sum_{\nu=n+1}^{\infty} (C_{\nu+1} - C_\nu), \qquad C_0 = 0, \end{aligned}$$

that is,

$$C_{n+1} = C - \sum_{\nu=n+1}^{\infty} (C_{\nu+1} - C_\nu) = C + O\left(\frac{1}{n}\right) \tag{1.5}$$

Hence as $n \to \infty$, $\log Z_n = C - 1/\lambda \log n + O(\frac{1}{n})$ so $Z_n = An^{-1/\lambda}(1 + O(\frac{1}{n})) = An^{-1/\lambda} + O(n^{-\operatorname{Re}(1/\lambda)-1})$. If $\operatorname{Re}(1/\lambda) \ge 1$, $\lambda = 1$, the partial sums of $\sum_{n=1}^{\infty} n^{-1/\lambda}$ are bounded and $\sum_{n=1}^{\infty} n^{-\operatorname{Re}(1/\lambda)-1} < \infty$, so the partial sums of $\sum_{n=1}^{\infty} Z_n$ are bounded. If $0 < \operatorname{Re}(1/\lambda) < 1$ or $\lambda = 1$ then the partial sums of $\sum_{n=1}^{\infty} n^{-1/\lambda}$ are unbounded, but we still have $\sum_{n=1}^{\infty} n^{-\operatorname{Re}(1/\lambda)-1} < \infty$. If $\operatorname{Re}(1/\lambda) \leq 0$ then

$$\sum_{n=1}^{N} n^{-1/\lambda} \asymp N^{1-1/\lambda} \Big/ \left(1 - \frac{1}{\lambda}\right) \tag{1.6}$$

Now

$$\sum_{n=1}^{N} n^{-\operatorname{Re}(1/\lambda)-1} = \begin{cases} O(N^{-\operatorname{Re}(1/\lambda)}), & \text{if } \operatorname{Re}(1/\lambda) < 0, \\ O(\log N), & \text{if } \operatorname{Re}(1/\lambda) = 0. \end{cases}$$

Using (1.6) we see that the partial sums of $\sum_{n=1}^{\infty} n^{-1/\lambda}$ are unbounded although $\sum_{n=1}^{\infty} n^{-\operatorname{Re}(1/\lambda)-1} < \infty$, and hence we conclude that the partial sums of $\sum_{n=1}^{\infty} Z_n$ are bounded if and only if $\operatorname{Re}(1/\lambda) \geq 1$.

2. Determination of the spectrum of C_1 on bv_0

2.1. THEOREM. *The eigenvalues of* $C_1^* \in B(bv_0^*)$, *that is,* $C_1^t \in B(bs)$, *are all* $\lambda \in \mathbb{C}$ *satisfying* $|\lambda - \frac{1}{2}| \leq \frac{1}{2}$.

PROOF. Suppose $C_1^t x = \lambda x$, $x \in bs$, $x \neq \theta$ where θ is the zero sequence. Then as in Lemma 1.4,

$$C_1^* = C_1^t = \begin{pmatrix} 1 & \frac{1}{2} & \frac{1}{3} & \frac{1}{4} & \cdots & \\ 0 & \frac{1}{2} & \frac{1}{3} & \frac{1}{4} & \cdots & \\ 0 & 0 & \frac{1}{3} & \frac{1}{4} & \cdots & \\ & \cdots & \cdots & \cdots & & \end{pmatrix}$$

and solving the system of equations

$$\begin{aligned} x_0 + \tfrac{1}{2}x_1 + \tfrac{1}{3}x_2 + \cdots &= x_0 \\ \tfrac{1}{2}x_1 + \tfrac{1}{3}x_2 + \cdots &= x_1 \\ \tfrac{1}{3}x_2 + \cdots &= x_2 \\ &\cdots \\ \frac{1}{n}x_{n-1} + \frac{1}{n+1}x_n + \cdots &= x_{n-1} \\ \frac{1}{n+1}x_n + \cdots &= x_n \\ &\cdots \end{aligned}$$

we obtain

$$x_1 = \left(1 - \frac{1}{\lambda}\right) x_0$$

$$x_2 = \left(1 - \frac{1}{\lambda}\right)\left(1 - \frac{1}{2\lambda}\right) x_0$$

$$x_3 = \left(1 - \frac{1}{\lambda}\right)\left(1 - \frac{1}{2\lambda}\right)\left(1 - \frac{1}{3}\lambda\right) x_0. \tag{2.2}$$

$$\dots$$

$$x_N = \prod_{n=1}^{N}\left(1 - \frac{1}{n\lambda}\right) x_0.$$

By Lemma 1.6, $(x_N)_1^\infty \in bs$ if and only if $\mathrm{Re}(1/\lambda) \geq 1$, that is, $|\lambda - \frac{1}{2}| \leq \frac{1}{2}$. Hence the result follows.

2.2. Theorem. *Let $C_1 : bv_0 \to bv_0$. Then the spectrum of C_1 is*

$$\sigma(C_1) = \{\lambda \in \mathbb{C} : |\lambda - \tfrac{1}{2}| \leq \tfrac{1}{2}\}.$$

2.3. Definition (Weighted mean method). The weighted mean method is a matrix $A = (a_{nk})$ with

$$a_{nk} = p_k | P_n, \quad P_n = \sum_{k=0}^{n} p_k \neq 0.$$

2.4. Lemma. *If $(M, p) = (N, p)$ is a regular (conservative) weighted mean method then $(M, p) = (N, p)$ is absolutely regular (conservative).*

(See [1], [14] for further details.)

Proof. Since (N, p) is a regular (conservative) mean method we have by the Kojima-Schur conditions

$$|P_n| \to \infty \quad \text{as } n \to \infty, \tag{2.3}$$

where $P_n = \sum_{\nu=0}^{n} p_\nu$ and

$$P_n^* = \sum_{\nu=0}^{n} |p_\nu| = O(P_n). \tag{2.4}$$

We need to prove that (N, p) is absolutely regular (conservative), that is, that

$$P_{k-1} \sum_{n=k}^{\infty} \left| \frac{1}{P_n} - \frac{1}{P_{n-1}} \right| \leq M. \tag{2.5}$$

Let $P_n^* = \sum_{\nu=0}^{n} |p_\nu|$. Then (2.4) becomes $P_n^* \le K|P_n|$ for all $n \ge 1$ (K some constant). Thus

$$|P_{k-1}| \sum_{n=k}^{\infty} \frac{|p_n|}{|P_n|\,|P_{n-1}|} \le |P_{k-1}| \sum_{n=k}^{\infty} \frac{|P_n| K^2}{P_n^* \cdot P_{n-1}^*}$$
$$\le K^2 |P_{k-1}| \sum_{n=k}^{\infty} \left(\frac{1}{P_{n-1}^*} - \frac{1}{P_n^*} \right).$$

Since $|P_n| \to \infty$ as $n \to \infty$ by (2.3), we have that $P_n^* \to \infty$ (since $P_n^* \ge |P_n|$), therefore $\sum_{n=k}^{\infty}(1/P_{n-1}^* - 1/P_n^*) = 1/P_{k-1}^*$ and so (2.5) follows, provided that $|P_{k-1}|/P_{k-1}^* \le M$ for some M. But $M = 1$ will do and the result follows.

We now prove Theorem 2.2.

PROOF. By virtue of Theorem 2.1 and the fact that $\sigma(C_1) = \sigma(C_1^*)$ (see Theorem 1.1), it is enough to prove that $B = (C_1 - \lambda I)^{-1} \in B(bv_0)$ for all $|\lambda - \frac{1}{2}| > \frac{1}{2}$, that is, that Q is absolutely regular where $B = -I/\lambda - Q/\lambda(\lambda - 1)$ except when λ is the reciprocal of a positive integer, $B = (C - \lambda I)^{-1} = I/\lambda - Q/\lambda(\lambda - 1)$, where $Q = (q_{nk})$, $q_{nk} = A_{k-1}^{-1/\lambda}/A_{n-1}^{1-1/\lambda}$,

$$A_n^\alpha = \binom{n+\alpha}{n} = \frac{(\alpha+n)\cdots(\alpha+1)}{n!}$$

is a Hausdorff matrix (μ, μ_n),

$$\mu_n = \frac{1}{\lambda}\left(-1 - \frac{\frac{1}{\lambda}}{(n+1) - \frac{1}{\lambda}} \right).$$

It is also clear that Q is the Hausdorff matrix $(\mu, (1 - \frac{1}{\lambda})/((n+1) - \frac{1}{\lambda}))$. The proof of this is trivial (see Rhoades [11]).

Now Q is a regular Hausdorff transformation when $\mathrm{Re}(1/\lambda) < 1$. To see this we simply check the regularity conditions, namely:

(i) $\lim_{n\to\infty} q_{nk} = \lim_{n\to\infty} A_{n-1}^{-1/\lambda}/A_{n-1}^{1-1/\lambda} = 0$ since

$$|q_{nk}| = |A_{k-1}^{-1/\lambda}/A_{n-1}^{1-1/\lambda}| = |A_{k-1}^{-1/\lambda}| \cdot O(n^{\alpha-1})$$

and $\alpha = \mathrm{Re}(1/\lambda) < 1$, whence $q_{nk} \to 0$ as $n \to \infty$;

(ii) $\sum_{k=1}^{n} A_{k-1}^{-1/\lambda} = A_{n-1}^{1-1/\lambda}$, and therefore $\lim_{n\to\infty} \sum_{k=1}^{n} q_{nk} = 1$;

(iii) $\sum_{k=1}^{n} A_{k-1}^{-1/\lambda} = \sum_{k=1}^{n} O(k^{-\alpha}) = O(n^{1-\alpha}) \asymp O(|A_{n-1}^{1-1/\lambda}|)$ and therefore $\sum_{k=1}^{n} |A_{k-1}^{-1/\lambda}| = O(|A_{n-1}^{1-1/\lambda}|)$.

It is clear that $Q = (q_{nk})$ is a weighted mean method (matrix) $(N, A_{k-1}^{-1/\lambda})$ with $\sum_{k=1}^{n} A_{k-1}^{-1/\lambda} = A_{n-1}^{1-1/\lambda}$. Since Q is a weighted mean method and a regular Hausdorff method, theorem 2.2 follows.

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